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Research article

On fuzzy soft β-continuity and β-irresoluteness: some new results

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Abstract: In this paper, we first introduced the concept of r-fuzzy soft β-closed sets in fuzzy soft topological spaces based on the sense of Šostak and investigated some properties of them. Also, we defined the closure and interior operators with respect to the classes of r-fuzzy soft β-closed and r-fuzzy soft β-open sets and studied some of their properties. Moreover, the concept of r-fuzzy soft β-connected sets was introduced and characterized with the help of fuzzy soft β-closure operators. Thereafter, some properties of a fuzzy soft β-continuity were studied. Also, we introduced and studied the concepts of fuzzy soft almost (weakly) β-continuous functions, which are weaker forms of a fuzzy soft β-continuity. The relationships between these classes of functions were specified with the help of some illustrative examples. Finally, we explored new types of fuzzy soft functions called fuzzy soft β-irresolute (strongly β-irresolute, β-irresolute open, β-irresolute closed, and β-irresolute homeomorphism) functions and discussed some properties of them. Also, we showed that fuzzy soft strongly β-irresolute $\Rightarrow$ fuzzy soft β-irresolute $\Rightarrow$ fuzzy soft β-continuity, but the converse may not be true.

Keywords: fuzzy soft set; fuzzy soft topological space; r-fuzzy soft β-closed (β-open) set; fuzzy soft β-closure (β-interior) operator; fuzzy soft β-continuity; fuzzy soft almost (weakly) β-continuity; fuzzy soft β-irresolute (strongly β-irresolute) function; connectedness
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1. Introduction and preliminaries

The theory of soft set was pioneered by Molodtsov [1], which is a completely novel approach for modeling uncertainty and vagueness. He demonstrated some applications of this theory in solving some practical problems in engineering, economics, medical science, social science, etc. The concept of soft sets was used to define soft topological spaces in [2]. The study in [2] was particularly important in the development of the field of soft topology (for more details, see [3–6]). Generalizations of soft open sets play an effective role in soft topology through their use to improve on some known results or to open the door to redefine and investigate some of the soft topological concepts such as soft continuity [7], soft connectedness [8, 9], soft separation axioms [10, 11], etc. Akdag and Ozkan [12] initiated and studied the concept of soft α-open sets on soft topological spaces. The concept of soft β-open sets was studied by the authors of [13, 14], and some properties of soft β-continuity were investigated. Also, the concepts of somewhere dense and Q-sets were defined and studied by the authors of [15, 16]. Al-shami et al. [17] initiated the concept of weakly soft semi-open sets and studied its main properties. Also, Al-shami et al. [18] defined and studied the concept of weakly soft β-open sets. Kaur et al. [19] introduced a new approach to studying soft continuous mappings using an induced mapping based on soft sets. Al Ghour and Al-Mufarrij [20] defined new concepts of mappings over soft topological spaces: soft somewhat-r-continuity and soft somewhat-r-openness. Ameen et al. [21] explored more properties of soft somewhere dense continuity.

The notion of fuzzy soft sets was introduced by Maji et al. [22], which combines fuzzy sets [23] and soft sets [1]. Based on fuzzy topologies in the sense of Šostak [24], the notion of fuzzy soft topology is defined and some properties such as fuzzy soft continuity, fuzzy soft interior (closure) set, and fuzzy soft subspace are introduced in [25, 26]. The notion of r-fuzzy soft regularly open sets was defined and studied by Çetkin and Aygün [27]. In addition, the notions of r-fuzzy soft β-open (resp., pre-open) sets were introduced by Taha [28]. A new approach to studying separation and regularity axioms via fuzzy soft sets was introduced by the author of [29, 30] based on the paper by Aygünoğlu et al. [25].

The main contribution of this study is arranged as follows:

• In Section 2, we are going to present the notions of fuzzy soft β-closure (β-interior) operators in fuzzy soft topological spaces based on the article by Aygünoğlu et al. [25] and study some properties of them. Also, the concept of r-fuzzy soft β-connected sets was defined and studied.

• In Section 3, we investigate some properties of a fuzzy soft β-continuity. Moreover, we explore and study the notions of fuzzy soft almost (weakly) β-continuous functions, which are weaker forms of fuzzy soft β-continuous functions. Also, we show that fuzzy soft β-continuity $\Rightarrow$ fuzzy soft almost β-continuity $\Rightarrow$ fuzzy soft weakly β-continuity, but the converse may not be true.

• In Section 4, we introduce the notions of fuzzy soft β-irresolute (resp., strongly β-irresolute, β-irresolute open, β-irresolute closed, and β-irresolute homeomorphism) functions between two fuzzy soft topological spaces (V, η_F) and (U, τ_E) and investigate some properties of these functions. Additionally, the relationships between these classes of functions are considered with the help of some examples.

• In the end, we close this study with some conclusions and open a door to suggest some future papers in Section 5.

In this study, nonempty sets will be denoted by V, U, etc. F is the set of all parameters for V and $B \subseteq F$. The family of all fuzzy sets on V is denoted by I^V (where $I_\circ = (0, 1]$, $I = [0, 1]$), and for $t \in I$, $\underline{t}(v) = t$, for all $v \in V$.

The following notions and results will be used in the next sections:

Definition 1.1. *[25, 31, 32]* A fuzzy soft set g_B on V is a function from F to I^V such that $g_B(k)$ is a fuzzy set on V, for each $k \in B$ and $g_B(k) = \underline{0}$, if $k \notin B$. The family of all fuzzy soft sets on V is denoted by $\widetilde{(V, F)}$. In [33], the difference between two fuzzy soft sets g_B and f_A is a fuzzy soft set, defined as follows, for each $k \in F$:

$$(g_B \,\overline{\sqcap}\, f_A)(k) = \begin{cases} \underline{0}, & \text{if } g_B(k) \leq f_A(k), \\ g_B(k) \wedge (f_A(k))^c, & \text{otherwise.} \end{cases}$$

Definition 1.2. *[34]* A fuzzy soft point k_{v_t} on V is a fuzzy soft set, defined as follows:

$$k_{v_t}(e) = \begin{cases} v_t, & \text{if } \quad e = k, \\ \underline{0}, & \text{if } \quad e \in F - \{k\}, \end{cases}$$

where v_t is a fuzzy point on V. k_{v_t} is said to belong to a fuzzy soft set g_B, denoted by $k_{v_t} \tilde{\in} g_B$, if $t \leq g_B(k)(v)$. The family of all fuzzy soft points on V is denoted by $\widetilde{P_t(V)}$.

Definition 1.3. *[35]* A fuzzy soft point $k_{v_t} \in \widetilde{P_t(V)}$ is called a soft quasi-coincident with $g_B \in \widetilde{(V, F)}$, denoted by $k_{v_t} \nabla g_B$, if $t + g_B(k)(v) > 1$. A fuzzy soft set $f_A \in \widetilde{(V, F)}$ is called a soft quasi-coincident with $g_B \in \widetilde{(V, F)}$, denoted by $f_A \nabla g_B$, if there is $k \in F$ and $v \in V$ such that, $f_A(k)(v) + g_B(k)(v) > 1$, if f_A is not soft quasi-coincident with g_B, $f_A \overline{\nabla} g_B$.

