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Research article

Extension operators of circular intuitionistic fuzzy sets with triangular norms and conorms: Exploring a domain radius

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Abstract: The circular intuitionistic fuzzy set (*CIFS*) extends the concept of *IFS*, representing each set element with a circular area on the *IFS* interpretation triangle (*IFIT*). Each element in *CIFS* is characterized not only by membership and non-membership degrees but also by a radius, indicating the imprecise areas of these degrees. While some basic operations have been defined for *CIFS*, not all have been thoroughly explored and generalized. The radius domain has been extended from $[0, 1]$ to $[0, \sqrt{2}]$. However, the operations on the radius domain are limited to *min* and *max*. We aimed to address these limitations and further explore the theory of *CIFS*, focusing on operations for membership and non-membership degrees as well as radius domains. First, we proposed new radius operations on *CIFS* with a domain $[0, \psi]$, where $\psi \in [1, \sqrt{2}]$, called a radius algebraic product (*RAP*) and radius algebraic sum (*RAS*). Second, we developed basic operators for generalized union and intersection operations on *CIFS* based on triangular norms and conorms, investigating their algebraic properties. Finally, we explored negation and modal operators based on proposed radius conditions and examined their characteristics. This research contributes to a more explicit understanding of the properties and capabilities of *CIFS*, providing valuable insights into its potential applications, particularly in decision-making theory.

Keywords: circular intuitionistic fuzzy sets; radius operations; negation operators; modal operators
Mathematics Subject Classification: 03E72,47S40

1. Introduction

Fuzzy set (*FS*) [1] is designed to handle the problem of uncertainty. It assigns a value from 0 to 1 to an object, where higher value indicates a higher degree of membership, and vice versa. In some cases, representation requires not only the membership degree ($\mathcal{M}$) but also the non-membership degree ($\mathcal{N}$), with the relationship between the two being $\mathcal{M} + \mathcal{N} = 1$. The development of fuzzy sets led to the concept of intuitionistic fuzzy sets (*IFS*) [2], introduced in 1983 as an extension of *FS*. Since then, *IFS* has been extensively studied and modified. In general, *IFS* characterizes each element by the degrees of membership and non-membership, with $\mathcal{M} + \mathcal{N} \leq 1$. Research and development on *IFS* can be classified into theoretical and applied development. Theoretical development includes algebraic aspects such as subring fuzzy [4], and further advancements in homomorphism intuitionistic fuzzy subrings [5, 6]. Additionally, *IFS* properties have been expanded, including the development of operators based on *t*-norm and conorm and algebraic laws [7, 8], as well as extensions into complex sets [9], distance metrics, similarity, and distance measures [10, 11, 14], among others. On the other hand, applicative development involves solving decision-making problems using *IFS* [12, 13, 15].

In 1989, the representation of *IFS* membership and non-membership, initially crisp values ($\mathcal{M}, \mathcal{N}$), was extended to interval values, giving rise to what is known as interval-valued IFS (*IVIFS*) [16]. This set is characterized by the values of $\mathcal{M}$ and $\mathcal{N}$ being in interval form, with an element in *IVIFS* represented as an ordered pair of membership and non-membership intervals. Research on *IVIFS* has delved into various aspects, including basic operations, modal operators, and algebraic laws [17, 18], determination of cosine similarity measure based on weighted reduced *IFS* [19], accuracy and score functions [20], and *IVIF*-confidence intervals [21]. In application side, there has also been research in decision-making, such as [22], and its integration with the *DEMATEL* Method combined with Choquet integral [23].

Atanassov introduced another extension of *IFS*, distinct from *IVIFS*, known as circular *IFS* (*CIFS*). This set expands upon *IFS* by considering ($\mathcal{M}, \mathcal{N}$) as the center and incorporating the radius as a measure of imprecision. For ($\mathcal{M}, \mathcal{N}$) within *IFS* interpretation triangle (*IFIT*), the difference between *CIFS* and *IVIFS* lies in the form of interpretation, where *IVIFS* has a rectangle interpretation, and *CIFS* has a circle interpretation. The *CIFS* theory is at an early stage of development. Several studies have begun to expand on *CIFS*; however, most of the research focuses on applications that have previously been carried out on *IFS* or *IVIFS*, and not much theoretical research has been done on it. For example, case studies of multi-criteria decision-making *(MCDM)* [26, 28], comparing *IVIFS* and *CIFS* for present worth analysis [32], upgrading the *TOPSIS* method [34], demonstrating *CIF-TOPSIS* with vague membership functions [27], extending the *VIKOR* method, design *CIF-ELECTRE III* for group decision analysis [40], develop *CIF-TODIM* method [43], *CIF-EDAS* method [44], *CIF-PROMETHEE* method [45], upgrade *CIF-AHP* method [46] and integrating with others [36, 37]. In terms of theoretical research, some studies have been conducted, such as some distance measures in *CIFS* [25, 33, 39], similarity and entropy measures [41], divergence measures for *CIFS* [35], circular *q*-rung orthopair fuzzy set theory [3], circular phytagorean fuzzy set [42] and generalized *CIFS* [38].

Several studies on *CIFS*, particularly theoretical ones, have often overlooked the novel characteristics of the radius domain. Aspects such as the range $r \in [0, \sqrt{2}]$, the intrinsic correlation where smaller radius values enhance the clarity of the *CIFS* information, and the constraints posed by previously defined operators have not been adequately emphasized. Furthermore, certain limitations may have served as motivation for the development of this paper. First, the radius domain, which is in the interval $[0, \sqrt{2}]$, makes the operator on the radius not belong to the t-norm or conorm category. In [24] and [25], the operations on radius are limited to *min* and *max*. Operators used in fuzzy set structures and their extensions are categorized as t-norm and conorm. It is necessary to have a radius operator that has equivalent properties to the t-norm and conorm operators. The radius operator should also be an interval extension of t-norm and conorm operators. Second, Atanassov [24] proposed basic operations such as union, intersection, algebraic sum, algebraic product, and arithmetic mean in *CIFS*. Generalization is possible by classifying basic operators into t-norm and conorm. There is a need for the definition of generalized operators from the previously proposed operators. Third, the proposed unary operators such as negation and modal operators in *CIFS* introduced by Atanassov [24] do not significantly impact radius. The *CIFS* negation operator produces comparable results to the *IFS* negation operator, indicating no significant difference between *CIFS* and *IFS*. The objectives of this paper are described based on the three problems and constraints:

(1) To develop new radius operations for *CIFS* with domain $[0, \psi]$, where $\psi \in [1, \sqrt{2}]$, and justify their properties as well as some special domains.

(2) To propose generalized union and intersection operators in *CIFS*, base them on t-norm and t-conorm categories and subsequently verify their properties.

(3) To identify and propose negation and modal operators based on the radius interval condition, and examine their relationship with the existing operators.

To accomplish these objectives, we begin with establishing fundamental definitions such as *IFS*, *IVIFS*, and *CIFS*, along with their basic relations and operations in Section 2. Section 3, the generalized intersection and union are introduced based on t-norm (conorm) with conditions of membership, non-membership, and radius in the interval $[0, 1]$. Several algebraic properties have been demonstrated, such as commutative, associative, and De Morgan's laws. Furthermore, the distributive property is proved for special cases namely algebraic sum-product, and arithmetic mean types by modifying the operators on the radius. Section 4 defines the generalized radius operations on the interval $[0, \psi]$, where $\psi \in [1, \sqrt{2}]$, and provides a proof of its algebraic properties. After defining the radius operation, Section 5 introduces another form of the negation operator in *CIFS*, accompanied by a proof of the De Morgan's law. Finally, the integration of the negation operator with the modal "necessity" and "possibility" operators is discussed, along with an examination of their advanced properties. Conclusions and discussion of further research are given in Section 6.

2. Preliminaries

In this section, the basic definitions of *IFS*, *IVIFS*, and *CIFS* are provided. Let X be a finite set, any $x \in X$, $\mathcal{M}(x)$ is defined as membership degree and $\mathcal{N}(x)$ is defined as non-membership degree of x.

2.1. Intuitionistic fuzzy sets

Definition 2.1. [2] An Intuitionistic fuzzy set *(IFS)* $\mathcal{A}$ in X is defined as an object of the form:

$$\mathcal{A} = \{\langle x, \mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x)\rangle \mid x \in X\},$$

where $\mathcal{M}_{\mathcal{A}} : X \to [0,1]$ and $\mathcal{N}_{\mathcal{A}} : X \to [0,1]$ and satisfies $0 \le \mathcal{M}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) \le 1$ for every $x \in X$.

Note that, in this case $\mathcal{M}_{\mathcal{A}}(x)$ and $\mathcal{N}_{\mathcal{A}}(x)$ are given as a single values in $[0,1]$. Further extension of this set is when the values of membership and non-membership functions are presented in unit interval $[0,1]$. For any $x \in X$, $\mathcal{M}^*(x)$ is interval of membership degree and $\mathcal{N}^*(x)$ is interval of non-membership degree for x. Let $Int([0,1])$ are represented as interval in $[0,1]$.

2.2. Interval-valued intuitionistic fuzzy sets

Definition 2.2. [16] An interval-valued *IFS (IVIFS)* $\mathcal{A}^*$ in X is defined as an object of the form:

$$\mathcal{A}^* = \{\langle x, \mathcal{M}^*_{\mathcal{A}}(x), \mathcal{N}^*_{\mathcal{A}}(x)\rangle \mid x \in X\},$$

where $\mathcal{M}^*_{\mathcal{A}} : X \to Int([0,1])$ and $\mathcal{N}^*_{\mathcal{A}} : X \to Int([0,1])$ are defined by $\mathcal{M}^*_{\mathcal{A}}(x) = \left[{\mathcal{M}^*_{\mathcal{A}}}^L(x), {\mathcal{M}^*_{\mathcal{A}}}^U(x)\right]$, $\mathcal{N}^*_{\mathcal{A}}(x) = \left[{\mathcal{N}^*_{A}}^L(x), {\mathcal{N}^*_{\mathcal{A}}}^U(x)\right]$, such that $0 \le {\mathcal{M}^*_{\mathcal{A}}}^U(x) + {\mathcal{N}^*_{A}}^U(x) \le 1$ for every $x \in X$.

It can be observed that under the *IFIT* [16], the *IFS* forms a single point of intersection between $\mathcal{M}_{\mathcal{A}}(x)$ and $\mathcal{N}_{\mathcal{A}}(x)$ for every $x \in X$, while for the *IVIFS*, its membership and non-membership functions form a rectangular area. Atanassov [24] then introduced another extension of *IFS*, instead of a rectangular area as in *IVIFS*, a circular area is proposed, called *CIFS* (*CIFS*). Under this new set, the point of intersection between $\mathcal{M}_{\mathcal{A}}(x)$ and $\mathcal{N}_{\mathcal{A}}(x)$ in *IFS* can be represented by a circular area with radius r and the center as the intersection point.

2.3. Circular intuitionistic fuzzy sets

Definition 2.3. [24, 25] A circular *IFS (CIFS)* $\mathcal{A}_r$ in X is defined as $\mathcal{A}_r = \{\langle x, \mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x); r\rangle \mid x \in X\}$, where $\mathcal{M}_{\mathcal{A}} : X \to [0,1]$ and $\mathcal{N}_{\mathcal{A}} : X \to [0,1]$ satisfy $0 \le \mathcal{M}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) \le 1$ and $r \in [0, \sqrt{2}]$ is the radius of the circle around each element $x \in X$.

Moreover, the function $\mathcal{H}_{\mathcal{A}_r}$ where $\mathcal{H}_{\mathcal{A}_r}(x) = 1 - \mathcal{M}_{\mathcal{A}_r}(x) - \mathcal{N}_{\mathcal{A}_r}(x) \in [0,1]$ corresponds to the degree of indeterminacy (uncertainty). It is clear that if $r = 0$, then $\mathcal{A}_0$ is an *IFS* (i.e., a single point), but for $r > 0$, it cannot be represented by an *IFS*. Let $L^* = \{(p,q) | p, q \in [0,1] \text{ and } p + q \le 1\}$, then $\mathcal{A}_r$ can be written in the form $\mathcal{A}_r = \{\langle x, O_r(\mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x))\rangle | x \in X\}$ where,

$$O_r(\mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x)) = \{\langle p, q\rangle | p, q \in [0,1] \text{ and } \sqrt{(\mathcal{M}_{\mathcal{A}}(x) - p)^2 + (\mathcal{N}_{\mathcal{A}}(x) - q)^2} \le r\} \cap L^*.$$

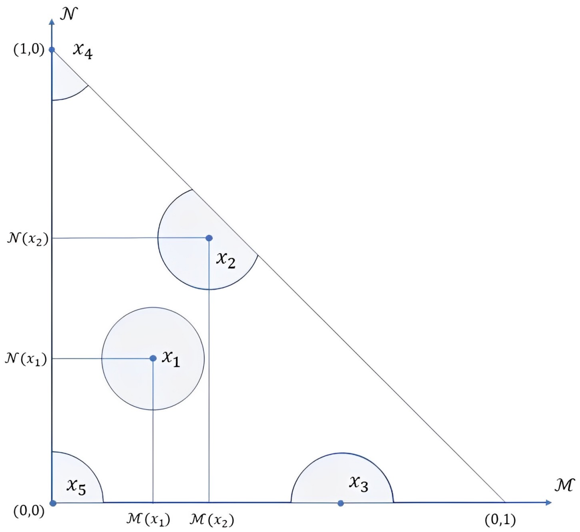

Figure 1. Intepretating circular intuitionistic fuzzy sets.

