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Research article

Separable algebras in multitensor C*-categories are unitarizable

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Abstract: S. Carpi et al. (Comm. Math. Phys., 402 (2023), 169–212) proved that every connected (i.e., haploid) Frobenius algebra in a tensor C*-category is unitarizable (i.e., isomorphic to a special C*-Frobenius algebra). Building on this result, we extend it to the non-connected case by showing that an algebra in a multitensor C*-category is unitarizable if and only if it is separable.

Keywords: multitensor C*-category; separable algebra; unitarily separable algebra; C*-Frobenius algebra; Q-system
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1. Introduction

Separable algebras in tensor categories are a natural generalization of finite-dimensional (associative unital) semisimple algebras over $\mathbb{C}$. Let $\mathcal{C}$ be a tensor category, see e.g., [19, 51]. If $\mathcal{C}$ happens to be in addition unitary i.e., C*, see e.g., [6, 52], the main result of this note, Theorem 4.13, states that every separable algebra is "unitarizable" i.e., it is isomorphic to a "unitarily" separable algebra, and the converse holds trivially. For the precise notions, see Definitions 3.3, 4.1 and 4.2. By Theorem 4.13, every statement involving separable algebras living in a tensor or multitensor C*-category has a "unitary" counterpart.

On the one hand, unitarily separable algebras also appear in the literature under the name of special C*-Frobenius algebras [6] or Q-systems [45, 46, 48]. Their study was initially motivated by the applications to operator algebras, in particular to the construction and classification of finite-index subfactors [37, 38, 55, 57, 58]. See [20] for an introduction to the subject, [31] for an overview, and [4], and references therein, for recent classification results. Since [47], Q-systems also play a pivotal role in the construction and classification of finite-index extensions of algebraic quantum field theories [33] in arbitrary spacetime dimensions, and of one-dimensional conformal field theories in the (completely) unitary vertex operator algebra framework [12, 39] as well, since [30]. Recently, Q-systems have been employed in the study of "quantum symmetries" (tensor category actions, generalizing ordinary group symmetries) of C*-algebras [10, 11, 14, 21].

On the other hand, separable algebras have a priori no inbuilt unitarity. Together with an additional commutativity assumption with respect to a given braiding, since [15], they are also often called étale algebras. These objects, typically assuming connectedness, are studied in relation to Ocneanu's quantum subgroups [54]. See [26] for recent results and a detailed account on their classification program. As for (commutative irreducible) Q-systems in the algebraic quantum field theory framework, connected étale algebras can be used to describe (local irreducible) extensions of vertex operator algebras [35], see also [13, 40]. Notably, they describe all rational 2D conformal field theories maximally extending a given tensor product of (isomorphic) chiral subtheories. See [22–25, 60] in the Euclidean setting, [34, 41] in the full vertex operator algebra setting, [5, 7] for the algebraic quantum field theory setting, and [3] for the Wightman quantum field theory setting. See also [42] for a proof of functoriality of the [22] construction when varying the given chiral subtheory.

The proof of our main result, Theorem 4.13, strongly relies on Theorem 3.2 in [8]. In the connected (i.e., haploid) case, the notions of separable algebra, Frobenius algebra, and isomorphic to unitarily separable algebra (i.e., isomorphic to special C*-Frobenius algebra = Q-system) all coincide by Lemma 4.10 below and by Theorem 3.2, see also Remark 3.3, in [8]. In the non-connected case, we first decompose a separable algebra A in $\mathcal{C}$ into indecomposable ones, Lemma 4.8, then unitarize the category of right A-modules in $\mathcal{C}$, Lemma 4.11. Last, we show that the unitarized category is equivalent to the modules over a unitarily separable algebra in $\mathcal{C}$ to which A is isomorphic, Proposition 4.12. This leads to Theorem 4.13.

We point out that the semisimplicity of $\mathcal{C}$ (or of the tensor or multitensor subcategory generated by A) is implicitly used in Theorem 3.2 in [8]. Here, we need it to exploit the separability of A via Proposition 4.3. Thus, a possible generalization of Theorem 4.13 to the case of non-semisimple monoidal C*-categories $\mathcal{C}$ should require a different idea, possibly "internal" to the C*-algebra $\mathcal{C}(A, A)$, on how to show directly that a separable algebra is isomorphic in $\mathcal{C}$ to a unitarily separable one.

2. Preliminaries

A **C*-category** is a generalization of a C*-algebra of operators acting between different Hilbert spaces instead of one. The objects $X, Y, Z, \dots$ of $\mathcal{C}$ can be thought of as the Hilbert spaces, the morphisms $f, g, h, \dots$ of $\mathcal{C}$ as the bounded linear operators. Formally, it is a $\mathbb{C}$-linear category $\mathcal{C}$ ([19, 49]) equipped with an involutive contravariant anti-linear endofunctor $* : \mathcal{C} \to \mathcal{C}$ (sometimes called *dagger* or *adjoint*) and a family of norms $\|\cdot\|$ on morphisms such that

- the endofunctor $*$ is the identity on objects (we use $f^* \in \mathcal{C}(Y, X)$ to denote the image of the morphism $f \in \mathcal{C}(X, Y)$),
- the hom space $\mathcal{C}(X, Y)$ is a Banach space for every $X, Y \in \mathcal{C}$,
- $\|gf\| \leq \|g\|\|f\|$, $\|f^*f\| = \|f\|^2$, $f^*f \geq 0$, for every $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$.

In particular, a C*-category with one object is a unital C*-algebra (see [28]).

In the following, we use 1_X to denote the identity morphism in $\mathcal{C}(X, X)$. For a morphism $f \in \mathcal{C}(X, Y)$ we will occasionally write $f : X \to Y$ if the environment category $\mathcal{C}$ is clear from the context.

