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Research article

Results pertaining to fixed points in ordered metric spaces with auxiliary functions and application to integral equation

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Abstract: This paper delves into fixed point findings within a complete partially ordered b-metric space, focusing on mappings that adhere to weakly contractive conditions in the presence of essential topological characteristics. These findings represent modifications of established results and further extend analogous outcomes in the existing literature. The conclusions are substantiated by illustrative examples that strengthen the conclusion of the paper.

Keywords: weakly contractive conditions; b-metric space; fixed point; compatible mappings; coincidence points; coupled coincidence points
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1. Introduction

The concept of fixed point theory has seen diverse extensions beyond the classical idea of a metric space. One notable variation is the b-metric space, also known as a metric type space according to certain authors. Bakhtin [1] introduced and extensively investigated this space, and it has been extensively employed by Czerwik in subsequent study [2]. Furthermore, Czerwik extended the classical Banach contraction principle in the context of a b-metric space that is complete. Following this, a collection of papers has been devoted to refining fixed point results for both single and multi-valued operators in b-metric space, taking into account diverse topological properties. Interested readers can explore some of these contributions in [3–12] and related references.

Bhaskar and Lakshmikantham [13] were the pioneers who introduced coupled fixed points to specific maps in partially ordered metric space. Their findings were employed to investigate solutions' existence and uniqueness for boundary value problems. Subsequently, Lakshmikantham and Ćirić [14] extended this work by introducing coupled coincidence, which is coupled common fixed point results for nonlinear contractive maps with mixed monotone properties in partially ordered metric space that is complete, thereby generalizing the findings from [13]. Since then, various authors have further generalized and improved the fixed point results in both partially ordered metric spaces and partially ordered b-metric spaces. Significant contributions have been made by the authors of papers [15–36]. Recent studies by Mituku et al. [37] and Rao et al. [38–42] have yielded results related to fixed points, coincidence points and coupled coincidence points of self-maps that adhere to generalized weak contractions in partially ordered b-metric spaces, taking into account necessary topological properties. The Banach fixed point theorem plays an essential role in the fields of optimization and inverse problems some of such works can be seen in [43–45].

This study was designed to demonstrate various types of results, including those on fixed points, coincidence points, coupled coincidence points and coupled common fixed points of the maps that fulfill the (ϕ, ψ)-weak contraction condition with auxiliary functions within a partially ordered b-metric space that is complete. These outcomes expand and generalize the findings from [13,14,37] and several analogous results found in the literature, as presented in [38,46]. To reinforce the validity of the results, the paper includes illustrative examples.

2. Preliminaries

The forthcoming results often rely on the following definitions.

Definition 1. [38] A function ω: $\mathscr{L} \times \mathscr{L} \to [0, +\infty)$, with $\mathscr{L} \neq \emptyset$ is termed as a b-metric, if it adheres to the specified properties for $\mathscr{o}_1, \mathscr{o}_2, \mathscr{o}_3 \in \mathscr{L}$ and $\mathrm{s} \geq 1 \in \mathbb{R}$:

(a) $\omega(\mathscr{o}_1, \mathscr{o}_2) = 0$ if and only if $\mathscr{o}_1 = \mathscr{o}_2$.

(b) $\omega(\mathscr{o}_1, \mathscr{o}_2) = \omega(\mathscr{o}_2, \mathscr{o}_1)$.

(c) $\omega(\mathscr{o}_1, \mathscr{o}_2) \leq \mathrm{s}\,(\omega(\mathscr{o}_1, \mathscr{o}_3) + \omega(\mathscr{o}_3, \mathscr{o}_2))$.

Subsequently, the triple $(\mathscr{L}, \omega, \mathrm{s})$ is referred to as a b-metric space. If the pair $(\mathscr{L}, \preceq)$ continues as a partially ordered set (p.o.s), then $(\mathscr{L}, \omega, s, \preceq)$ is termed a partially ordered b-metric space (p.o.m.s).

Definition 2. [46] A sequence $\{\mathscr{o}_n\}$ in a b-metric space $(\mathscr{L}, \omega, \mathrm{s})$ adheres to the following conditions:

(1) It converges to $\mathscr{o}$ if $\lim_{n\to+\infty} \omega(\mathscr{o}_n, \mathscr{o}) = 0$, denoted as $\lim_{n\to+\infty} \mathscr{o}_n = \mathscr{o}$,

(2) It is considered a Cauchy sequence in $\mathscr{L}$ when $\lim_{n,m\to+\infty} \omega(\mathscr{o}_n, \mathscr{o}_m) = 0$.

Completeness of $(\mathscr{L}, \omega, s)$ is affirmed when all Cauchy sequences within it converge.

Definition 3. In cases in which the metric ω achieves completeness, the term complete partially ordered b-metric space (c.m.s) is applied to characterize the structure $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$.

Definition 4. [46] Suppose that $\mathscr{U}$ and $\mathfrak{g}$ are two self mappings defined on a partially ordered set $(\mathscr{L}, \preceq)$. In this scenario, the following holds:

(1) A map $\mathscr{U}$ is considered monotone non-decreasing, if for every pair $\mathscr{o}$ and $\mathfrak{a}$ in $\mathscr{L}$ such that $\mathscr{o} \leq \mathfrak{a}$, the condition $\mathscr{U}(\mathscr{o}) \leq \mathscr{U}(\mathfrak{a})$ is satisfied.

(2) When $\mathfrak{g}(\mathscr{o}) = \mathscr{U}(\mathscr{o})$ or $\mathfrak{g}(\mathscr{o}) = \mathscr{U}(\mathscr{o}) = \mathscr{o}$ for $\mathscr{o} \in \mathscr{L}$, then $\mathscr{o}$ is referred to as being a coincidence point or a common fixed point of $\mathfrak{g}$ and $\mathscr{U}$.

(3) $\mathfrak{g}$ and $\mathscr{U}$ are considered commuting if $\mathfrak{g}(\mathscr{U}(\mathscr{o})) = \mathscr{U}(\mathfrak{g}(\mathscr{o})), \forall \mathscr{o} \in \mathscr{L}$ holds.

(4) The mappings $\mathscr{U}$ and $\mathfrak{g}$ are known to be compatible, if $\lim_{n\to+\infty} \omega(\mathscr{U}\mathfrak{g}\mathfrak{o}_n, \mathfrak{g}\mathscr{U}\mathscr{o}_n) = 0$ whenever a sequence $\{\mathscr{o}_n\} \in \mathscr{L}$ satisfies that $\lim_{n\to+\infty} \mathscr{U}\mathscr{o}_n = \lim_{n\to+\infty} \mathfrak{g}\mathscr{o}_n = \mathscr{o}$, for some $\mathscr{o} \in \mathscr{L}$.

(5) If $\mathfrak{g}(\mathscr{U}(\mathscr{o})) = \mathscr{U}(\mathfrak{g}(\mathscr{o}))$ whenever $\mathscr{U}(\mathscr{o}) = \mathfrak{g}(\mathscr{o})$ then the maps $\mathfrak{g}$, $\mathscr{U}$ are referred to be weakly compatible.

(6) $\mathscr{U}$ is referred to as monotonically $\mathfrak{g}$ non-decreasing if for all $\mathscr{o}, \mathfrak{a} \in \mathscr{L}$, the condition $\mathfrak{g}(\mathscr{o}) \leq \mathfrak{g}(\mathfrak{a})$ implies that $\mathscr{U}(\mathscr{o}) \leq \mathscr{U}(\mathfrak{a})$.

(7) A set $\mathscr{L} \neq \emptyset$ is well-ordered if either $\mathscr{o} \leq \mathfrak{a}$ or $\mathfrak{a} \leq \mathscr{o}$ for every pair $\mathscr{o}, \mathfrak{a} \in \mathscr{L}$.

