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*Research article*

# Approximation of fixed points for a new class of generalized non-expansive mappings in Banach spaces

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**Abstract:** In this paper, we first introduced a new class of generalized non-expansive mappings, which was larger than the class satisfying the condition $B_{\gamma,\mu}$. Also, we proposed a new iterative process to approximate the fixed point of the mapping we introduced in this work, then we prove convergence theorems for these mappings by using our iteration process. Lastly, a numerical example was given to show the efficiency of this new iteration process. Our results were the extension and generalization of many known results in the literature in fixed point theory.

**Keywords:** fixed point; generalized non-expansive mappings; uniformly convex Banach space
**Mathematics Subject Classification:** 47H09, 47H10, 47J25, 54H25

## 1. Introduction

The theory of fixed points plays a very important role in nonlinear analysis. In the recent years, the generalization of non-expansive mappings with different applications has been reviewed by many authors (see [2, 5, 9, 11, 12, 15, 18, 20, 21, 23, 32]), and the references therein. In this context, different new mapping classes have been developed with interesting properties in the following years.

In 2008, Suzuki [30] defined a different class of generalized non-expansive mappings, which is known as Suzuki's generalized non-expansive mapping and is also referred as condition $(C)$. Suzuki proved that the mappings satisfying the condition $(C)$ are weaker than non-expansive and also obtained few results related to the existence of fixed points for such mappings. Many authors have contributed to the literature by generalizing the Suzuki's generalized non-expansive mapping (see [6, 10, 19, 24]). For approximation of fixed points and non-expansive mapping we refer to [8, 14, 26, 28] or to the recent updates [13, 33] in the fixed point theory. For more fixed point results with iterative techniques for single and multivalued mappings we refer to [4, 17]. Also, for the sake of application on fractional evolution equations and partial differential equations, we suggest [16, 22, 29].

Later, in 2018, Patir et al. [25] introduced another generalization of non-expansive mappings, called the condition $B_{\gamma,\mu}$ and proved some weak and strong convergence results for this type of mappings in uniformly convex Banach spaces. A lot of authors have used various iterative methods reckoning fixed points of nonlinear mappings, which is a captivating problem of nonlinear analysis. The most renowned iterative method was improved by Picard, that is, $x_{n+1} = \Upsilon x_n$. It is well-known that Banach contraction principle uses the Picard iteration to approximate the unique fixed point of $\Upsilon$, where $\Upsilon$ is a contraction mapping. Nevertheless, for non-expansive generalized non-expansive mappings, the Picard iterative method may fail to converge to the fixed point in general. To calculate fixed points of these mappings, it is natural to investigate new iterative methods in the current literature.

In 2017, Ullah and Arshad [34] introduced the following iteration process, namely, the M* iteration process:

$$\begin{aligned} \varrho_0 &\in \Omega, \\ \varrho_{n+1} &= \Upsilon \upsilon_n, \\ \upsilon_n &= \Upsilon((1-\alpha_n)\varrho_n + \alpha_n \Upsilon \varpi_n), \\ \varpi_n &= (1-\beta_n)\varrho_n + \beta_n \Upsilon \varrho_n, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They proved some weak and strong convergence theorems for the fixed point of Suzuki generalized non-expansive mappings in uniformly convex Banach spaces.

Recently, Ali and Ali [7] introduced a new iteration process the called F iteration as follows:

$$\begin{aligned} \varrho_0 &\in \Omega, \\ \varrho_{n+1} &= \Upsilon \upsilon_n, \\ \upsilon_n &= \Upsilon \varpi_n, \\ \varpi_n &= \Upsilon((1-\alpha_n)\varrho_n + \alpha_n \Upsilon \varrho_n), \end{aligned} \tag{1.2}$$

where $\alpha_n \in (0,1)$. They showed that the F iteration process has a better rate of convergence when compared with the other iterations.

Abdeljawad et al. [1] showed that the so-called JA iteration process, which they proposed in 2020, has a convergence faster than the other iterations in the literature for mappings satisfying the condition $B_{\gamma,\mu}$ in the setting of uniformly convex Banach spaces. The iteration process reads as follows:

$$\begin{aligned} \varrho_0 &\in \Omega, \\ \varrho_{n+1} &= \Upsilon((1-\alpha_n)\Upsilon\varrho_n + \alpha_n \Upsilon \upsilon_n), \\ \upsilon_n &= \Upsilon \varpi_n, \\ \varpi_n &= (1-\beta_n)\varrho_n + \beta_n \Upsilon \varrho_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Motivated by the above work, in this paper, we introduce a new class of mappings, called the condition $B_{\gamma,\mu,\eta}$, which is wider than the classes satisfying the condition $B_{\gamma,\mu}$. Also, we propose a new iteration process to approximate fixed points of such mappings. We prove some weak and strong convergence results for mappings satisfying the condition $B_{\gamma,\mu,\eta}$ by using iteration process (3.1). Also, we compare the speed of the proposed iteration with abovementioned iteration processes by giving a numerical example.

## 2. Preliminaries

First, we give some basic definitions and a relevant lemma.

**Definition 2.1.** *[30] Let $\Upsilon$ be a mapping on a subset $\Omega$ of a Banach space $\mathcal{M}$, then $\Upsilon$ is said to satisfy the condition $(C)$ if*

$$\frac{1}{2}\|\varrho - \Upsilon\varrho\| \leq \|\varrho - \upsilon\| \text{ implies } \|\Upsilon\varrho - \Upsilon\upsilon\| \leq \|\varrho - \upsilon\|$$

for all $\varrho, \upsilon \in \Omega$.

**Definition 2.2.** *[25] Let $\Omega$ be a nonempty subset of a Banach space $\mathcal{M}$. Let $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ be such that $2\mu \leq \gamma$. A mapping $\Upsilon : \Omega \longrightarrow \mathcal{M}$ is said to satisfy the condition $B_{\gamma,\mu}$ on $\Omega$ if, for all $\varrho, \upsilon$ in $\Omega$,*

$$\begin{aligned} \gamma\|\varrho - \Upsilon\varrho\| &\leq \|\varrho - \upsilon\| + \mu\|\upsilon - \Upsilon\upsilon\| \textit{ implies} \\ \|\Upsilon\varrho - \Upsilon\upsilon\| &\leq (1-\gamma)\|\varrho - \upsilon\| + \mu(\|\varrho - \Upsilon\upsilon\| + \|\upsilon - \Upsilon\varrho\|). \end{aligned}$$

**Definition 2.3.** *[21] Let $\Omega$ be a nonempty subset of a Banach space $\mathcal{M}$ and $\{\varrho_n\}$ be a bounded sequence in $\mathcal{M}$. For $\varrho \in \mathcal{M}$.*

The asymptotic radius of $\{\varrho_n\}$ at $\varrho$ is defined by

$$r(\varrho, \{\varrho_n\}) = \lim\sup_{n\to+\infty} \|\varrho_n - \varrho\|.$$

The asymptotic radius of $\{\varrho n\}$ relative to $\Omega$ is defined by

$$r(\Omega, \{\varrho_n\}) = \inf\{r(\varrho, \{\varrho_n\} : \varrho \in \Omega\},$$

and the asymptotic center of $\{\varrho_n\}$ relative to $\Omega$ is defined by

$$A(\Omega, \{\varrho_n\}) = \{\varrho \in \Omega : r(\varrho, \{\varrho_n\}) = r(\Omega, \{\varrho_n\})\}.$$

We note that if $\Omega$ is weakly compact, the asymptotic center $A(\Omega, \{\varrho_n\})$ is nonempty. If $\mathcal{M}$ is uniformly convex, then, $A(\Omega, \{\varrho_n\})$ has exactly one point.