Definition 1.4. *[25]* A function $\eta : F \longrightarrow [0, 1]^{\widetilde{(V,F)}}$ is said to be a fuzzy soft topology on V if it satisfies the following, for each $k \in F$:

(1) $\eta_k(\Phi) = \eta_k(\widetilde{F}) = 1$,

(2) $\eta_k(g_B \sqcap f_A) \geq \eta_k(g_B) \wedge \eta_k(f_A)$, for each $g_B, f_A \in \widetilde{(V, F)}$,

(3) $\eta_k(\sqcup_{\delta \in \Delta}(g_B)_\delta) \geq \wedge_{\delta \in \Delta} \eta_k((g_B)_\delta)$, for each $(g_B)_\delta \in \widetilde{(V, F)}, \delta \in \Delta$.

Thus, (V, η_F) is said to be a fuzzy soft topological space (briefly, FSTS) based on the sense of Šostak [24].

Definition 1.5. *[25]* Let (V, η_F) and (U, τ_E) be an FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$ is said to be fuzzy soft continuous if $\eta_k(\varphi_\psi^{-1}(g_B)) \geq \tau_e(g_B)$ for each $g_B \in \widetilde{(U, E)}$, $k \in F$, and $(e = \psi(k)) \in E$.

Definition 1.6. *[26,27]* In an FSTS (V, η_F), for each $g_B \in \widetilde{(V, F)}$, $k \in F$, and $r \in I_0$, we define the fuzzy soft operators C_η and $I_\eta : F \times \widetilde{(V, F)} \times I_\circ \to \widetilde{(V, F)}$ as follows:

$C_\eta(k, g_B, r) = \sqcap \{f_A \in \widetilde{(V, F)} : g_B \sqsubseteq f_A,\ \eta_k(f_A^c) \geq r\}$,

$I_\eta(k, g_B, r) = \sqcup \{f_A \in \widetilde{(V, F)} : f_A \sqsubseteq g_B,\ \eta_k(f_A) \geq r\}$.

Definition 1.7. Let (V, η_F) be an FSTS and $r \in I_0$. A fuzzy soft set $g_B \in \widetilde{(V, F)}$ is said to be r-fuzzy soft β-open [28] (resp., pre-open [28], semi-open [36], and regularly open [27]) if $g_B \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, g_B, r), r), r)$ (resp., $g_B \sqsubseteq I_\eta(k, C_\eta(k, g_B, r), r)$, $g_B \sqsubseteq C_\eta(k, I_\eta(k, g_B, r), r)$, and $g_B = I_\eta(k, C_\eta(k, g_B, r), r)$) for each $k \in F$.

Definition 1.8. *[27]* Let (V, η_F) be an FSTS and $r \in I_0$. A fuzzy soft set $g_B \in \widetilde{(V, F)}$ is said to be an r-fuzzy soft regularly closed if $g_B = C_\eta(k, I_\eta(k, g_B, r), r)$ for each $k \in F$.

Remark 1.1. *[28]* From the previous definition, we can summarize the relationships among different types of fuzzy soft sets as in the next diagram.

$$\begin{array}{ccc} \textit{regularly open set} & & \\ \Downarrow & & \\ \textit{pre} - \textit{open set} & \Rightarrow & \beta - \textit{open set} \end{array}$$

Definition 1.9. *[36]* Let (V, η_F) and (U, τ_E) be an FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ is said to be a fuzzy soft almost (resp., weakly) continuous if, for each $k_{v_t} \in \widetilde{P_t(V)}$ and each $g_B \in \widetilde{(U,E)}$ with $\tau_e(g_B) \geq r$ containing $\varphi_\psi(k_{v_t})$, there is $f_A \in \widetilde{(V,F)}$ with $\eta_k(f_A) \geq r$ containing, k_{v_t} such that $\varphi_\psi(f_A) \sqsubseteq I_\tau(e, C_\tau(e, g_B, r), r)$ (resp., $\varphi_\psi(f_A) \sqsubseteq C_\tau(e, g_B, r)$).

Remark 1.2. *[36]* From Definitions 1.5 and 1.9, we have: fuzzy soft continuity $\Rightarrow$ fuzzy soft almost continuity $\Rightarrow$ fuzzy soft weakly continuity.

The basic results and definitions that we need in the next sections are found in [25, 26].

2. Some properties of r-fuzzy soft β-closed sets

Here, we introduce the concept of r-fuzzy soft β-closed sets in FSTSs based on the sense of Šostak [24] and investigate some properties of them. Also, we define and study the concepts of fuzzy soft β-closure (β-interior) operators. Moreover, the concept of r-fuzzy soft β-connected sets is defined and characterized.

Definition 2.1. Let (V, η_F) be an FSTS. A fuzzy soft set $g_B \in \widetilde{(V,F)}$ is said to be an r-fuzzy soft β-closed if $I_\eta(k, C_\eta(k, I_\eta(k, g_B, r), r), r) \sqsubseteq g_B$ for each $k \in F$ and $r \in I_0$.

Proposition 2.1. Let (V, η_F) be an FSTS, $g_B \in \widetilde{(V,F)}$, $k \in F$, and $r \in I_0$, then we have

(1) g_B is an r-fuzzy soft β-closed set if g_B^c is r-fuzzy soft β-open [28].

(2) Any intersection of r-fuzzy soft β-closed sets is r-fuzzy soft β-closed.

(3) Any union of r-fuzzy soft β-open sets is r-fuzzy soft β-open.

Proof. Follows from Definitions 1.7 and 2.1.

Proposition 2.2. Let (V, η_F) be an FSTS, $g_B, f_A \in \widetilde{(V,F)}$, $k \in F$, and $r \in I_0$. If g_B is an r-fuzzy soft pre-open set such that $g_B \sqsubseteq f_A \sqsubseteq C_\eta(k, I_\eta(k, g_B, r), r)$, f_A is r-fuzzy soft β-open.

Proof. Since g_B is r-fuzzy soft pre-open and $g_B \sqsubseteq f_A$, then $g_B \sqsubseteq I_\eta(k, C_\eta(k, g_B, r), r) \sqsubseteq I_\eta(k, C_\eta(k, f_A, r), r)$. Since $f_A \sqsubseteq C_\eta(k, I_\eta(k, g_B, r), r)$, then $f_A \sqsubseteq C_\eta(k, I_\eta(k, I_\eta(k, C_\eta(k, f_A, r), r), r), r) = C_\eta(k, I_\eta(k, C_\eta(k, f_A, r), r), r)$, so f_A is r-fuzzy soft β-open.

Proposition 2.3. Let (V, η_F) be an FSTS, $g_B, f_A \in \widetilde{(V,F)}$, $k \in F$, and $r \in I_0$. If g_B is an r-fuzzy soft pre-closed set such that $g_B \sqsupseteq f_A \sqsupseteq I_\eta(k, C_\eta(k, g_B, r), r)$, f_A is r-fuzzy soft β-closed.

Proof. Easily proved by a similar way in Proposition 2.2.

Definition 2.2. In an FSTS (V, η_F), for each $g_B \in \widetilde{(V,F)}$, $k \in F$, and $r \in I_0$, we define a fuzzy soft operator $\beta C_\eta : F \times \widetilde{(V,F)} \times I_\circ \to \widetilde{(V,F)}$ as follows: $\beta C_\eta(k, g_B, r) = \sqcap \{f_A \in \widetilde{(V,F)} : g_B \sqsubseteq f_A,\ f_A$ is r-fuzzy soft β-closed$\}$.

Theorem 2.1. In an FSTS (V, η_F), for each $f_A, g_B \in \widetilde{(V,F)}$, $k \in F$, and $r \in I_0$, the operator $\beta C_\eta : F \times \widetilde{(V,F)} \times I_\circ \to \widetilde{(V,F)}$ satisfies the following properties:

(1) $\beta C_\eta(k, \Phi, r) = \Phi$.