Definition 5. [14,15] Take a partially ordered set $(\mathscr{L}, \leq)$, and let us assume that there are two mappings $\mathscr{U}\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ and $\mathfrak{g}\colon \mathscr{L} \to \mathscr{L}$. In this scenario, the following holds:

(1) A map $\mathscr{U}$ is considered to exhibit a mixed $\mathfrak{g}$-monotone property, if it is increasing and decreasing $\mathfrak{g}$-monotone in its first and second arguments. In formal terms, for all $\mathscr{o}, \mathfrak{a} \in \mathscr{L}$,

$$\mathscr{o}_1, \mathscr{o}_2 \in \mathscr{L}, \quad \mathfrak{g}\mathscr{o}_1 \leq \mathfrak{g}\mathscr{o}_2 \Rightarrow \mathscr{U}(\mathscr{o}_1, \mathfrak{a}) \leq \mathscr{U}(\mathscr{o}_2, \mathfrak{a})$$

and

$$\mathfrak{a}_1, \mathfrak{a}_2 \in \mathscr{L}, \quad \mathfrak{g}\mathfrak{a}_1 \leq \mathfrak{g}\mathfrak{a}_2 \Rightarrow \mathscr{U}(\mathscr{o}, \mathfrak{a}_1) \geq \mathscr{U}(\mathscr{o}, \mathfrak{a}_2).$$

In the specific scenario in which $\mathfrak{g} = I$, I is an identity mapping, $\mathscr{U}$ is described as having the mixed monotone property.

(2) $(\mathscr{o}, \mathfrak{a}) \in \mathscr{L} \times \mathscr{L}$ with $\mathscr{U}(\mathscr{o}, \mathfrak{a}) = \mathfrak{g}(\mathscr{o})$ and $\mathscr{U}(\mathfrak{a}, \mathscr{o}) = \mathfrak{g}(\mathfrak{a})$ is referred to as a coupled coincidence point for $\mathscr{U}$ and $\mathfrak{g}$. It is important to emphasize that if $\mathfrak{g} = I$, then $(\mathscr{o}, \mathfrak{a})$ is denoted as a coupled fixed point of $\mathscr{U}$.

(3) If

$$\mathscr{U}(\mathscr{o}, \mathscr{o}) = \mathfrak{g}(\mathscr{o}) = \mathscr{o},$$

then $\mathscr{o} \in \mathscr{L}$ is designated as a common fixed point for $\mathscr{U}$, $\mathfrak{g}$.

(4) If

$$\mathscr{U}(\mathfrak{g}(\mathscr{o}), \mathfrak{g}(\mathfrak{a})) = \mathfrak{g}(\mathscr{U}(\mathscr{o}, \mathfrak{a}))$$

for all $\mathscr{o}, \mathfrak{a} \in \mathscr{L}$, then $\mathscr{U}$, $\mathfrak{g}$ are commutative.

(5) $\mathscr{U}$ and $\mathfrak{g}$ are termed compatible, if

$$\lim_{n\to+\infty} \omega(\mathfrak{g}(\mathscr{U}(\mathscr{o}_n, \mathfrak{a}_n)), \mathscr{U}(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathfrak{a}_n)) = 0$$

and

$$\lim_{n\to+\infty} \omega(\mathfrak{g}(\mathscr{U}(\mathfrak{a}_n, \mathscr{o}_n)), \mathscr{U}(\mathfrak{g}\mathfrak{a}_n, \mathfrak{g}\mathscr{o}_n)) = 0,$$

whenever the sequences $\{\mathscr{o}_n\}$ and $\{\mathfrak{a}_n\}$ in $\mathscr{L}$ satisfy

$$\lim_{n\to+\infty} \mathscr{U}(\mathscr{o}_n, \mathfrak{a}_n) = \lim_{n\to+\infty} \mathfrak{g}(\mathscr{o}_n) = \mathscr{o}$$

and

$$\lim_{n\to+\infty} \mathscr{U}(\mathfrak{a}_n, \mathscr{o}_n) = \lim_{n\to+\infty} \mathfrak{g}(\mathfrak{a}_n) = \mathfrak{a}$$

for some $\mathscr{o}, \mathfrak{a} \in \mathscr{L}$.

The upcoming lemma is regularly utilized in our results within a b-metric space to verify the convergence of the sequences.

Lemma 6. *[15] Consider a b-metric space* $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ *with* $\mathrm{s} \geq 1$. *Assuming that the sequences* $\{\mathscr{o}_n\}$ *and* $\{\mathfrak{a}_n\}$ *b-converge to* $\mathscr{o}$ *and* $\mathfrak{a}$ *respectively, then*

$$\frac{1}{\mathrm{s}^2}\omega(\mathscr{o}, \mathfrak{a}) \leq \lim_{n\to+\infty} \inf \omega(\mathscr{o}_n, \mathfrak{a}_n) \leq \lim_{n\to+\infty} \sup \omega(\mathscr{o}_n, \mathfrak{a}_n) \leq \mathrm{s}^2\omega(\mathscr{o}, \mathfrak{a}).$$

In particular, when $\mathscr{o} = \mathfrak{a}$, *the limit as n approaches infinity of* $\omega(\mathscr{o}_n, \mathfrak{a}_n)$ *is zero. Also, for every* $\mathfrak{d} \in \mathscr{L}$, *the subsequent inequalities are applicable:*

$$\frac{1}{s}\omega(\mathscr{o}, \mathfrak{d}) \leq \liminf_{n\to+\infty} \omega(\mathscr{o}_n, \mathfrak{d}) \leq \limsup_{n\to+\infty} \omega(\mathscr{o}_n, \mathfrak{d}) \leq s\omega(\mathscr{o}, \mathfrak{d}).$$

The subsequent distance functions are utilized consistently in the present work.

(i) A continuous, non-decreasing self-map ϕ over $[0, +\infty)$ is referred to as an alternating distance function if it satisfies the condition $\phi(\mathrm{t}) = 0$ if and only if $\mathrm{t} = 0$. Refer to all such functions as Φ.

(ii) Ψ represents the collection of self-maps ψ over $[0, +\infty)$ with the characteristics that ψ is lower semi-continuous and $\psi(\mathrm{t}) = 0$ if and only if $\mathrm{t} = 0$.

3. Results

For a self-map $\mathscr{U}$ over a partially ordered b-metric space $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ let

$$\mathrm{G}(\mathscr{o}, \mathfrak{a}) = \max\left\{\frac{\omega(\mathfrak{a}, \mathscr{U}\mathfrak{a})\,[1 + \omega(\mathscr{o}, \mathscr{U}\mathscr{o})]}{1 + \omega(\mathscr{o}, \mathfrak{a})}, \frac{\omega(\mathscr{o}, \mathscr{U}\mathscr{o})\,\omega(\mathfrak{a}, \mathscr{U}\mathfrak{a})}{1 + \omega(\mathscr{o}, \mathfrak{a})}, \frac{\omega(\mathscr{o}, \mathscr{U}\mathfrak{a}) + \omega(\mathfrak{a}, \mathscr{U}\mathscr{o})}{2\mathrm{s}}, \omega(\mathscr{o}, \mathfrak{a})\right\} \quad (3.1)$$

and

$$\mathrm{H}(\mathscr{o}, \mathfrak{a}) = \max\left\{\frac{\omega(\mathfrak{a}, \mathscr{U}\mathfrak{a})\,[1 + \omega(\mathscr{o}, \mathscr{U}\mathscr{o})]}{1 + \omega(\mathscr{o}, \mathfrak{a})}, \omega(\mathscr{o}, \mathfrak{a})\right\}. \quad (3.2)$$

Then, for $\phi \in \Phi$ and $\psi \in \Psi$, $\mathscr{U}$ is called a (ϕ, ψ)-generalized contraction, if it adheres to the following condition:

$$\phi(\mathrm{s}\omega(\mathscr{U}\mathscr{o}, \mathscr{U}\mathfrak{a})) \leq \phi(\mathrm{G}(\mathscr{o}, \mathfrak{a})) - \psi(\mathrm{H}(\mathscr{o}, \mathfrak{a})). \quad (3.3)$$

From here, we begin with the below theorem regarding fixed points in a b-metric space that is ordered.