**Definition 2.4.** *[31] A Banach space $\mathcal{M}$ is said to satisfy the Opial property if for any sequence $\{\varrho_n\}$ in $\mathcal{M}$ with $\varrho_n \rightharpoonup \varpi$, we have*

$$\lim_{n\to+\infty}\inf\|\varrho_n-\varpi\|<\lim_{n\to+\infty}\inf\|\varrho_n-\upsilon\|,$$

for all $\upsilon\in\mathcal{M}$ with $\upsilon\neq\varpi$.

**Lemma 2.1.** *[27] Let $\mathcal{M}$ be uniformly convex Banach space and $\{t_n\}$ be a sequence in $[a,b]$ for some $a,b\in(0,1)$. Suppose that the sequences $\{\varrho_n\}$ and $\{\upsilon_n\}$ in $\mathcal{M}$ are such that $\limsup_{n\longrightarrow+\infty}\|\varrho_n\|\leq r$, $\limsup_{n\longrightarrow+\infty}\|\upsilon_n\|\leq r$ and $\limsup_{n\longrightarrow+\infty}\|(1-t_n)\varrho_n+t_n\upsilon_n\|=r$ for some $r\geq 0$. Then, $\lim_{n\to+\infty}\|\varrho_n-\upsilon_n\|=0$.*

## 3. Main results

In this section, we first define a new class of non-expansive mappings, called the condition $B_{\gamma,\mu,\eta}$. Later, we introduce a new iteration process and we prove some convergence theorem for mappings satisfying the condition $B_{\gamma,\mu,\eta}$ by this iteration.

**Definition 3.1.** *Let $\Omega$ be a nonempty subset of a Banach space $\mathcal{M}$. Let $\gamma\in[0,1]$ and $\mu,\eta\in[0,\frac{1}{2}]$ such that $2\mu+2\eta\leq\gamma$, then a mapping $\Upsilon:\Omega\longrightarrow\mathcal{M}$ is said to satisfy the condition $B_{\gamma,\mu,\eta}$ on $\Omega$ if for all $\varrho,\upsilon$ in $\Omega$,*

$$\begin{aligned}\gamma\|\varrho-\Upsilon\varrho\| &\leq \|\varrho-\upsilon\|+\mu\|\upsilon-\Upsilon\upsilon\|+\eta\|\varrho-\Upsilon\upsilon\| \text{ implies}\\ \|\Upsilon\varrho-\Upsilon\upsilon\| &\leq (1-\gamma)\|\varrho-\upsilon\|+\mu(\|\varrho-\Upsilon\upsilon\|+\|\upsilon-\Upsilon\varrho\|)\\ &\quad+\eta(\|\varrho-\Upsilon\varrho\|+\|\upsilon-\Upsilon\upsilon\|).\end{aligned}$$

**Example 3.1.** *Let $\Upsilon:[0,4]\longrightarrow\mathbb{R}$ be defined by*

$$\Upsilon\varrho=\begin{cases}0, & \text{if } \varrho\neq 4,\\ 1, & \text{if } \varrho=4.\end{cases}$$

Choose $\gamma=1$, $\mu=\frac{1}{4}$ and $\eta=\frac{1}{4}$, then $\Upsilon$ satisfies the condition $B_{\gamma,\mu,\eta}$. We consider different cases as follows:

(i) For $\varrho\neq 4$, $\upsilon\neq 4$, we have $\|\Upsilon\varrho-\Upsilon\upsilon\|=0$. Obviously, $\Upsilon$ satisfies the condition $B_{\gamma,\mu,\eta}$.

(ii) For $\varrho\neq 4$, $\upsilon=4$, we have $\|\Upsilon\varrho-\Upsilon\upsilon\|=1$, and for $\gamma=1$, $\mu=\frac{1}{4}$, and $\eta=\frac{1}{4}$.

$$\begin{aligned}&(1-\gamma)\|\varrho-\upsilon\|+\mu(\|\varrho-\Upsilon\upsilon\|+\|\upsilon-\Upsilon\varrho\|)+\eta(\|\varrho-\Upsilon\varrho\|+\|\upsilon-\Upsilon\upsilon\|)\\ =\;&\frac{1}{4}\|\varrho-1\|+\frac{1}{4}\|\varrho\|+\frac{7}{4}\\ >\;&1=\|\Upsilon\varrho-\Upsilon\upsilon\|.\end{aligned}$$

Thus, $\Upsilon$ satisfies the condition $B_{\gamma,\mu,\eta}$.

(iii) For $\varrho=4$, $\upsilon=4$, we have $\|\Upsilon\varrho-\Upsilon\upsilon\|=0$ and, again, $\Upsilon$ satisfies the condition $B_{\gamma,\mu,\eta}$.

**Proposition 3.1.** *Let $\Upsilon:\Omega\longrightarrow\mathcal{M}$ be a mapping satisfying the condition $B_{\gamma,\mu,\eta}$ for a nonempty subset $\Omega$ of a Banach space $\mathcal{M}$, then the following holds.*

(i) If $\Upsilon$ satisfies condition $(C)$, then $\Upsilon$ satisfies condition $B_{\gamma,\mu,\eta}$ for $\gamma=\mu=\eta=0$.

(ii) If $\Upsilon$ is a mapping satisfying the condition $B_{\gamma,\mu,\eta}$ and $(\Upsilon) \neq \emptyset$, then $\Upsilon$ is quasi-non-expansive mapping.

*Proof.* (i) From the definition of condition ($C$),

$$\frac{1}{2}\|\varrho - \Upsilon\varrho\| \leq \|\varrho - \upsilon\| \Longrightarrow \|\Upsilon\varrho - \Upsilon\upsilon\| \leq \|\varrho - \upsilon\|,$$

and it is easily seen that

$$\begin{aligned}\|\Upsilon\varrho - \Upsilon\upsilon\| \leq\ & (1-\gamma)\|\varrho - \upsilon\| + \mu(\|\varrho - \Upsilon\upsilon\| + \|\upsilon - \Upsilon\varrho\|) \\ & + \eta(\|\varrho - \Upsilon\varrho\| + \|\upsilon - \Upsilon\upsilon\|),\end{aligned}$$

for $\gamma = \mu = \eta = 0$. Hence, $\Upsilon$ satisfies condition $B_{\gamma,\mu,\eta}$.