(2) $g_B \sqsubseteq \beta C_\eta(k, g_B, r) \sqsubseteq C_\eta(k, g_B, r)$.

(3) $\beta C_\eta(k, g_B, r) \sqsubseteq \beta C_\eta(k, f_A, r)$, if $g_B \sqsubseteq f_A$.

(4) $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r) = \beta C_\eta(k, g_B, r)$.

(5) $\beta C_\eta(k, g_B \sqcup f_A, r) \sqsupseteq \beta C_\eta(k, g_B, r) \sqcup \beta C_\eta(k, f_A, r)$.
(6) g_B is r-fuzzy soft β-closed if $\beta C_\eta(k, g_B, r) = g_B$.
(7) $\beta C_\eta(k, C_\eta(k, g_B, r), r) = C_\eta(k, g_B, r)$.

Proof. (1), (2), (3), and (6) are easily proved from the definition of βC_η.

(4) From (2) and (3), $\beta C_\eta(k, g_B, r) \sqsubseteq \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$. Now, we show that $\beta C_\eta(k, g_B, r) \sqsupseteq \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$. Suppose that $\beta C_\eta(k, g_B, r)$ does not contain $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$, then there is $v \in V$ and $t \in (0, 1)$ such that $\beta C_\eta(k, g_B, r)(k)(v) < t < \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)(k)(v)$. (A)

Since $\beta C_\eta(k, g_B, r)(k)(v) < t$, by the definition of βC_η, there is h_F as an r-fuzzy soft β-closed with $g_B \sqsubseteq h_F$ such that $\beta C_\eta(k, g_B, r)(k)(v) \leq h_F(k)(v) < t$. Since $g_B \sqsubseteq h_F$, we have $\beta C_\eta(k, g_B, r) \sqsubseteq h_F$. Again, by the definition of βC_η, $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r) \sqsubseteq h_F$. Hence, $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r)(k)(v) \leq h_F(k)(v) < t$, which is a contradiction for (A). Thus, $\beta C_\eta(k, g_B, r) \sqsupseteq \beta C_\eta(k, \beta C_\eta(k, g_B, r), r)$, then $\beta C_\eta(k, \beta C_\eta(k, g_B, r), r) = \beta C_\eta(k, g_B, r)$.

(5) Since $g_B \sqsubseteq g_B \sqcup f_A$ and $f_A \sqsubseteq g_B \sqcup f_A$, then by (3) we have $\beta C_\eta(k, g_B, r) \sqsubseteq \beta C_\eta(k, g_B \sqcup f_A, r)$ and $\beta C_\eta(k, f_A, r) \sqsubseteq \beta C_\eta(k, g_B \sqcup f_A, r)$. Thus, $\beta C_\eta(k, g_B \sqcup f_A, r) \sqsupseteq \beta C_\eta(k, g_B, r) \sqcup \beta C_\eta(k, f_A, r)$.

(7) Since $C_\eta(k, g_B, r)$ is an r-fuzzy soft β-closed set, then by (6) we have $\beta C_\eta(k, C_\eta(k, g_B, r), r) = C_\eta(k, g_B, r)$.

Definition 2.3. Let (V, η_F) be an FSTS, $r \in I_0$, and $g_B, f_A \in \widetilde{(V, F)}$, then we have:

(1) Two fuzzy soft sets g_B and f_A are called r-fuzzy soft β-separated if $g_B \,\overline{\nabla}\, \beta C_\eta(k, f_A, r)$ and $f_A \,\overline{\nabla}\, \beta C_\eta(k, g_B, r)$ for each $k \in F$.

(2) Any fuzzy soft set which cannot be expressed as the union of two r-fuzzy soft β-separated sets is called an r-fuzzy soft β-connected.

Theorem 2.2. In an FSTS (V, η_F), we have:

(1) If f_A and $g_B \in \widetilde{(V, F)}$ are r-fuzzy soft β-separated and h_C, $t_D \in \widetilde{(V, F)}$ such that $h_C \sqsubseteq f_A$ and $t_D \sqsubseteq g_B$, then h_C and t_D are r-fuzzy soft β-separated.

(2) If $f_A \,\overline{\nabla}\, g_B$ are either both r-fuzzy soft β-open or both are r-fuzzy soft β-closed, then f_A and g_B are r-fuzzy soft β-separated.

(3) If f_A and g_B are either both r-fuzzy soft β-open or both r-fuzzy soft β-closed, then $f_A \sqcap g_B^c$ and $g_B \sqcap f_A^c$ are r-fuzzy soft β-separated.

Proof. (1) and (2) are obvious.

(3) Let f_A and g_B be an r-fuzzy soft β-open. Since $f_A \sqcap g_B^c \sqsubseteq g_B^c$, $\beta C_\eta(k, f_A \sqcap g_B^c, r) \sqsubseteq g_B^c$, and $\beta C_\eta(k, f_A \sqcap g_B^c, r) \,\overline{\nabla}\, g_B$, then $\beta C_\eta(k, f_A \sqcap g_B^c, r) \,\overline{\nabla}\, (g_B \sqcap f_A^c)$.

Again, since $g_B \sqcap f_A^c \sqsubseteq f_A^c$, $\beta C_\eta(k, g_B \sqcap f_A^c, r) \sqsubseteq f_A^c$, and $\beta C_\eta(k, g_B \sqcap f_A^c, r) \,\overline{\nabla}\, f_A$, then $\beta C_\eta(k, g_B \sqcap f_A^c, r) \,\overline{\nabla}\, (f_A \sqcap g_B^c)$. Thus, $f_A \sqcap g_B^c$ and $g_B \sqcap f_A^c$ are r-fuzzy soft β-separated. The other case follows similar lines.

Theorem 2.3. In an FSTS (V, η_F), $f_A, g_B \in \widetilde{(V, F)}$ are r-fuzzy soft β-separated if there exist two r-fuzzy soft β-open sets h_C and t_D such that $f_A \sqsubseteq h_C$, $g_B \sqsubseteq t_D$, $f_A \,\overline{\nabla}\, t_D$, and $g_B \,\overline{\nabla}\, h_C$.

Proof. $(\Rightarrow)$ Let f_A and $g_B \in \widetilde{(V, F)}$ be r-fuzzy soft β-separated, $f_A \sqsubseteq (\beta C_\eta(k, g_B, r))^c = h_C$ and $g_B \sqsubseteq (\beta C_\eta(k, f_A, r))^c = t_D$, where t_D and h_C are r-fuzzy soft β-open, then $t_D \,\overline{\nabla}\, \beta C_\eta(k, f_A, r)$ and $h_C \,\overline{\nabla}\, \beta C_\eta(k, g_B, r)$. Thus, $g_B \,\overline{\nabla}\, h_C$ and $f_A \,\overline{\nabla}\, t_D$. Hence, we obtain the required result.

($\Leftarrow$) Let h_C and t_D be r-fuzzy soft β-open such that $g_B \sqsubseteq t_D$, $f_A \sqsubseteq h_C$, $g_B \overline{\nabla} h_C$, and $f_A \overline{\nabla} t_D$, then $g_B \sqsubseteq h_C^c$ and $f_A \sqsubseteq t_D^c$. Hence, $\beta C_\eta(k, g_B, r) \sqsubseteq h_C^c$ and $\beta C_\eta(k, f_A, r) \sqsubseteq t_D^c$, then $\beta C_\eta(k, g_B, r) \overline{\nabla} f_A$ and $\beta C_\eta(k, f_A, r) \overline{\nabla} g_B$. Thus, g_B and f_A are r-fuzzy soft β-separated. Hence, we obtain the required result.