Theorem 7. *A non-decreasing self-map* $\mathscr{U}$ *on a c.m.s* $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ *is continuous and satisfies the conditions of a* (ϕ, ψ)*-generalized contraction with respect to* $\preceq$. *If there exists* $\mathscr{o}_0 \in \mathscr{L}$ *with* $\mathscr{o}_0 \preceq \mathscr{U}\mathscr{o}_0$, *then* $\mathscr{U}$ *admits a fixed point in* $\mathscr{L}$.

Proof. If a particular $\eth_0 \in \mathscr{L}$ satisfies that $\mathscr{U}\eth_0 = \eth_0$, the proof is concluded. Assuming that $\eth_0 \prec \mathscr{U}\eth_0$, we form a sequence $\{\eth_n\} \subset \mathscr{L}$ through recursive definitions, where $\eth_{n+1} = \mathscr{U}\eth_n$ for $n \geq 0$. By leveraging the non-decreasing characteristic of $\mathscr{U}$, we infer, through an inductive argument, that

$$\eth_0 \prec \mathscr{U}\eth_0 = \eth_1 \preceq \cdots \preceq \eth_n \preceq \mathscr{U}\eth_n = \eth_{n+1} \preceq \cdots . \tag{3.4}$$

If there exists $n_0 \in \mathbb{N}$ such that $\eth_{n_0} = \eth_{n_0+1}$ as indicated by (3.4), it implies that $\eth_{n_0}$ serves as a fixed point for $\mathscr{U}$ and therefore, no further proof is necessary. Let us consider the scenario in which $\eth_n \neq \eth_{n+1}, \forall n \geq 1$ holds. Applying the condition (3.3), we can conclude that

$$\begin{aligned}\phi(\omega(\eth_n, \eth_{n+1})) = \phi(\omega(\mathscr{U}\eth_{n-1}, \mathscr{U}\eth_n)) &\leq \phi(s\omega(\mathscr{U}\eth_{n-1}, \mathscr{U}\eth_n))\\ &\leq \phi(\mathrm{G}(\eth_{n-1}, \eth_n)) - \psi(\mathrm{H}(\eth_{n-1}, \eth_n)).\end{aligned} \tag{3.5}$$

From (3.5), we acquire

$$\omega(\eth_n, \eth_{n+1}) = \omega(\mathscr{U}\eth_{n-1}, \mathscr{U}\eth_n) \leq \frac{1}{\mathrm{s}}\mathrm{G}(\eth_{n-1}, \eth_n), \tag{3.6}$$

where

$$\begin{aligned}\mathrm{G}(\eth_{n-1}, \eth_n) =& \max\{\frac{\omega(\eth_n, \mathscr{U}\eth_n)\,[1 + \omega(\eth_{n-1}, \mathscr{U}\eth_{n-1})]}{1 + \omega(\eth_{n-1}, \eth_n)}, \frac{\omega(\eth_{n-1}, \mathscr{U}\eth_{n-1})\,\omega(\eth_n, \mathscr{U}\eth_n)}{1 + \omega(\eth_{n-1}, \eth_n)}\\ &\frac{\omega(\eth_{n-1}, \mathscr{U}\eth_n) + \omega(\eth_n, \mathscr{U}\eth_{n-1})}{2\mathrm{s}}, \omega(\eth_{n-1}, \eth_n)\}\\ =& \max\{\omega(\eth_n, \eth_{n+1}), \frac{\omega(\eth_{n-1}, \eth_n)\,\omega(\eth_n, \eth_{n+1})}{1 + \omega(\eth_{n-1}, \eth_n)}, \frac{\omega(\eth_{n-1}, \eth_{n+1}) + \omega(\eth_n, \eth_n)}{2\mathrm{s}},\\ &\omega(\eth_{n-1}, \eth_n)\}\\ \leq& \max\{\omega(\eth_n, \eth_{n+1}), \frac{\omega(\eth_{n-1}, \eth_n) + \omega(\eth_n, \eth_{n+1})}{2}, \omega(\eth_{n-1}, \eth_n)\}\\ \leq& \max\{\omega(\eth_n, \eth_{n+1}), \omega(\eth_{n-1}, \eth_n)\}.\end{aligned} \tag{3.7}$$

If

$$\max\{\omega(\eth_n, \eth_{n+1}), \omega(\eth_{n-1}, \eth_n)\} = \omega(\eth_n, \eth_{n+1})$$

for some $n \geq 1$, then from (3.6), it follows that

$$\omega(\eth_n, \eth_{n+1}) \leq \frac{1}{\mathrm{s}}\omega(\eth_n, \eth_{n+1}),$$

which leads to a contradiction. Consequently,

$$\max\{\omega(\eth_n, \eth_{n+1}), \omega(\eth_{n-1}, \eth_n)\} = \omega(\eth_{n-1}, \eth_n)$$

for $n \geq 1$. So, we infer from (3.6) that

$$\omega(\eth_n, \eth_{n+1}) \leq \frac{1}{\mathrm{s}}\omega(\eth_{n-1}, \eth_n).$$

Given that $\frac{1}{\mathrm{s}} \in (0, 1)$, it follows, according to [1, 3, 4, 7], that the sequence $\{\eth_n\}$ is a Cauchy sequence.

Therefore, $\mathscr{o}_n \to \mathrm{v}$ in $\mathscr{L}$ by its completeness.

Additionally, the continuity of $\mathscr{U}$ results in

$$\mathscr{U}\mathrm{v} = \mathscr{U}(\lim_{n\to+\infty} \mathscr{o}_n) = \lim_{n\to+\infty} \mathscr{U}\mathscr{o}_n = \lim_{n\to+\infty} \mathscr{o}_{n+1} = \mathrm{v}. \tag{3.8}$$

Consequently, v constitutes a fixed point of $\mathscr{U}$ within the set $\mathscr{L}$. □

By easing the conditions for continuity on a mapping $\mathscr{U}$ as per Theorem 7, the subsequent result is obtained.

Theorem 8. *Assume that a non-decreasing self-maping $\mathscr{U}$ on a c.m.s $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$, where ω is continuous in each variable satisfying the conditions of a (ϕ, ψ)-generalized contraction with respect to $\preceq$ and there exists $\mathscr{o}_0 \in \mathscr{L}$ such that $\mathscr{o}_0 \preceq \mathscr{U}\mathscr{o}_0$. If $\mathscr{L}$ satisfies the condition that, for every non-decreasing sequence $\{\mathscr{o}_n\}$ in $\mathscr{L}$ such that $\mathscr{o}_n \to \mathrm{v} \in \mathscr{L}$, then $\mathscr{o}_n \preceq \mathrm{v}, \forall n \geq 1$, i.e., $\mathrm{v} = \sup_{n\in\mathbb{N}} \mathscr{o}_n$, then $\mathscr{U}$ possesses a fixed point in $\mathscr{L}$.*

Proof. We form a non-decreasing Cauchy sequence $\{\mathscr{o}_n\}$ within $\mathscr{L}$ that converges to $\mathrm{v} \in \mathscr{L}$ by applying Theorem 7. Therefore, based on the provided hypotheses, it follows that $\mathscr{o}_n \preceq \mathrm{v}$ for every $n \in \mathbb{N}$, leading to the conclusion that $\mathrm{v} = \sup_{n\in\mathbb{N}} \mathscr{o}_n$.