(ii) Let $\varpi$ be a fixed point of $\Upsilon$. For all $\varrho \in \Omega$, we have

$$\gamma\|\varpi - \Upsilon\varpi\| \leq \|\varpi - \varrho\| + \mu\|\varrho - \Upsilon\varrho\| + \eta\|\varrho - \Upsilon\upsilon\|.$$

From condition $B_{\gamma,\mu,\eta}$,

$$\begin{aligned}\|\Upsilon\varpi - \Upsilon\varrho\| &\leq (1-\gamma)\|\varpi - \varrho\| + \mu(\|\varpi - \Upsilon\varrho\| + \|\varrho - \Upsilon\varpi\|) \\ &\quad + \eta(\|\varpi - \Upsilon\varpi\| + \|\varrho - \Upsilon\varrho\|) \\ &= (1-\gamma)\|\varpi - \varrho\| + \mu(\|\varpi - \Upsilon\varrho\| + \|\varrho - \varpi\|) + \eta\|\varrho - \Upsilon\varrho\| \\ &\Longrightarrow \|\varpi - \Upsilon\varrho\| \leq (1-\gamma+\mu+\eta)\|\varrho - \varpi\| + (\mu+\eta)\|\varpi - \Upsilon\varrho\| \\ &\Longrightarrow \|\varpi - \Upsilon\varrho\| \leq \left(\frac{1-\gamma+\mu+\eta}{1-\mu-\eta}\right)\|\varrho - \varpi\|.\end{aligned}$$

Since $2\mu + 2\eta \leq \gamma$, we obtain:

$$\|\varpi - \Upsilon\varrho\| \leq \|\varpi - \varrho\|,$$

and this completes the proof.

**Remark 3.1.** *The converse of Proposition 3.1 does not hold in general, i.e., if a mapping is quasi-non-expansive, it does not need to satisfy condition* $B_{\gamma,\mu,\eta}$.

We now discuss some properties of mappings satisfying the condition $B_{\gamma,\mu,\eta}$.

**Proposition 3.2.** *Let* $\Omega$ *be a nonempty subset of a Banach space* $\mathcal{M}$. *Let* $\Upsilon : \Omega \longrightarrow \Omega$ *satisfy the condition* $B_{\gamma,\mu,\eta}$, *then for all* $\varrho, \upsilon \in \Omega$ *and for* $c \in [0,1]$.

(i) $\left\|\Upsilon\varrho - \Upsilon^2\varrho\right\| \leq \|\varrho - \Upsilon\varrho\|$,

(ii) at least one of the following holds:

(a) $\frac{c}{2}\|\varrho - \Upsilon\varrho\| \leq \|\varrho - \upsilon\|$.

(b) $\frac{c}{2}\left\|\Upsilon\varrho - \Upsilon^2\varrho\right\| \leq \|\Upsilon\varrho - \upsilon\|$.

The condition (a) implies

$$\|\Upsilon\varrho - \Upsilon\upsilon\| \leq (1 - \frac{c}{2})\|\varrho - \upsilon\| + \mu(\|\varrho - \Upsilon\upsilon\| + \|\upsilon - \Upsilon\varrho\|)$$

$$+\eta\left(\|\varrho-\Upsilon\varrho\|+\|\upsilon-\Upsilon\upsilon\|\right).$$

The condition (b) implies

$$\begin{aligned}\left\|\Upsilon^2\varrho-\Upsilon\upsilon\right\| &\le (1-\frac{c}{2})\|\Upsilon\varrho-\upsilon\|+\mu\left(\|\Upsilon\varrho-\Upsilon\upsilon\|+\left\|\upsilon-\Upsilon^2\varrho\right\|\right)\\ &+\eta\left(\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\|\upsilon-\Upsilon\upsilon\|\right).\end{aligned}$$

(iii) $\|\varrho-\Upsilon\upsilon\|\le(3-c)\|\varrho-\Upsilon\varrho\|+(1-\frac{c}{2})\|\varrho-\upsilon\|+(\mu+\eta)(2\|\varrho-\Upsilon\varrho\|+2\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\|\varrho-\Upsilon\upsilon\|+\|\upsilon-\Upsilon\varrho\|)$.

*Proof.* (i) We have, for all $\varrho\in\Omega$,

$$\gamma\|\varrho-\Upsilon\varrho\|\le\|\varrho-\Upsilon\varrho\|+\mu\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\eta\left\|\varrho-\Upsilon^2\upsilon\right\|.$$

By the condition $B_{\gamma,\mu,\eta}$,

$$\begin{aligned}\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\| &\le (1-\gamma)\|\varrho-\Upsilon\varrho\|+\mu\left\|\varrho-\Upsilon^2\varrho\right\|\\ &+\eta\left(\|\varrho-\Upsilon\varrho\|+\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|\right)\\ &\le (1-\gamma)\|\varrho-\Upsilon\varrho\|+\mu\|\varrho-\Upsilon\varrho\|+\mu\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|\\ &+\eta\|\varrho-\Upsilon\varrho\|+\eta\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|\\ &\Longrightarrow \left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|\le\left(\frac{1-\gamma+\mu+\eta}{1-\mu-\eta}\right)\|\varrho-\Upsilon\varrho\|\le\|\varrho-\Upsilon\varrho\|.\end{aligned}$$

(ii) We suppose, on the contrary, that $\frac{c}{2}\|\varrho-\Upsilon\varrho\|>\|\varrho-\upsilon\|$ and $\frac{c}{2}\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|>\|\Upsilon\varrho-\upsilon\|$ for some $\varrho,\upsilon\in\Omega$.
Now,

$$\begin{aligned}\|\varrho-\Upsilon\varrho\| &\le \|\varrho-\upsilon\|+\|\upsilon-\Upsilon\varrho\|\\ &< \frac{c}{2}\|\varrho-\Upsilon\varrho\|+\frac{c}{2}\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|\\ &\le \frac{c}{2}\|\varrho-\Upsilon\varrho\|+\frac{c}{2}\|\varrho-\Upsilon\varrho\|\\ &\le c\|\varrho-\Upsilon\varrho\|.\end{aligned}$$

For $c\le 1$, we get $\|\varrho-\Upsilon\varrho\|<\|\varrho-\Upsilon\varrho\|$, which is not possible. Thus, at least one of (a) and (b) holds.