Previously, Atanassov identified five possible forms of the circle (see Figure 1), two of which have a center inside *IFIT* and the other three have a center on *IFIT* (x_3 is on the coordinate-X or Y, x_4 is on the maximum limit of the coordinate-X or Y, and x_5 is on $(0,0)$). In the following, some related properties of *CIFS* are provided, such as the relations, operations, and modal operators.

Definition 2.4. [24] Let $\mathcal{A}_r$ and $\mathcal{B}_s$ be *CIFSs*, for each $x \in X$, the relations between $\mathcal{A}_r$ and $\mathcal{B}_s$ are as follows:

- $\mathcal{A}_r \subset_\rho \mathcal{B}_s$ iff $(r < s)$ and $(\mathcal{M}_\mathcal{A}(x) = \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) = \mathcal{N}_\mathcal{B}(x))$.
- $\mathcal{A}_r \subset_\nu \mathcal{B}_s$ iff $(r = s)$ and one of the following conditions is fulfilled,
  - $\mathcal{M}_\mathcal{A}(x) < \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) \geq \mathcal{N}_\mathcal{B}(x)$,
  - $\mathcal{M}_\mathcal{A}(x) \leq \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) > \mathcal{N}_\mathcal{B}(x)$,
  - $\mathcal{M}_\mathcal{A}(x) < \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) > \mathcal{N}_\mathcal{B}(x)$.
- $\mathcal{A}_r \subset \mathcal{B}_s$ iff $(r < s)$ and one of the following conditions is fulfilled,
  - $\mathcal{M}_\mathcal{A}(x) < \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) \geq \mathcal{N}_\mathcal{B}(x)$,
  - $\mathcal{M}_\mathcal{A}(x) \leq \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) > \mathcal{N}_\mathcal{B}(x)$,
  - $\mathcal{M}_\mathcal{A}(x) < \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) > \mathcal{N}_\mathcal{B}(x)$.
- $\mathcal{A}_r \subseteq_\rho \mathcal{B}_s$ iff $(r < s)$ and satisfied $(\mathcal{M}_\mathcal{A}(x) = \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) = \mathcal{N}_\mathcal{B}(x))$.
- $\mathcal{A}_r \subseteq_\nu \mathcal{B}_s$ iff $(r = s)$ and satisfied $(\mathcal{M}_\mathcal{A}(x) \leq \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) \geq \mathcal{N}_\mathcal{B}(x))$.
- $\mathcal{A}_r \subseteq \mathcal{B}_s$ iff $(r \leq s)$ and satisfied $(\mathcal{M}_\mathcal{A}(x) \leq \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) \geq \mathcal{N}_\mathcal{B}(x))$.
- $\mathcal{A}_r =_\rho \mathcal{B}_s$ iff $(r = s)$.
- $\mathcal{A}_r =_\nu \mathcal{B}_s$ iff $(\mathcal{M}_\mathcal{A}(x) = \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) = \mathcal{N}_\mathcal{B}(x))$.
- $\mathcal{A}_r = \mathcal{B}_s$ iff $(r = s)$ and satisfied $(\mathcal{M}_\mathcal{A}(x) = \mathcal{M}_\mathcal{B}(x)$ and $\mathcal{N}_\mathcal{A}(x) = \mathcal{N}_\mathcal{B}(x))$.

Definition 2.5. [24] Let $\mathcal{A}_r, \mathcal{B}_s$ are *CIFSs* and operation radius $\propto \in \{\min, \max\}$. The negation, intersection, union, algebraic product, algebraic sum, and arithmetic mean operators between $\mathcal{A}_r$ and

$\mathcal{B}_s$ respectively as follows:

$$\neg\mathcal{A}_r = \{\langle x, \mathcal{N}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{A}}(x); r\rangle | x \in X\}.$$
$$\mathcal{A}_r \cap_\propto \mathcal{B}_s = \{\langle x, \min(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \max(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \propto (r,s)\rangle | x \in X\}.$$
$$\mathcal{A}_r \cup_\propto \mathcal{B}_s = \{\langle x, \max(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \min(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \propto (r,s)\rangle | x \in X\}.$$
$$\mathcal{A}_r \circ_\propto \mathcal{B}_s = \{\langle x, \mathcal{M}_{\mathcal{A}}(x) \cdot \mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{B}}(x) - \mathcal{N}_{\mathcal{A}}(x) \cdot \mathcal{N}_{\mathcal{B}}(x); \propto (r,s)\rangle | x \in X\}.$$
$$\mathcal{A}_r +_\propto \mathcal{B}_s = \{\langle x, \mathcal{M}_{\mathcal{A}}(x) + \mathcal{M}_{\mathcal{B}}(x) - \mathcal{M}_{\mathcal{A}}(x) \cdot \mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x) \cdot \mathcal{N}_{\mathcal{B}}(x); \propto (r,s)\rangle | x \in X\}.$$
$$\mathcal{A}_r @_\propto \mathcal{B}_s = \{\langle x, \frac{\mathcal{M}_{\mathcal{A}}(x) + \mathcal{M}_{\mathcal{B}}(x)}{2}, \frac{\mathcal{N}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{B}}(x)}{2}; \propto (r,s)\rangle | x \in X\}.$$

Definition 2.6. [24] Let $\mathcal{A}_r$ be *CIFS*, then modal operator "necessity" and "possibility" of $\mathcal{A}_r$ have the form:

$$\begin{aligned} \Box\mathcal{A}_r &= \{\langle x, \mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x); r\rangle | x \in X\} \\ &= \{\langle x, O_r(\mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x))\rangle | x \in X\}, \\ \Diamond\mathcal{A}_r &= \{\langle x, 1 - \mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x); r\rangle | x \in X\} \\ &= \{\langle x, O_r(1 - \mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x))\rangle | x \in X\}. \end{aligned}$$

2.4. Triangular norms (conorms)

The operators used in membership functions of *FS* has several criteria that must be fulfilled, as well as non-membership functions in *IFS*. Operators such as minimum, maximum, algebraic product, and algebraic sum are included in the triangular norms or conorms as the following:

Definition 2.7. [29, 30] A triangular norm (briefly t-norm) is binary operation $\mathcal{T}$ on the unit interval $[0,1]$ with definition $\mathcal{T} : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$:

(T1) $\mathcal{T}(x,y) = \mathcal{T}(y,x)$,
(T2) $\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z)$,
(T3) $\mathcal{T}(x,y) \le \mathcal{T}(x,z)$ whenever $y \le z$,
(T4) $\mathcal{T}(x,1) = x$.

Definition 2.8. [29, 30] A triangular conorm (t-conorm for short) is binary operation $\mathcal{S}$ on the unit interval $[0,1]$ with definition $\mathcal{S} : [0,1]^2 \to [0,1]$ which satisfies for all $x, y, z \in [0,1]$, (T1-T3) and

(S4) $\mathcal{S}(x,0) = x$.

Some examples of functions under t-norm or t-conorm include:

$$\mathcal{T}_M(x,y) = \min(x,y) \qquad \mathcal{S}_M(x,y) = \max(x,y)$$
$$\mathcal{T}_P(x,y) = xy \qquad \mathcal{S}_P(x,y) = x + y - xy$$
$$\mathcal{T}_L(x,y) = \max(x+y-1, 0) \qquad \mathcal{S}_L(x,y) = \min(x+y, 1)$$
$$\mathcal{T}_D(x,y) = \begin{cases} 0 & if (x,y) \in [0,1[^2 \\ \min(x,y) & otherwise \end{cases} \qquad \mathcal{S}_D(x,y) = \begin{cases} 1 & if (x,y) \in]0,1]^2 \\ \max(x,y) & otherwise \end{cases}$$

The operator $\mathcal{S}$ is also called the dual of $\mathcal{T}$ and has the relation $\mathcal{T}(x,y) = 1 - \mathcal{S}(1-x, 1-y)$ while the classification on both of them is in accordance with the prevailing properties. In this paper, we focus on two operators, the operator $\mathcal{T}_P$ is called product t-norm (algebraic product) and the operator $\mathcal{S}_P$ is called probabilistic sum (algebraic sum). Both operators satisfy the properties of monotone, continuous, strictly, cancellation law, and archimedean. Moreover, algebraic properties on its members such as idempotent, nilpotent and zero divisor can also be shown.

Remark 1 : In Definition 2.5, the operations on $\mathcal{M}$ and $\mathcal{N}$ fall under the category of t-norm, allowing them to be extended and applied using various forms of t-norm or conorm. This differs from the radius, which operates within the domain interval $[0, \sqrt{2}]$*, where the operation used is not a t-norm or conorm. The operations applied to the radius are constrained to* min *and* max*. Expanding the radius from the interval* $[0, 1]$ *to* $[0, \sqrt{2}]$ *presents an opportunity to extend the operations on t-norm (conorm) into new operations. To maintain generality, the following proposes the extension of the algebraic sum and product operators into the interval* $[0, \sqrt{2}]$.

3. Extension of radius operations in interval $[0, \psi]$ with $\psi \in [1, \sqrt{2}]$

In early research, Attanasov [24] introduced the radius domain in the interval $[0, 1]$ and the min and max operators as initial operations. In this case, the selection of min and max operators can refer to the category type of t-norm and t-conorm functions so that the operators on the radius can be projected using t-norm and t-conorm operators. Furthermore, the radius interval is expanded to $[0, \sqrt{2}]$ to ecompass the entire *IFIT* region within the circle range [25]. This expansion causes the categories of min and max operations to no longer be seen as t-norm and t-conorm functions and has an impact on the limitations of operators other than min and max.

In this section, the operators extension algebraic product, $\mathcal{T}^*$ and extension algebraic sum, $\mathcal{S}^*$ are defined on the interval $[0, \psi]$ with $\psi \in [1, \sqrt{2}]$. These operators are based on the algebraic product and algebraic sum on the t-norm and t-conorm. The $\mathcal{T}^*$ and $\mathcal{S}^*$ operators are expected to have similar properties and structure to the algebraic product and sum.

Definition 3.1. Let $\psi \in [1, \sqrt{2}]$ and $\mathcal{T}^*, \mathcal{S}^* : [0, \psi]^2 \to [0, \psi]$ where for $a, b \in [0, \psi]$ can be defined as $\mathcal{T}^*(a, b) = \frac{ab}{\psi}$ and $\mathcal{S}^*(a, b) = a + b - \frac{ab}{\psi}$.

Next, we will show the properties related to the t-norm axioms of the operator in Definition 2.4.

Theorem 3.1. Let $\psi \in [1, \sqrt{2}]$ and for $a, b \in [0, \psi]$, $\mathcal{T}^*(a, b) = \frac{ab}{\psi}$ and $\mathcal{S}^*(a, b) = a + b - \frac{ab}{\psi}$. The operations $\mathcal{T}^*$ and $\mathcal{S}^*$ satisfy the following properties:

(i) $\mathcal{T}^*$ and $\mathcal{S}^*$ are commutative.

(ii) $\mathcal{T}^*$ and $\mathcal{S}^*$ are associative.

(iii) $\mathcal{T}^*$ and $\mathcal{S}^*$ are monotonic.

(iv) The neutral element in $\mathcal{T}^*$ is ψ and the neutral element in $\mathcal{S}^*$ is 0.

(v) $\mathcal{T}^*(a, b) = \psi - \mathcal{S}^*(\psi - a, \psi - b)$ and $\mathcal{S}^*(a, b) = \psi - \mathcal{T}^*(\psi - a, \psi - b)$.

Proof. Let $\psi \in [1, \sqrt{2}]$ and $a, b \in [0, \psi]$ such that,

(i) $\mathcal{T}^*(a, b) = \frac{ab}{\psi} = \frac{ba}{\psi} = \mathcal{T}^*(b, a)$ and $\mathcal{S}^*(a, b) = a + b - \frac{ab}{\psi} = b + a - \frac{ba}{\psi} = \mathcal{S}^*(b, a)$. So it is proven that $\mathcal{T}^*$ and $\mathcal{S}^*$ are commutative.

(ii) Let $c \in [0, \psi]$, then

$\mathcal{T}^*(a, \mathcal{T}^*(b, c)) = \mathcal{T}^*(a, \frac{bc}{\psi}) = \frac{a\frac{bc}{\psi}}{\psi} = \frac{\frac{ab}{\psi}c}{\psi} = \mathcal{T}^*(\frac{ab}{\psi}, c) = \mathcal{T}^*(\mathcal{T}^*(a, b), c)$ and

$$\begin{aligned}\mathcal{S}^*(a, \mathcal{T}^*(b, c)) &= \mathcal{S}^*\left(a, b + c - \tfrac{bc}{\psi}\right)\\ &= a + \left(b + c - \tfrac{bc}{\psi}\right) - \tfrac{a\left(b+c-\frac{bc}{\psi}\right)}{\psi}\\ &= a + b + c - \tfrac{bc}{\psi} - \tfrac{ab}{\psi} - \tfrac{ac}{\psi} + \tfrac{abc}{\psi^2}\\ &= a + b - \tfrac{ab}{\psi} + c - \tfrac{\left(a+b-\frac{ab}{\psi}\right)c}{\psi}\\ &= \mathcal{S}^*(a + b - \tfrac{ab}{\psi}, c)\\ &= \mathcal{S}^*(\mathcal{S}^*(a, b), c).\end{aligned}$$

Hence, it is proved that $\mathcal{T}^*$ and $\mathcal{S}^*$ are associative.