Subsequently, we demonstrate that v serves as a fixed point of $\mathscr{U}$ in $\mathscr{L}$. Let us assume the contrary, i.e., $\mathscr{U}\mathrm{v} \neq \mathrm{v}$; then,

$$\begin{aligned} \mathrm{G}(\mathscr{o}_n, \mathrm{v}) = \max\{&\frac{\omega(\mathrm{v}, \mathscr{U}\mathrm{v})\,[1 + \omega(\mathscr{o}_n, \mathscr{U}\mathscr{o}_n)]}{1 + \omega(\mathscr{o}_n, \mathrm{v})}, \frac{\omega(\mathscr{o}_n, \mathscr{U}\mathscr{o}_n)\,\omega(\mathrm{v}, \mathscr{U}\mathrm{v})}{1 + \omega(\mathscr{o}_n, \mathrm{v})}, \\ &\frac{\omega(\mathscr{o}_n, \mathscr{U}\mathrm{v}) + \omega(\mathrm{v}, \mathscr{U}\mathscr{o}_n)}{2\mathrm{s}}, \omega(\mathscr{o}_n, \mathrm{v})\} \end{aligned} \tag{3.9}$$

and

$$\mathrm{H}(\mathscr{o}_n, \mathrm{v}) = \max\{\frac{\omega(\mathrm{v}, \mathscr{U}\mathrm{v})\,[1 + \omega(\mathscr{o}_n, \mathscr{U}\mathscr{o}_n)]}{1 + \omega(\mathscr{o}_n, \mathrm{v})}, \omega(\mathscr{o}_n, \mathrm{v})\}. \tag{3.10}$$

Taking the limit as $n \to \infty$ in (3.9) and (3.10), and by using the continuity in each variable of ω, we obtain

$$\lim_{n\to+\infty} \mathrm{G}(\mathscr{o}_n, \mathrm{v}) = \max\{\omega(\mathrm{v}, \mathscr{U}\mathrm{v}), 0, \frac{\omega(\mathrm{v}, \mathscr{U}\mathrm{v})}{\mathrm{s}}, 0\} = \omega(\mathrm{v}, \mathscr{U}\mathrm{v}) \tag{3.11}$$

and

$$\lim_{n\to+\infty} \mathrm{H}(\mathscr{o}_n, \mathrm{v}) = \max\{\omega(\mathrm{v}, \mathscr{U}\mathrm{v}), 0\} = \omega(\mathrm{v}, \mathscr{U}\mathrm{v}). \tag{3.12}$$

Considering $\mathscr{o}_n \preceq \mathrm{v}$, for all n, we can derive from condition (3.3) that

$$\phi(\omega(\mathscr{o}_{n+1}, \mathscr{U}\mathrm{v})) = \phi(\omega(\mathscr{U}\mathscr{o}_n, \mathscr{U}\mathrm{v}) \leq \phi(\mathrm{s}\omega(\mathscr{U}\mathscr{o}_n, \mathscr{U}\mathrm{v}) \leq \phi(\mathrm{G}(\mathscr{o}_n, \mathrm{v})) - \psi(\mathrm{H}(\mathscr{o}_n, \mathrm{v})). \tag{3.13}$$

Allowing n to tend to infinity and employing (3.11) and (3.12), we obtain

$$\phi(\omega(\mathrm{v}, \mathscr{U}\mathrm{v})) \leq \phi(\omega(\mathrm{v}, \mathscr{U}\mathrm{v})) - \psi(\omega(\mathrm{v}, \mathscr{U}\mathrm{v})) < \phi(\omega(\mathrm{v}, \mathscr{U}\mathrm{v})), \tag{3.14}$$

which is absurd, unless $\mathscr{U}\mathrm{v} = \mathrm{v}$, indicating that $\mathscr{U}$ possesses a fixed point v in $\mathscr{L}$. □

We will now present a condition that is adequate to guarantee the existence of a unique fixed point in Theorems 7 and 8.

$$\text{For any } \varrho, \mathfrak{a} \in \mathscr{L}, \text{there exists some } w \in \mathscr{L} \text{ that is comparable to both } \varrho, \mathfrak{a}. \tag{3.15}$$

Theorem 9. *Consider a non-decreasing self-mapping $\mathscr{U}$ on a c.m.s $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ that satisfies the conditions of a (ϕ, ψ)-generalized contraction with respect to $\preceq$ and there exists $\varrho_0 \in \mathscr{L}$ such that $\varrho_0 \preceq \mathscr{U}\varrho_0$. If $\mathscr{L}$ satisfies the condition* (3.15) *in Theorem 7 (or Theorem 8), then $\mathscr{U}$ has a unique fixed point in $\mathscr{L}$.*

Proof. We infer the existence of a nonempty set of fixed points for $\mathscr{U}$ based on Theorem 7 (or Theorem 8). Assuming that ϱ^*, $\mathfrak{a}^*$ are two fixed points for $\mathscr{U}$, we assert that $\varrho^* = \mathfrak{a}^*$.

If $\varrho^* \neq \mathfrak{a}^*$, according to the given assumptions, we obtain

$$\phi(\omega(\mathscr{U}\varrho^*, \mathscr{U}\mathfrak{a}^*)) \leq \phi(\mathrm{s}\omega(\mathscr{U}\varrho^*, \mathscr{U}\mathfrak{a}^*)) \leq \phi(\mathrm{G}(\varrho^*, \mathfrak{a}^*)) - \psi(\mathrm{H}(\varrho^*, \mathfrak{a}^*)). \tag{3.16}$$

Consequently, we acquire

$$\omega(\varrho^*, \mathfrak{a}^*) = \omega(\mathscr{U}\varrho^*, \mathscr{U}\mathfrak{a}^*) \leq \frac{1}{\mathrm{s}}\mathrm{G}(\varrho^*, \mathfrak{a}^*), \tag{3.17}$$

where

$$\begin{aligned}
\mathrm{G}(\varrho^*, \mathfrak{a}^*) =& \max\{\frac{\omega(\mathfrak{a}^*, \mathscr{U}\mathfrak{a}^*)\,[1 + \omega(\varrho^*, \mathscr{U}\varrho^*)]}{1 + \omega(\varrho^*, \mathfrak{a}^*)}, \frac{\omega(\varrho^*, \mathscr{U}\varrho^*)\,\omega(\mathfrak{a}^*, \mathscr{U}\mathfrak{a}^*)}{1 + \omega(\varrho^*, \mathfrak{a}^*)}, \\
& \frac{\omega(\varrho^*, \mathscr{U}\mathfrak{a}^*) + \omega(\mathfrak{a}^*, \mathscr{U}\varrho^*)}{2\mathrm{s}}, \omega(\varrho^*, \mathfrak{a}^*)\} \\
=& \max\{\frac{\omega(\mathfrak{a}^*, \mathfrak{a}^*)\,[1 + \omega(\varrho^*, \varrho^*)]}{1 + \omega(\varrho^*, \mathfrak{a}^*)}, \frac{\omega(\varrho^*, \varrho^*)\,\omega(\mathfrak{a}^*, \mathfrak{a}^*)}{1 + \omega(\varrho^*, \mathfrak{a}^*)}, \\
& \frac{\omega(\varrho^*, \mathfrak{a}^*) + \omega(\mathfrak{a}^*, \varrho^*)}{2\mathrm{s}}, \omega(\varrho^*, \mathfrak{a}^*)\} \\
=& \max\{0, 0, \frac{\omega(\varrho^*, \mathfrak{a}^*)}{\mathrm{s}}, \omega(\varrho^*, \mathfrak{a}^*)\} \\
=& \omega(\varrho^*, \mathfrak{a}^*).
\end{aligned} \tag{3.18}$$