(iii) From (ii)-(a) and (b),

$$\begin{aligned}\|\varrho-\Upsilon\upsilon\| &\le \|\varrho-\Upsilon\varrho\|+\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\left\|\Upsilon^2\varrho-\Upsilon\upsilon\right\|\\ &\le \|\varrho-\Upsilon\varrho\|+(1-\frac{c}{2})\|\Upsilon\varrho-\varrho\|+\mu\left\|\varrho-\Upsilon^2\varrho\right\|\\ &+\eta\left(\|\varrho-\Upsilon\varrho\|+\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|\right)+(1-\frac{c}{2})\|\Upsilon\varrho-\upsilon\|\\ &+\mu\left(\|\Upsilon\varrho-\Upsilon\upsilon\|+\left\|\upsilon-\Upsilon^2\varrho\right\|\right)+\eta\left(\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\|\upsilon-\Upsilon\upsilon\|\right)\\ &\le \|\varrho-\Upsilon\varrho\|+(1-\frac{c}{2})\|\Upsilon\varrho-\varrho\|+\mu\|\varrho-\Upsilon\varrho\|+\mu\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|\end{aligned}$$

$$
\begin{aligned}
&+\eta\,\|\varrho-\Upsilon\varrho\|+\eta\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+(1-\frac{c}{2})\,\|\Upsilon\varrho-\varrho\| \\
&+(1-\frac{c}{2})\,\|\varrho-\upsilon\|+\mu\,\|\Upsilon\varrho-\varrho\|+\mu\,\|\varrho-\Upsilon\upsilon\| \\
&+\mu\,\|\upsilon-\Upsilon\varrho\|+\mu\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\eta\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\eta\,\|\upsilon-\Upsilon\upsilon\| \\
\leq\;& (3-c)\,\|\varrho-\Upsilon\varrho\|+(1-\frac{c}{2})\,\|\varrho-\upsilon\| \\
&+\mu(2\,\|\varrho-\Upsilon\varrho\|+\|\varrho-\Upsilon\upsilon\|+\|\upsilon-\Upsilon\varrho\|+2\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|) \\
&+\eta(2\,\|\varrho-\Upsilon\varrho\|+2\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\|\upsilon-\Upsilon\varrho\|+\|\varrho-\Upsilon\upsilon\|) \\
\Longrightarrow\;& \|\varrho-\Upsilon\upsilon\|\leq(3-c)\,\|\varrho-\Upsilon\varrho\|+(1-\frac{c}{2})\,\|\varrho-\upsilon\| \\
&+(\mu+\eta)(2\,\|\varrho-\Upsilon\varrho\|+2\left\|\Upsilon\varrho-\Upsilon^2\varrho\right\|+\|\varrho-\Upsilon\upsilon\|+\|\upsilon-\Upsilon\varrho\|).
\end{aligned}
$$

In this part, we introduce a new iteration process to approximate fixed points of mappings satisfying the condition $B_{\gamma,\mu,\eta}$.

Let $\Omega$ be a Banach space and $\Upsilon:\Omega\longrightarrow\Omega$ be a self-mapping on $\Omega$. We define our iteration process as follows:

$$
\begin{aligned}
\varrho_0 &\in \Omega, \\
\varrho_{n+1} &= \Upsilon\upsilon_n, \\
\upsilon_n &= \Upsilon\left((1-\alpha_n)\,\varpi_n+\alpha_n\Upsilon\varpi_n\right), \\
\varpi_n &= \Upsilon\left((1-\beta_n)\,\Upsilon\varrho_n+\beta_n\Upsilon^2\varrho_n\right),
\end{aligned}
\tag{3.1}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Now, we prove following results which we will use in the next proofs.

**Lemma 3.1.** *Let $\Omega$ be a nonempty closed and convex subset of a uniformly convex Banach space $\mathcal{M}$ and $\Upsilon$ be a self-mapping on $\Omega$ satisfying the condition $B_{\gamma,\mu,\eta}$. For $\varrho_0\in\Omega$, let $\{\varrho_n\}$ be the sequence in $\Omega$ defined by the iteration process (3.1), then $\lim_{n\to+\infty}\|\varrho_n-\varpi\|$ exists for all $\varpi\in(\Upsilon)$.*

*Proof.* Let $(\Upsilon)\neq\varnothing$ and let $\varpi\in(\Upsilon)$. By Proposition 3.1,

$$
\begin{aligned}
\|\Upsilon\varrho_n-\varpi\| &\leq \|\varrho_n-\varpi\| \\
\left\|\Upsilon^2\varrho_n-\varpi\right\| &\leq \|\Upsilon\varrho_n-\varpi\|\leq\|\varrho_n-\varpi\|.
\end{aligned}
\tag{3.2}
$$

Also,

$$
\begin{aligned}
\|\Upsilon\varpi_n-\varpi\| &\leq \|\varpi_n-\varpi\| \\
&= \left\|\Upsilon\left((1-\beta_n)\,\Upsilon\varrho_n+\beta_n\Upsilon^2\varrho_n\right)-\varpi\right\| \\
&\leq \left\|(1-\beta_n)\,\Upsilon\varrho_n+\beta_n\Upsilon^2\varrho_n-\varpi\right\| \\
&\leq (1-\beta_n)\,\|\Upsilon\varrho_n-\varpi\|+\beta_n\left\|\Upsilon^2\varrho_n-\varpi\right\| \\
&\leq (1-\beta_n)\,\|\varrho_n-\varpi\|+\beta_n\,\|\varrho_n-\varpi\| \\
&= \|\varrho_n-\varpi\|,
\end{aligned}
\tag{3.3}
$$

and

$$\begin{aligned} \|\Upsilon \upsilon_n - \varpi\| &\leq \|\upsilon_n - \varpi\| \qquad (3.4) \\ &= \|\Upsilon((1-\alpha_n)\varpi_n + \alpha_n \Upsilon \varpi_n) - \varpi\| \\ &\leq \|(1-\alpha_n)\varpi_n + \alpha_n \Upsilon \varpi_n - \varpi\| \\ &\leq (1-\alpha_n)\|\varpi_n - \varpi\| + \alpha_n \|\Upsilon \varpi_n - \varpi\| \\ &\leq (1-\alpha_n)\|\varpi_n - \varpi\| + \alpha_n \|\varpi_n - \varpi\| \\ &= \|\varpi_n - \varpi\| \\ &\leq \|\varrho_n - \varpi\|. \qquad (3.5) \end{aligned}$$

Using Eqs (3.2)–(3.4), we get

$$\begin{aligned} \|\varrho_{n+1} - \varpi\| &= \|\Upsilon \upsilon_n - \varpi\| \\ &\leq \|\upsilon_n - \varpi\| \\ &\leq \|\varpi_n - \varpi\| \\ &\leq \|\varrho_n - \varpi\|, \end{aligned}$$

which shows that sequence $\{\|\varrho_n - \varpi\|\}$ is nonincreasing and bounded. Hence, $\lim_{n\to+\infty} \|\varrho_n - \varpi\|$ exists for all $\varpi \in (\Upsilon)$.

**Theorem 3.2.** *Let $\Omega$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{M}$. Let $\Upsilon$ be a self-mapping on $\Omega$ satisfying the condition $B_{\gamma,\mu,\eta}$. Let $\{\varrho_n\}$ be a sequence in $\Omega$ defined by the iteration process (3.1), then $(\Upsilon) \neq \emptyset$ if and only if $\{\varrho_n\}$ is bounded and* $\lim_{n\to+\infty} \|\Upsilon\varrho_n - \varrho_n\| = 0$.

*Proof.* Let $(\Upsilon) \neq \emptyset$ and $\varpi \in (\Upsilon)$. By Lemma 3.1, $\lim_{n\to+\infty} \|\varrho_n - \varpi\|$ exists and $\{\varrho_n\}$ is bounded. Suppose $\lim_{n\to+\infty} \|\varrho_n - \varpi\| = p$.