(iii) Let $c \in [0, \psi]$ where $b \leq c$, then $\mathcal{T}^*(a, b) = \frac{ab}{\psi} \leq \frac{ac}{\psi} = \mathcal{T}^*(a, c)$ and $\mathcal{S}^*(a, b) = a + b - \frac{ab}{\psi} \leq a + c - \frac{ac}{\psi} = \mathcal{S}^*(a, c)$. So $\mathcal{T}^*$ and $\mathcal{S}^*$ have monotonic.

(iv) Suppose the neutral element in $\mathcal{T}^*$ is $e' \in [0, \psi]$, then $\mathcal{T}^*(a, e') = \mathcal{T}^*(e', a) = a$. It means $\frac{ae'}{\psi} = a$ and obtained $e' = \psi$. Similarly, suppose the neutral element in $\mathcal{S}^*$ is e'', then $\mathcal{S}^*(a, e'') = \mathcal{S}^*(e'', a) = a$ and we get $e'' = 0$.

(v) It can be shown that $\psi - \mathcal{S}^*(\psi - a, \psi - b) = \mathcal{T}^*(a, b)$ and also $\psi - \mathcal{T}^*(\psi - a, \psi - b) = \mathcal{S}^*(a, b)$.

$$\begin{aligned}\psi - \mathcal{S}^*(\psi - a, \psi - b) &= \psi - \left([\psi - a] + [\psi - b] - \tfrac{[\psi-a][\psi-b]}{\psi}\right)\\ &= \psi - (\psi - a + \psi - b - \psi + a + b - \tfrac{ab}{\psi})\\ &= \tfrac{ab}{\psi} = \mathcal{T}^*(a, b)\end{aligned}$$

$$\begin{aligned}\psi - \mathcal{T}^*(\psi - a, \psi - b) &= \psi - \left(\tfrac{[\psi-a][\psi-b]}{\psi}\right)\\ &= \psi - \left(\psi - a - b + \tfrac{ab}{\psi}\right)\\ &= a + b - \tfrac{ab}{\psi} = \mathcal{S}^*(a, b).\end{aligned}$$

□

If $\psi = 1$, the domain interval $[0, 1]$ is obtained so that $\mathcal{T}^*(a, b) = ab/1 = ab = \mathcal{T}_p$ (product t-norm) and $\mathcal{S}^*(a, b) = a + b - \frac{ab}{1} = a + b - ab = \mathcal{S}_p$ (probabilistic sum t-conorm). Moreover, if $\psi = \sqrt{2}$, the domain interval is $[0, \sqrt{2}]$ such that $\mathcal{T}^*(a, b) = \frac{ab}{\sqrt{2}}$ and $\mathcal{S}^*(a, b) = a + b - \frac{ab}{\sqrt{2}}$. It can be seen that the variable ψ is an index of expansion that occurs in the domain interval from 1 to $\sqrt{2}$. This raises the question whether the properties and characters that apply $\mathcal{T}_p$ also apply to $\mathcal{T}^*$ and $\mathcal{S}_p$ also apply to $\mathcal{S}^*$. The properties and characters in question are monotone, continuous, strict, cancellation law and archimedean.

Remark 2. The combination of monotonicity and commutativity leads to the non-decreasing property of $\mathcal{T}^*$. *If* $a \leq b$ *and* $c \leq d$, *then we get* $\mathcal{T}^*(a, c) \leq \mathcal{T}^*(a, d) = \mathcal{T}^*(d, a) \leq \mathcal{T}^*(d, b) = \mathcal{T}^*(b, d)$. *The same effect occurs for* $\mathcal{S}^*$, *such that* $\mathcal{S}^*(a, c) \leq \mathcal{S}^*(b, d)$.

Theorem 3.2. Given $\mathcal{T}^*$ and $\mathcal{S}^*$ in Definition 3.1, both operations satisfy the properties:

(i) $\mathcal{T}^*$ and $\mathcal{S}^*$ are continuous for every $(a, b) \in [0, \psi]^2$ with $\psi \in [1, \sqrt{2}]$.

(ii) $\mathcal{T}^*$ and $\mathcal{S}^*$ are strictly monotone.

(iii) $\mathcal{T}^*$ and $\mathcal{S}^*$ satisfy cancellation law.

(iv) $\mathcal{T}^*$ and $\mathcal{S}^*$ are archimedean.

Proof. The proof will be done on the operator $\mathcal{T}^*$ (The same assumption is used for $\mathcal{S}^*$).

(i) It will be proved that for every $(a, b) \in [0, \psi]^2$ with $\psi \in [1, \sqrt{2}]$ and $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for every $(a, b) \in [0, \psi]^2$ with $\sqrt{(x-a)^2 + (y-b)^2} < \delta_\epsilon$ holds $|\frac{xy}{\psi} - \frac{ab}{\psi}| < \epsilon$. Let $\psi \in [1, \sqrt{2}], (a, b) \in [0, \psi]^2$ and $\epsilon > 0$. Note that for every $(x, y) \in [0, \psi]^2$ holds $0 < a, y < \psi$, then

$$
\begin{aligned}
|\tfrac{xy}{\psi} - \tfrac{ab}{\psi}| &= \tfrac{1}{\psi}|xy - ab| \\
&= \tfrac{1}{\psi}\sqrt{(xy-ab)^2} \\
&= \tfrac{1}{\psi}\sqrt{(xy-ay+ay-ab)^2} \\
&= \tfrac{1}{\psi}\sqrt{(x-a)^2y^2 + 2ay(x-a)(y-b) + a^2(y-b)^2} \\
&< \tfrac{1}{\psi}\sqrt{(x-a)^2\psi^2 + 2\psi(x-a)(y-b) + \psi^2(y-b)^2} \\
&= \sqrt{(x-a)^2 + 2(x-a)(x-b) + (y-b)^2} \\
&= \sqrt{((x-a)+(y-b))^2} \\
&= |(x-a)+(y-b)| \\
&\le |x-a| + |y-b| \\
&= \sqrt{(x-a)^2} + \sqrt{(y-b)^2} \\
&\le \sqrt{(x-a)^2+(y-b)^2} + \sqrt{(x-a)^2+(y-b)^2} \\
&= 2\sqrt{(x-a)^2+(y-b)^2}.
\end{aligned}
$$

By taking $\delta_\epsilon = \frac{1}{2}\epsilon$, then for every $(x, y) \in [0, \psi]^2$ with $\sqrt{(x-a)^2 + (y-b)^2} < \delta_\epsilon$ satisfied $|\frac{xy}{\psi} - \frac{ab}{\psi}| < 2\sqrt{(x-a)^2 + (y-b)^2} < 2\delta_\epsilon = 2(\frac{1}{2}\epsilon) = \epsilon$.

(ii) Let $\psi \in [1, \sqrt{2}]$ and $a, b, c \in [0, \psi]$ with $b < c$. By using the monotonicity property in Theorem 3.1, it is obtained that $\mathcal{T}^*(a, b) < \mathcal{T}^*(a, c)$ and also $\mathcal{S}^*(a, b) < \mathcal{S}^*(a, c)$. Thus $\mathcal{T}^*$ and $\mathcal{S}^*$ are strictly monotone.

(iii) It will be proved that for every $a, b, c \in [0, \psi]$ with $\psi \in [1, \sqrt{2}]$, if $\mathcal{T}^*(a, b) = \mathcal{T}^*(a, c)$ then $a = 0$ or $b = c$. Let $\psi \in [1, \sqrt{2}]$ and $a, b, c \in [0, \psi]$ that satisfy $\mathcal{T}^*(a, b) = \mathcal{T}^*(a, c)$. Hence $\frac{a(b-c)}{\psi} = 0$, Otherwise $a = 0$ or $b = c$. Analogously using $\mathcal{S}^*$ operator, if $\mathcal{S}^*(a, b) = \mathcal{S}^*(a, c)$ then,

$$
\begin{aligned}
(b-c) - \frac{a(b-c)}{\psi} &= 0 \\
(b-c)(1 - \frac{a}{\psi}) &= 0.
\end{aligned}
$$

The result is $b = c$ or $a = \psi$.

(iv) We will prove for every $(a, b) \in]0, \psi[^2$ with $\psi \in [1, \sqrt{2}]$, there exist $n \in \mathbb{N}$ such that $a^{(n)}_{\mathcal{T}^*} < b$. Let $\psi \in [0, \sqrt{2}]$, $(a, b) \in]0, \psi[^2$ then

$$a^{(n)}_{\mathcal{T}^*} = \underbrace{\mathcal{T}^*(a, \mathcal{T}^*(a, ...(\mathcal{T}^*(a, a))))}_{\text{n times}} = \frac{a^n}{\psi^{n-1}}.$$

Using contradiction, if $\frac{a^n}{\psi^{n-1}} - b \geq 0$ then

$$\frac{a^n}{\psi^{n-1}} - b = \psi\left(\frac{a}{\psi}\right)^n - b \geq b\left(\frac{a}{\psi}\right)^n - b = b\left(\left(\frac{a}{\psi}\right)^n - 1\right) \geq 0.$$

Since $0 < a < \psi$, then $\left(\frac{a}{\psi}\right)^n - 1 < 0$ and also $b \neq 0$ such that equation $b\left(\left(\frac{a}{\psi}\right)^n - 1\right) \geq 0$ is false. So it is proved that there exist $a^{(n)}_{\mathcal{T}^*} < b$. Similarly with the operator $\mathcal{S}^*$ then,

$$a^{(n)}_{\mathcal{S}^*} = \underbrace{\mathcal{T}^*(a, \mathcal{T}^*(a, ...(\mathcal{T}^*(a, a))))}_{\text{n times}} = \psi - \psi(1 - \frac{a}{\psi})^n.$$

Analogously, by using contradiction, if it holds that $\psi - \psi\left(1 - \frac{a}{\psi}\right)^n - b \geq 0$ then,

$$\psi - \psi\left(1 - \frac{a}{\psi}\right)^n - b \geq b - \psi\left(1 - \frac{a}{\psi}\right)^n - b = -\psi\left(1 - \frac{a}{\psi}\right)^n \geq 0.$$

Considering $0 < a < \psi$, it follows $\left(1 - \frac{a}{\psi}\right)^n \geq 0$ such that $-\psi\left(1 - \frac{a}{\psi}\right)^n \geq 0$ is false and it is proved for $a^{(n)}_{\mathcal{S}^*} < b$. □

Theorem 3.3. Let $\mathcal{T}^*$ and $\mathcal{S}^*$ operator , $\psi \in [1, \sqrt{2}]$ and $a \in [0, \psi]$ then applies:

(i) Element 0 and ψ are idempotent element in $\mathcal{T}^*$ and $\mathcal{S}^*$.

(ii) The operator $\mathcal{T}^*$ and $\mathcal{S}^*$ have no nilpotent elements and zero divisor.

Proof. Let $\psi \in [1, \sqrt{2}]$,

(i) Element $a \in [0, \psi]$ is called idempotent of $\mathcal{T}^*$(or $\mathcal{S}^*$) iff $\mathcal{T}^*(a, a) = a$ (or $\mathcal{S}^*(a, a) = a$). Start with $\mathcal{T}^*(a, a) = a$ such that it is obtained,

$$\begin{aligned} \mathcal{T}^*(a, a) &= a \\ \frac{a^2}{\psi} &= a \\ a - \frac{a^2}{\psi} &= 0 \\ \frac{a(\psi - a)}{\psi} &= 0. \end{aligned}$$

This leads to $a = 0$ or $a = \psi$. Now for operator $\mathcal{S}^*(a,a) = a$ we get,

$$\begin{aligned}\mathcal{S}^*(a,a) &= 0\\ 2a - \frac{a^2}{\psi} &= a\\ a - \frac{a^2}{\psi} &= 0\\ \frac{a(\psi - a)}{\psi} &= 0.\end{aligned}$$

This also means that $a = 0$ or $a = \psi$. Therefore, 0 and ψ for $\psi \in [1, \sqrt{2}]$ are idempotent elements for both $\mathcal{T}^*$ and $\mathcal{S}^*$.

(ii) Element $a \in]0, \psi[$ is called nilpotent of $\mathcal{T}^*$ (or $\mathcal{S}^*$) iff there exist some $n \in \mathbb{N}$ such that $a^{(n)}_{\mathcal{T}^*} = 0$ (or $a^{(n)}_{\mathcal{S}^*} = 0$). Starting with operator $\mathcal{T}^*$ we have $a^{(n)}_{\mathcal{T}^*} = \frac{a^n}{\psi^{n-1}} = 0$ such that $a = 0$ is obtained. Similarly for operator $\mathcal{S}^*$, we have

$$\begin{aligned}a^n_{\mathcal{S}^*} = \psi - \psi\left(1 - \frac{a}{\psi}\right)^n &= 0\\ \psi\left(1 - \frac{a}{\psi}\right)^n &= \psi\\ \left(1 - \frac{a}{\psi}\right)^n &= 1.\end{aligned}$$

The above equation holds if only $a = \psi$ is obtained for n odd or even. Since $a \in]0, \psi[$, both operators $\mathcal{T}^*$ and $\mathcal{S}^*$ have no nilpotent element. Furthermore, element $a \in]0, \psi[$ is called zero divisor of $\mathcal{T}^*$ (or $\mathcal{S}^*$) iff there exist some $b \in]0, \psi[$ such that $\mathcal{T}^*(a, b) = 0$ (or $\mathcal{S}^*(a, b) = 0$). Analogously with nilpotent, we have only $a = 0$ or $a = \psi$ that satisfies $\mathcal{T}^*(a, b) = 0$ and $\mathcal{S}^*(a, b) = 0$, so the operators $\mathcal{T}^*$ and $\mathcal{S}^*$ also have no zero divisor. □

At the end of this section, we will show the relation between $\mathcal{T}^*$ and $\mathcal{T}_p$ from the algebraic side using the concept of semigroup. Given interval sets $I = [0, 1]$ and $I^* = [0, \psi]$ with $\psi \in [1, \sqrt{2}]$, and also operator $\mathcal{T}_p : I^2 \to I$ and operator $\mathcal{T}^* : [I^*]^2 \to I^*$ defined respectively:

$$\begin{aligned}\mathcal{T}_p(a, b) &= ab(\forall a, b \in I),\\ \mathcal{T}^*(a, b) &= \frac{a^* b^*}{\psi}(\forall a, b \in I^*).\end{aligned}$$

It can be shown that $(I, \mathcal{T}_p)$ and $(I^*, \mathcal{T}^*)$ are both commutative semigroups with neutral elements.