A morphism f in a C*-category is called *unitary* (resp. *self-adjoint*) if $f^* = f^{-1}$ (resp. $f^* = f$). Let $\mathcal{C}$ and $\mathcal{D}$ be two C*-categories. A **$*$-functor** from $\mathcal{C}$ to $\mathcal{D}$ is a linear functor such that $F(f^*) = F(f)^*$ for every morphism f.

A **multitensor C*-category** is an abelian rigid ([17, 48]) monoidal category $(\mathcal{C}, \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \mathbb{1})$ equipped a C*-category structure satisfying the following conditions:

- the tensor unit $\mathbb{1}$ of $\mathcal{C}$ is semisimple, i.e., $\mathcal{C}(\mathbb{1}, \mathbb{1})$ is finite-dimensional,
- $\otimes$ is a bilinear functor and $(f \otimes g)^* = f^* \otimes g^*$ for every morphisms f, g,
- the associator and the left/right unitor constraints are unitary.

If $\mathcal{C}(\mathbb{1}, \mathbb{1}) \simeq \mathbb{C}$, i.e., if $\mathbb{1}$ is simple, then $\mathcal{C}$ is called a **tensor C*-category**. By Proposition 8.16 in [27], every multitensor C*-category $\mathcal{C}$ is semisimple and locally finite. Moreover, by Mac Lane's coherence theorem, $\mathcal{C}$ is equivalent to a strict multitensor C*-category, i.e., where the associator and the left/right unitors are identities (see [6, 19]). From now on, unless otherwise specified, we use $\mathcal{C}$ to denote a (strict) multitensor C*-category.

Remark 2.1. The tensor unit $\mathbb{1}$ of $\mathcal{C}$ is a direct sum of simple objects $\oplus_{i=1}^n \mathbb{1}_i$. Note that $\mathcal{C} \simeq \oplus_{ij}\mathcal{C}_{ij}$, where $\mathcal{C}_{ij} := \mathbb{1}_i \otimes \mathcal{C} \otimes \mathbb{1}_j$ (see Remark 4.3.4 in [19]). Let τ be the linear functional on $\mathcal{C}(\mathbb{1}, \mathbb{1})$ defined by

$$\tau\left(\sum_i a_i 1_{\mathbb{1}_i}\right) := \sum_i a_i.$$

Let $X \in \mathcal{C}$. We have $X \simeq \oplus_{ij}X_{ij}$ and $\overline{X} \simeq \oplus_{ij}\overline{X}_{ji}$, where $X_{ij} := \mathbb{1}_i \otimes X \otimes \mathbb{1}_j$ and $\overline{X}$, $\overline{X}_{ij}$ denote the dual (or conjugate) objects of X, X_{ij} respectively. Namely, for every $i, j \in \{1, \dots, n\}$, there exists (see below) a solution $(\gamma_{ij} \in \mathcal{C}(\mathbb{1}_j, \overline{X}_{ij} \otimes X_{ij}), \overline{\gamma}_{ij} \in \mathcal{C}(\mathbb{1}_i, X_{ij} \otimes \overline{X}_{ij}))$ of the conjugate equations

$$(\overline{\gamma}_{ij}^* \otimes 1_{X_{ij}})(1_{X_{ij}} \otimes \gamma_{ij}) = 1_{X_{ij}}, \quad (\gamma_{ij}^* \otimes 1_{\overline{X}_{ij}})(1_{\overline{X}_{ij}} \otimes \overline{\gamma}_{ij}) = 1_{\overline{X}_{ij}},$$

which is unique up to unitaries, and such that

$$\tau\left(\gamma_{ij}^*(1_{\overline{X}_{ij}} \otimes f)\gamma_{ij}\right) = \tau\left(\overline{\gamma}_{ij}^*(f \otimes 1_{\overline{X}_{ij}})\overline{\gamma}_{ij}\right) \tag{2.1}$$

for every $f \in \mathcal{C}(X_{ij}, X_{ij})$. The *scalar dimension* of X_{ij} ([27, 48]) is then $d_{X_{ij}} = \tau(\gamma_{ij}^*\gamma_{ij}) = \tau(\overline{\gamma}_{ij}^*\overline{\gamma}_{ij})$.

For the convenience of the reader, we sketch proof of this well-known fact when $i \neq j$ (the case where $i = j$ can be proved similarly). Let $\{Z_s\}_s$ be a set of representatives of simple objects in $\mathcal{C}_{ij}$. Since $\dim \mathcal{C}(\mathbb{1}_j, \overline{Z}_s \otimes Z_s) = \dim \mathcal{C}(\mathbb{1}_i, Z_s \otimes \overline{Z}_s) = 1$, we can choose a solution of the conjugate equations $(\gamma_s, \overline{\gamma}_s)$ such that $\tau(\gamma_s^*\gamma_s) = \tau(\overline{\gamma}_s^*\overline{\gamma}_s)$, i.e., $\|\gamma_s\| = \|\overline{\gamma_s}\|$ (as in Definition 3.4 in [48]). For non-simple $X_{ij} \in \mathcal{C}_{ij}$, let $\{u_{s,k}\}_k$ (resp. $\{\overline{u}_{s,k}\}_k$) be a basis of $\mathcal{C}(Z_s, X_{ij})$ (resp. $\mathcal{C}(\overline{Z}_s, \overline{X}_{ij})$) such that $u_{s,l}^* u_{s,k} = \delta_{k,l}1_{Z_s}$ (resp. $\overline{u}_{s,l}^*\overline{u}_{s,k} = \delta_{k,l}1_{\overline{Z}_s}$). Let

$$\gamma_{ij} := \sum_s \sum_k (\overline{u}_{s,k} \otimes u_{s,k})\gamma_s, \quad \overline{\gamma}_{ij} := \sum_s \sum_k (u_{s,k} \otimes \overline{u}_{s,k})\overline{\gamma}_s,$$

as before Lemma 3.7 in [48], or before Lemma 8.23 in [27], then $(\gamma_{ij}, \overline{\gamma}_{ij})$ is a solution of the conjugate equations that satisfies the Eq (2.1). Indeed,

$$\tau\left(\gamma_{ij}^*(1_{\overline{X}_{ij}} \otimes u_{s,k}u_{s,l}^*)\gamma_{ij}\right) = \delta_{k,l}\tau(\gamma_s^*\gamma_s) = \delta_{k,l}\tau(\overline{\gamma}_s^*\overline{\gamma}_s) = \tau\left(\overline{\gamma}_{ij}^*(u_{s,k}u_{s,l}^* \otimes 1_{\overline{X}_{ij}})\overline{\gamma}_{ij}\right).$$