Theorem 2.4. In an FSTS (V, η_F), if $g_B \in \widetilde{(V, F)}$ is r-fuzzy soft β-connected such that $g_B \sqsubseteq f_A \sqsubseteq \beta C_\eta(k, g_B, r)$, then f_A is r-fuzzy soft β-connected.

Proof. Suppose that f_A is not r-fuzzy soft β-connected, then there is r-fuzzy soft β-separated sets h_C^* and $t_D^* \in \widetilde{(V, F)}$ such that $f_A = h_C^* \sqcup t_D^*$. Let $h_C = g_B \sqcap h_C^*$ and $t_D = g_B \sqcap t_D^*$, then $g_B = t_D \sqcup h_C$. Since $h_C \sqsubseteq h_C^*$ and $t_D \sqsubseteq t_D^*$, by Theorem 2.2(1), h_C and t_D are r-fuzzy soft β-separated, which is a contradiction. Thus, f_A is r-fuzzy soft β-connected, as required.

Theorem 2.5. In an FSTS (V, η_F), for each $g_B \in \widetilde{(V, F)}$, $k \in F$, and $r \in I_0$, we define a fuzzy soft operator $\beta I_\eta : F \times \widetilde{(V, F)} \times I_\circ \to \widetilde{(V, F)}$ as follows: $\beta I_\eta(k, g_B, r) = \sqcup \{f_A \in \widetilde{(V, F)} : f_A \sqsubseteq g_B, \text{and } f_A \text{ is } r\text{-fuzzy soft } \beta\text{-open}\}$. For each $f_A, g_B \in \widetilde{(V, F)}$, the operator βI_η satisfies the following properties:

(1) $\beta I_\eta(k, \widetilde{F}, r) = \widetilde{F}$.
(2) $I_\eta(k, g_B, r) \sqsubseteq \beta I_\eta(k, g_B, r) \sqsubseteq g_B$.
(3) $\beta I_\eta(k, g_B, r) \sqsubseteq \beta I_\eta(k, f_A, r)$, if $g_B \sqsubseteq f_A$.
(4) $\beta I_\eta(k, \beta I_\eta(k, g_B, r), r) = \beta I_\eta(k, g_B, r)$.
(5) $\beta I_\eta(k, g_B, r) \sqcap \beta I_\eta(k, f_A, r) \sqsupseteq \beta I_\eta(k, g_B \sqcap f_A, r)$.
(6) g_B is r-fuzzy soft β-open if $\beta I_\eta(k, g_B, r) = g_B$.
(7) $\beta I_\eta(k, g_B^c, r) = (\beta C_\eta(k, g_B, r))^c$.

Proof. (1), (2), (3), and (6) are easily proved from the definition of βI_η.

(4) and (5) are easily proved by a similar way in Theorem 2.1.

(7) For each $g_B \in \widetilde{(V, F)}$, $k \in F$, and $r \in I_0$, we have $\beta I_\eta(k, g_B^c, r) = \sqcup\{f_A \in \widetilde{(V, F)} : f_A \sqsubseteq g_B^c,\ f_A \text{ is } r\text{-fuzzy soft } \beta\text{-open}\} = [\sqcap\{f_A^c \in \widetilde{(V, F)} : g_B \sqsubseteq f_A^c, \text{ and } f_A^c \text{ is } r\text{-fuzzy soft } \beta\text{-closed}\}]^c = (\beta C_\eta(k, g_B, r))^c$.

3. Weaker forms of fuzzy soft β-continuity

Here, we investigate some properties of fuzzy soft β-continuity. As a weaker form of fuzzy soft β-continuity, the concepts of fuzzy soft almost (weakly) β-continuous functions are introduced and some properties are given.

Definition 3.1. Let (V, η_F) and (U, τ_E) be FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$ is said to be a fuzzy soft β-continuous if $\varphi_\psi^{-1}(g_B)$ is an r-fuzzy soft β-closed set for each $g_B \in \widetilde{(U, E)}$ with $\tau_e(g_B^c) \geq r$, $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_o$.

Theorem 3.1. Let (V, η_F) and (U, τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in \widetilde{(U, E)}$, $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft β-continuous.
(2) For each $g_B \in \widetilde{(U, E)}$ with $\tau_e(g_B) \geq r$, $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft β-open.
(3) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.
(4) $\varphi_\psi^{-1}(I_\tau(e, g_B, r)) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B), r)$.
(5) $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

Proof. (1) ⇔ (2) Follows from Proposition 2.1 and $\varphi_\psi^{-1}(g_B^c) = (\varphi_\psi^{-1}(g_B))^c$.

(1) ⇒ (3) Let $g_B \in \widetilde{(U,E)}$; hence, by (1), $\varphi_\psi^{-1}(C_\tau(e, g_B, r))$ is r-fuzzy soft β-closed, then we obtain $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

(3) ⇔ (4) Follows from Theorem 2.5.

(3) ⇒ (5) Let $g_B \in \widetilde{(U,E)}$; hence, by (3), we obtain $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

(5) ⇒ (2) Let $g_B \in \widetilde{(U,E)}$ with $\tau_e(g_B) \geq r$. By (5), we obtain $(\varphi_\psi^{-1}(g_B))^c = \varphi_\psi^{-1}(g_B^c) \sqsupseteq I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B^c), r), r), r) = (C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r))^c$, then $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$, so $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft β-open.

Lemma 3.1. Every fuzzy soft continuous function [25] is fuzzy soft β-continuous.

Proof. Follows from Definition 1.5 and Theorem 3.1.

Remark 3.1. The converse of Lemma 3.1 is not true, as shown by Example 3.1.

Example 3.1. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$, and define $g_F, h_F \in \widetilde{(V,F)}$ as follows: $g_F = \{(k_1, \{\frac{v_1}{0.2}, \frac{v_2}{0.3}, \frac{v_3}{0.2}\}), (k_2, \{\frac{v_1}{0.2}, \frac{v_2}{0.3}, \frac{v_3}{0.2}\})\}$, $h_F = \{(k_1, \{\frac{v_1}{0.3}, \frac{v_2}{0.4}, \frac{v_3}{0.8}\}), (k_2, \{\frac{v_1}{0.3}, \frac{v_2}{0.4}, \frac{v_3}{0.8}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \longrightarrow [0,1]^{\widetilde{(V,F)}}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if } \; n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{2}, & \text{if } \; n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if } \; n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{3}, & \text{if } \; n_F = h_F, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \longrightarrow (V, \tau_F)$ is fuzzy soft β-continuous, but it is not fuzzy soft continuous.

Definition 3.2. Let (V, η_F) and (U, τ_E) be FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ is said to be fuzzy soft almost (resp., weakly) β-continuous if for each $k_{v_t} \in \widetilde{P_t(V)}$ and each $f_A \in \widetilde{(U,E)}$ with $\tau_e(f_A) \geq r$ containing $\varphi_\psi(k_{v_t})$, there is $g_B \in \widetilde{(V,F)}$ that is an r-fuzzy soft β-open set containing k_{v_t}, such that $\varphi_\psi(g_B) \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r)$ (resp., $\varphi_\psi(g_B) \sqsubseteq C_\tau(e, f_A, r)$), $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_\circ$.

Lemma 3.2. (1) Every fuzzy soft β-continuous function is fuzzy soft almost β-continuous.

(2) Every fuzzy soft almost β-continuous function is fuzzy soft weakly β-continuous.

Proof. Follows from Definition 3.2 and Theorem 3.1.

Remark 3.2. The converse of Lemma 3.2 is not true, as shown by Examples 3.2 and 3.3.