From Eqs (3.2)–(3.4), we obtain

$$\lim_{n\to+\infty} \sup \|\upsilon_n - \varpi\| \leq \lim_{n\to+\infty} \sup \|\varrho_n - \varpi\| = p. \qquad (3.6)$$

and

$$\lim_{n\to+\infty} \sup \|\Upsilon\varpi_n - \varpi\| \leq \lim_{n\to+\infty} \sup \|\varpi_n - \varpi\| \leq \lim_{n\to+\infty} \sup \|\varrho_n - \varpi\| = p. \qquad (3.7)$$

By iteration (3.1), we get

$$\|\varrho_{n+1} - \varpi\| = \|\Upsilon\upsilon_n - \varpi\| \leq \|\upsilon_n - \varpi\|,$$

so that

$$p = \lim_{n\to+\infty} \inf \|\varrho_{n+1} - \varpi\| \leq \lim_{n\to+\infty} \inf \|\upsilon_n - \varpi\|. \qquad (3.8)$$

Thus, from (3.6) and (3.8),

$$\lim_{n\to+\infty} \|\upsilon_n - \varpi\| = p.$$

Consider,

$$\lim_{n\to+\infty} \|\upsilon_n - \varpi\| = \lim_{n\to+\infty} \|\Upsilon((1-\alpha_n)\varpi_n + \alpha_n \Upsilon\varpi_n) - \varpi\|$$

$$\begin{aligned}
&\leq \lim_{n\to+\infty} \|(1-\alpha_n)(\varpi_n-\varpi)+\alpha_n(\Upsilon\varpi_n-\varpi)\| \\
&\leq \lim_{n\to+\infty} \|\varpi_n-\varpi\|.
\end{aligned}$$

i.e., $\lim_{n\to+\infty} \|(1-\alpha_n)(\varpi_n-\varpi)+\alpha_n(\Upsilon\varpi_n-\varpi)\| = p$.

Using (3.7) and Lemma 2.1, we get

$$\lim_{n\to+\infty} \|\varpi_n-\Upsilon\varpi_n\| = 0. \tag{3.9}$$

Next,

$$\begin{aligned}
\|\upsilon_n-\Upsilon\varpi_n\| &= \|\Upsilon((1-\alpha_n)\varpi_n+\alpha_n\Upsilon\varpi_n)-\Upsilon\varpi_n\| \\
&\leq \|(1-\alpha_n)\varpi_n+\alpha_n\Upsilon\varpi_n-\varpi_n\| \\
&\leq \alpha_n\|\Upsilon\varpi_n-\varpi_n\|,
\end{aligned}$$

which gives with (3.9),

$$\lim_{n\to+\infty} \|\upsilon_n-\Upsilon\varpi_n\| = 0. \tag{3.10}$$

Now,

$$\|\varpi_n-\upsilon_n\| \leq \|\varpi_n-\Upsilon\varpi_n\|+\|\Upsilon\varpi_n-\upsilon_n\|.$$

Therefore, we have

$$\lim_{n\to+\infty} \|\varpi_n-\upsilon_n\| = 0. \tag{3.11}$$

Finally, by using (3.10) and (3.11), we obtain:

$$\begin{aligned}
\|\Upsilon\varrho_{n+1}-\varrho_{n+1}\| &= \|\Upsilon\varrho_{n+1}-\Upsilon\upsilon_n\| \\
&\leq \|\varrho_{n+1}-\upsilon_n\| \\
&= \|\Upsilon\upsilon_n-\upsilon_n\| \\
&\leq \|\Upsilon\upsilon_n-\Upsilon\varpi_n\|+\|\Upsilon\varpi_n-\upsilon_n\| \\
&\leq \|\upsilon_n-\varpi_n\|+\|\Upsilon\varpi_n-\upsilon_n\|.
\end{aligned}$$

Thus, we get

$$\lim_{n\to+\infty} \|\Upsilon\varrho_n-\varrho_n\| = 0.$$

Conversely, let $\{\varrho_n\}$ be bounded and $\lim_{n\to+\infty} \|\Upsilon\varrho_n-\varrho_n\| = 0$. Let $\varpi \in A(\Omega,\{\varrho_n\})$. By Proposition 3.2 (iii), for $\gamma = \frac{c}{2}$, $c \in [0,1]$,

$$\begin{aligned}
\|\varrho_n-\Upsilon\varpi\| &\leq (3-c)\|\varrho_n-\Upsilon\varrho_n\|+(1-\frac{c}{2})\|\varrho_n-\varpi\| \\
&\quad +(\mu+\eta)(2\|\varrho_n-\Upsilon\varrho_n\|+2\left\|\Upsilon\varrho_n-\Upsilon^2\varrho_n\right\|+\|\varrho_n-\Upsilon\varpi\|+\|\varpi-\Upsilon\varrho_n\|) \\
&\leq (3-c)\|\varrho_n-\Upsilon\varrho_n\|+(1-\frac{c}{2})\|\varrho_n-\varpi\| \\
&\quad +(\mu+\eta)(2\|\varrho_n-\Upsilon\varrho_n\|+2\left\|\Upsilon\varrho_n-\Upsilon^2\varrho_n\right\|+\|\varrho_n-\Upsilon\varpi\|+\|\varrho_n-\varpi\|+\|\varrho_n-\Upsilon\varrho_n\|)
\end{aligned}$$

$$\Longrightarrow \quad (1-\mu-\eta)\lim_{n\to+\infty}\sup\|\varrho_n-\Upsilon\varpi\| \leq \left(1-\frac{c}{2}+\mu+\eta\right)\lim_{n\to+\infty}\sup\|\varrho_n-\varpi\|$$

$$\Longrightarrow \lim_{n\to+\infty} \sup \|\varrho_n - \Upsilon\varpi\| \le \left(\frac{1-\frac{c}{2}+\mu+\eta}{1-\mu-\eta}\right) \lim_{n\to+\infty} \sup \|\varrho_n - \varpi\|,$$

where $\left(\frac{1-\frac{c}{2}+\mu+\eta}{1-\mu-\eta}\right) \le 1$ for $2\mu + 2\eta \le \gamma = \frac{c}{2}$. Hence, we have

$$r(\Upsilon\varpi, \{\varrho_n\}) \le r(\varpi, \{\varrho_n\}),$$

so $\Upsilon\varpi \in A(\Omega, \{\varrho_n\})$. Since $\mathcal{M}$ is a uniformly convex Banach space, which implies $A(\Omega, \{\varrho_n\})$ contains a single point, we obtain $\Upsilon\varpi = \varpi$, which is $\varpi \in (\Upsilon)$, so $(\Upsilon)(\Upsilon) \neq \emptyset$.

Thereinafter, we prove some convergence results for mappings satisfying the condition $B_{\gamma,\mu,\eta}$ by using iteration process (3.1).