Theorem 3.4. Semigroup $(I, \mathcal{T}_p)$ isomorphic to $(I^*, \mathcal{T}^*)$ is denoted $(I, \mathcal{T}_p) \cong (I^*, \mathcal{T}^*)$.

Proof. It will be shown that there exists a bijective function $\xi : I \to I^*$ such that for all $a, b \in I$,

$$\xi(\mathcal{T}_p(a, b)) = \mathcal{T}^*(\xi(a), \xi(b)). \tag{3.1}$$

Define the function $\xi : I \to I^*$ by $\xi(a) = a\psi$ for $\psi \in [1, \sqrt{2}]$. This implies that for all $a, b \in I$ holds,

$$\xi(\mathcal{T}_p(a,b)) = \xi(ab) = ab\psi = \frac{a\psi . b\psi}{\psi} = \mathcal{T}^*(\xi(a), \xi(b)).$$

Furthermore, the function ξ is injective because $\forall a, b \in I$ if $\xi(a) = \xi(b)$ then $a = b$ holds. Moreover, the function ξ is also surjective, because for any $a^* \in I^*$ there exist $a = \frac{a^*}{\psi} \in I$ such that $\xi(a) = a\psi = (\frac{a^*}{\psi})\psi = a^*$ for any $\psi \in [1, \sqrt{2}]$. Since the function ξ is injective, surjective and satisfies Eq (3.1), then $(I, \mathcal{T}_p) \cong (I^*, \mathcal{T}^*)$. □

Similarly, the operators $\mathcal{S}_p : I^2 \to I$ and $\mathcal{S}^* : [I^*] \to I^*$ are defined respectively:

$$\mathcal{S}_p(a,b) = a + b - ab(\forall a, b \in I),$$
$$\mathcal{S}^*(a,b) = a + b - \frac{a^*b^*}{\psi}(\forall a, b \in I^*).$$

Theorem 3.5. Semigroup $(I, \mathcal{S}_p)$ isomorphic to $(I^*, \mathcal{S}^*)$ is denoted $(I, \mathcal{S}_p) \cong (I^*, \mathcal{S}^*)$.

Proof. Straightforward. □

With the existence of operators $\mathcal{T}^*$ and $\mathcal{S}^*$ that have the same structure, character, and properties as $\mathcal{T}_p$ and $\mathcal{S}_p$, they can be used as alternative operators of radius instead of the previously defined min and max. Henceforth, in the case of *CIFS* using the value $\psi = \sqrt{2}$, the operator $\mathcal{T}^*$ is called radius algebraic product (*RAP*) and is denoted $\otimes$, while the operator $\mathcal{S}^*$ is called radius algebraic sum (*RAS*) and is denoted $\oplus$.

4. Generalized operators for CIFS

This section begins by defining the generalized operators for *CIFS* based on *t*-norm (conorm). These generalizations include the membership ($\mathcal{M}$), non-membership ($\mathcal{N}$), with the range [0, 1] and radius, r with the range $[0, \sqrt{2}]$. Due to the difference in interval domains, this generalization process is applied to $\mathcal{M}$ and $\mathcal{N}$ which belong to the *t*-norm (conorm) category [29, 30]. While the radius uses min, max, *RAP*, and *RAS*. We focus on algebraic product and algebraic sum for alternative operations on the radius and studying their related properties. Moreover, the arithmetic mean can also be applied to directly affect the radius or be consistent with the operations $\mathcal{M}_{\mathcal{A}}(x)$ and $\mathcal{N}_{\mathcal{A}}(x)$. The first step is to define the generalization of intersection and union in *CIFS* using *t*-norm (conorm).

Definition 4.1. Let $\mathcal{A}_r, \mathcal{B}_s$ are *CIFSs* in X with $r, s \in [0, 1]$. The generalized intersection ($\widetilde{\cap}$) and generalized union ($\widetilde{\cup}$) of *CIFS*s can be presented as follows:

$$(\mathcal{A}_r)\widetilde{\cap}_{\mathcal{T},\mathcal{S},\propto}(\mathcal{B}_s) = \{\langle x, \mathcal{T}(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \mathcal{S}(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \propto (r,s) | x \in X\rangle\},$$
$$(\mathcal{A}_r)\widetilde{\cup}_{\mathcal{S},\mathcal{T},\propto}(\mathcal{B}_s) = \{\langle x, \mathcal{S}(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \mathcal{T}(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \propto (r,s) | x \in X\rangle\},$$

where $\mathcal{T}$ denotes *t*-norm and $\mathcal{S}$ denotes *t*-conorm, while $\propto \in \{\min, \max, \otimes, \oplus\}$.

It is clear that the Definition 4.1 will be defined for radius $r, s \in [0, \sqrt{2}]$. It can be seen that generalized intersection $(\mathcal{A}_r)\widetilde{\cap}_{\mathcal{T},\mathcal{S},\propto}(\mathcal{B}_s)$ is a *CIFS* in X if $\mathcal{T}(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)) + \mathcal{S}(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)) \leq 1$. Since $\mathcal{A}_r$ and $\mathcal{B}_s$ are *CIFSs*, obtained $\mathcal{N}_{\mathcal{A}}(x) \leq 1 - \mathcal{M}_{\mathcal{A}}(x)$ and $\mathcal{N}_{\mathcal{B}}(x) \leq 1 - \mathcal{M}_{\mathcal{B}}(x)$. Then,

$\mathcal{S}(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)) \leq \mathcal{S}(1-\mathcal{M}_{\mathcal{A}}(x), 1-\mathcal{M}_{\mathcal{B}}(x))$, which is equivalent to $\mathcal{S}^*(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)) = 1-\mathcal{S}(1-\mathcal{M}_{\mathcal{A}}(x), 1-\mathcal{M}_{\mathcal{B}}(x)) \leq 1-\mathcal{S}(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x))$, where the equality holds if $\mathcal{A}$ and $\mathcal{B}$ are fuzzy sets. So, if $\mathcal{T}(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)) \leq \mathcal{S}^*(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x))$, then $(\mathcal{A}_r)\widetilde{\cap}_{\mathcal{T},\mathcal{S},\propto}(\mathcal{B}_s)$ is a *CIFS* in X. Analogously, generalized union $(\mathcal{A}_r)\widetilde{\cup}_{\mathcal{S},\mathcal{T},\propto}(\mathcal{B}_s)$ is a *CIFS* when $\mathcal{S}(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)) \leq \mathcal{T}^*(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x))$, where $\mathcal{T}^*(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)) = 1-\mathcal{T}(1-\mathcal{M}_{\mathcal{A}}(x), 1-\mathcal{M}_{\mathcal{B}}(x))$. Thus, $(\mathcal{A}_r)\widetilde{\cup}_{\mathcal{S},\mathcal{T},\propto}(\mathcal{B}_s)$ is a *CIFS* in X, the so called t-conorm (or dual t-norm $\mathcal{T}$).

The properties of these operators are also a generalization of the basic properties of *CIFS*. The properties like commutative, associative and De'Morgan law can be derived from the generalized intersection and union in *CIFS*. However, for distributive properties of generalized union and intersection do not always apply, therefore it is necessary to focus on the type of t-norm (conorm) to be used. For example, if the t-norm used is min then, $\mathcal{T}(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)) = \min(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x))$ and $\mathcal{S}(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)) = \max(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x))$ so for each $x \in X$ can be written:

$$\begin{aligned}
(\mathcal{A}_r)\widetilde{\cap}_{\min,\max,\min}(\mathcal{B}_s) &= \{\langle x, \min(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \max(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \min(r, s)\rangle\} \\
&= \mathcal{A}_r \cap_{\min} \mathcal{B}_s, \\
(\mathcal{A}_r)\widetilde{\cap}_{\min,\max,\max}(\mathcal{B}_s) &= \{\langle x, \min(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \max(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \max(r, s)\rangle\} \\
&= \mathcal{A}_r \cap_{\max} \mathcal{B}_s.
\end{aligned}$$

The same thing happened for generalized union (t-conorm). This time, if the t-norm used is an algebraic product, then the t-conorm used is an algebraic sum, and the generalized intersection and union becomes:

$$(\mathcal{A}_r)\widetilde{\cap}_{\circ,+,\otimes}(\mathcal{B}_s) = \{\langle x, \mathcal{M}_{\mathcal{A}}(x).\mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{B}}(x) - \mathcal{N}_{\mathcal{A}}(x).\mathcal{N}_{\mathcal{B}}(x); r.s\rangle\} = \mathcal{A}_r \circ_{\otimes} \mathcal{B}_s,$$

$$(\mathcal{A}_r)\widetilde{\cap}_{\circ,+,\oplus}(\mathcal{B}_s) = \{\langle x, \mathcal{M}_{\mathcal{A}}(x).\mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{B}}(x) - \mathcal{N}_{\mathcal{A}}(x).\mathcal{N}_{\mathcal{B}}(x); r + s - r.s\rangle\} = \mathcal{A}_r \circ_{\oplus} \mathcal{B}_s,$$

$$(\mathcal{A}_r)\widetilde{\cup}_{+,\circ,\otimes}(\mathcal{B}_s) = \{\langle x, \mathcal{M}_{\mathcal{A}}(x) + \mathcal{M}_{\mathcal{B}}(x) - \mathcal{M}_{\mathcal{A}}(x).\mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x).\mathcal{N}_{\mathcal{B}}(x); r.s\rangle\} = \mathcal{A}_r +_{\otimes} \mathcal{B}_s,$$

$$(\mathcal{A}_r)\widetilde{\cup}_{+,\circ,\oplus}(\mathcal{B}_s) = \{\langle x, \mathcal{M}_{\mathcal{A}}(x) + \mathcal{M}_{\mathcal{B}}(x) - \mathcal{M}_{\mathcal{A}}(x).\mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x).\mathcal{N}_{\mathcal{B}}(x); r + s - r.s\rangle\} = \mathcal{A}_r +_{\oplus} \mathcal{B}_s.$$

In the previous researchers [24], the operations used on the radius are only min and max. From Definition 2.5 instead of using min and max as radius operations in *CIFS*, it can be generalized to other t-norm (conorm) operators. We focus on the algebraic sum and product for radius operations. In addition to the algebraic product and sum form, the definition of the mean arithmetic operator will also be applied to the radius. Therefore, the definition $\propto$ in this paper includes min, max, $\otimes, \oplus$, and @. Thus, Definition 2.5 can be expanded to:

Definition 4.2. Let $\mathcal{A}_r, \mathcal{B}_s$ are *CIFS* in X, radius operations $\propto \in \{\min, \max, \oplus, \otimes, @\}$, and $r, s \in [0, \sqrt{2}]$.

The algebraic product, algebraic sum and arithmetic mean operator of *CIFSs* can be presented as follows:

$$\begin{aligned}
\mathcal{A}_r \circ_{\propto} \mathcal{B}_s &= \{\langle x, \mathcal{M}_{\mathcal{A}}(x).\mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{B}}(x) - \mathcal{N}_{\mathcal{A}}(x).\mathcal{N}_{\mathcal{B}}(x); \propto (r, s)\rangle | x \in X\}, \\
\mathcal{A}_r +_{\propto} \mathcal{B}_s &= \{\langle x, \mathcal{M}_{\mathcal{A}}(x) + \mathcal{M}_{\mathcal{B}}(x) - \mathcal{M}_{\mathcal{A}}(x).\mathcal{M}_{\mathcal{B}}(x), \mathcal{N}_{\mathcal{A}}(x).\mathcal{N}_{\mathcal{B}}(x); \propto (r, s)\rangle | x \in X\}, \\
\mathcal{A}_r @_{\propto} \mathcal{B}_s &= \{\langle x, \frac{\mathcal{M}_{\mathcal{A}}(x) + \mathcal{M}_{\mathcal{B}}(x)}{2}, \frac{\mathcal{N}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{B}}(x)}{2}; \propto (r, s)\rangle | x \in X\}.
\end{aligned}$$

The following example will show the difference between the min, max, $\otimes, \oplus$ and @ on radius part.