Therefore from (3.17), we derive that

$$\omega(\varrho^*, \mathfrak{a}^*) \leq \frac{1}{\mathrm{s}}\omega(\varrho^*, \mathfrak{a}^*) < \omega(\varrho^*, \mathfrak{a}^*). \tag{3.19}$$

This results in a contradiction. Thus, we conclude that $\varrho^* = \mathfrak{a}^*$, completing the proof. □

Consider a p.o.m.s $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ with $\mathrm{s} > 1$ and let $\mathscr{U}, \mathfrak{g}\colon \mathscr{L} \to \mathscr{L}$ be two mappings. Define:

$$\begin{aligned}
\mathrm{G}_{\mathfrak{g}}(\varrho, \mathfrak{a}) =& \max\{\frac{\omega(\mathfrak{g}\mathfrak{a}, \mathscr{U}\mathfrak{a})\,[1 + \omega(\mathfrak{g}\varrho, \mathscr{U}\varrho)]}{1 + \omega(\mathfrak{g}\varrho, \mathfrak{g}\mathfrak{a})}, \frac{\omega(\mathfrak{g}\varrho, \mathscr{U}\varrho)\,\omega(\mathfrak{g}\mathfrak{a}, \mathscr{U}\mathfrak{a})}{1 + \omega(\mathfrak{g}\varrho, \mathfrak{g}\mathfrak{a})}, \\
& \frac{\omega(\mathfrak{g}\varrho, \mathscr{U}\mathfrak{a}) + \omega(\mathfrak{g}\mathfrak{a}, \mathscr{U}\varrho)}{2s}, \omega(\mathfrak{g}\varrho, \mathfrak{g}\mathfrak{a})\}
\end{aligned} \tag{3.20}$$

and

$$\mathrm{H}_{\mathfrak{g}}(\varrho, \mathfrak{a}) = \max\{\frac{\omega(\mathfrak{g}\mathfrak{a}, \mathscr{U}\mathfrak{a})\,[1 + \omega(\mathfrak{g}\varrho, \mathscr{U}\varrho)]}{1 + \omega(\mathfrak{g}\varrho, \mathfrak{g}\mathfrak{a})}, \omega(\mathfrak{g}\varrho, \mathfrak{g}\mathfrak{a})\}. \tag{3.21}$$

Definition 10. A function $\mathscr{U}$: $\mathscr{L} \to \mathscr{L}$, where $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ is a p.o.m.s satisfying the following condition is termed as a (ϕ, ψ)-generalized contraction by a self-mapping $\mathfrak{g}$ over $\mathscr{L}$:

$$\phi(s\omega(\mathscr{U}\mathscr{o}, \mathscr{U}\mathfrak{a})) \leq \phi(\mathrm{G}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a})) - \psi(\mathrm{H}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a})), \tag{3.22}$$

where $\phi \in \Phi$, $\psi \in \Psi$, $\mathscr{o}, \mathfrak{a} \in \mathscr{L}$ with $\mathfrak{g}\mathscr{o} \preceq \mathfrak{g}\mathfrak{a}$ and where $\mathrm{G}\mathfrak{g}(\mathscr{o}, \mathfrak{a})$ and $\mathrm{H}\mathfrak{g}(\mathscr{o}, \mathfrak{a})$ are expressed as per (3.20) and (3.21), respectively.

Theorem 11. *The mappings $\mathscr{U}$ and $\mathfrak{g}$ in Definition 10 possess a coincidence point over a c.m.s. $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ if there is an element $\mathscr{o}_0 \in \mathscr{L}$ such that $\mathfrak{g}\mathscr{o}_0 \preceq \mathscr{U}\mathscr{o}_0$ when $\mathscr{U}$, $\mathfrak{g}$ are continuous, $\mathscr{U}\mathscr{L} \subseteq \mathfrak{g}\mathscr{L}$ and $\mathscr{U}$ is compatible with $\mathfrak{g}$ and a monotone $\mathfrak{g}$-non-decreasing mapping.*

Proof. Utilizing the proof technique outlined in [18, Theorem 2.2], we construct two sequences $\{\mathscr{o}_n\}$ and $\{\mathfrak{a}_n\}$ in $\mathscr{L}$ such that

$$\mathfrak{a}_n = \mathscr{U}\mathscr{o}_n = \mathfrak{g}\mathscr{o}_{n+1}, \quad \forall\, n \geq 0, \tag{3.23}$$

for which

$$\mathfrak{g}\mathscr{o}_0 \preceq \mathfrak{g}\mathscr{o}_1 \preceq \cdots \preceq \mathfrak{g}\mathscr{o}_n \preceq \mathfrak{g}\mathscr{o}_{n+1} \preceq \cdots . \tag{3.24}$$

Again from [18], we can assume that $\mathfrak{a}_n \neq \mathfrak{a}_{n+1}$ for all $n \neq 0$. For this case we have to show that

$$\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \leq \lambda\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n) \tag{3.25}$$

for all $n \geq 1$, where $\lambda \in [0, \frac{1}{\mathrm{s}}]$.

Now, employing (3.22) and utilizing (3.23) and (3.24), we obtain that

$$\begin{aligned}\phi(\mathrm{s}\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})) &= \phi(\mathrm{s}\omega(\mathscr{U}\mathscr{o}_n, \mathscr{U}\mathscr{o}_{n+1})) \\ &\leq \phi(\mathrm{G}_{\mathfrak{g}}(\mathscr{o}_n, \mathscr{o}_{n+1})) - \psi(\mathrm{H}_{\mathfrak{g}}(\mathscr{o}_n, \mathscr{o}_{n+1})),\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}\mathrm{G}_{\mathfrak{g}}(\mathscr{o}_n, \mathscr{o}_{n+1}) &= \max\{\frac{\omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathscr{U}\mathscr{o}_{n+1})\,[1 + \omega(\mathfrak{g}\mathscr{o}_n, \mathscr{U}\mathscr{o}_n)]}{1 + \omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1})}, \\ &\quad \frac{\omega(\mathfrak{g}\mathscr{o}_n, \mathscr{U}\mathscr{o}_n)\,\omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathscr{U}\mathscr{o}_{n+1})}{1 + \omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1})}, \\ &\quad \frac{\omega(\mathfrak{g}\mathscr{o}_n, \mathscr{U}\mathscr{o}_{n+1}) + \omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathscr{U}\mathscr{o}_n)}{2\mathrm{s}}, \omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1})\} \\ &= \max\{\frac{\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})\,[1 + \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)]}{1 + \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)}, \frac{\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)\,\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})}{1 + \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)}, \\ &\quad \frac{\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_{n+1}) + \omega(\mathfrak{a}_n, \mathfrak{a}_n)}{2\mathrm{s}}, \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)\} \\ &\leq \max\{\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1}), \frac{\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n) + \omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})}{2}, \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)\} \\ &\leq \max\{\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n), \omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})\}\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\mathrm{H}_{\mathfrak{g}}(\mathscr{o}_n, \mathscr{o}_{n+1}) &= \max\{\frac{\omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathscr{U}\mathscr{o}_{n+1})\,[1 + \omega(\mathfrak{g}\mathscr{o}_n, \mathscr{U}\mathscr{o}_n)]}{1 + \omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1})}, \omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1})\} \\ &= \max\{\frac{\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})\,[1 + \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)]}{1 + \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)}, \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n)\} \\ &= \max\{\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n), \omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})\}.\end{aligned} \tag{3.28}$$

Consequently, based on (3.26)–(3.28), we obtain

$$\phi(\mathrm{s}\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})) \leq \phi(\max\{\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n), \omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})\}) - \psi(\max\{\omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n), \omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})\}). \tag{3.29}$$