**Theorem 3.3.** *Let $\Omega$ be a compact and convex subset of a Banach space $\mathcal{M}$. Let $\Upsilon$ be a self-mapping on $\Omega$ satisfying the condition $B_{\gamma,\mu,\eta}$ for $\mu + \eta < 1$ and $2\mu < \gamma$. Let $\{\varrho_n\}$ be a sequence in $\Omega$ as defined by the iteration process (3.1), then $\{\varrho_n\}$ converges strongly to a fixed point of $\Upsilon$.*

*Proof.* Since $\Omega$ is compact, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ and $\varpi \in \Omega$ such that $\{\varrho_{n_j}\}$ converges to $\varpi$.

From Proposition 3.2 (ii), for $\gamma = \frac{c}{2}$, $c \in [0,1]$

$$\gamma \left\|\varrho_{n_j} - \Upsilon\varrho_{n_j}\right\| \le \left\|\varrho_{n_j} - \varpi\right\| \le \left\|\varrho_{n_j} - \varpi\right\| + \mu \|\varpi - \Upsilon\varpi\| + \eta \left\|\varrho_{n_j} - \Upsilon\varpi\right\|.$$

By the condition $B_{\gamma,\mu,\eta}$,

$$\begin{aligned} \left\|\Upsilon\varrho_{n_j} - \Upsilon\varpi\right\| &\le (1-\gamma)\left\|\varrho_{n_j} - \varpi\right\| + \mu\left(\left\|\varrho_{n_j} - \Upsilon\varpi\right\| + \left\|\varpi - \Upsilon\varrho_{n_j}\right\|\right) \\ &\quad + \eta\left(\left\|\varrho_{n_j} - \Upsilon\varrho_{n_j}\right\| + \|\varpi - \Upsilon\varpi\|\right) \\ &\le (1-\gamma)\left\|\varrho_{n_j} - \varpi\right\| + \mu\left(\left\|\varrho_{n_j} - \Upsilon\varpi\right\| + \left\|\varpi - \varrho_{n_j}\right\| + \left\|\varrho_{n_j} - \Upsilon\varrho_{n_j}\right\|\right) \\ &\quad + \eta\left(\left\|\varrho_{n_j} - \Upsilon\varrho_{n_j}\right\| + \|\varpi - \Upsilon\varpi\|\right). \end{aligned}$$

Taking limit as $n_j \longrightarrow +\infty$ and using Theorem 3.2, we have

$$(1-\mu-\eta)\|\varpi - \Upsilon\varpi\| \le 0,$$

which is $\Upsilon\varpi = \varpi$. This shows that $\varpi$ is a fixed point of $\Upsilon$.

Now, we show that $\{\varrho_n\}$ converges to $\varpi$. From Lemma 3.1, $\lim_{n\to+\infty} \|\varrho_n - \varpi\|$ exists. Let's say $\lim_{n\to+\infty} \|\varrho_n - \varpi\| = u$.

Next,

$$\begin{aligned} \|\varrho_n - \Upsilon\varpi\| &\le \|\varrho_n - \Upsilon\varrho_n\| + \|\Upsilon\varrho_n - \Upsilon\varpi\| \\ &\le \|\varrho_n - \Upsilon\varrho_n\| + (1-\gamma)\|\varrho_n - \varpi\| + \mu(\|\varrho_n - \Upsilon\varpi\| + \|\varpi - \Upsilon\varrho_n\|) \\ &\quad + \eta(\|\varrho_n - \Upsilon\varrho_n\| + \|\varpi - \Upsilon\varpi\|) \\ &\le \|\varrho_n - \Upsilon\varrho_n\| + (1-\gamma)\|\varrho_n - \varpi\| \\ &\quad + \mu(\|\varrho_n - \Upsilon\varpi\| + \|\varpi - \varrho_n\| + \|\varrho_n - \Upsilon\varrho_n\|) \end{aligned}$$

$$
\begin{aligned}
&+\eta\left(\|\varrho_n - \Upsilon\varrho_n\| + \|\varpi - \Upsilon\varpi\|\right) \\
\Longrightarrow\quad &(\gamma - 2\mu)\|\varrho_n - \varpi\| \le (1 + \mu + \eta)\|\varrho_n - \Upsilon\varrho_n\|.
\end{aligned}
$$

Taking limit as $n \longrightarrow +\infty$, we get $(\gamma - 2\mu)\, u \le 0$. Since $\gamma - 2\mu$, we obtain $u = 0$ which is $\lim_{n\to+\infty}\|\varrho_n - \varpi\| = 0$. Hence, $\{\varrho_n\}$ converges strongly to $\varpi$.

**Theorem 3.4.** *Let $\Omega$ be a nonempty subset of a Banach space $\mathcal{M}$ having the Opial property. Let $\Upsilon$ be a self-mapping on $\Omega$ satisfying the condition $B_{\gamma,\mu,\eta}$. If $\{\varrho_n\}$ is a sequence in $\mathcal{M}$ such that $\{\varrho_n\} \rightharpoonup \varpi$ and* $\lim_{n\to+\infty}\|\Upsilon\varrho_n - \varrho_n\| = 0$, *then $\Upsilon\varpi = \varpi$.*

*Proof.* By Proposition 3.2 (ii), for $\gamma = \frac{c}{2}$,

$$
\gamma\|\varrho_n - \Upsilon\varrho_n\| \le \|\varrho_n - \varpi\| \le \|\varrho_n - \varpi\| + \mu\|\varpi - \Upsilon\varpi\| + \eta\|\varrho_n - \Upsilon\varpi\|.
$$

So, by the condition $B_{\gamma,\mu,\eta}$,

$$
\begin{aligned}
\|\Upsilon\varrho_n - \Upsilon\varpi\| \;\le\;& (1-\gamma)\|\varrho_n - \varpi\| + \mu\left(\|\varrho_n - \Upsilon\varpi\| + \|\varpi - \Upsilon\varrho_n\|\right) && (3.12)\\
&+\eta\left(\|\varrho_n - \Upsilon\varrho_n\| + \|\varpi - \Upsilon\varpi\|\right). && (3.13)
\end{aligned}
$$

Using by (3.12), we have

$$
\begin{aligned}
\|\varrho_n - \Upsilon\varpi\| \;\le\;& \|\varrho_n - \Upsilon\varrho_n\| + \|\Upsilon\varrho_n - \Upsilon\varpi\| \\
\le\;& \|\varrho_n - \Upsilon\varrho_n\| + (1-\gamma)\|\varrho_n - \varpi\| + \mu\left(\|\varrho_n - \Upsilon\varpi\| + \|\varpi - \Upsilon\varrho_n\|\right) \\
&+\eta\left(\|\varrho_n - \Upsilon\varrho_n\| + \|\varpi - \Upsilon\varpi\|\right) \\
\le\;& \|\varrho_n - \Upsilon\varrho_n\| + (1-\gamma)\|\varrho_n - \varpi\| \\
&+\mu\left(\|\varrho_n - \Upsilon\varpi\| + \|\varpi - \varrho_n\| + \|\varrho_n - \Upsilon\varrho_n\|\right) \\
&+\eta\left(\|\varrho_n - \Upsilon\varrho_n\| + \|\varpi - \varrho_n\| + \|\varrho_n - \Upsilon\varpi\|\right).
\end{aligned}
$$