Let $(\omega \in \mathcal{C}(\mathbb{1}, \overline{X}_{ij} \otimes X_{ij}), \overline{\omega} \in \mathcal{C}(\mathbb{1}, X_{ij} \otimes \overline{X}_{ij}))$ be a solution of the conjugate equations that satisfies the Eq (2.1). Then, there exists an invertible morphism $h \in \mathcal{C}(X_{ij}, X_{ij})$ such that $\omega = (1_{\overline{X}_{ij}} \otimes h)\gamma_{ij}$ and $\overline{\omega} = ((h^*)^{-1} \otimes 1_{\overline{X}_{ij}})\overline{\gamma}_{ij}$. By choosing a different basis of $\mathcal{C}(Z_s, X_{ij})$, we may assume that $h = \sum_s \sum_k a_{s,k} u_{s,k} u^*_{s,k}$, where $a_{s,k} > 0$. Then, the condition that $(\omega, \overline{\omega})$ fulfills the Eq (2.1) implies that $h = 1_{X_{ij}}$. In other words, the solution of the conjugate equations that satisfies the Eq (2.1) is unique up to unitaries (see Lemmas 3.3 and 3.7 in [48], and cf. Lemma 8.35 in [27], for more details).

Let $\gamma_X := \oplus_{ij}\gamma_{ij}$ and $\overline{\gamma}_X := \oplus_{ij}\overline{\gamma}_{ij}$. Note that these are not the *standard solutions* of the conjugate equations defined in [27], where the Perron-Frobenius data of the *matrix dimension* enter as numerical prefactors for each i, j (see Definitions 8.25 and 8.29 therein), unless the tensor unit is simple (as in Section 3 of [48]) and they coincide with the standard solutions of [48]. In particular, the "loop" or "bubble" morphisms $\gamma_X^*\gamma_X$ and $\overline{\gamma}_X^*\overline{\gamma}_X$ will neither be scalar in $\mathcal{C}(\mathbb{1}, \mathbb{1})$, nor equal, nor will $(\gamma_X, \overline{\gamma}_X)$ be *spherical* (resp. *minimal*) in the sense of Theorem 8.39 (resp. Theorem 8.44) in [27].

With the $(\gamma_X, \overline{\gamma}_X)$ defined above, we have

$$(\gamma_Y^* \otimes 1_{\overline{X}})\,(1_{\overline{Y}} \otimes g \otimes 1_{\overline{X}})\,(1_{\overline{Y}} \otimes \overline{\gamma}_X) = (1_{\overline{X}} \otimes \overline{\gamma}_Y^*)\,(1_{\overline{X}} \otimes g \otimes 1_{\overline{Y}})\,(\gamma_X \otimes 1_{\overline{Y}})$$

and

$$\tau\left(\gamma_X^*(1_{\overline{X}} \otimes hg)\gamma_X\right) = \tau\left(\overline{\gamma}_X^*(hg \otimes 1_{\overline{X}})\overline{\gamma}_X\right) = \tau\left(\gamma_Y^*(1_{\overline{Y}} \otimes gh)\gamma_Y\right)$$

for every $g \in \mathcal{C}(X, Y)$, $h \in \mathcal{C}(Y, X)$, and $X, Y \in \mathcal{C}$. Moreover, if a solution of the conjugate equations $(\omega \in \mathcal{C}(\mathbb{1}, \overline{X} \otimes X), \overline{\omega} \in \mathcal{C}(\mathbb{1}, X \otimes \overline{X}))$ fulfills

$$\tau\left(\omega^*(1_{\overline{X}} \otimes g)\,\omega\right) = \tau\left(\overline{\omega}^*(g \otimes 1_{\overline{X}})\overline{\omega}\right), \quad \forall g \in \mathcal{C}(X, X),$$

then there exists a unitary $u \in \mathcal{C}(X, X)$ (or $\overline{u} \in \mathcal{C}(\overline{X}, \overline{X})$) such that $\omega = (1_{\overline{X}} \otimes u)\gamma_X$ and $\overline{\omega} = (u \otimes 1_{\overline{X}})\overline{\gamma}_X$ (or $\omega = (\overline{u} \otimes 1_X)\gamma_X$ and $\overline{\omega} = (1_X \otimes \overline{u})\overline{\gamma}_X$).

Based on these observations, it is not hard to check that $\mathcal{C}$ endowed with the pivotal duality $\{(\overline{X}, \gamma_X, \overline{\gamma}_X)\}_{X \in \mathcal{C}}$ is a *pivotal category* (see, e.g., Section 1.7 in [61] for the definition of pivotal category).

3. Algebras and modules in multitensor C*-categories

We recall below the natural generalization of the notion of finite-dimensional unital associative algebra (in the tensor category of finite-dimensional complex vector spaces $\mathbf{Vec}_{\text{f.d.},\mathbb{C}}$). Let $\mathcal{C}$ be a strict multitensor C*-category.