Example 1: Let $\mathcal{A}_r$ and $\mathcal{B}_s$ are *CIFS* which are defined as follows:

$$\mathcal{A}_r = \{\langle x, 0.52, 0.10; 0.3\rangle, \langle y, 0.24, 0.65; 0.3\rangle, \langle z, 0.40, 0.57; 0.3\rangle\},$$
$$\mathcal{B}_s = \{\langle x, 0.32, 0.68; 1.2\rangle, \langle y, 0.73, 0.11; 1.2\rangle, \langle z, 0.63, 0.20; 1.2\rangle\}.$$

Using the algebraic product as the operator for membership and non-membership, compare the radius values for each operation $\propto$. If $\propto=$ min, then,

$$\mathcal{A}_r \circ_{\min} \mathcal{B}_s = \{\langle x, 0.17, 0.71; \mathbf{0.3}\rangle, \langle y, 0.18, 0.69; \mathbf{0.3}\rangle, \langle z, 0.25, 0.66; \mathbf{0.3}\rangle\},$$

and if $\propto= max$, the another result will be,

$$\mathcal{A}_r \circ_{max} \mathcal{B}_s = \{\langle x, 0.17, 0.71; \mathbf{1.2}\rangle, \langle y, 0.18, 0.69; \mathbf{1.2}\rangle, \langle z, 0.25, 0.66; \mathbf{1.2}\rangle\}.$$

A significant difference occurs in the radius while the membership and non-membership values are the same. So for $\mathcal{A}_r \circ_{\otimes} \mathcal{B}_s$ has a radius value of **0.254**, $\mathcal{A}_r \circ_{\oplus} \mathcal{B}_s$ obtains a radius value of **1.245** and $\mathcal{A}_r \circ_{@} \mathcal{B}_s$ obtains a radius value of **0.75**. The next thing is to know the relation of each operation radius.

Theorem 4.1. Let $\mathcal{A}_r$ and $\mathcal{B}_s$ are *CIFS*, $\phi \in \{\cap, \cup, +, \circ, @\}$ and $r, s \in [0, 1]$ the following equation holds,

$$\mathcal{A}_r \phi_{\otimes} \mathcal{B}_s \subseteq_\rho \mathcal{A}_r \phi_{\min} \mathcal{B}_s \subseteq_\rho \mathcal{A}_r \phi_{@} \mathcal{B}_s \subseteq_\rho \mathcal{A}_r \phi_{\max} \mathcal{B}_s \subseteq_\rho \mathcal{A}_r \phi_{\oplus} \mathcal{B}_s.$$

Proof. The proof of this theorem is focused on the radius value. Since the relation $\subseteq_\rho$ is in Definition 2.4, if $\mathcal{A}_r \subseteq_\rho \mathcal{B}_s$ then $r \leq s$ applies to radius. Furthermore, using the operating properties of the t-norm and conorm obtained for each $r, s \in [0, 1]$,

$$\frac{rs}{\sqrt{2}} \leq \min\{r, s\} \leq \frac{r+s}{2} \leq \max\{r, s\} \leq r + s - \frac{rs}{\sqrt{2}}.$$

So it is proved for the theorem. □

Theorem 4.2. Let $\mathcal{A}_r$ and $\mathcal{B}_s$ are *CIFS* , $\phi \in \{\cap, \cup, +, \circ, @\}, \propto \in \{\otimes, \oplus, @\}$ and $r, s \in [0, \sqrt{2}]$, then the following statements are true:

(i) $(\mathcal{A}_r \phi_{\min} \mathcal{B}_s) \cap_\propto (\mathcal{A}_r \phi_{\max} \mathcal{B}_s) = \mathcal{A}_r \cap_\propto \mathcal{B}_s$.

(ii) $(\mathcal{A}_r \phi_{\min} \mathcal{B}_s) \cup_\propto (\mathcal{A}_r \phi_{\max} \mathcal{B}_s) = \mathcal{A}_r \cup_\propto \mathcal{B}_s$.

(iii) $(\mathcal{A}_r \phi_{\min} \mathcal{B}_s) @_\propto (\mathcal{A}_r \phi_{\max} \mathcal{B}_s) = \mathcal{A}_r @_\propto \mathcal{B}_s$.

Proof. In proving points (i), (ii), and (iii) using a similar method. Since for let $\mathcal{A}_r, \mathcal{B}_s$ are *CIFS* (i.e. point 1),

$$\begin{aligned}
&(\mathcal{A}_r \phi_{\min} \mathcal{B}_s) \cap_\propto (\mathcal{A}_r \phi_{\max} \mathcal{B}_s) \\
&\quad = \{\langle x, \phi(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \phi(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \min(r, s)\rangle\} \cap_\propto \{\langle x, \phi(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \\
&\qquad \phi(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \max(r, s)\rangle\} \\
&\quad = \{\langle x, \phi(\mathcal{M}_{\mathcal{A}}(x), \mathcal{M}_{\mathcal{B}}(x)), \phi(\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{B}}(x)); \propto (\min(r, s), \max(r, s))\rangle\}.
\end{aligned}$$

If $\propto = \oplus$, then $\propto (\min(r,s), max(r,s)) = \min(r,s) + max(r,s) - \frac{[\min(r,s)\cdot max(r,s)]}{\sqrt{2}} = r + s - \frac{rs}{\sqrt{2}}$. The same is true if $\propto = \otimes$ and @ are selected. A similar proof was also made for $\phi = \cap, \cup, \circ, @$ and for points 2 and 3. □

Because the algebraic product and sum are a type of t-norm (conorm), it is proven that they are commutative, associative and De'Morgan law. However, further investigation is needed for the arithmetic mean operator combined with the algebraic product or sum,

Theorem 4.3. (*Commutative Law*) Let $\mathcal{A}_r$ and $\mathcal{B}_s$ are *CIFSs*, $\propto \in \{\otimes, \oplus, @\}$ and $r, s \in [0, \sqrt{2}]$ then satisfied:

$$\mathcal{A}_r +_@ \mathcal{B}_s = \mathcal{B}_s +_@ \mathcal{A}_r, \mathcal{A}_r \circ_@ \mathcal{B}_s = \mathcal{B}_s \circ_@ \mathcal{A}_r, \text{ and } \mathcal{A}_r @_\propto \mathcal{B}_s = \mathcal{B}_s @_\propto \mathcal{A}_r,$$

Proof. Let say $\mathcal{A}_r +_@ \mathcal{B}_s = C_t$, then radius part is $t = \frac{r+s}{2} = \frac{s+r}{2}$ and the equation $C_t = \mathcal{B}_s +_@ \mathcal{A}_r$ is proven. Next for $\mathcal{A}_r @_\otimes \mathcal{B}_s$ obtained,

$$\begin{aligned}
\mathcal{A}_r @_\otimes \mathcal{B}_s &= \{\langle x, \frac{\mathcal{M}_\mathcal{A}(x) + \mathcal{M}_\mathcal{B}(x)}{2}, \frac{\mathcal{N}_\mathcal{A}(x) + \mathcal{N}_\mathcal{B}(x)}{2}; \frac{rs}{\sqrt{2}} \rangle\} \\
&= \{\langle x, \frac{\mathcal{M}_\mathcal{B}(x) + \mathcal{M}_\mathcal{A}(x)}{2}, \frac{\mathcal{N}_\mathcal{B}(x) + \mathcal{N}_\mathcal{A}(x)}{2}; \frac{sr}{\sqrt{2}} \rangle\} \\
&= \mathcal{B}_s @_\otimes \mathcal{A}_r.
\end{aligned}$$

It can be seen that operator $@_\otimes$ is also commutative, so it is also proven for $\mathcal{A}_r @_\oplus \mathcal{B}_s$ and $\mathcal{A}_r @_@ \mathcal{B}_s$. □

Remark 3. Using the fact that for any $r, s, t \in [0, 1]$ *holds* $\frac{r+s}{2} + t \neq r + \frac{s+t}{2}$ *then the arithmetic mean is* ***not associative****. This means that for every* $\phi \in \{\cap, \cup, +, \circ\}$ *and* $\propto \in \{\min, \max, \otimes, \oplus\}$ *applies,*

$$\mathcal{A}_r \phi_@ (\mathcal{B}_s \phi_@ C_t) \neq (\mathcal{A}_r \phi_@ \mathcal{B}_s) \phi_@ C_t \text{ and } \mathcal{A}_r @_\propto (\mathcal{B}_s @_\propto C_t) \neq (\mathcal{A}_r @_\propto \mathcal{B}_s) @_\propto C_t.$$

operators, we will show some properties that apply to *RAP*, *RAS*, and arithmetic mean on the radius operator.

Theorem 4.4. Let $r, s, t \in [0, \sqrt{2}]$, then the relations between operators $\otimes, \oplus$ and @ satisfied,

(i) $\otimes(r, \otimes(s,t)) \geq \otimes(\otimes(r,s), \otimes(r,t))$.

(ii) $\oplus(r, \oplus(s,t)) \leq \oplus(\oplus(r,s), \oplus(r,t))$.

(iii) $\otimes(r, \oplus(s,t)) \leq \oplus(\otimes(r,s), \otimes(r,t))$.

(iv) $\oplus(r, \otimes(s,t)) \geq \otimes(\oplus(r,s), \oplus(r,t))$.

(v) $\otimes(r, @(s,t)) = @(\otimes(r,s), \otimes(r,t))$.

(vi) $\oplus(r, @(s,t)) = @(\oplus(r,s), \oplus(r,t))$.

(vii) $@(r, \oplus(s,t)) \leq \oplus(@(r,s), @(r,t))$.

(viii) $@(r, \otimes(s,t)) \geq \otimes(@(r,s), @(r,t))$.

(ix) $@(r, @(s,t)) = @(@(r,s), @(r,t))$.

Proof. Let $r, s, t \in [0, \sqrt{2}]$ real number. To prove that the left side is smaller or equal to the right side, the difference between the left and right sides is negative and vice versa.

(i) From left side, $\otimes(r, \otimes(s,t)) = \otimes(r, \frac{st}{\sqrt{2}}) = \frac{r\frac{st}{\sqrt{2}}}{\sqrt{2}} = \frac{rst}{2}$, and from right side we have, $\otimes(\otimes(r,s), \otimes(r,t)) = \otimes(\frac{rs}{\sqrt{2}}, \frac{rt}{\sqrt{2}}) = \frac{r^2st}{2\sqrt{2}}$. Then the difference between the two is,

$$\left[\frac{rst}{2}\right] - \left[\frac{r^2st}{2\sqrt{2}}\right] = \frac{\sqrt{2}rst - r^2st}{2\sqrt{2}} = \frac{rst(\sqrt{2}-r)}{2\sqrt{2}} \geq 0.$$

It means that $\otimes(r, \otimes(s,t)) \geq \otimes(\otimes(r,s), \otimes(r,t))$.

(ii) The value of the left side is, $\oplus(r, \oplus(s,t)) = \oplus\left(r, s + t - \frac{st}{\sqrt{2}}\right)$

$$= r + (s + t - \frac{st}{\sqrt{2}}) - \frac{r(s+t-\frac{st}{\sqrt{2}})}{\sqrt{2}}$$
$$= r + s + t - \frac{rs}{\sqrt{2}} - \frac{rt}{\sqrt{2}} - \frac{st}{\sqrt{2}} + \frac{rst}{2},$$

while from the right side, $\oplus(\oplus(r,s), \oplus(r,t)) = \otimes\left(r + s - \frac{rs}{\sqrt{2}}, r + t - \frac{rt}{\sqrt{2}}\right)$

$$= \left(r + s - \frac{rs}{\sqrt{2}}\right) + \left(r + t - \frac{rt}{\sqrt{2}}\right) - \frac{\left(r+s-\frac{rs}{\sqrt{2}}\right)\left(r+t-\frac{rt}{\sqrt{2}}\right)}{\sqrt{2}}$$
$$= r + s - \frac{rs}{\sqrt{2}} + r + t - \frac{rs}{\sqrt{2}} + \frac{r^2}{\sqrt{2}} + \frac{rt}{\sqrt{2}} - \frac{r^2t}{2} + \frac{rs}{\sqrt{2}} + \frac{st}{\sqrt{2}} - \frac{rst}{2} - \frac{r^2s}{2} - \frac{rst}{2} + \frac{r^2st}{2\sqrt{2}}.$$

The difference between the two is,

$$[r+s+t-\frac{st}{\sqrt{2}} - \frac{rs}{\sqrt{2}} - \frac{rt}{\sqrt{2}} + \frac{rst}{2}] - [r+s-\frac{rs}{\sqrt{2}} + r + t - \frac{rs}{\sqrt{2}} + \frac{r^2}{\sqrt{2}} + \frac{rt}{\sqrt{2}} - \frac{r^2t}{2} + \frac{rs}{\sqrt{2}} + \frac{st}{\sqrt{2}} - \frac{rst}{2} - \frac{r^2s}{2} - \frac{rst}{2} + \frac{r^2st}{2\sqrt{2}}]$$
$$= \frac{3rst}{2} - r - \frac{r^2}{\sqrt{2}} - \frac{2rt}{\sqrt{2}} + \frac{r^2t}{2} - \frac{2st}{\sqrt{2}} + \frac{r^2s}{2} - \frac{r^2st}{2\sqrt{2}}$$
$$= \frac{3\sqrt{2}rst - 2\sqrt{2}r - 2\sqrt{2}r^2 - 4rt + \sqrt{2}r^2t - 4st + \sqrt{2}r^2s - r^2st}{2\sqrt{2}}$$
$$= \frac{3\sqrt{2}rst - r^2st - 2\sqrt{2}r - 2r^2 + \sqrt{2}r^2(s+t) - 4t(r+s)}{2\sqrt{2}}.$$

Since, $s + t - \frac{st}{\sqrt{2}} \leq \sqrt{2}$ and $r + s - \frac{rs}{\sqrt{2}} \leq \sqrt{2}$ then,

$$\leq 3\sqrt{2}rst - r^2st - 2\sqrt{2}r - 2r^2 + \sqrt{2}r^2(\sqrt{2} - \frac{st}{\sqrt{2}}) - 4t(r+s)$$
$$= 3\sqrt{2}rst - 2r^2st - 2\sqrt{2}r - 4t(r+s)$$

$$\leq 3\sqrt{2}rst - 2r^2st - 2\sqrt{2}r - 4t(\sqrt{2} + \tfrac{rs}{\sqrt{2}})$$

$$= \sqrt{2}rst - 2r^2st - 2\sqrt{2}r - 4\sqrt{2}t$$

$$= -2r^2st + \sqrt{2}r(st-2) - 4\sqrt{2}t \leq 0.$$

So it is proven that $\oplus(r, \oplus(s,t)) \leq \oplus(\oplus(r,s), \oplus(r,t))$.