Example 3.2. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$, and define $f_F, g_F, h_F \in \widetilde{(V,F)}$ as follows: $f_F = \{(k_1, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.5}\}), (k_2, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.5}\})\}$, $g_F = \{(k_1, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\}), (k_2, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\})\}$, $h_F = \{(k_1, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.6}\}), (k_2, \{\frac{v_1}{0.5}, \frac{v_2}{0.5}, \frac{v_3}{0.6}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \longrightarrow [0,1]^{\widetilde{(V,F)}}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if} \quad n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{2}{3}, & \text{if} \quad n_F = f_F, \\ \frac{1}{2}, & \text{if} \quad n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if} \quad n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{2}{3}, & \text{if} \quad n_F = f_F, \\ \frac{1}{3}, & \text{if} \quad n_F = h_F, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \longrightarrow (V, \tau_F)$ is fuzzy soft almost β-continuous, but it is not fuzzy soft β-continuous.

Example 3.3. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$, and define $g_F, h_F \in \widetilde{(V, F)}$ as follows: $g_F = \{(k_1, \{\frac{v_1}{0.6}, \frac{v_2}{0.2}, \frac{v_3}{0.4}\}), (k_2, \{\frac{v_1}{0.6}, \frac{v_2}{0.2}, \frac{v_3}{0.4}\})\}$, $h_F = \{(k_1, \{\frac{v_1}{0.3}, \frac{v_2}{0.2}, \frac{v_3}{0.5}\}), (k_2, \{\frac{v_1}{0.3}, \frac{v_2}{0.2}, \frac{v_3}{0.5}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \longrightarrow [0, 1]^{\widetilde{(V,F)}}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if} \quad n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{2}, & \text{if} \quad n_F = g_F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_k(n_F) = \begin{cases} 1, & \text{if} \quad n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{3}, & \text{if} \quad n_F = h_F, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (V, \eta_F) \longrightarrow (V, \tau_F)$ is fuzzy soft weakly β-continuous, but it is not fuzzy soft almost β-continuous.

Lemma 3.3. (1) Every fuzzy soft almost continuous function is fuzzy soft almost β-continuous.
(2) Every fuzzy soft weakly continuous function is fuzzy soft weakly β-continuous.

Proof. Follows from Definitions 1.9 and 3.2.

Remark 3.3. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft continuity as in the next diagram.

$$\begin{array}{ccc} \text{fuzzy soft continuity} & \Rightarrow & \text{fuzzy soft } \beta\text{-continuity} \\ \Downarrow & & \Downarrow \\ \text{fuzzy soft almost continuity} & \Rightarrow & \text{fuzzy soft almost } \beta\text{-continuity} \\ \Downarrow & & \Downarrow \\ \text{fuzzy soft weakly continuity} & \Rightarrow & \text{fuzzy soft weakly } \beta\text{-continuity} \end{array}$$

Theorem 3.2. Let (V, η_F) and (U, τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in \widetilde{(U, E)}$, $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_\circ$:
(1) φ_ψ is fuzzy soft almost β-continuous.

(2) $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft β-open, for each g_B is r-fuzzy soft regularly open.
(3) $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft β-closed, for each g_B is r-fuzzy soft regularly closed.
(4) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$, for each g_B is r-fuzzy soft β-open.
(5) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$, for each g_B is r-fuzzy soft semi-open.
(6) $\beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r) \sqsupseteq \varphi_\psi^{-1}(g_B)$, for each g_B with $\tau_e(g_B) \geq r$.

Proof. (1) ⇒ (2) Let $k_{v_t} \in \widetilde{P_t(V)}$ and $g_B \in \widetilde{(U, E)}$ be an r-fuzzy soft regularly open set containing $\varphi_\psi(k_{v_t})$. Hence, by (1), $f_A \in \widetilde{(V, F)}$ is an r-fuzzy soft β-open set containing k_{v_t} such that $\varphi_\psi(f_A) \sqsubseteq I_\tau(e, C_\tau(e, g_B, r), r)$.

Thus, $f_A \sqsubseteq \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)) = \varphi_\psi^{-1}(g_B)$ and $k_{v_t} \tilde{\in} f_A \sqsubseteq \varphi_\psi^{-1}(g_B)$, then $k_{v_t} \tilde{\in} C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$ and $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$. Therefore, $\varphi_\psi^{-1}(g_B)$ is an r-fuzzy soft β-open set.

(2) ⇒ (3) Let g_B be an r-fuzzy soft regularly closed set. Hence, by (2), $\varphi_\psi^{-1}(g_B^c) = (\varphi_\psi^{-1}(g_B))^c$ is an r-fuzzy soft β-open set, then $\varphi_\psi^{-1}(g_B)$ is an r-fuzzy soft β-closed set.

(3) ⇒ (4) Let g_B be an r-fuzzy soft β-open set. Since $C_\tau(e, g_B, r)$ is an r-fuzzy soft regularly closed set, by (3), $\varphi_\psi^{-1}(C_\tau(e, g_B, r))$ is an r-fuzzy soft β-closed set. Since $\varphi_\psi^{-1}(g_B) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$, then we have $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.

(4) ⇒ (5) This is obvious from every r-fuzzy soft semi-open set that is an r-fuzzy soft β-open set.

(5) ⇒ (3) Let g_B be an r-fuzzy soft regularly closed set. Hence, g_B is an r-fuzzy soft semi-open set, then by (5), $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r)) = \varphi_\psi^{-1}(g_B)$. Therefore, $\varphi_\psi^{-1}(g_B)$ is an r-fuzzy soft β-closed set.

(3) ⇒ (6) Let $g_B \in \widetilde{(U, E)}$ with $\tau_e(g_B) \geq r$ and $k_{v_t} \tilde{\in} \varphi_\psi^{-1}(g_B)$, then we have $k_{v_t} \tilde{\in} \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r))$. Since $[I_\tau(e, C_\tau(e, g_B, r), r)]^c$ is an r-fuzzy soft regularly closed set, by (3), $\varphi_\psi^{-1}([I_\tau(e, C_\tau(e, g_B, r), r)]^c)$ is an r-fuzzy soft β-closed set. Thus, $\varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r))$ is an r-fuzzy soft β-open set and $k_{v_t} \tilde{\in} \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$, then $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$.

(6) ⇒ (1) Let $k_{v_t} \in \widetilde{P_t(V)}$ and $g_B \in \widetilde{(U, E)}$ with $\tau_e(g_B) \geq r$ containing $\varphi_\psi(k_{v_t})$; hence, by (6), $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$.

Since $k_{v_t} \tilde{\in} \varphi_\psi^{-1}(g_B)$, then we obtain $k_{v_t} \tilde{\in} \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r) = f_A$ (say). Hence, $f_A \in \widetilde{(V, F)}$ is an r-fuzzy soft β-open set containing k_{v_t} such that $\varphi_\psi(f_A) \sqsubseteq I_\tau(e, C_\tau(e, g_B, r), r)$. Therefore, φ_ψ is fuzzy soft almost β-continuous.

In a similar way, we can prove the following theorem.

Theorem 3.3. Let (V, η_F) and (U, τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in \widetilde{(U, E)}$, $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_\circ$:
(1) φ_ψ is fuzzy soft weakly β-continuous.
(2) $C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(C_\tau(e, g_B, r)), r), r), r) \sqsupseteq \varphi_\psi^{-1}(g_B)$, if $\tau_e(g_B) \geq r$.
(3) $I_\eta(k, C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, g_B, r)), r), r), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\tau_e(g_B^c) \geq r$.
(4) $\beta C_\eta(k, \varphi_\psi^{-1}(I_\tau(e, g_B, r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\tau_e(g_B^c) \geq r$.
(5) $\beta C_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_\tau(e, g_B, r))$.
(6) $\beta I_\eta(k, \varphi_\psi^{-1}(C_\tau(e, I_\tau(e, g_B, r), r)), r) \sqsupseteq \varphi_\psi^{-1}(I_\tau(e, g_B, r))$.
(7) $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(C_\tau(e, g_B, r)), r)$, if $\tau_e(g_B) \geq r$.