Definition 3.1. An **algebra in** $\mathcal{C}$ is a triple (A, m, ι), where A is an object in $\mathcal{C}$, $m \in \mathcal{C}(A \otimes A, A)$ is the "multiplication" morphism, $\iota \in \mathcal{C}(\mathbb{1}, A)$ is the "unit" morphism, fulfilling the associativity and unit laws

$$m(m \otimes 1_A) = m(1_A \otimes m), \quad m(\iota \otimes 1_A) = 1_A = m(1_A \otimes \iota).$$

Definition 3.2. Two algebras (A, m, ι) and (A', m', ι') in $\mathcal{C}$ are said to be **isomorphic** if there is an invertible (not necessarily unitary) morphism $t \in \mathcal{C}(A, A')$ such that $tm = m'(t \otimes t)$ and $t\iota = \iota'$.

Definition 3.3. An algebra (A, m, ι) in $\mathcal{C}$ is called a **C*-Frobenius algebra** if m^* is a left (or equivalently right) A-module morphism such that

$$(m \otimes 1_A)(1_A \otimes m^*) = m^* m = (1_A \otimes m)(m^* \otimes 1_A). \tag{3.1}$$

An algebra (A, m, ι) in $\mathcal{C}$ is called **special** if the multiplication is a coisometry*

$$mm^* = 1_A.$$

Definition 3.4. Forgetting the C* structure, an algebra (A, m, ι) in $\mathcal{C}$ endowed with a **coalgebra** structure $(A, \Delta \in \mathcal{C}(A, A \otimes A), \varepsilon \in \mathcal{C}(A, \mathbb{1}))$ (not necessarily $\Delta = m^*$, $\varepsilon = \iota^*$) fulfilling the coassociativity and counit laws, is called a **Frobenius algebra** if the analogue of (3.1) holds with m^* replaced by Δ (see [1,22,62]).

The following crucial results proven in [6, 22, 48] assuming $\mathcal{C}(\mathbb{1}, \mathbb{1}) \simeq \mathbb{C}$, see in particular Chapter 3 in [6], also hold for multitensor C*-categories, cf. Section 2.2 in [32].

Proposition 3.5. *Let (A, m, ι) be an algebra in $\mathcal{C}$.*

- *If (A, m, ι) is special, then it is a C*-Frobenius algebra.*
- *If (A, m, ι) is a C*-Frobenius algebra, then it is isomorphic to a special one.*

Example 3.6. Recall, e.g., from Section 2 in [1] and Section 2.1 in [53], that a C*-Frobenius algebra in $\mathbf{Hilb}_{\text{f.d.},\mathbb{C}}$, the tensor C*-category of finite-dimensional Hilbert spaces, is just an ordinary finite-dimensional C*-algebra with a Frobenius structure. Forgetting the C* structure, a Frobenius algebra in the tensor category $\mathbf{Vec}_{\text{f.d.},\mathbb{C}}$ of finite-dimensional vector spaces is a finite-dimensional Frobenius algebra.

We shall use module categories (and their unitary version, C*-module categories recalled below) over multitensor C*-categories. See [56] or Chapter 7 in [19] for the definitions of module category over a monoidal category $\mathcal{C}$ and module functor.

Definition 3.7. A left **C*-module category** over a multitensor C*-category $\mathcal{C}$ is a left $\mathcal{C}$-module category $(\mathcal{M}, \odot : \mathcal{C} \times \mathcal{M} \to \mathcal{M})$ which is also a C*-category, such that

- $\odot$ is bilinear and $(f \odot g)^* = f^* \odot g^*$ for every morphisms $f \in \mathcal{C}$, $g \in \mathcal{M}$,
- the associator and the unitor constraints are unitary.

Right C*-module categories and C*-bimodule categories are defined similarly.

Typical examples of left (resp. right) $\mathcal{C}$-module categories (not necessarily C*) come from considering right (resp. left) modules over an algebra (A, m, ι) in $\mathcal{C}$. We use $\mathbf{RMod}_{\mathcal{C}}(A)$ (resp. $\mathbf{LMod}_{\mathcal{C}}(A)$) to denote the category of right (resp. left) A-modules in $\mathcal{C}$.

Definition 3.8. Let (A, m, ι) be a special C*-Frobenius algebra in $\mathcal{C}$. As for algebras, a right A**-module** $(X, r \in \mathcal{C}(X \otimes A, X))$ in $\mathcal{C}$ is called **special** if

$$rr^* = 1_X.$$

We denote by $\mathbf{sRMod}_{\mathcal{C}}(A)$ the category of special right A-modules in $\mathcal{C}$. The definition for left A-modules is analogous.

*or, in a different convention, a scalar multiple of a coisometry, cf. [2, 6, 29, 50, 53]. Also, note that we do neither ask $\iota^*\iota$ to be $1_{\mathbb{1}}$, nor a multiple of $1_{\mathbb{1}}$, and that the latter condition is automatic if the tensor unit $\mathbb{1}$ is simple.

By the arguments of Chapter 3 in [6], cf. Section 2.2 in [32], we have

Proposition 3.9. *Let (A, m, ι) be a special C*-Frobenius algebra in $\mathcal{C}$. Then $\mathbf{sRMod}_{\mathcal{C}}(A)$ is a left C*-module category over $\mathcal{C}$, where the involution and norms are inherited from $\mathcal{C}$.*

More generally, given a right A-module $(X, r \in \mathcal{C}(X \otimes A, X))$, then $(X, r' := h^{-1}r(h \otimes 1_A))$ is a special right A-module, where $h := \sqrt{rr^*}$, and h^{-1} is a right A-module isomorphism from (X, r) to (X, r'). Moreover, $\mathbf{RMod}_{\mathcal{C}}(A)$ is a left C-module category over $\mathcal{C}$ with the following C*-structure*

- $f \in \mathbf{RMod}_{\mathcal{C}}(A)(X, Y) \mapsto h_X^2 f^* h_Y^{-2} \in \mathbf{RMod}_{\mathcal{C}}(A)(Y, X)$,
- $|||f||| := \left\| h_Y^{-1} f h_X \right\|$, $f \in \mathbf{RMod}_{\mathcal{C}}(A)(X, Y)$,

where $h_X := \sqrt{r_X r_X^*}$ and $h_Y := \sqrt{r_Y r_Y^*}$ are defined respectively from the right A-module actions of X and Y. The embedding $\mathbf{sRMod}_{\mathcal{C}}(A) \hookrightarrow \mathbf{RMod}_{\mathcal{C}}(A)$ is an equivalence of left C-module categories.*

4. Separable algebras are unitarizable

In this section, we prove our main theorem.