For $n \in \mathbb{N}$, $0 < \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n) \leq \omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})$; then, (3.29) becomes

$$\phi(\mathrm{s}\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})) \leq \phi(\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})) - \psi(\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})) < \phi(\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1})),$$

or equivalently

$$\mathrm{s}\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \leq \omega(\mathfrak{a}_n, \mathfrak{a}_{n+1}).$$

This is a contradiction. Hence from (3.29) we obtain that

$$\mathrm{s}\omega(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \leq \omega(\mathfrak{a}_{n-1}, \mathfrak{a}_n). \tag{3.30}$$

Therefore, (3.25) is valid, where $\lambda = \frac{1}{\mathrm{s}} \in [0, \frac{1}{\mathrm{s}}]$. because $\mathrm{s} > 1$, it follows that $0 \leq \lambda = \frac{1}{\mathrm{s}} < 1$. Now combining (3.25) and Lemma 3.1 of [8], we deduce that

$$\{\mathfrak{a}_n\} = \{\mathscr{U}\mathscr{o}_n\} = \{\mathfrak{g}\mathscr{o}_{n+1}\}$$

forms a convergent Cauchy sequence in $\mathscr{L}$ which converges to some $\mathrm{v} \in \mathscr{L}$ due to the completeness of $(\mathscr{L}, \omega)$. i.e.,

$$\lim_{n\to+\infty} \mathscr{U}\mathscr{o}_n = \lim_{n\to+\infty} \mathfrak{g}\mathscr{o}_{n+1} = \mathrm{v}.$$

Hence, considering the compatibility of $\mathscr{U}$ and $\mathfrak{g}$, we conclude that

$$\lim_{n\to+\infty} \omega(\mathfrak{g}\mathscr{U}\mathscr{o}_n, \mathscr{U}\mathfrak{g}\mathscr{o}_n) = 0. \tag{3.31}$$

Furthermore, utilizing the continuity properties of both $\mathscr{U}$ and $\mathfrak{g}$, we acquire

$$\lim_{n\to+\infty} \mathfrak{g}\mathscr{U}\mathscr{o}_n = \mathfrak{g}\mathrm{v}, \qquad \lim_{n\to+\infty} \mathscr{U}\mathfrak{g}\mathscr{o}_n = \mathscr{U}\mathrm{v}. \tag{3.32}$$

Moreover, by employing the triangle inequality and utilizing Eqs (3.31) and (3.32), we derive

$$\frac{1}{\mathrm{s}}\omega(\mathscr{U}\mathrm{v}, \mathfrak{g}\mathrm{v}) \leq \omega(\mathscr{U}\mathrm{v}, \mathscr{U}(\mathfrak{g}\mathscr{o}_n)) + \mathrm{s}\omega(\mathscr{U}(\mathfrak{g}\mathscr{o}_n), \mathfrak{g}(\mathscr{U}\mathscr{o}_n)) + \mathrm{s}\omega(\mathfrak{g}(\mathscr{U}\mathscr{o}_n), \mathfrak{g}\mathrm{v}). \tag{3.33}$$

In conclusion, as $n \to +\infty$ in (3.33), we obtain that $\omega(\mathscr{U}\mathrm{v}, \mathfrak{g}\mathrm{v}) = 0$. Therefore, v is a coincidence point of $\mathscr{U}$ and $\mathfrak{g}$. Hence, we have the result. □

By relaxing the continuity requirements imposed on $\mathfrak{g}$ and $\mathscr{U}$ as stated in Theorem 11, we arrive at the following result:

Theorem 12. *Consider an increasing sequence $\{\mathfrak{g}\mathscr{o}_n\} \subset \mathscr{L}$ of Theorem 11 such that*

(i) $\lim_{n\to+\infty} \mathfrak{g}\mathscr{o}_n = \mathfrak{g}\mathscr{o}$ *in* $\mathfrak{g}\mathscr{L}$*;*

(ii) $\mathfrak{g}\mathscr{L}$ *is a closed subset of* $\mathscr{L}$*;*

(iii) $\mathfrak{g}\mathscr{o}_n \preceq \mathfrak{g}\mathscr{o}$ *and* $\mathfrak{g}\mathscr{o} \preceq \mathfrak{g}(\mathfrak{g}\mathscr{o})$ *for* $n \in \mathbb{N}$.

Hence, a coincidence point exists for weakly compatible mappings $\mathscr{U}$, $\mathfrak{g}$ if $\mathfrak{g}o_0 \preceq \mathscr{U}o_0$ for $o_0 \in \mathscr{L}$. Additionally, at coincidence points $\mathscr{U}$, $\mathfrak{g}$ are commutative; then a common fixed point in $\mathscr{L}$ exists for $\mathscr{U}$ and $\mathfrak{g}$.

Proof. The sequence

$$\{\mathfrak{a}_n\} = \{\mathscr{U}o_n\} = \{\mathfrak{g}o_{n+1}\}$$

forms a Cauchy sequence, as established in the proof of Theorem 11. Because of the closed nature of $\mathfrak{g}\mathscr{L}$, we have

$$\lim_{n\to+\infty} \mathscr{U}o_n = \lim_{n\to+\infty} \mathfrak{g}o_{n+1} = \mathfrak{g}\mathrm{v} \quad \text{for } \mathrm{v} \in \mathscr{L}.$$

Therefore, based on the given assumptions, it follows that $\mathfrak{g}o_n \preceq \mathfrak{g}\mathrm{v}$, for all $n \geq 1$. The subsequent step establishes that v serves as a coincidence point for both $\mathscr{U}$ and $\mathfrak{g}$.

According to (3.22), it follows that

$$\phi(\mathrm{s}\omega(\mathscr{U}o_n, \mathscr{U}o)) \leq \phi(\mathrm{G}_{\mathfrak{g}}(o_n, o)) - \psi(\mathrm{H}_{\mathfrak{g}}(o_n, o)), \tag{3.34}$$

where

$$\begin{aligned}
\mathrm{G}_{\mathfrak{g}}(o_n, \mathrm{v}) = \max\{&\frac{\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v})\,[1 + \omega(\mathfrak{g}o_n, \mathscr{U}o_n)]}{1 + \omega(\mathfrak{g}o_n, \mathfrak{g}\mathrm{v})}, \frac{\omega(\mathfrak{g}o_n, \mathscr{U}o_n)\,\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v})}{1 + \omega(\mathfrak{g}o_n, \mathfrak{g}\mathrm{v})},\\
&\frac{\omega(\mathfrak{g}o_n, \mathscr{U}\mathrm{v}) + \omega(\mathfrak{g}\mathrm{v}, \mathscr{U}o_n)}{2\mathrm{s}}, \omega(\mathfrak{g}o_n, \mathfrak{g}\mathrm{v})\}\\
\to& \max\{\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v}), 0, \frac{\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v})}{2\mathrm{s}}, 0\}\\
=&\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v}) \;\text{ as }\; n \to +\infty
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{H}_{\mathfrak{g}}(o_n, \mathrm{v}) &= \max\{\frac{\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v})\,[1 + \omega(\mathfrak{g}o_n, \mathscr{U}o_n)]}{1 + \omega(\mathfrak{g}o_n, \mathfrak{g}\mathrm{v})}, \omega(\mathfrak{g}o_n, \mathfrak{g}\mathrm{v})\}\\
&\to \max\{\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v}), 0\}\\
&= \omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v}) \;\text{ as }\; n \to +\infty.
\end{aligned}$$

Consequently, (3.34) transforms into the following:

$$\phi(\mathrm{s}\lim_{n\to+\infty} \omega(\mathscr{U}o_n, \mathscr{U}o)) \leq \phi(\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v})) - \psi(\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v})) < \phi(\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v})). \tag{3.35}$$

As a result, we obtain

$$\lim_{n\to+\infty} \omega(\mathscr{U}o_n, \mathscr{U}o) < \frac{1}{\mathrm{s}}\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v}). \tag{3.36}$$