(iii) Analogously to the previous, for the left side value is obtained,

$$\otimes(r, \oplus(s,t)) = \otimes\left(r, s+t-\tfrac{st}{\sqrt{2}}\right) = \frac{r\left(s+t-\frac{st}{\sqrt{2}}\right)}{\sqrt{2}} = \tfrac{rs}{\sqrt{2}} + \tfrac{rt}{\sqrt{2}} - \tfrac{rst}{2},$$

while the right side is obtained,

$$\oplus(\otimes(r,s), \otimes(r,t)) = \oplus\left(\tfrac{rs}{\sqrt{2}}, \tfrac{rt}{\sqrt{2}}\right) = \tfrac{rs}{\sqrt{2}} + \tfrac{rt}{\sqrt{2}} = \frac{\left(\frac{rs}{\sqrt{2}}\right)\left(\frac{rt}{\sqrt{2}}\right)}{\sqrt{2}} = \tfrac{rs}{\sqrt{2}} + \tfrac{rt}{\sqrt{2}} - \tfrac{r^2st}{2\sqrt{2}}.$$

The difference is,

$$[\tfrac{rs}{\sqrt{2}} + \tfrac{rt}{\sqrt{2}} - \tfrac{rst}{2}] - [\tfrac{rs}{\sqrt{2}} + \tfrac{rt}{\sqrt{2}} - \tfrac{r^2st}{2\sqrt{2}}] = \tfrac{r^2st - \sqrt{2}rst}{2\sqrt{2}} = \tfrac{rst(r-\sqrt{2})}{2\sqrt{2}} \leq 0.$$

So it is clear that $\otimes(r, \oplus(s,t)) \leq \oplus(\otimes(r,s), \otimes(r,t))$.

(iv) Similarly, from left side we have,

$$\oplus(r, \otimes(s,t)) = \oplus(r, \tfrac{st}{\sqrt{2}}) = r + \tfrac{st}{\sqrt{2}} - \frac{r\frac{st}{\sqrt{2}}}{\sqrt{2}} = r + \tfrac{st}{\sqrt{2}} - \tfrac{rst}{2},$$

from right side,

$$\otimes(\oplus(r,s), \oplus(r,t)) = \otimes\left(r+s-\tfrac{rs}{\sqrt{2}}, r+t-\tfrac{rt}{\sqrt{2}}\right)$$

$$= \frac{\left(r+s-\frac{rs}{\sqrt{2}}\right)\left(r+t-\frac{rt}{\sqrt{2}}\right)}{\sqrt{2}}$$

$$= \tfrac{r^2}{\sqrt{2}} + \tfrac{rt}{\sqrt{2}} - \tfrac{r^2t}{2} + \tfrac{rs}{\sqrt{2}} + \tfrac{st}{\sqrt{2}} - \tfrac{rst}{2} - \tfrac{r^2s}{2} - \tfrac{rst}{2} + \tfrac{r^2st}{2\sqrt{2}}.$$

So the difference between left and right sides is,

$$[r + \tfrac{st}{\sqrt{2}} - \tfrac{rst}{2}] - [\tfrac{r^2}{\sqrt{2}} + \tfrac{rt}{\sqrt{2}} - \tfrac{r^2t}{2} + \tfrac{rs}{\sqrt{2}} + \tfrac{st}{\sqrt{2}} - \tfrac{rst}{2} - \tfrac{r^2s}{2} - \tfrac{rst}{2} + \tfrac{r^2st}{2\sqrt{2}}]$$

$$= r - \tfrac{r^2}{\sqrt{2}} - \tfrac{rt}{\sqrt{2}} + \tfrac{r^2t}{2} - \tfrac{rs}{\sqrt{2}} + \tfrac{r^2s}{2} + \tfrac{rst}{2} - \tfrac{r^2st}{2\sqrt{2}}$$

$$= \tfrac{r(\sqrt{2}-r)((2-st)-\sqrt{2}(s+t))}{2\sqrt{2}}.$$

It is clear that $r \geq 0$, $\sqrt{2} - r \geq 0$, then for $(2-st) - \sqrt{2}(s+t)$ will be investigated as follows,

$$(2-st) - \sqrt{2}(s+t) = 2 - st - \sqrt{2}s - \sqrt{2}t$$

$$= \sqrt{2}(\sqrt{2} - \tfrac{st}{\sqrt{2}} - s - t)$$

$$= \sqrt{2}\left(\sqrt{2} - \left(s + t - \tfrac{st}{\sqrt{2}}\right)\right).$$

Because $s+t-\frac{st}{\sqrt{2}} \leq \sqrt{2}$ for $s,t \in [0, \sqrt{2}]$, then obtained that information $(2-st)-\sqrt{2}(s+t) \geq 0$. So it is proven that $\frac{r(\sqrt{2}-r)((2-st)-\sqrt{2}(s+t))}{2\sqrt{2}} \geq 0$.

(v) To prove the similarity of left and right sides is as follows: $\frac{r(\frac{s+t}{2})}{\sqrt{2}} = \frac{r(s+t)}{2\sqrt{2}} = \frac{\frac{rs}{\sqrt{2}}+\frac{rt}{\sqrt{2}}}{2} = @(\otimes(r,s), \otimes(r,t))$.

(vi) Likewise with point (v), it is obtained,

$$\oplus(r, @(s,t)) = \oplus\left(r, \frac{s+t}{2}\right) = r + \frac{s+t}{2} - \frac{r\left(\frac{s+t}{2}\right)}{\sqrt{2}} = \frac{\left(r+s-\frac{rs}{\sqrt{2}}\right)+\left(r+t-\frac{rt}{\sqrt{2}}\right)}{2} = @(\oplus(r,s), \oplus(r,t)),$$

(vii) The left side value is,

$$@(r, \oplus(s,t)) = \frac{r+\left(s+t-\frac{st}{\sqrt{2}}\right)}{2} = \frac{r+s+t-\frac{st}{\sqrt{2}}}{2} = \frac{2\sqrt{2}r+2\sqrt{2}s+2\sqrt{2}t-2st}{4\sqrt{2}},$$

while its right side is obtained,

$$\begin{aligned}\oplus(@(r,s), @(r,t)) &= \left(\frac{r+s}{2}\right) + \left(\frac{r+t}{2}\right) - \frac{\left(\frac{r+s}{2}\right)\left(\frac{r+t}{2}\right)}{\sqrt{2}}\\ &= \frac{2r+s+t}{2} - \left(\frac{r^2+rt+rs+st}{4\sqrt{2}}\right)\\ &= \frac{4\sqrt{2}r+2\sqrt{2}s+2\sqrt{2}t-r^2-rt-rs-st}{4\sqrt{2}}.\end{aligned}$$

So the difference between left side and right side is,

$$\begin{aligned}&\left[\frac{2\sqrt{2}r+2\sqrt{2}s+2\sqrt{2}t-2st}{4\sqrt{2}}\right] - \left[\frac{4\sqrt{2}r+2\sqrt{2}s+2\sqrt{2}t-r^2-rt-rs-st}{4\sqrt{2}}\right]\\ &= \frac{-st-2\sqrt{2}r+r^2+rt+rs}{4\sqrt{2}}\\ &= \frac{r(r+s+t-2\sqrt{2})-st}{4\sqrt{2}}.\end{aligned}$$

Remember that $s+t-\frac{st}{\sqrt{2}} \leq \sqrt{2}$

$$\begin{aligned}&\leq \frac{r\left(r+\sqrt{2}+\frac{st}{\sqrt{2}}-2\sqrt{2}\right)-st}{4\sqrt{2}}\\ &\leq \frac{(r-\sqrt{2})\left(r+\frac{st}{\sqrt{2}}\right)}{4\sqrt{2}} \leq 0.\end{aligned}$$

It is obtained that $@(r, \oplus(s,t)) \leq \oplus(@(r,s), @(r,t))$.

(viii) From left side,

$$@(r, \otimes(s,t)) = \frac{r+\frac{st}{\sqrt{2}}}{2} = \frac{\sqrt{2}r+st}{2\sqrt{2}}.$$

From right side,

$$\otimes(@(r,s), @(r,t)) = \frac{\left(\frac{r+s}{2}\right)\left(\frac{r+t}{2}\right)}{\sqrt{2}} = \frac{r^2+rt+rs+st}{4\sqrt{2}}.$$

Then the difference is,

$$\left[\frac{\sqrt{2}r+st}{2\sqrt{2}}\right] - \left[\frac{r^2+rt+rs+st}{4\sqrt{2}}\right]$$

$$= \frac{2\sqrt{2}r+st-r^2-rt-rs}{4\sqrt{2}}$$

$$= \frac{r(\sqrt{2}-r)+\sqrt{2}r+st-rt-rs}{4\sqrt{2}}.$$

Remember that $s+t-\frac{st}{\sqrt{2}} \leq \sqrt{2}$ and $st \geq \frac{st}{\sqrt{2}}$,

$$\geq \frac{r(\sqrt{2}-r)+\sqrt{2}r+(s+t-\sqrt{2})-rt-rs}{4\sqrt{2}}$$

$$\geq \frac{r(\sqrt{2}-r)+(1-r)(s+t-\sqrt{2})}{4\sqrt{2}} \geq 0.$$

It is obtained that the value of the left side is greather than equal to the right side for radius, so $@(r,\otimes(s,t)) \geq \otimes(@(r,s),@(r,t))$.

(ix) The same as point (v) is obtained,

$$@(r,@(s,t)) = @(r,\tfrac{s+t}{2}) = \frac{r+\frac{s+t}{2}}{2} = \frac{\frac{r+s}{2}+\frac{r+t}{2}}{2} = @(@(r,s),@(r,t)).$$

Thus, it is proven for the distributive relation between the operators $\otimes, \oplus$ and $@$. □

The distributive properties of *CIFS* have been shown in previous research [24], but the radius operations are only limited to min and max. Furthermore, the distributive properties in algebraic addition, product, and arithmetic mean in *CIFS* will be shown along with additional operators on radius.

Theorem 4.5. (*Distributive Law*) Let $\mathcal{A}_r, \mathcal{B}_s$ and C_t are *CIFSs*, $\phi \in \{+, \circ, @\}$ and $r, s, t \in [0, \sqrt{2}]$ then,

(i) $\mathcal{A}_r \circ_\otimes (\mathcal{B}_s +_\oplus C_t) \subset (\mathcal{A}_r \circ_\otimes \mathcal{B}_s) +_\oplus (\mathcal{A}_r \circ_\otimes C_t)$.

(ii) $\mathcal{A}_r \circ_\otimes (\mathcal{B}_s @_\oplus C_t) \subset_\rho (\mathcal{A}_r \circ_\otimes \mathcal{B}_s) @_\oplus (\mathcal{A}_r \circ_\otimes C_t)$.

(iii) $\mathcal{A}_r \circ_\otimes (\mathcal{B}_s @_\otimes C_t) \subset_\rho (\mathcal{A}_r \circ_\otimes \mathcal{B}_s) @_\otimes (\mathcal{A}_r \circ_\otimes C_t)$.

(iv) $\mathcal{A}_r \circ_\oplus (\mathcal{B}_s +_\oplus C_t) \subset_\rho (\mathcal{A}_r \circ_\oplus \mathcal{B}_s) +_\oplus (\mathcal{A}_r \circ_\oplus C_t)$.

(v) $\mathcal{A}_r \circ_\oplus (\mathcal{B}_s @_\oplus C_t) \subset_\rho (\mathcal{A}_r \circ_\oplus \mathcal{B}_s) @_\oplus (\mathcal{A}_r \circ_\oplus C_t)$.

(vi) $\mathcal{A}_r \circ_\oplus (\mathcal{B}_s @_\otimes C_t) \supset_\rho (\mathcal{A}_r \circ_\oplus \mathcal{B}_s) @_\otimes (\mathcal{A}_r \circ_\oplus C_t)$.

(vii) $\mathcal{A}_r \circ_@ (\mathcal{B}_s @_\otimes C_t) =_\nu (\mathcal{A}_r \circ_@ \mathcal{B}_s) @_\otimes (\mathcal{A}_r \circ_@ C_t)$.

(viii) $\mathcal{A}_r \circ_@ (\mathcal{B}_s @_\oplus C_t) =_\nu (\mathcal{A}_r \circ_@ \mathcal{B}_s) @_\oplus (\mathcal{A}_r \circ_@ C_t)$.

(ix) $\mathcal{A}_r \circ_\phi (\mathcal{B}_s +_@ C_t) \subset_\nu (\mathcal{A}_r \circ_\phi \mathcal{B}_s) +_@ (\mathcal{A}_r \circ_\phi C_t)$.

(x) $\mathcal{A}_r \circ_\phi (\mathcal{B}_s @_@ C_t) = (\mathcal{A}_r \circ_\phi \mathcal{B}_s) @_@ (\mathcal{A}_r \circ_\phi C_t)$.