Definition 4.1. An algebra (A, m, ι) in $\mathcal{C}$ is called **separable** if the multiplication $m \in \mathcal{C}(A \otimes A, A)$ splits as a morphism of A-A-bimodules in $\mathcal{C}$, i.e., if there is an A-A-bimodule morphism $f \in \mathcal{C}(A, A \otimes A)$ such that $mf = 1_A$.

Clearly, every (not necessarily special) C*-Frobenius algebra in $\mathcal{C}$ is separable. Indeed, by Proposition 3.5, it is isomorphic to a special algebra in $\mathcal{C}$ (Definition 3.3), namely $mm^* = 1_A$ holds up to isomorphism of algebras, hence it is separable.

Moreover, a special C*-Frobenius algebra, which is also called a **Q-system** after [46] (see also [6, 8, 11, 48, 50] and references therein), can be viewed as a "unitarily" separable algebra. The following definition is motivated by this fact.

Definition 4.2. A (Frobenius) algebra in $\mathcal{C}$ is **unitarizable** if it is (not necessarily unitarily) isomorphic to a special C*-Frobenius algebra in $\mathcal{C}$.

Our main result (Theorem 4.13) states that every separable algebra in $\mathcal{C}$ is unitarizable.

By the proof of Proposition 7.8.30 in [19], cf. Section 3 in [56], Section 2.3 in [15], Section 2.4 in [36], Section 4 in [43], the following characterization of separability for algebras in (not necessarily C*) multitensor categories holds.

Proposition 4.3. *Let (A, m_A, ι_A), (B, m_B, ι_B) be separable algebras in $\mathcal{C}$. Then the categories $\mathbf{RMod}_{\mathcal{C}}(A)$, $\mathbf{LMod}_{\mathcal{C}}(A)$, and $\mathbf{BiMod}_{\mathcal{C}}(A|B)$ (A-B-bimodules in $\mathcal{C}$) are semisimple.*

In particular, an algebra (C, m_C, ι_C) in $\mathcal{C}$ is separable if and only if $\mathbf{BiMod}_{\mathcal{C}}(C|C)$ is semisimple.

Let (A, m, ι) be an algebra in $\mathcal{C}$, $(X, r) \in \mathbf{RMod}_{\mathcal{C}}(A)$, and $(Y, l) \in \mathbf{LMod}_{\mathcal{C}}(A)$. We recall, e.g. from Section 7.8 in [19] *tensor product* of X and Y over A is the object $X \otimes_A Y \in \mathcal{C}$ defined as the co-equalizer of the diagram

$$X \otimes A \otimes Y \underset{1_X \otimes l}{\overset{r \otimes 1_Y}{\rightrightarrows}} X \otimes Y \longrightarrow X \otimes_A Y.$$

The following result follows from Proposition 7.11.1 in [19].

Proposition 4.4. *Let (A, m_A, ι_A), (B, m_B, ι_B) be algebras in $\mathcal{C}$ such that $\mathbf{RMod}_{\mathcal{C}}(A)$, $\mathbf{RMod}_{\mathcal{C}}(B)$ are semisimple. Then, the category* $\mathrm{Fun}_{\mathcal{C}|}(\mathbf{RMod}_{\mathcal{C}}(A), \mathbf{RMod}_{\mathcal{C}}(B))$ *of left $\mathcal{C}$-module functors is equivalent to* $\mathbf{BiMod}_{\mathcal{C}}(A|B)$.

The equivalence is given by

$$X \mapsto - \otimes_A X : \mathbf{BiMod}_{\mathcal{C}}(A|B) \to \mathrm{Fun}_{\mathcal{C}|}(\mathbf{RMod}_{\mathcal{C}}(A), \mathbf{RMod}_{\mathcal{C}}(B)).$$

Definition 4.5. A separable algebra (A, m_A, ι_A) in $\mathcal{C}$ is called **indecomposable** if $\mathbf{RMod}_{\mathcal{C}}(A)$ is an indecomposable left $\mathcal{C}$-module category, i.e., if it is not equivalent to a direct sum of non-zero left $\mathcal{C}$-module categories.

Definition 4.6. An algebra (A, m_A, ι_A) is called **connected** (or **haploid**) if $\dim(\mathcal{C}(\mathbb{1}, A)) = 1$, i.e., if A is a simple object in $\mathbf{RMod}_{\mathcal{C}}(A)$.

Lemma 4.7. *Let $\mathcal{C} \simeq \oplus_{ij}\mathcal{C}_{ij}$ be the decomposition as in Remark 2.1. Then (A, m_A, ι_A) is a connected algebra in $\mathcal{C}$ if and only if there exists exactly one $j \in \{1, \ldots, n\}$ such that $A = A_{jj}$ is a connected algebra contained in the tensor C*-category $\mathcal{C}_{jj}$ with tensor unit $\mathbb{1}_j$.*

Proof. Recall $\mathbb{1} = \oplus_{i=1}^n \mathbb{1}_i$. By connectedness, there is only one j such that $\mathcal{C}(\mathbb{1}_j, A) \neq 0$, and $\dim(\mathcal{C}(\mathbb{1}_j, A)) = 1$. Moreover, every A_{kl} must be zero unless $k = l = j$. □

The following result is well-known, we sketch the proof for the reader's convenience.