Moreover, through the triangular inequality, we can express the following:

$$\frac{1}{\mathrm{s}}\omega(\mathfrak{g}\mathrm{v}, \mathscr{U}\mathrm{v}) \leq \omega(\mathfrak{g}\mathrm{v}, \mathscr{U}o_n) + \omega(\mathscr{U}o_n, \mathscr{U}\mathrm{v}). \tag{3.37}$$

Subsequently, if $\mathfrak{g}\mathrm{v} \neq \mathscr{U}\mathrm{v}$, a contradiction arises from (3.36) and (3.37). Therefore, $\mathfrak{g}\mathrm{v} = \mathscr{U}\mathrm{v}$. Assuming that $\mathfrak{g}\mathrm{v} = \mathscr{U}\mathrm{v} = \rho$, which indicates the commutative propertyof $\mathscr{U}$ and $\mathfrak{g}$ at ρ, we have that

$\mathcal{U}\rho = \mathcal{U}(\mathfrak{g}\mathrm{v}) = \mathfrak{g}(\mathcal{U}\mathrm{v}) = \mathfrak{g}\rho$. Given that $\mathfrak{g}\mathrm{v} = \mathfrak{g}(\mathfrak{g}\mathrm{v}) = \mathfrak{g}\rho$, applying (3.34) with $\mathfrak{g}\mathrm{v} = \mathcal{U}\mathrm{v}$ and $\mathfrak{g}\rho = \mathcal{U}\rho$ yields

$$\phi(\mathrm{s}\omega(\mathcal{U}\mathrm{v}, \mathcal{U}\rho)) \leq \phi(\mathrm{G}_{\mathfrak{g}}(\mathrm{v}, \rho)) - \psi(\mathrm{H}_{\mathfrak{g}}(\mathrm{v}, \rho)) < \phi(\omega(\mathcal{U}\mathrm{v}, \mathcal{U}\rho)),$$

or equivalently,

$$\mathrm{s}\omega(\mathcal{U}\mathrm{v}, \mathcal{U}\rho) \leq \omega(\mathcal{U}\mathrm{v}, \mathcal{U}\rho),$$

which results in a contradiction, if $\mathcal{U}\mathrm{v} \neq \mathcal{U}\rho$. Therefore, it follows that $\mathcal{U}\mathrm{v} = \mathcal{U}\rho = \rho$. Consequently, $\mathcal{U}\mathrm{v} = \mathfrak{g}\rho = \rho$, showing that ρ is a common fixed point for both $\mathcal{U}$ and $\mathfrak{g}$. □

Definition 13. A mapping $\mathcal{U}\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$, where $(\mathscr{L}, \omega, s, \preceq)$ is a c.m.s with respect to the function $\mathfrak{g}\colon \mathscr{L} \to \mathscr{L}$ is denoted as a (ϕ, ψ)-generalized contractive mapping if

$$\phi(\mathrm{s}^k\omega(\mathcal{U}(\sigma, \mathfrak{a}), \mathcal{U}(\rho, \mathfrak{d}))) \leq \phi(\mathrm{G}_{\mathfrak{g}}(\sigma, \mathfrak{a}, \rho, \mathfrak{d})) - \psi(\mathrm{H}_{\mathfrak{g}}(\sigma, \mathfrak{a}, \rho, \mathfrak{d})), \tag{3.38}$$

where $k > 2$, $\mathfrak{g}\sigma \preceq \mathfrak{g}\rho$, for $\sigma, \mathfrak{a}, \rho, \mathfrak{d} \in \mathscr{L}$ and

$$\begin{aligned}\mathrm{G}_{\mathfrak{g}}(\sigma, \mathfrak{a}, \rho, \mathfrak{d}) = \max\{&\frac{\omega(\mathfrak{g}\rho, \mathcal{U}(\rho, \mathfrak{d}))\,[1 + \omega(\mathfrak{g}\sigma, \mathcal{U}(\sigma, \mathfrak{a}))]}{1 + \omega(\mathfrak{g}\sigma, \mathfrak{g}\rho)}, \frac{\omega(\mathfrak{g}\sigma, \mathcal{U}(\sigma, \mathfrak{a}))\,\omega(\mathfrak{g}\rho, \mathcal{U}(\rho, \mathfrak{d}))}{1 + \omega(\mathfrak{g}\sigma, \mathfrak{g}\rho)},\\ &\frac{\omega(\mathfrak{g}\sigma, \mathcal{U}(\rho, \mathfrak{d})) + \omega(\mathfrak{g}\rho, \mathcal{U}(\sigma, \mathfrak{a}))}{2\mathrm{s}}, \omega(\mathfrak{g}\sigma, \mathfrak{g}\rho)\}\end{aligned}$$

and

$$\mathrm{H}_{\mathfrak{g}}(\sigma, \mathfrak{a}, \rho, \mathfrak{d}) = \max\{\frac{\omega(\mathfrak{g}\rho, \mathcal{U}(\rho, \mathfrak{d}))\,[1 + \omega(\mathfrak{g}\sigma, \mathcal{U}(\sigma, \mathfrak{a}))]}{1 + \omega(\mathfrak{g}\sigma, \mathfrak{g}\rho)}, \omega(\mathfrak{g}\sigma, \mathfrak{g}\rho)\}.$$

Theorem 14. *A coupled coincidence point for the mappings $\mathcal{U}$, $\mathfrak{g}$ exists in Definition 13 over a c.m.s $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ when there exists $(\sigma_0, \mathfrak{a}_0) \in \mathscr{L} \times \mathscr{L}$ with $f\sigma_0 \preceq \mathcal{U}(\sigma_0, \mathfrak{a}_0)$, $\mathfrak{g}\mathfrak{a}_0 \succeq \mathcal{U}(\mathfrak{a}_0, \sigma_0)$, $\mathcal{U}(\mathscr{L} \times \mathscr{L}) \subseteq \mathfrak{g}(\mathscr{L})$, $\mathcal{U}$ and commutes with $\mathfrak{g}$ and has a mixed $\mathfrak{g}$-monotone property.*

Proof. Based on the given assumptions and by following the argument presented in the proof of [18, Theorem 2.2], we form two sequences $\{\sigma_n\}$, $\{\mathfrak{a}_n\}$ within the space $\mathscr{L}$ such that

$$\mathfrak{g}\sigma_{n+1} = \mathcal{U}(\sigma_n, \mathfrak{a}_n), \quad \mathfrak{g}\mathfrak{a}_{n+1} = \mathcal{U}(\mathfrak{a}_n, \sigma_n) \quad \text{for all } n \geq 0.$$

Specifically, the sequences $\{\mathfrak{g}\sigma_n\}$ and $\{\mathfrak{g}\mathfrak{a}_n\}$ are, respectively, non-decreasing and non-increasing in $\mathscr{L}$. Substituting $\sigma = \sigma_n, \mathfrak{a} = \mathfrak{a}_n, \rho = \sigma_{n+1}$ and $\mathfrak{d} = \mathfrak{a}_{n+1}$ into (3.38), we obtain

$$\begin{aligned}\phi(\mathrm{s}^k\omega(\mathfrak{g}\sigma_{n+1}, \mathfrak{g}\sigma_{n+2})) &= \phi(\mathrm{s}^k\omega(\mathcal{U}(\sigma_n, \mathfrak{a}_n), \mathcal{U}(\sigma_{n+1}, \mathfrak{a}_{n+1})))\\ &\leq \phi(\mathrm{G}_{\mathfrak{g}}(\sigma_n, \mathfrak{a}_n, \sigma_{n+1}, \mathfrak{a}_{n+1})) - \psi(\mathrm{H}_{\mathfrak{g}}(\sigma_n, \mathfrak{a}_n, \sigma_{n+1}, \mathfrak{a}_{n+1})),\end{aligned} \tag{3.39}$$

where

$$\mathrm{G}_{\mathfrak{g}}(\sigma_n, \mathfrak{a}_n, \sigma_{n+1}, \mathfrak{a}_{n+1}) \leq \max\{\omega(\mathfrak{g}\sigma_n, \mathfrak{g}\sigma_{n+1}), \omega(\mathfrak{g}\sigma_{n+1}, \mathfrak{g}\sigma_{n+2})\} \tag{3.40}$$

and

$$\mathrm{H}_{\mathfrak{g}}(\sigma_n, \mathfrak{a}_n, \sigma_{n+1}, \mathfrak{a}_{n+1}) = \max\{\omega(\mathfrak{g}\sigma_n, \mathfrak{g}\sigma_{n+1}), \omega(\mathfrak{g}\sigma_{n+1}, \mathfrak{g}\sigma_{n+2})\}. \tag{3.41}$$