(xi) $\mathcal{A}_r +_\otimes (\mathcal{B}_s \circ_\otimes C_t) \supset (\mathcal{A}_r +_\otimes \mathcal{B}_s) \circ_\otimes (\mathcal{A}_r +_\otimes C_t)$.

(xii) $\mathcal{A}_r +_\otimes (\mathcal{B}_s @_\oplus C_t) \subset_\rho (\mathcal{A}_r +_\otimes \mathcal{B}_s) @_\oplus (\mathcal{A}_r +_\otimes C_t)$.

(xiii) $\mathcal{A}_r +_\otimes (\mathcal{B}_s @_\otimes C_t) \supset_\rho (\mathcal{A}_r +_\otimes \mathcal{B}_s) @_\otimes (\mathcal{A}_r +_\otimes C_t)$.

(xiv) $\mathcal{A}_r +_\oplus (\mathcal{B}_s \circ_\otimes C_t) \supset (\mathcal{A}_r +_\oplus \mathcal{B}_s) \circ_\otimes (\mathcal{A}_r +_\oplus C_t)$.

(xv) $\mathcal{A}_r +_{\oplus} (\mathcal{B}_s \circ_{\otimes} C_t) \supset (\mathcal{A}_r +_{\oplus} \mathcal{B}_s) \circ_{\otimes} (\mathcal{A}_r +_{\oplus} C_t)$.

(xvi) $\mathcal{A}_r +_{\oplus} (\mathcal{B}_s @_{\oplus} C_t) \subset_{\rho} (\mathcal{A}_r +_{\oplus} \mathcal{B}_s) @_{\oplus} (\mathcal{A}_r +_{\oplus} C_t)$.

(xvii) $\mathcal{A}_r +_{\oplus} (\mathcal{B}_s @_{\otimes} C_t) \supset_{\rho} (\mathcal{A}_r +_{\oplus} \mathcal{B}_s) @_{\otimes} (\mathcal{A}_r +_{\oplus} C_t)$.

(xviii) $\mathcal{A}_r +_{@} (\mathcal{B}_s @_{\otimes} C_t) =_{\nu} (\mathcal{A}_r +_{@} \mathcal{B}_s) @_{\otimes} (\mathcal{A}_r +_{@} C_t)$.

(xix) $\mathcal{A}_r +_{@} (\mathcal{B}_s @_{\oplus} C_t) =_{\nu} (\mathcal{A}_r +_{@} \mathcal{B}_s) @_{\oplus} (\mathcal{A}_r +_{@} C_t)$.

(xx) $\mathcal{A}_r +_{\psi} (\mathcal{B}_s @_{@} C_t) = (\mathcal{A}_r +_{\psi} \mathcal{B}_s) @_{@} (\mathcal{A}_r +_{\psi} C_t)$.

(xxi) $\mathcal{A}_r +_{\psi} (\mathcal{B}_s \circ_{@} C_t) \supset_{\nu} (\mathcal{A}_r +_{\psi} \mathcal{B}_s) \circ_{@} (\mathcal{A}_r +_{\psi} C_t)$.

(xxii) $\mathcal{A}_r @_{\otimes} (\mathcal{B}_s +_{\oplus} C_t) \subset (\mathcal{A}_r @_{\otimes} \mathcal{B}_s) +_{\oplus} (\mathcal{A}_r @_{\otimes} C_t)$.

(xxiii) $\mathcal{A}_r @_{\otimes} (\mathcal{B}_s \circ_{\otimes} C_t) \supset (\mathcal{A}_r @_{\otimes} \mathcal{B}_s) \circ_{\otimes} (\mathcal{A}_r @_{\otimes} C_t)$.

(xxiv) $\mathcal{A}_r @_{\oplus} (\mathcal{B}_s +_{\oplus} C_t) \subset (\mathcal{A}_r @_{\oplus} \mathcal{B}_s) +_{\oplus} (\mathcal{A}_r @_{\oplus} C_t)$.

(xxv) $\mathcal{A}_r @_{\oplus} (\mathcal{B}_s \circ_{\otimes} C_t) \supset (\mathcal{A}_r @_{\oplus} \mathcal{B}_s) \circ_{\otimes} (\mathcal{A}_r @_{\oplus} C_t)$.

(xxvi) $\mathcal{A}_r @_{\psi} (\mathcal{B}_s +_{@} C_t) \subset_{\nu} (\mathcal{A}_r @_{\psi} \mathcal{B}_s) +_{@} (\mathcal{A}_r @_{\psi} C_t)$.

(xxvii) $\mathcal{A}_r @_{\psi} (\mathcal{B}_s \circ_{@} C_t) \subset_{\nu} (\mathcal{A}_r @_{\psi} \mathcal{B}_s) \circ_{@} (\mathcal{A}_r @_{\psi} C_t)$.

Proof. The proof of this theorem can be demonstrated by utilizing Theorem in Atanassov [24] and Theorem 4.4. □

Remark 4. The proof that has been carried out in Theorems 4.3 and 4.5 can also be applied to IFS ($r = 0$). If the radius is 0, then just ignore the radius relation in the relation operator. As an example of the distributive property points 2 and 7, if $r = s = t = 0$ then $\mathcal{A}_r \circ_{\otimes} (\mathcal{B}_s @_{\oplus} C_t) \subset (\mathcal{A}_r \circ_{\otimes} \mathcal{B}_s) @_{\oplus} (\mathcal{A}_r \circ_{\otimes} C_t)$ and $\mathcal{A}_r \circ_{@} (\mathcal{B}_s @_{\otimes} C_t) = (\mathcal{A}_r \circ_{@} \mathcal{B}_s) @_{\otimes} (\mathcal{A}_r \circ_{@} C_t)$.

5. Proposed modified negation operators on CIFS

In the original paper [24], Atanssov defined the negation operators on *CIFS* as redefined from *IFS* which only affects $\mathcal{M}$ and $\mathcal{N}$, but not radius. Next, we will define a type negation operator based on the radius condition.

Definition 5.1. Let $\mathcal{A}_r$ is *CIFS* and $r \in [0, \sqrt{2}]$ then, modified negation operator based on radius are the following:

$$\neg_2(\mathcal{A}_r) = \{\langle x, \mathcal{M}_{\mathcal{A}_r}(x), \mathcal{N}_{\mathcal{A}_r}(x);\ \sqrt{2} - r\rangle | x \in X\},$$
$$\neg_3(\mathcal{A}_r) = \{\langle x, \mathcal{N}_{\mathcal{A}_r}(x), \mathcal{M}_{\mathcal{A}_r}(x);\ \sqrt{2} - r\rangle | x \in X\}.$$

It is clear that the negation operator defined earlier in [24] is type-1 negation i.e. $\neg_1(\mathcal{A}_r) = \{\langle x, \mathcal{N}_{\mathcal{A}_r}(x), \mathcal{M}_{\mathcal{A}_r}(x); r\rangle\}$. The type-2 and type-3 negation operators satisfy the complement axioms, boundary conditions, monotonic descent, continuity, and involution properties. These changes are based on operations on the radius giving rise to some properties that apply to the definition.

Theorem 5.1. The following equalities are valid for *CIFSs* $\mathcal{A}_r$,

(i) $\neg_1(\neg_1(\mathcal{A}_r)) = \mathcal{A}_r$ likewise for $\neg_2$ and $\neg_3$.

(ii) $\neg_1(\neg_2(\mathcal{A}_r)) = \neg_3(\mathcal{A}_r)$.

(iii) $\neg_2(\neg_1(\mathcal{A}_r)) = \neg_3(\mathcal{A}_r)$.

(iv) $\mathcal{A}_r \subseteq_\rho \neg_2(\mathcal{A}_r) \Leftrightarrow r \leq \frac{\sqrt{2}}{2}$.

(v) $\mathcal{A}_r \supseteq_\rho \neg_2(\mathcal{A}_r) \Leftrightarrow r \geq \frac{\sqrt{2}}{2}$.

Proof. For points (i) until (iii), it is clearly proven by Definition 5.1. For the rest, it is sufficient to prove if $r \leq \sqrt{2} - r$ then $r \leq \frac{\sqrt{2}}{2}$ and vice versa. □

Theorem 5.2. (*De'Morgan Law*) The following equalities are valid for *CIFSs* $\mathcal{A}_r$ and $\mathcal{B}_s$ for $\phi \in \{\cap, \cup, +, \circ, @\}$ and $\propto \in \{\max, \min, \oplus, \otimes, @\}$,

(i) $\neg_1(\mathcal{A}_r @_\propto \mathcal{B}_s) = \neg_1(\mathcal{A}_r) @_\propto \neg_1(\mathcal{B}_s)$.

(ii) $\neg_2(\mathcal{A}_r \phi_@ \mathcal{B}_s) = \neg_2(\mathcal{A}_r) \phi_@ \neg_2(\mathcal{B}_s)$.

(iii) $\neg_1\left[\neg_1(\mathcal{A}_r \phi_\propto \mathcal{B}_s)\right] = \mathcal{A}_r \phi_\propto \mathcal{B}_s$.

(iv) $\neg_2\left[\neg_1(\mathcal{A}_r \cap_{\max/\min} \mathcal{B}_s)\right] = \neg_3(\mathcal{A}_r) \cup_{\min/\max} \neg_3(\mathcal{B}_s)$.

(v) $\neg_2\left[\neg_1(\mathcal{A}_r \cup_{\max/\min} \mathcal{B}_s)\right] = \neg_3(\mathcal{A}_r) \cap_{\min/\max} \neg_3(\mathcal{B}_s)$.

(vi) $\neg_2\left[\neg_1(\mathcal{A}_r +_{\max/\min} \mathcal{B}_s)\right] = \neg_3(\mathcal{A}_r) \circ_{\min/\max} \neg_3(\mathcal{B}_s)$.

(vii) $\neg_2\left[\neg_1(\mathcal{A}_r \circ_{\max/\min} \mathcal{B}_s)\right] = \neg_3(\mathcal{A}_r) +_{\min/\max} \neg_3(\mathcal{B}_s)$.

(viii) $\neg_2\left[\neg_1(\mathcal{A}_r @_{\max/\min} \mathcal{B}_s)\right] = \neg_3(\mathcal{A}_r) @_{\min/\max} \neg_3(\mathcal{B}_s)$.

(ix) $\neg_2\left[\neg_1(\mathcal{A}_r @_@ \mathcal{B}_s)\right] = \neg_3(\mathcal{A}_r) @_@ \neg_3(\mathcal{B}_s)$

Proof. It is clearly proven by Definitions 2.5 and 5.1. □

Next will be defined another modal operators "necessity" and "possibility". The previously defined modal operators [24] only affect membership or non-membership functions, not radius. This is the reason why the modal operators "necessity" and "possibility" also affect the radius (denoted $\Box_2$ and $\Diamond_2$) as follows:

Definition 5.2. Let $\mathcal{A}_r$ is *CIFS*, then modified modal operator based form radius are the following:

$$\Box_2 \mathcal{A}_r = \{\langle x, \mathcal{M}_\mathcal{A}(x), 1 - \mathcal{M}_\mathcal{A}(x); \sqrt{2} - r\rangle | x \in X\},$$
$$\Diamond_2 \mathcal{A}_r = \{\langle x, 1 - \mathcal{N}_\mathcal{A}(x), \mathcal{N}_\mathcal{A}(x); \sqrt{2} - r\rangle | x \in X\}.$$

Similar to the negation operator, the modal operators "necessity" and "possibility" [24] are symbolized by $\Box_1$ and $\Diamond_1$. Some properties derived from these modal operators are presented in next the theorem:

Theorem 5.3. The following equalities are valid for *CIFS* $\mathcal{A}_r$,

(i) $\Box_1 \mathcal{A}_r \subseteq_\nu \mathcal{A}_r$.

(ii) $\mathcal{A}_r \subseteq_\nu \Diamond_1\mathcal{A}_r$.

(iii) $\Diamond_1(\Box_1\mathcal{A}_r) = \Box_1\mathcal{A}_r$.

(iv) $\Box_1(\Diamond_1\mathcal{A}_r) = \Diamond_1\mathcal{A}_r$.

(v) $\Box_1(\Box_1...(\Box_1(\Box_1\mathcal{A}_r))) = \Box_1\mathcal{A}_r$.

(vi) $\Diamond_1(\Diamond_1...(\Diamond_1(\Diamond_1\mathcal{A}_r))) = \Diamond_1\mathcal{A}_r$.

(vii) $r \geq \frac{\sqrt{2}}{2} \Leftrightarrow \Box_2\mathcal{A}_r \subseteq \mathcal{A}_r$ and $r < \frac{\sqrt{2}}{2} \Leftrightarrow \Box_2\mathcal{A}_r \supset \mathcal{A}_r$.

(viii) $r \leq \frac{\sqrt{2}}{2} \Leftrightarrow \mathcal{A}_r \subseteq \Diamond_2\mathcal{A}_r$ and $r < \frac{\sqrt{2}}{2} \Leftrightarrow \mathcal{A}_r \supset \Diamond_2\mathcal{A}_r$.

(ix) $\Diamond_2(\Box_2\mathcal{A}_r) = \Box_1\mathcal{A}_r$.

(x) $\Box_2(\Diamond_2\mathcal{A}_{r_\neg})_\neg = \Diamond_1\mathcal{A}_r$.

(xi) $\Box_2(\Box_2\mathcal{A}_{r_\neg})_\neg = \Box_1\mathcal{A}_r$.

(xii) $\Diamond_2(\Diamond_2\mathcal{A}_r) = \Diamond_1\mathcal{A}_r$.