So, taking limit as $n \longrightarrow +\infty$ and using $\lim_{n\to+\infty}\|\Upsilon\varrho_n - \varrho_n\| = 0$, we get

$$
\|\varrho_n - \Upsilon\varpi\| \le \frac{1-\gamma+\mu+\eta}{1-\mu-\eta}\|\varrho_n - \varpi\|.
$$

Since $\frac{1-\gamma+\mu+\eta}{1-\mu-\eta} \le 1$, we obtain $\|\varrho_n - \Upsilon\varpi\| \le \|\varrho_n - \varpi\|$. Hence,

$$
\lim_{n\to+\infty}\inf\|\varrho_n - \Upsilon\varpi\| \le \lim_{n\to+\infty}\inf\|\varrho_n - \varpi\|. \tag{3.14}
$$

Now, accept $\Upsilon\varpi \neq \varpi$. Since $\varrho_n \rightharpoonup \varpi$, by the Opial property, we have

$$
\lim_{n\to+\infty}\inf\|\varrho_n - \varpi\| < \lim_{n\to+\infty}\inf\|\varrho_n - \Upsilon\varpi\|,
$$

which is a contradiction to (3.14). Hence, $\Upsilon\varpi = \varpi$.

**Theorem 3.5.** *Let $\Omega$ be a weakly compact convex subset of a Banach space $\mathcal{M}$ with the Opial property, and $\Upsilon$ be a self-mapping on $\Omega$ satisfying the condition $B_{\gamma,\mu,\eta}$. Let $\{\varrho_n\}$ be a sequence in $\Omega$ as defined by the iteration process (3.1), then $\{\varrho_n\}$ converges weakly to a fixed point of $\Upsilon$.*

*Proof.* By Theorem 3.2, $\lim_{n\to+\infty} \|\Upsilon\varrho_n - \varrho_n\| = 0$. Since $\Omega$ is weakly compact, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ and $\varpi \in \Omega$ such that $\{\varrho_{n_j}\}$ converges weakly to $\varpi$. Now, by Theorem 3.4, $\varpi$ is a fixed point of $\Upsilon$.

We accept that $\{\varrho_n\}$ does not converge weakly to $\varpi$. Then, there is a subsequence $\{\varrho_{n_k}\}$ of $\{\varrho_n\}$ and $u \in \Omega$ such that $\{\varrho_{n_k}\}$ converges weakly to $u$ and $u \neq \varpi$. Again, $\Upsilon u = u$ (by Theorem 3.4).

Now,

$$\begin{aligned}
\liminf_{n\to+\infty} \|\varrho_n - \varpi\| &= \liminf_{n_j\to+\infty} \left\|\varrho_{n_j} - \varpi\right\| \\
&< \liminf_{n_j\to+\infty} \left\|\varrho_{n_j} - u\right\| \text{ (by Opial property)} \\
&= \liminf_{n_k\to+\infty} \left\|\varrho_{n_k} - u\right\| \\
&< \liminf_{n_k\to+\infty} \left\|\varrho_{n_k} - \varpi\right\| \\
&= \liminf_{n\to+\infty} \|\varrho_n - \varpi\|,
\end{aligned}$$

which is a contradiction.

So, $\{\varrho_n\}$ converges weakly to $\varpi$.

## 4. Numerical example

Let $\Omega = [-\frac{1}{2}, \frac{1}{2}]$ be endowed with an absolute valued norm and $\Upsilon : \Omega \longrightarrow \Omega$ be a mapping defined by

$$\Upsilon\varrho = \begin{cases} \frac{\varrho-1}{3} & \text{if } \varrho \neq \frac{1}{2}, \\ 0 & \text{if } \varrho = \frac{1}{2}. \end{cases}$$

For $\gamma = 1$, $\mu = \frac{1}{4}$, and $\eta = \frac{1}{4}$, we prove that $\Upsilon$ satisfies the condition $B_{\gamma,\mu,\eta}$.

**Case 1.** For $\varrho \neq \frac{1}{2}$, $\upsilon \neq \frac{1}{2}$, we have $\|\Upsilon\varrho - \Upsilon\upsilon\| = \frac{1}{3}\|\varrho - \upsilon\|$, and

$$\begin{aligned}
&(1-\gamma)\|\varrho - \upsilon\| + \mu(\|\varrho - \Upsilon\upsilon\| + \|\upsilon - \Upsilon\varrho\|) + \eta(\|\varrho - \Upsilon\varrho\| + \|\upsilon - \Upsilon\upsilon\|) \\
= \ & \frac{1}{4}\left(\left\|\varrho - \left(\frac{\upsilon-1}{3}\right)\right\| + \left\|\upsilon - \left(\frac{\varrho-1}{3}\right)\right\|\right) + \frac{1}{4}\left(\left\|\varrho - \left(\frac{\varrho-1}{3}\right)\right\| + \left\|\upsilon - \left(\frac{\upsilon-1}{3}\right)\right\|\right) \\
= \ & \frac{1}{12}(\|3\varrho - \upsilon + 1\| + \|3\upsilon - \varrho + 1\|) + \frac{1}{12}(\|2\varrho + 1\| + \|2\upsilon + 1\|) \\
\geq \ & \frac{1}{12}\|4\varrho - 4\upsilon\| + \frac{1}{12}\|2\varrho - 2\upsilon\| \\
= \ & \frac{1}{2}\|\varrho - \upsilon\| \\
> \ & \frac{1}{3}\|\varrho - \upsilon\| = \|\Upsilon\varrho - \Upsilon\upsilon\|.
\end{aligned}$$

**Case 2.** For $\varrho \neq \frac{1}{2}$, $\upsilon = \frac{1}{2}$, we have $\|\Upsilon\varrho - \Upsilon\upsilon\| = \frac{1}{3}\|\varrho - 1\|$, and

$$(1-\gamma)\|\varrho - \upsilon\| + \mu(\|\varrho - \Upsilon\upsilon\| + \|\upsilon - \Upsilon\varrho\|) + \eta(\|\varrho - \Upsilon\varrho\| + \|\upsilon - \Upsilon\upsilon\|)$$

$$= \frac{1}{4}\left(\|\varrho - 0\| + \left\|\frac{1}{2} - \left(\frac{\varrho - 1}{3}\right)\right\|\right) + \frac{1}{4}\left(\left\|\varrho - \left(\frac{\varrho - 1}{3}\right)\right\| + \left\|\frac{1}{2} - 0\right\|\right)$$
$$= \frac{1}{4}\|\varrho\| + \frac{1}{24}\|5 - 2\varrho\| + \frac{1}{12}\|2\varrho + 1\| + \frac{1}{8}$$
$$> \frac{1}{3}\|\varrho - 1\| = \|\Upsilon\varrho - \Upsilon\upsilon\|.$$

**Case 3.** For $\varrho = \frac{1}{2}$, $\upsilon = \frac{1}{2}$, we have

$$(1-\gamma)\|\varrho - \upsilon\| + \mu(\|\varrho - \Upsilon\upsilon\| + \|\upsilon - \Upsilon\varrho\|) + \eta(\|\varrho - \Upsilon\varrho\| + \|\upsilon - \Upsilon\upsilon\|) \geq 0 = \|\Upsilon\varrho - \Upsilon\upsilon\|.$$

In all the above cases, we have $\|\Upsilon\varrho - \Upsilon\upsilon\| \leq (1-\gamma)\|\varrho - \upsilon\| + \mu(\|\varrho - \Upsilon\upsilon\| + \|\upsilon - \Upsilon\varrho\|) + \eta(\|\varrho - \Upsilon\varrho\| + \|\upsilon - \Upsilon\upsilon\|)$, hence, $\Upsilon$ satisfies the condition $B_{\gamma,\mu,\eta}$.