Let $\mathcal{P}$ and $\mathcal{Q} : F \times \widetilde{(V, F)} \times I_\circ \to \widetilde{(V, F)}$ be operators on $\widetilde{(V, F)}$, and $\mathcal{R}$ and $\mathcal{S} : E \times \widetilde{(U, E)} \times I_\circ \to \widetilde{(U, E)}$ be operators on $\widetilde{(U, E)}$.

Definition 3.3. *[36]* Let (V, η_F) and (U, τ_E) be FSTSs. $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ is said to be a fuzzy soft $(\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S})$-continuous function if $\mathcal{P}[k, \varphi_\psi^{-1}(\mathcal{S}(e, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{Q}[k, \varphi_\psi^{-1}(\mathcal{R}(e, g_B, r)), r] = \Phi$ for each $g_B \in \widetilde{(U,E)}$ with $\tau_e(g_B) \geq r$, $k \in F$, and $(e = \psi(k)) \in E$.

In (2014), Aygünoğlu et al. [25] defined the notion of fuzzy soft continuous functions: $\eta_k(\varphi_\psi^{-1}(f_A)) \geq \tau_e(f_A)$, for each $f_A \in \widetilde{(U,E)}$, $k \in F$, and $(e = \psi(k)) \in E$. We can see that Definition 3.3 generalizes the concept of fuzzy soft continuous functions when we choose $\mathcal{P}$ = identity operator, $\mathcal{Q}$ = interior operator, $\mathcal{R}$ = identity operator, and $\mathcal{S}$ = identity operator.

A historical justification of Definition 3.3:
(1) In Section 3, we introduced the notion of fuzzy soft β-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r), r)$, for each $g_B \in \widetilde{(U,E)}$ with $\tau_e(g_B) \geq r$. Here, $\mathcal{P}$ = identity operator, $\mathcal{Q}$ = closure interior closure operator, $\mathcal{R}$ = identity operator, and $\mathcal{S}$ = identity operator.
(2) In Section 3, we introduced the notion of fuzzy soft almost β-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(I_\tau(e, C_\tau(e, g_B, r), r)), r)$, for each $g_B \in \widetilde{(U,E)}$ with $\tau_e(g_B) \geq r$. Here, $\mathcal{P}$ = identity operator, $\mathcal{Q}$ = β-interior operator, $\mathcal{R}$ = interior closure operator, and $\mathcal{S}$ = identity operator.
(3) In Section 3, we introduced the notion of fuzzy soft weakly β-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(C_\tau(e, g_B, r)), r)$, for each $g_B \in \widetilde{(U,E)}$ with $\tau_e(g_B) \geq r$. Here, $\mathcal{P}$ = identity operator, $\mathcal{Q}$ = β-interior operator, $\mathcal{R}$ = closure operator, and $\mathcal{S}$ = identity operator.

4. Fuzzy soft β-irresoluteness

Here, we introduce the concepts of fuzzy soft β-irresolute (resp., strongly β-irresolute, β-irresolute open, β-irresolute closed, and β-irresolute homeomorphism) functions between two FSTSs (V, η_F) and (U, τ_E) and study some of its features. Also, we show that fuzzy soft strongly β-irresolute $\Rightarrow$ fuzzy soft β-irresolute $\Rightarrow$ fuzzy soft β-continuity, but the converse may not be true.

Definition 4.1. Let (V, η_F) and (U, τ_E) be FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ is said to be a fuzzy soft β-irresolute (resp., strongly β-irresolute) if $\varphi_\psi^{-1}(g_B)$ is an r-fuzzy soft β-open (resp., semi-open) set for each $g_B \in \widetilde{(U,E)}$ r-fuzzy soft β-open set and $r \in I_o$.

Lemma 4.1. (1) Every fuzzy soft strongly β-irresolute function is fuzzy soft β-irresolute.
(2) Every fuzzy soft β-irresolute function is fuzzy soft β-continuous.

Proof. Follows from Definition 4.1 and Theorem 3.1.

Remark 4.1. The converse of Lemma 4.1 is not true, as shown by Examples 4.1 and 4.2.

Example 4.1. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$, and define $f_F, g_F \in \widetilde{(V,F)}$ as follows: $f_F = \{(k_1, \{\frac{v_1}{0.3}, \frac{v_2}{0.3}, \frac{v_3}{0.3}\}), (k_2, \{\frac{v_1}{0.3}, \frac{v_2}{0.3}, \frac{v_3}{0.3}\})\}$, $g_F = \{(k_1, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\}), (k_2, \{\frac{v_1}{0.4}, \frac{v_2}{0.4}, \frac{v_3}{0.4}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \longrightarrow [0,1]^{\widetilde{(V,F)}}$ as follows: $\forall k \in F$,

$$\eta_k(n_F) = \begin{cases} 1, & \text{if} \quad n_F \in \{\Phi, \widetilde{F}\}, \\ \frac{1}{2}, & \text{if} \quad n_F = g_F, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_k(n_F)=\begin{cases}1, & \text{if} \quad n_F\in\{\Phi,\widetilde{F}\},\\ \frac{1}{2}, & \text{if} \quad n_F=f_F,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (V,\eta_F) \longrightarrow (V,\tau_F)$ is fuzzy soft β-irresolute, but it is not fuzzy soft strongly β-irresolute.

Example 4.2. Let $V = \{v_1, v_2, v_3\}$, $F = \{k_1, k_2\}$, and define $f_F, g_F \in \widetilde{(V,F)}$ as follows: $f_F = \{(k_1,\{\frac{v_1}{0.5},\frac{v_2}{0.5},\frac{v_3}{0.5}\}),(k_2,\{\frac{v_1}{0.5},\frac{v_2}{0.5},\frac{v_3}{0.5}\})\}$, $g_F = \{(k_1,\{\frac{v_1}{0.4},\frac{v_2}{0.4},\frac{v_3}{0.4}\}),(k_2,\{\frac{v_1}{0.4},\frac{v_2}{0.4},\frac{v_3}{0.4}\})\}$. Define fuzzy soft topologies $\eta_F, \tau_F : F \longrightarrow [0,1]^{\widetilde{(V,F)}}$ as follows: $\forall k\in F$,

$$\eta_k(n_F)=\begin{cases}1, & \text{if} \quad n_F\in\{\Phi,\widetilde{F}\},\\ \frac{1}{3}, & \text{if} \quad n_F=f_F,\\ \frac{1}{2}, & \text{if} \quad n_F=g_F,\\ 0, & \text{otherwise,}\end{cases}$$

$$\tau_k(n_F)=\begin{cases}1, & \text{if} \quad n_F\in\{\Phi,\widetilde{F}\},\\ \frac{1}{3}, & \text{if} \quad n_F=f_F,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (V,\eta_F) \longrightarrow (V,\tau_F)$ is fuzzy soft β-continuous, but it is not fuzzy soft β-irresolute.

Remark 4.2. From the previous results, we have: fuzzy soft strongly β-irresolute $\Rightarrow$ fuzzy soft β-irresolute $\Rightarrow$ fuzzy soft β-continuity.