Lemma 4.8. *Let (A, m, ι) be a separable algebra in $\mathcal{C}$. Then A is a direct sum of indecomposable separable algebras.*

Proof. Note that $\mathbf{RMod}_{\mathcal{C}}(A)$ is indecomposable if and only if the identity functor $\mathrm{id} = -\otimes_A A$ associated with the trivial bimodule A is a simple object in $\mathrm{Fun}_{\mathcal{C}|}(\mathbf{RMod}_{\mathcal{C}}(A), \mathbf{RMod}_{\mathcal{C}}(A))$. By Proposition 4.4,

$$\mathbf{BiMod}_{\mathcal{C}}(A|A)(A, A) \simeq \mathrm{Fun}_{\mathcal{C}|}(\mathbf{RMod}_{\mathcal{C}}(A), \mathbf{RMod}_{\mathcal{C}}(A))(\mathrm{id}, \mathrm{id}).$$

Assume that $\dim(\mathbf{BiMod}_{\mathcal{C}}(A|A)(A, A)) > 1$. Recall from Proposition 4.3 that $\mathbf{BiMod}_{\mathcal{C}}(A|A)$ is semisimple. Let p be a non-trivial idempotent in $\mathbf{BiMod}_{\mathcal{C}}(A|A)(A, A)$, i.e., $1_A - p \neq 0$, $p^2 = p$, and let B be the image of p. Then B is a separable algebra with multiplication and unit given by $vm(w \otimes w)$ and $v\iota$, where $v : A \to B$ and $w : B \to A$ are A-A-bimodule morphisms such that $vw = 1_B$ and $wv = p$. Note that $f : B \to B$ is a B-B-bimodule morphism with the previous algebra structure on B if and only if $wfv : A \to A$ is an A-A-bimodule morphism. Thus $\dim(\mathbf{BiMod}_{\mathcal{C}}(B|B)(B, B)) < \dim(\mathbf{BiMod}_{\mathcal{C}}(A|A)(A, A))$. This implies that A is a direct sum of indecomposable separable algebras. □

Remark 4.9. If, in addition, the category $\mathcal{C}$ is *braided* and the separable algebra (A, m, ι) is *commutative* in the sense of Definition 1.1 in [40], cf. Definition 4.20 in [6], then $\mathbf{BiMod}_{\mathcal{C}}(A|A)$ and $\mathbf{RMod}_{\mathcal{C}}(A)$ can be identified. Hence, by the previous proof, A is a direct sum of connected separable algebras, cf. Remark 3.2 in [15].

Lemma 4.10. *Let (A, m, ι) be a connected separable algebra in $\mathcal{C}$. Then A can be promoted to a Frobenius algebra.*

Proof. By Lemma 4.7, we may assume that $\mathcal{C}$ is a tensor C*-category. Recall the conventions in Remark 2.1. $\overline{A}$ is a right A-module with right A-action given by

$$\overline{A} \otimes A \xrightarrow{1_{\overline{A}\otimes A}\otimes\overline{\gamma}_A} \overline{A} \otimes A \otimes A \otimes \overline{A} \xrightarrow{1_{\overline{A}}\otimes m\otimes 1_{\overline{A}}} \overline{A} \otimes A \otimes \overline{A} \xrightarrow{\gamma_A^*\otimes 1_{\overline{A}}} \overline{A}.$$

Let $f : A \to \overline{A}$ be the non-zero right A-module morphism defined by

$$f := A \xrightarrow{1_A\otimes\overline{\gamma}_A} A \otimes A \otimes \overline{A} \xrightarrow{(\iota^* m)\otimes 1_{\overline{A}}} \overline{A}.$$

Since $\mathbf{RMod}_{\mathcal{C}}(A)$ is semisimple by Proposition 4.3, A is a simple right A-module by connectedness, and $d_A = d_{\overline{A}}$ (where d_A is the scalar dimension [27] of A in $\mathcal{C}$, or equivalently the dimension [48] in $\mathcal{C}_{jj}$, cf. Lemma 4.7), f is invertible in $\mathcal{C}$. Hence, by Lemma 3.7 in [22], A can be promoted to a Frobenius algebra. □

Let $(\mathcal{M}, \odot)$ be a left $\mathcal{C}$-module category. Then $\mathcal{M}$ is said to be *enriched* in $\mathcal{C}$ if the functor $C \mapsto \mathcal{M}(C \odot X, Y) : \mathcal{C} \to \mathbf{Vec}_{\mathrm{f.d.},\mathbb{C}}$ is *representable* for every $X, Y \in \mathcal{M}$, i.e., there exists an object $[X, Y] \in \mathcal{C}$ such that

$$\mathcal{M}(- \odot X, Y) \simeq \mathcal{C}(-, [X, Y]).$$

The object $[X, Y]$ is called the *internal hom* from X to Y. In particular, $[X, -] : \mathcal{M} \to \mathcal{C}$ is the right adjoint of the functor $- \odot X : \mathcal{C} \to \mathcal{M}$.

If $\mathcal{M} = \mathbf{RMod}_{\mathcal{C}}(A)$, where A is a separable algebra in $\mathcal{C}$, then $\mathcal{M}$ is enriched in $\mathcal{C}$. More explicitly, the internal hom $[X, Y]$ is given by $\overline{X \otimes_A \overline{Y}}$. We refer the reader to Section 7 in [19] or Section 2 in [44] for basic facts about internal homs.