(xiii) $\Box_2(\Box_2(\Box_2\mathcal{A}_r)) = \Box_2\mathcal{A}_r$.

(xiv) $\Diamond_2(\Diamond_2(\Diamond_2\mathcal{A}_r)) = \Diamond_2\mathcal{A}_r$.

(xv) $\underbrace{\Box_2(\Box_2...(\Box_2\mathcal{A}_r))}_{n \text{ factor}} = \begin{cases} \Box_1\mathcal{A}_r \text{ , for n even number,} \\ \Box_2\mathcal{A}_r \text{ , for n odd number.} \end{cases}$

(xvi) $\underbrace{\Diamond_2(\Diamond_2...(\Diamond_2\mathcal{A}_r))}_{n \text{ factor}} = \begin{cases} \Diamond_1\mathcal{A}_r \text{ , for n even number,} \\ \Diamond_2\mathcal{A}_r \text{ , for n odd number.} \end{cases}$

Proof. This proof will be carried out at each point,

(i) It is clear that the difference between $\Box_1\mathcal{A}_r$ and $\mathcal{A}_r$ lies in the value of $\mathcal{N}_{\Box_1\mathcal{A}_r} \leq \mathcal{N}_{\mathcal{A}_r}$, so it holds $\Box_1\mathcal{A}_r \subseteq_\rho \mathcal{A}_r$.

(ii) Same with point (i), the difference is in the value $\mathcal{M}_{\Diamond_1\mathcal{A}_r} \geq \mathcal{M}_{\mathcal{A}_r}$ so it happens $\mathcal{A}_r \subseteq_\rho \Diamond_1\mathcal{A}_r$.

(iii) It is clear by Definition 5.2,

$$\begin{aligned} \Diamond_1(\Box_1\mathcal{A}_r) &= \Diamond_1\{\langle x, \mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x), r\rangle\} \\ &= \{\langle x, 1 - (1 - \mathcal{M}_{\mathcal{A}}(x)), 1 - \mathcal{M}_{\mathcal{A}}(x), r\rangle\} \\ &= \{\langle x, \mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x), r\rangle\} \\ &= \Box_1\mathcal{A}_r. \end{aligned}$$

(iv) Same with point (iii),

$$\begin{aligned} \Box_1(\Diamond_1\mathcal{A}_r) &= \Box\{\langle x, 1 - \mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x), r\rangle\} \\ &= \{\langle x, 1 - \mathcal{N}_{\mathcal{A}}(x), 1 - (1 - \mathcal{N}_{\mathcal{A}}(x)), r\rangle\} \\ &= \{\langle x, 1 - \mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x), r\rangle\} \\ &= \Diamond_1\mathcal{A}_r. \end{aligned}$$

(v) Let start for $n = 1$, it's clear. For $n = 2, 3, 4, ...$ it will be,

$$\begin{aligned}\Box_1(\Box_1\mathcal{A}_r) &= \Box_1\{\langle x, \mathcal{M}_{\mathcal{A}_r}(x), 1 - \mathcal{M}_{\mathcal{A}_r}(x), r\rangle\}\\ &= \{\langle x, \mathcal{M}_{\mathcal{A}_r}(x), 1 - \mathcal{M}_{\mathcal{A}_r}(x), r\rangle\}\\ &= \Box_1\mathcal{A}_r,\end{aligned}$$

recursively get,

$$\Box_1(\Box_1...(\Box_1(\Box_1\mathcal{A}_r))) = \Box_1(\Box_1...(\Box_1\mathcal{A}_r)) = ... = (\Box_1(\Box_1\mathcal{A}_r)) = \Box_1\mathcal{A}_r.$$

(vi) Same with point (v) and get,

$$\Diamond_1(\Diamond_1...(\Diamond_1(\Diamond_1\mathcal{A}_r))) = \Diamond_1(\Diamond_1...(\Diamond_1\mathcal{A}_r)) = ... = (\Diamond_1(\Diamond_1\mathcal{A}_r)) = \Diamond_1\mathcal{A}_r.$$

(vii) From left side, if $r \geq \frac{\sqrt{2}}{2}$ then value in $\Box_2\mathcal{A}_r$ is $\sqrt{2} - r \leq \frac{\sqrt{2}}{2}$ and the value in $\mathcal{A}_r \geq \frac{\sqrt{2}}{2}$. In addition, membership and non-membership grades are obtained $\Box_2 A_r = \langle x, \mathcal{M}_{\mathcal{A}_r}(x), 1 - \mathcal{M}_{\mathcal{A}_r}(x); \sqrt{2} - r\rangle$. The fact that $\mathcal{N}_{\mathcal{A}_r}(x) = 1 - \mathcal{M}_{\mathcal{A}_r} - \mathcal{H}_{\mathcal{A}_r}$, then $\mathcal{N}_{\mathcal{A}_r} \leq 1 - \mathcal{M}_{\mathcal{A}_r}(x)$ so proved that $\Box_2\mathcal{A}_r \subseteq \mathcal{A}_r$. From right side, if $\Box_2\mathcal{A}_r \subseteq \mathcal{A}_r$ then, $\frac{\sqrt{2}}{2} \leq r$. Same method for $\Box_2\mathcal{A}_r \supset \mathcal{A}_r$ iff $r < \frac{\sqrt{2}}{2}$.

(viii) From left side, if $r \leq \frac{\sqrt{2}}{2}$ then value in $\Diamond_2\mathcal{A}_r$ is $\sqrt{2} - r \geq \frac{\sqrt{2}}{2}$ and the value in $\mathcal{A}_r \leq \frac{\sqrt{2}}{2}$. In addition, membership and non-membership grades are obtained $\Diamond_2\mathcal{A}_r = \langle x, 1 - \mathcal{N}_{\mathcal{A}_r}(x), \mathcal{N}_{\mathcal{A}_r}(x); \sqrt{2} - r\rangle$. The fact that $\mathcal{M}_{\mathcal{A}_r}(x) = 1 - \mathcal{N}_{\mathcal{A}_r} - \mathcal{H}_{\mathcal{A}_r}$, then $\mathcal{M}_{\mathcal{A}_r} \leq 1 - \mathcal{N}_{\mathcal{A}_r}(x)$ so proved that $\mathcal{A}_r \subseteq \Diamond_2\mathcal{A}_r$. From right side, if $\mathcal{A}_r \subseteq \Diamond_2\mathcal{A}_r$ then $r \leq \frac{\sqrt{2}}{2}$ same method for $\mathcal{A}_r \supset \Diamond_2\mathcal{A}_r$ iff $r > \frac{\sqrt{2}}{2}$.

(ix) Using the modified two negation combination on the modal operator according to Definition 5.2,

$$\begin{aligned}\Diamond_2(\Box_2\mathcal{A}_r) &= \Diamond_2\{\langle x, \mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x), \sqrt{2} - r\rangle\}\\ &= \{\langle x, 1 - (1 - \mathcal{M}_{\mathcal{A}}(x)), 1 - \mathcal{M}_{\mathcal{A}}(x), \sqrt{2} - (\sqrt{2} - r)\rangle\}\\ &= \{\langle x, \mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x), r\rangle\}\\ &= \Box_1\mathcal{A}_r.\end{aligned}$$

(x) Same using combination from Definition 5.2,

$$\begin{aligned}\Box_2(\Diamond_2\mathcal{A}_r) &= \Diamond_2\{\langle x, 1 - \mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x), \sqrt{2} - r\rangle\}\\ &= \{\langle x, 1 - \mathcal{N}_{\mathcal{A}}(x), 1 - (1 - \mathcal{N}_{\mathcal{A}}(x)), \sqrt{2} - (\sqrt{2} - r)\rangle\}\\ &= \{\langle x, 1 - \mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x), r\rangle\}\\ &= \Diamond_1\mathcal{A}_r.\end{aligned}$$

(xi) Equivalent with the previous,

$$\begin{aligned}\Box_2(\Box_2\mathcal{A}_r) &= \Box_2(\langle x, \mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x); \sqrt{2} - r\rangle)\\ &= \langle x, \mathcal{M}_{\mathcal{A}}(x), 1 - \mathcal{M}_{\mathcal{A}}(x); \sqrt{2} - (\sqrt{2} - r)\rangle\\ &= \Box_1\mathcal{A}_r.\end{aligned}$$

(xii) Equivalent with the previous,

$$\begin{aligned}\Diamond_2(\Diamond_2\mathcal{A}_r) &= \Diamond_2(\langle x, 1-\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x);\ \sqrt{2}-r\rangle)\\ &= \langle x, 1-\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x);\ \sqrt{2}-(\sqrt{2}-r)\rangle\\ &= \Diamond_1\mathcal{A}_r.\end{aligned}$$

(xiii) From point (xi),

$$\begin{aligned}\Box_2(\Box_2(\Box_2\mathcal{A}_r)) &= \Box_2(\Box_2\mathcal{A}_r)\\ &= \Box_2(\langle x, \mathcal{M}_{\mathcal{A}}(x), 1-\mathcal{M}_{\mathcal{A}}; r\rangle)\\ &= (\langle x, \mathcal{M}_{\mathcal{A}}(x), 1-\mathcal{M}_{\mathcal{A}};\ \sqrt{2}-r\rangle)\\ &= \Box_2\mathcal{A}_r.\end{aligned}$$

(xiv) From point (xii),

$$\begin{aligned}\Diamond_2(\Diamond_2(\Diamond_2\mathcal{A}_r))) &= \Diamond_2(\Diamond_2\mathcal{A}_r)\\ &= \Diamond_2(\langle x, 1-\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}; r\rangle)\\ &= (\langle x, 1-\mathcal{N}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}};\ \sqrt{2}-r\rangle)\\ &= \Diamond_2\mathcal{A}_r.\end{aligned}$$

(xv) From points (xi) and (xiii), it's clear to proved that,

$$\underbrace{\Box_2(\Box_2...(\Box_2\mathcal{A}_r))}_{n factor} = \begin{cases}\Box_1\mathcal{A}_r \text{ , for } n \text{ even number,}\\ \Box_2\mathcal{A}_r \text{ , for } n \text{ odd number.}\end{cases}$$

(xvi) From points (xii) and (xiv), it's clear to proved that,

$$\underbrace{\Diamond_2(\Diamond_2...(\Diamond_2 A_r))}_{n factor} = \begin{cases}\Diamond_1\mathcal{A}_r \text{ , for } n \text{ even number,}\\ \Diamond_2\mathcal{A}_r \text{ , for } n \text{ odd number.}\end{cases}$$

□

Remark 5. Similarly with Remark 4, we can also apply the negation operators to IFS. Let $\mathcal{A}_0 = \{\langle x, \mathcal{M}_{\mathcal{A}_r}(x), \mathcal{N}_{\mathcal{A}_r}(x); 0\rangle | x \in X\}$ *and* $\mathcal{A}_{\sqrt{2}} = \{\langle x, \mathcal{M}_{\mathcal{A}_r}(x), \mathcal{N}_{\mathcal{A}_r}(x);\ \sqrt{2}\rangle | x \in X\}$ *then,*

$$\begin{aligned}&\neg_1(\mathcal{A}_r) = \neg\mathcal{A}, \textit{ for any IFS } \mathcal{A},\\ &\neg_2(\mathcal{A}_r) = \mathcal{A}_{\sqrt{2}}, \textit{ and}\\ &\neg_3(\mathcal{A}_r) = \neg_1(\mathcal{A}_{\sqrt{2}}).\end{aligned}$$

Likewise for modal operators, so it is obtained

$$\begin{aligned}&\Box_1\mathcal{A}_r = \Box\mathcal{A}; \Box_2\mathcal{A}_r = \mathcal{A}_{\sqrt{2}}, \textit{ and}\\ &\Diamond_1\mathcal{A}_r = \Diamond\mathcal{A}; \Diamond_2\mathcal{A}_r = \mathcal{A}_{\sqrt{2}}.\end{aligned}$$

Analogously for Theorems 5.1 and 5.3 just ignore the radius relation in the relation operator.

6. Conclusions and discussion

This research builds on Atanassov's work on theoretical *CIFS*, focusing on defining alternative operations for radius beyond minimum and maximum functions. These operations, namely Radius Algebraic Product (*RAP*) and Radius Algebraic Sum (*RAS*), leverage the properties of *t*-norm and *t*-conorm. Additionally, we introduce the arithmetic mean operator as a radius operator, distinct from traditional *t*-norm or *t*-conorm categories. These three operators share structural and characteristic similarities with algebraic product *t*-norm and probabilistic *t*-conorm, including algebraic properties, idempotence, nilpotence, and zero divisor elements.

Following their definition, we integrate these operations with those defined by Atanassov, extending them to generalize intersection and union based on *t*-norm (conorm). We explore various properties, from commutativity to associativity and distributivity, to assess the consistency of these operations. Furthermore, we propose alternative negation and modal operators beyond those defined by Atanassov, and examine related theorems.

This work contributes to the literature on *CIFS* theory, providing valuable tools for researchers. However, it also suggests avenues for further exploration, particularly in the realm of decision-making processes. Future research should delve into the multiplicative characteristics of *CIFS* operators, which serve as the foundation for developing aggregation operators such as Weighted Averaging (*WA*), Ordered WA (*OWA*), Weighted Geometric (*WG*), or Ordered WG (*OWG*). Flexibility in radius selection is crucial for decision-making agility. Moreover, the introduction of additional radius operators prompts innovation in *MCDM* methods such as *TOPSIS*, *AHP*, *ELECTRE*, *DEMATEL*, etc.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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