Now, we show that iteration process (3.1) converges to fixed point $\varpi = -\frac{1}{2}$ faster than the F-iteration, M* iteration, and JA iteration processes.

Choosing initial value $\varrho_1 = 0.5$ and parameters $\alpha_n = \beta_n = 1/4$, Table 1 and Figure 1 show the efficiency of the iteration process (3.1).

**Table 1.** Sequences generated by different iteration processes.

| $n$ | Iteration (3.1) | F iteration | $M^*$ iteration | JA iteration |
|---|---|---|---|---|
| 1 | 0.50000000000000000 | 0.50000000000000000 | 0.50000000000000000 | 0.50000000000000000 |
| 2 | −0.48713991769547325 | −0.46759259259259259 | −0.40856481481481481 | −0.36689814814814814 |
| 3 | −0.49988974552208053 | −0.49899977137631458 | −0.49167488283036122 | −0.48788115855052583 |
| 4 | −0.49999905474555967 | −0.49996912874618254 | −0.49924200322066560 | −0.49889658696679170 |
| 5 | −0.49999999189596673 | −0.49999904718352415 | −0.49993098486114084 | −0.49989953492444554 |
| 6 | −0.49999999993052097 | −0.49999997059208407 | −0.49999371621420881 | −0.49999085271688624 |
| 7 | −0.49999999999940432 | −0.49999999909234827 | −0.49999942786518259 | −0.49999916714551896 |
| 8 | −0.49999999999999489 | −0.49999999997198605 | −0.49999994790747804 | −0.49999992416911360 |
| 9 | −0.49999999999999995 | −0.49999999999913537 | −0.49999999525700803 | −0.49999999309564460 |
| 10 | −0.49999999999999999 | −0.49999999999997331 | −0.49999999956815350 | −0.49999999937136270 |
| 11 | −0.50000000000000000 | −0.49999999999999917 | −0.49999999996068064 | −0.49999999994276296 |
| 12 | −0.50000000000000000 | −0.49999999999999997 | −0.49999999999641999 | −0.49999999999478860 |
| 13 | −0.50000000000000000 | −0.49999999999999999 | −0.49999999999967404 | −0.49999999999952550 |
| 14 | −0.50000000000000000 | −0.50000000000000000 | −0.49999999999997032 | −0.49999999999995679 |
| 15 | −0.50000000000000000 | −0.50000000000000000 | −0.49999999999999729 | −0.49999999999999606 |

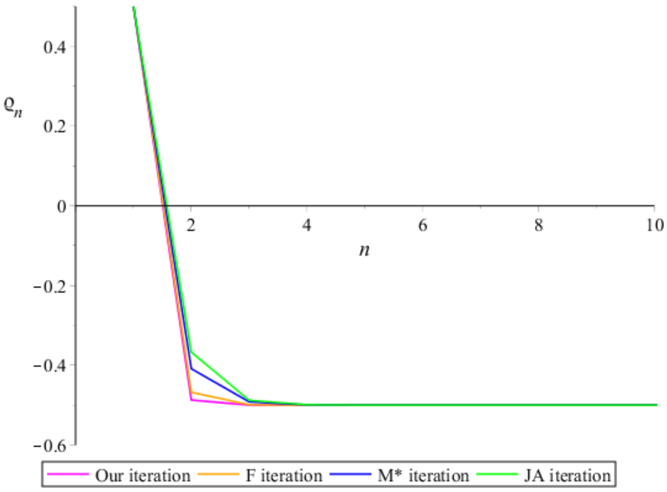


**Figure 1.** The efficiency of the iteration process (3.1).

In Table 2, we find numbers of iterations used to approximate fixed point $\varpi = -\frac{1}{2}$ for different initial values and the same parameters using numerical example.

**Table 2.** Influence of initial points for various iteration procecesses with $\alpha_n = \frac{1}{4}$ and $\beta_n = \frac{1}{4}$.

| | Initial points | | | |
|---|---|---|---|---|
| Iterations | −0.4 | −0.2 | 0.1 | 0.3 |
| Iteration (3.1) | 10 | 10 | 11 | 11 |
| F iteration | 13 | 13 | 14 | 14 |
| $M^*$ iteration | 18 | 18 | 19 | 19 |
| JA iteration | 18 | 18 | 19 | 19 |

Changing initial values and parameters, numbers of iterations used to approximate fixed point can be seen in Tables 3 and 4.

In any situation, it can be easily seen that iteration process (3.1) converges faster than the other iteration processes.

**Table 3.** Impact of parameters for different iteration processes.

| | Initial points | | | | |
|---|---|---|---|---|---|
| Iterations | −0.4 | −0.2 | 0.1 | 0.3 | 0.5 |
| | | For $\alpha_n = \frac{n+1}{2n+3}, \beta_n = \frac{n}{3n+1}$ | | | |
| Iteration (3.1) | 10 | 10 | 10 | 10 | 10 |
| F iteration | 13 | 13 | 13 | 13 | 13 |
| $M^*$ iteration | 17 | 17 | 17 | 17 | 17 |
| JA iteration | 17 | 17 | 17 | 17 | 18 |

**Table 4.** Impact of parameters for different iteration processes.

| Iterations | Initial points −0.4 | −0.2 | 0.1 | 0.3 | 0.5 |
|---|---|---|---|---|---|
| | | For $\alpha_n = \left(\frac{n+1}{n+1}\right)^{1/3}, \beta_n = \left(\frac{n+2}{2n+9}\right)^{2/5}$ | | | |
| Iteration (3.1) | 9 | 9 | 9 | 9 | 9 |
| F-iteration | 11 | 12 | 12 | 12 | 12 |
| $M^*$ iteration | 13 | 13 | 13 | 13 | 13 |
| JA iteration | 13 | 13 | 13 | 13 | 13 |

## 5. Conclusions

We have introduced a new class of generalized non-expansive mappings, which extends the class satisfying the condition $B_{\gamma,\mu}$. A new iterative process to approximate the fixed point of the newly introduced mapping has been followed and a related convergence theorem has been proved. Finally, a simple example has been given to illustrate the iterated process via the new class of the defined mappings. In fact, due to newly defined generalized class of non-expansive mappings, our result is considered to be an extension and generalization of many known fixed point results in the literature.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.


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## Conflict of interest

The authors declare no conflicts of interest.

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