Lemma 4.11. *Let (A, m_A, ι_A) be an indecomposable separable algebra in $\mathcal{C}$. Then there exists a connected special C*-Frobenius algebra (B, m_B, ι_B) in $\mathcal{C}$ such that* $\mathbf{RMod}_{\mathcal{C}}(A)$ *and* $\mathbf{RMod}_{\mathcal{C}}(B)$ *are equivalent as left $\mathcal{C}$-module categories.*

In particular, $\mathbf{RMod}_{\mathcal{C}}(A)$ *is equivalent to a left C*-module category over $\mathcal{C}$.*

Proof. Let X be a non-zero simple object in $\mathbf{RMod}_{\mathcal{C}}(A)$. By Proposition 4.3 and by the proof of Theorem 3.1 in [56] (cf. Theorem 2.1.7 in [44]), the internal hom $[X, X]$ in $\mathbf{RMod}_{\mathcal{C}}(A)$ is a connected (by the simplicity of X) algebra in $\mathcal{C}$ such that $\mathbf{RMod}_{\mathcal{C}}(A)$ and $\mathbf{RMod}_{\mathcal{C}}([X, X])$ are equivalent. Note that $\mathbf{RMod}_{\mathcal{C}}(A)$ and $\mathbf{RMod}_{\mathcal{C}}([X, X])$ are both semisimple. Since

$$\mathrm{Fun}_{\mathcal{C}|}(\mathbf{RMod}_{\mathcal{C}}([X, X]), \mathbf{RMod}_{\mathcal{C}}([X, X])) \simeq \mathrm{Fun}_{\mathcal{C}|}(\mathbf{RMod}_{\mathcal{C}}(A), \mathbf{RMod}_{\mathcal{C}}(A)),$$

from Propositions 4.3 and 4.4 it follows that A separable implies that $[X, X]$ is separable. By Lemma 4.10, $[X, X]$ can be promoted to a connected Frobenius algebra. Then, $[X, X]$ is isomorphic to a special C*-Frobenius algebra B in $\mathcal{C}$ by Lemma 4.7 and by Theorem 3.2, cf. Remark 3.3, in [8]. We conclude that $\mathbf{RMod}_{\mathcal{C}}(A)$ is equivalent to $\mathbf{RMod}_{\mathcal{C}}(B)$. The latter is a left C*-module category over $\mathcal{C}$ by Proposition 3.9. □

The following result is of independent interest and it should be compared with Lemma 2.18 in [29] for $\mathcal{M} = \mathbf{RMod}_{\mathcal{C}}(A)$, and Theorem A.1 in [53].

Proposition 4.12. *Let* $(\mathcal{M}, \odot)$ *be an indecomposable left* C^**-module over* $\mathcal{C}$ *which is enriched in* $\mathcal{C}$*. For every non-zero object* X *in* $\mathcal{M}$*, the internal hom* $[X, X]$ *is isomorphic (up to rescaling) to a special* C^**-Frobenius algebra in* $\mathcal{C}$*.*

Proof. By Proposition 2.3 in [59], we may choose the right adjoint $[X, -] : \mathcal{M} \to \mathcal{C}$ of the $*$-functor $- \odot X : \mathcal{C} \to \mathcal{M}$ to be a $*$-functor. For every $C \in \mathcal{C}$ and $Y \in \mathcal{M}$, we treat $\mathcal{C}(C, [X, Y])$ as the Hilbert space with inner product given by

$$\langle f_1 | f_2 \rangle := \tau\,(\gamma_C^*(1_{\overline{C}} \otimes f_1^* f_2)\gamma_C),$$

where γ_C and τ are defined in Remark 2.1. Fix a faithful tracial state Tr on $\mathcal{M}(X, X)$. We treat $\mathcal{M}(C \odot X, Y)$ as the Hilbert space with inner product defined by

$$\langle g_1 | g_2 \rangle := \mathrm{Tr}\,(((\gamma_C^* \otimes 1_X)(1_{\overline{C}} \odot g_1^*))\,((1_{\overline{C}} \odot g_2)(\gamma_C \otimes 1_X))).$$

By the enrichment assumption, $\mathcal{C}(-, [X, -])$ and $\mathcal{M}(- \odot X, -)$ are equivalent bilinear $*$-functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{M} \to \mathbf{Hilb}_{\mathrm{f.d.},\mathbb{C}}$, i.e., $\mathcal{C}(f, [1_X, g])^* = \mathcal{C}(f^*, [1_X, g^*])$ and $\mathcal{M}(f \odot 1_X, g)^* = \mathcal{M}(f^* \odot 1_X, g^*)$ for every $f \in \mathcal{C}(C_2, C_1)$ and $g \in \mathcal{M}(Y_1, Y_2)$. By considering the polar decomposition of natural isomorphisms, we may assume that the natural isomorphism $\mathcal{C}(-, [X, -]) \simeq \mathcal{M}(- \odot X, -)$ is componentwise unitary, i.e., $\mathcal{C}(C, [X, Y]) \simeq \mathcal{M}(C \odot X, Y)$ is unitary for every $C \in \mathcal{C}$ and $Y \in \mathcal{M}$.

Note that $[X, -]$ is a left $\mathcal{C}$-module functor with the $\mathcal{C}$-module structure $\alpha_{C,Y} : C \otimes [X, Y] \xrightarrow{\sim} [X, C \odot Y]$ defined by the following natural isomorphism

$$\begin{aligned}\mathcal{C}(B, C \otimes [X, Y]) &\xrightarrow{\sim} \mathcal{C}(\overline{C} \otimes B, [X, Y]) \xrightarrow{\sim} \mathcal{M}((\overline{C} \otimes B) \odot X, Y) \\ &\xrightarrow{\sim} \mathcal{M}(\overline{C} \odot (B \odot X), Y) \xrightarrow{\sim} \mathcal{M}(B \odot X, C \odot Y) \xrightarrow{\sim} \mathcal{C}(B, [X, C \odot Y]),\end{aligned} \tag{4.1}$$

where the first and fourth morphisms are induced by the solution of conjugate equation $(\gamma_C, \overline{\gamma}_C)$ and the third morphism is induced by the module structure of $\mathcal{M}$ (see Section 7.12 in [19]). By the fact that the natural isomorphism $\mathcal{C}(-, [X, -]) \simeq \mathcal{M}(- \odot X, -)$ is componentwise unitary, it is not hard to check the the natural isomorphism (4.1) is unitary. Thus, $\alpha_{C,Y}$ is unitary.