Theorem 4.1. Let (V,η_F) and (U,τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in \widetilde{(U,E)}$, $k\in F$, $(e=\psi(k))\in E$, and $r\in I_\circ$:

(1) φ_ψ is fuzzy soft β-irresolute.

(2) For each g_B r-fuzzy soft β-closed, $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft β-closed.

(3) $\beta C_\eta(k,\varphi_\psi^{-1}(g_B),r)\sqsubseteq\varphi_\psi^{-1}(\beta C_\tau(e,g_B,r))$.

(4) $\varphi_\psi^{-1}(\beta I_\tau(e,g_B,r))\sqsubseteq\beta I_\eta(k,\varphi_\psi^{-1}(g_B),r)$.

(5) $I_\eta(k,C_\eta(k,I_\eta(k,\varphi_\psi^{-1}(g_B),r),r),r)\sqsubseteq\varphi_\psi^{-1}(\beta C_\tau(e,g_B,r))$.

Proof. (1) $\Leftrightarrow$ (2) Follows from Proposition 2.1 and $\varphi_\psi^{-1}(g_B^c)=(\varphi_\psi^{-1}(g_B))^c$.

(2) $\Rightarrow$ (3) Let $g_B\in\widetilde{(U,E)}$. Hence, by (2), $\varphi_\psi^{-1}(\beta C_\tau(e,g_B,r))$ is r-fuzzy soft β-closed, then we obtain $\beta C_\eta(k,\varphi_\psi^{-1}(g_B),r)\sqsubseteq\varphi_\psi^{-1}(\beta C_\tau(e,g_B,r))$.

(3) $\Leftrightarrow$ (4) Follows from Theorem 2.5.

(3) $\Rightarrow$ (5) Let $g_B \in \widetilde{(U,E)}$. Hence, by (3), we obtain $I_\eta(k,C_\eta(k,I_\eta(k,\varphi_\psi^{-1}(g_B),r),r),r) \sqsubseteq \beta C_\eta(k,\varphi_\psi^{-1}(g_B),r)\sqsubseteq\varphi_\psi^{-1}(\beta C_\tau(e,g_B,r))$.

(5) $\Rightarrow$ (1) Let $g_B \in \widetilde{(U,E)}$ be an r-fuzzy soft β-open. Hence, by (5), we obtain $(\varphi_\psi^{-1}(g_B))^c = \varphi_\psi^{-1}(g_B^c) \sqsupseteq I_\eta(k,C_\eta(k,I_\eta(k,\varphi_\psi^{-1}(g_B^c),r),r),r) = (C_\eta(k,I_\eta(k,C_\eta(k,\varphi_\psi^{-1}(g_B),r),r),r))^c$, then $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k,I_\eta(k,C_\eta(k,\varphi_\psi^{-1}(g_B),r),r),r)$, so $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft β-open. Hence, φ_ψ is fuzzy soft β-irresolute.

Theorem 4.2. Let (V,η_F) and (U,τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ be a fuzzy soft function. The following statements are equivalent for each $g_B \in \widetilde{(U,E)}$, $k\in F$, $(e=\psi(k))\in E$, and $r\in I_\circ$:

(1) φ_ψ is fuzzy soft strongly β-irresolute.

(2) For each g_B r-fuzzy soft β-closed, $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft semi-closed.

(3) $I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

Proof. (1) $\Leftrightarrow$ (2) Follows from Proposition 2.1 and $\varphi_\psi^{-1}(g_B^c) = (\varphi_\psi^{-1}(g_B))^c$.

(2) $\Rightarrow$ (3) Let $g_B \in \widetilde{(U, E)}$. Hence, by (2), $\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$ is r-fuzzy soft semi-closed, then we obtain $I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

(3) $\Rightarrow$ (1) Let $g_B \in \widetilde{(U, E)}$ be an r-fuzzy soft β-open. Hence, by (3), we obtain $(\varphi_\psi^{-1}(g_B))^c = \varphi_\psi^{-1}(g_B^c) \sqsupseteq I_\eta(k, C_\eta(k, \varphi_\psi^{-1}(g_B^c), r), r) = (C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r))^c$, then $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\eta(k, I_\eta(k, \varphi_\psi^{-1}(g_B), r), r)$, so $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft semi-open. Hence, φ_ψ is fuzzy soft strongly β-irresolute.

Proposition 4.1. Let (V, η_F), (U, τ_E), and (W, γ_H) be FSTSs and $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$, $\varphi^*_{\psi^*} : \widetilde{(U, E)} \longrightarrow \widetilde{(W, H)}$ be two fuzzy soft functions, then the composition $\varphi^*_{\psi^*} \circ \varphi_\psi$ is fuzzy soft β-irresolute (resp., strongly β-irresolute and β-continuous) if φ_ψ is fuzzy soft β-irresolute (resp., strongly β-irresolute and β-irresolute) and $\varphi^*_{\psi^*}$ is fuzzy soft β-irresolute (resp., β-irresolute and β-continuous).

Proof. Follows from Definition 4.1 and Theorem 3.1.

Proposition 4.2. Let (V, η_F), (U, τ_E), and (W, γ_H) be FSTSs and $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$, $\varphi^*_{\psi^*} : \widetilde{(U, E)} \longrightarrow \widetilde{(W, H)}$ be two fuzzy soft functions, then the composition $\varphi^*_{\psi^*} \circ \varphi_\psi$ is fuzzy soft almost β-continuous if φ_ψ is fuzzy soft β-continuous (resp., β-irresolute and β-continuous) and $\varphi^*_{\psi^*}$ is fuzzy soft almost continuous (resp., almost β-continuous and continuous).

Proof. Follows from the above definitions.

Definition 4.2. Let (V, η_F) and (U, τ_E) be FSTSs and $r \in I_o$. A fuzzy soft function $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$ is said to be a fuzzy soft β-irresolute open (resp., closed) if $\varphi_\psi(f_A)$ is an r-fuzzy soft β-open (resp., β-closed) set for each $f_A \in \widetilde{(V, F)}$ which is an r-fuzzy soft β-open (resp., β-closed) set.

Theorem 4.3. Let (V, η_F) and (U, τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V, F)} \longrightarrow \widetilde{(U, E)}$ be a fuzzy soft function. The following statements are equivalent for each $f_A \in \widetilde{(V, F)}$, $g_B \in \widetilde{(U, E)}$, $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft β-irresolute open.

(2) $\varphi_\psi(\beta I_\eta(k, f_A, r)) \sqsubseteq \beta I_\tau(e, \varphi_\psi(f_A), r)$.

(3) $\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(\beta I_\tau(e, g_B, r))$.

(4) For each g_B and each f_A r-fuzzy soft β-closed with $\varphi_\psi^{-1}(g_B) \sqsubseteq f_A$, $h_C \in \widetilde{(U, E)}$ is r-fuzzy soft β-closed with $g_B \sqsubseteq h_C$ such that $\varphi_\psi^{-1}(h_C) \sqsubseteq f_A$.

Proof. (1) $\Rightarrow$ (2) Since $\varphi_\psi(\beta I_\eta(k, f_A, r)) \sqsubseteq \varphi_\psi(f_A)$, by (1), $\varphi_\psi(\beta I_\eta(k, f_A, r))$ is r-fuzzy soft β-open. Hence, $\varphi_\psi(\beta I_\eta(k, f_A, r)) \sqsubseteq \beta I_\tau(e, \varphi_\psi(f_A), r)$.

(2) $\Rightarrow$ (3) Put $f_A = \varphi_\psi^{-1}(g_B)$. By (2), $\varphi_\psi(\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r)) \sqsubseteq \beta I_\tau(e, \varphi_\psi(\varphi_\psi^{-1}(g_B)), r) \sqsubseteq \beta I_\tau(e, g_B, r)$, then $\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(\beta I_\tau(e, g_B, r))$.