The evaluation $\mathrm{ev}_Y : [X, Y] \odot X \to Y$ is obtained as the image of $1_{[X,Y]}$ under the natural isomorphism $\mathcal{C}([X, Y], [X, Y]) \simeq \mathcal{M}([X, Y] \odot X, Y)$. Let $\mathrm{ev}_Y = h_Y u_Y$ be the polar decomposition of ev_Y, where $h_Y := \sqrt{\mathrm{ev}_Y \mathrm{ev}_Y^*}$. Since $\alpha_{C,Y}$ is the unique morphism such that the following diagram commutes

$$\begin{array}{ccc} (C \otimes [X, Y]) \odot X & \xrightarrow{\sim} & C \odot ([X, Y] \odot X) \\ \downarrow{\scriptstyle \alpha_{C,Y}} & & \downarrow{\scriptstyle 1_C \odot \mathrm{ev}_Y} \\ [X, C \odot Y] \odot X & \xrightarrow{\mathrm{ev}_{C \odot Y}} & C \odot Y, \end{array}$$

by the uniqueness of the polar decomposition, we have $1_C \odot h_Y = h_{C \odot Y}$. In particular, $h_Y : Y \to Y$ is a left $\mathcal{C}$-module natural isomorphism of the identity functor $\mathrm{Id}_{\mathcal{M}}$ to itself. Since $\mathcal{M}$ is indecomposable, there exist $\lambda > 0$ such that $h_Y = \lambda 1_Y$ for every Y. Since the multiplication of $m : [X, X] \otimes [X, X] \to [X, X]$ is defined by

$$[X, X] \otimes [X, X] \xrightarrow{\alpha_{[X,X],X}} [X, [X, X] \odot X] \xrightarrow{[1_X, \mathrm{ev}_X]} [X, X],$$

(see Section 3.2 in [56]) we have $mm^* = \lambda^2 1_{[X,X]}$. Hence $[X, X]$ can be rescaled to a special C*-Frobenius algebra. □

Summing up, we can state and prove our main result.

Theorem 4.13. *An algebra in a multitensor C^*-category $\mathcal{C}$ is isomorphic to a special C^*-Frobenius algebra if and only if it is separable.*

Proof. By Lemma 4.8, we only need to show that every indecomposable separable algebra (A, m_A, ι_A) in $\mathcal{C}$ is isomorphic to a special C^*-Frobenius algebra. Recall that $\mathbf{RMod}_{\mathcal{C}}(A)$ is equivalent to a left C^*-module category over $\mathcal{C}$, denoted by $\mathcal{M}$, by Lemma 4.11. Let $F : \mathbf{RMod}_{\mathcal{C}}(A) \to \mathcal{M}$ be the equivalence of left $\mathcal{C}$-module categories. The algebra A seen as an object of $\mathbf{RMod}_{\mathcal{C}}(A)$ equals $[A, A]$, see e.g., Remark 3.5 in [56], hence it is isomorphic to $[F(A), F(A)]$. The latter is isomorphic to a special C^*-Frobenius algebra by Proposition 4.12, hence A is, and the proof is complete. □

For fusion C^*-categories $\mathcal{C}$, the following is stated as Corollary 3.8 in [8], as a consequence of Theorem 3.2 therein.

Corollary 4.14. *Let $\mathcal{M}$ be a finite semisimple left module category over a multi-fusion C^*-category $\mathcal{C}$. Then $\mathcal{M}$ is equivalent to $\mathbf{RMod}_{\mathcal{C}}(A)$ for a special C^*-Frobenius algebra A.*

Therefore, every finite semisimple left module category $\mathcal{M}$ over a multi-fusion C^*-category $\mathcal{C}$ admits a unique unitary structure (up to unitary module equivalence).

Proof. By Corollary 7.10.5 in [19], $\mathcal{M}$ is equivalent to $\mathbf{RMod}_{\mathcal{C}}(B)$, where B is an algebra in $\mathcal{C}$. Since $\mathcal{M}$ is semisimple, $\mathbf{BiMod}_{\mathcal{C}}(B|B) \simeq \mathrm{Fun}_{\mathcal{C}|}(\mathbf{RMod}_{\mathcal{C}}(B), \mathbf{RMod}_{\mathcal{C}}(B))$ is semisimple by Theorem 2.18 in [18]. Then B is separable by Proposition 4.3, and $\mathbf{RMod}_{\mathcal{C}}(B)$ is equivalent to $\mathbf{RMod}_{\mathcal{C}}(A)$ for a special C^*-Frobenius algebra A by Theorem 4.13. The uniqueness statement follows from Corollary 9 in [59], see also Theorem 1 and Remark 4 therein. □

We conclude with an application of Theorem 4.13 which justifies Remark 4.2 in [32]. The *idempotent completion* of a locally idempotent complete bicategory $\mathbf{B}$, introduced in Definition A.5.1 in [16], is the bicategory whose objects are *separable* algebras in $\mathbf{B}$, whose 1-morphisms are bimodules, and whose 2-morphisms are bimodule maps. By Proposition A.5.4 in [16], there exists a canonical fully faithful bifunctor from $\mathbf{B}$ into its idempotent completion. $\mathbf{B}$ is called *idempotent complete* if this bifunctor is a biequivalence. By combining the straightforward generalization of Theorem 4.13 to algebras in (rigid) semisimple C^*-bicategories and Lemma 4.1 in [32], we have the following result.

Corollary 4.15. *The rigid C^*-bicategory of finite direct sums of II_1 factors, finite Connes' bimodules and intertwiners is idempotent complete.*

This result is also stated with a different but equivalent terminology in [11]. By Theorem 4.13, at least for (rigid) semisimple C^*-bicategories, the terminology of *Q-system completion* used in Definition 3.34 in [11] coincides with the previously mentioned idempotent completion of [16].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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