(3) $\Rightarrow$ (4) Let $g_B \in \widetilde{(U, E)}$ and $f_A \in \widetilde{(V, F)}$ be an r-fuzzy soft β-closed with $\varphi_\psi^{-1}(g_B) \sqsubseteq f_A$. Since $f_A^c \sqsubseteq \varphi_\psi^{-1}(g_B^c)$, $f_A^c = \beta I_\eta(k, f_A^c, r) \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B^c), r)$. Hence, by (3), $f_A^c \sqsubseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B^c), r) \sqsubseteq \varphi_\psi^{-1}(\beta I_\tau(e, g_B^c, r))$. Thus, $f_A \sqsupseteq (\varphi_\psi^{-1}(\beta I_\tau(e, g_B^c, r)))^c = \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$. Hence, $\beta C_\tau(e, g_B, r) \in \widetilde{(U, E)}$ is r-fuzzy soft β-closed with $g_B \sqsubseteq \beta C_\tau(e, g_B, r)$ such that $\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r)) \sqsubseteq f_A$.

(4) $\Rightarrow$ (1) Let $w_D \in \widetilde{(V,F)}$ be r-fuzzy soft β-open. Put $g_B = (\varphi_\psi(w_D))^c$ and $f_A = w_D^c$, $\varphi_\psi^{-1}(g_B) = \varphi_\psi^{-1}((\varphi_\psi(w_D))^c) \sqsubseteq f_A$. Hence, by (4), $h_C \in \widetilde{(U,E)}$ is r-fuzzy soft β-closed with $g_B \sqsubseteq h_C$ such that $\varphi_\psi^{-1}(h_C) \sqsubseteq f_A = w_D^c$. Thus, $\varphi_\psi(w_D) \sqsubseteq \varphi_\psi(\varphi_\psi^{-1}(h_C^c)) \sqsubseteq h_C^c$. On the other hand, since $g_B \sqsubseteq h_C$, $\varphi_\psi(w_D) = g_B^c \sqsupseteq h_C^c$. Thus, $\varphi_\psi(w_D) = h_C^c$, so $\varphi_\psi(w_D)$ is r-fuzzy soft β-open, then φ_ψ is fuzzy soft β-irresolute open.

In a similar way, we can prove the following theorem.

Theorem 4.4. Let (V, η_F) and (U, τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ be a bijective fuzzy soft function. The following statements are equivalent for each $f_A \in \widetilde{(V,F)}$, $g_B \in \widetilde{(U,E)}$, $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft β-irresolute closed.

(2) $\beta C_\tau(e, \varphi_\psi(f_A), r) \sqsubseteq \varphi_\psi(\beta C_\eta(k, f_A, r))$.

(3) $\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r)) \sqsubseteq \beta C_\eta(k, \varphi_\psi^{-1}(g_B), r)$.

(4) For each g_B and each f_A r-fuzzy soft β-open with $\varphi_\psi^{-1}(g_B) \sqsubseteq f_A$, $h_C \in \widetilde{(U,E)}$ is r-fuzzy soft β-open with $g_B \sqsubseteq h_C$ such that $\varphi_\psi^{-1}(h_C) \sqsubseteq f_A$.

Proposition 4.3. Let (V, η_F) and (U, τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ be a bijective fuzzy soft function. φ_ψ is fuzzy soft β-irresolute closed if φ_ψ is fuzzy soft β-irresolute open.

Proof. It is easily proved from:

$\varphi_\psi^{-1}(\beta C_\tau(e, g_B, r)) \sqsubseteq \beta C_\eta(k, \varphi_\psi^{-1}(g_B), r)$

$\Longleftrightarrow \varphi_\psi^{-1}(\beta I_\tau(e, g_B^c, r)) \sqsupseteq \beta I_\eta(k, \varphi_\psi^{-1}(g_B^c), r)$.

Definition 4.3. Let (V, η_F) and (U, τ_E) be FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ is said to be a fuzzy soft β-irresolute homeomorphism if φ_ψ is bijective and both of φ_ψ and φ_ψ^{-1} are β-irresolute.

From the above theorems, we obtain the following corollary.

Corollary 4.1. Let (V, η_F) and (U, τ_E) be FSTSs and $\varphi_\psi : \widetilde{(V,F)} \longrightarrow \widetilde{(U,E)}$ be a bijective fuzzy soft function. The following statements are equivalent for each $f_A \in \widetilde{(V,F)}$, $g_B \in \widetilde{(U,E)}$, $k \in F$, $(e = \psi(k)) \in E$, and $r \in I_\circ$:

(1) φ_ψ is fuzzy soft β-irresolute homeomorphism.

(2) φ_ψ is fuzzy soft β-irresolute and fuzzy soft β-irresolute open.

(3) φ_ψ is fuzzy soft β-irresolute and fuzzy soft β-irresolute closed.

(4) $\varphi_\psi(\beta I_\eta(k, f_A, r)) = \beta I_\tau(e, \varphi_\psi(f_A), r)$.

(5) $\varphi_\psi(\beta C_\eta(k, f_A, r)) = \beta C_\tau(e, \varphi_\psi(f_A), r)$.

(6) $\beta I_\eta(k, \varphi_\psi^{-1}(g_B), r) = \varphi_\psi^{-1}(\beta I_\tau(e, g_B, r))$.

(7) $\beta C_\eta(k, \varphi_\psi^{-1}(g_B), r) = \varphi_\psi^{-1}(\beta C_\tau(e, g_B, r))$.

5. Conclusions and future work

This article is laid out as follows:

(**1**) In Section 2, fuzzy soft β-closure (β-interior) operators are introduced and studied in FSTSs based on the article by Aygünoğlu et al. [25]. Moreover, the concept of r-fuzzy soft β-connected sets is defined and characterized.

(**2**) In Section 3, some properties of a fuzzy soft β-continuity are investigated. As a weaker form of the notion of fuzzy soft β-continuous functions, the notions of fuzzy soft almost (weakly) β-continuous functions are introduced and some properties are obtained. Also, we show that fuzzy soft β-continuity $\Rightarrow$ fuzzy soft almost β-continuity $\Rightarrow$ fuzzy soft weakly β-continuity, but the converse may not be true. Furthermore, we have the following results:

- Fuzzy soft $(id_V, C_\eta(I_\eta(C_\eta)), id_U, id_U)$-continuous function is fuzzy soft β-continuous.
- Fuzzy soft $(id_V, \beta I_\eta, I_\tau(C_\tau), id_U)$-continuous function is fuzzy soft almost β-continuous.
- Fuzzy soft $(id_V, \beta I_\eta, C_\tau, id_U)$-continuous function is fuzzy soft weakly β-continuous.

(**3**) In Section 4, the notions of fuzzy soft β-irresolute (resp., strongly β-irresolute, β-irresolute open, β-irresolute closed, and β-irresolute homeomorphism) functions are introduced between two FSTSs (V, η_F) and (U, τ_E), and some properties of these functions are investigated. Additionally, we show that fuzzy soft strongly β-irresolute $\Rightarrow$ fuzzy soft β-irresolute $\Rightarrow$ fuzzy soft β-continuity, but the converse may not be true.

In upcoming manuscripts, we shall investigate the notions given here in the frames of fuzzy soft r-minimal structures [28]. Also, we will use the fuzzy soft β-closure operator to introduce some new separation axioms on FSTS based on the article by Aygünoğlu et al. [25].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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