Therefore from (3.39), we have

$$\begin{aligned}\phi(\mathrm{s}^k\omega(\mathfrak{g}\sigma_{n+1}, \mathfrak{g}\sigma_{n+2})) \leq &\phi(\max\{\omega(\mathfrak{g}\sigma_n, \mathfrak{g}\sigma_{n+1}), \omega(\mathfrak{g}\sigma_{n+1}, \mathfrak{g}\sigma_{n+2})\})\\ &- \psi(\max\{\omega(\mathfrak{g}\sigma_n, \mathfrak{g}\sigma_{n+1}), \omega(\mathfrak{g}\sigma_{n+1}, \mathfrak{g}\sigma_{n+2})\}).\end{aligned} \tag{3.42}$$

Similarly by taking $\mathscr{o} = \mathfrak{a}_{n+1}, \mathfrak{a} = \mathscr{o}_{n+1}, \rho = \mathscr{o}_n$ and $\mathfrak{d} = \mathscr{o}_n$ in (3.38), we get

$$\begin{aligned}\phi(\mathrm{s}^k\omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2})) \leq &\phi(\max\{\omega(\mathfrak{g}\mathfrak{a}_n, \mathfrak{g}\mathfrak{a}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2})\}) \\ &- \psi(\max\{\omega(\mathfrak{g}\mathfrak{a}_n, \mathfrak{g}\mathfrak{a}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2})\}).\end{aligned} \tag{3.43}$$

Given the relation,

$$\max\{\phi(d_1), \phi(d_2)\} = \phi\{\max\{d_1, d_2\}\}$$

for all $d_1, d_2 \in [0, +\infty)$, the combination of (3.42) and (3.43) yields

$$\begin{aligned}\phi(\mathrm{s}^k\Upsilon_n) \leq &\phi(\max\{\omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1}), \omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathfrak{g}\mathscr{o}_{n+2}), \omega(\mathfrak{g}\mathfrak{a}_n, \mathfrak{g}\mathfrak{a}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2})\}) \\ &- \psi(\max\{\omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1}), \omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathfrak{g}\mathscr{o}_{n+2}), \omega(\mathfrak{g}\mathfrak{a}_n, \mathfrak{g}\mathfrak{a}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2})\}),\end{aligned} \tag{3.44}$$

where

$$\Upsilon_n = \max\{\omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathfrak{g}\mathscr{o}_{n+2}), \omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2})\}. \tag{3.45}$$

Let us denote the following

$$\mathrm{K}_n = \max\{\omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1}), \omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathfrak{g}\mathscr{o}_{n+2}), \omega(\mathfrak{g}\mathfrak{a}_n, \mathfrak{g}\mathfrak{a}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2})\}. \tag{3.46}$$

As a result, by considering (3.42) to (3.45), we get

$$\mathrm{s}^k\Upsilon_n \leq \mathrm{K}_n. \tag{3.47}$$

Subsequently, we establish that

$$\Upsilon_n \leq \lambda\Upsilon_{n-1} \tag{3.48}$$

for all $n \geq 1$, with $\lambda = \frac{1}{\mathrm{s}^k} \in [0, 1)$.

Assume that if $\mathrm{K}_n = \Upsilon_n$, then according to (3.47), we derive $\mathrm{s}^k\Upsilon_n \leq \Upsilon_n$, which gives that $\Upsilon_n = 0$ because $\mathrm{s} > 1$. Consequently, (3.48) is satisfied. Alternatively, if

$$\mathrm{K}_n = \max\{\omega(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathscr{o}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}_n, \mathfrak{g}\mathfrak{a}_{n+1})\}$$

denoted as $\mathrm{K}_n = \Upsilon_{n-1}$, then (3.47) implies (3.48).

Now, based on (3.47), we can conclude that $\Upsilon_n \leq \lambda^n\Upsilon_0$, leading to

$$\omega(\mathfrak{g}\mathscr{o}_{n+1}, \mathfrak{g}\mathscr{o}_{n+2}) \leq \lambda^n\Upsilon_0 \quad \text{and} \quad \omega(\mathfrak{g}\mathfrak{a}_{n+1}, \mathfrak{g}\mathfrak{a}_{n+2}) \leq \lambda^n\Upsilon_0. \tag{3.49}$$

Consequently, in accordance with Lemma 3.1 from [8], the sequences $\{\mathfrak{g}\mathscr{o}_n\}$ and $\{\mathfrak{g}\mathfrak{a}_n\}$ are Cauchy sequences in $\mathscr{L}$. Thus, by implementing the remaining procedures detailed in [16, Theorem 2.2], we can demonstrate coincidence point existence for $\mathscr{U}$ and $\mathfrak{g}$ in $\mathscr{L}$. □

Corollary 15. *Consider a c.m.s $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ where $\mathrm{s} > 1$. A continuous mapping $\mathscr{U}\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ for a mixed monotone property. Assume the existence of $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\phi(\mathrm{s}^k\omega(\mathscr{U}(\mathscr{o}, \mathfrak{a}), \mathscr{U}(\rho, \mathfrak{d}))) \leq \phi(\mathrm{G}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a}, \rho, \mathfrak{d})) - \psi(\mathrm{H}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a}, \rho, \mathfrak{d}))$$

for all $\wp, \mathfrak{a}, \rho, \mathfrak{d} \in \mathscr{L}$ *satisfying that* $\wp \leq \rho$ *and* $\mathfrak{a} \geq \mathfrak{d}$*, with* $k > 2$ *and*

$$\begin{aligned}\mathrm{G}_{\mathfrak{g}}(\wp, \mathfrak{a}, \rho, \mathfrak{d}) = \max\{&\frac{\omega(\rho, \mathcal{U}(\rho, \mathfrak{d}))\,[1 + \omega(\wp, \mathcal{U}(\wp, \mathfrak{a}))]}{1 + \omega(\wp, \rho)}, \frac{\omega(\wp, \mathcal{U}(\wp, \mathfrak{a}))\, \omega(\rho, \mathcal{U}(\rho, \mathfrak{d}))}{1 + \omega(\wp, \rho)},\\ &\frac{\omega(\wp, \mathcal{U}(\rho, \mathfrak{d})) + \omega(\rho, \mathcal{U}(\wp, \mathfrak{a}))}{2\mathrm{s}}, \omega(\wp, \rho)\}\end{aligned}$$

and

$$\mathrm{H}_{\mathfrak{g}}(\wp, \mathfrak{a}, \rho, \mathfrak{d}) = \max\{\frac{\omega(\rho, \mathcal{U}(\rho, \mathfrak{d}))\,[1 + \omega(\wp, \mathcal{U}(\wp, \mathfrak{a}))]}{1 + \omega(\wp, \rho)}, \omega(\wp, \rho)\}.$$

Under those conditions, $\mathcal{U}$ *possesses a coupled fixed point in* $\mathscr{L}$ *if* $(\wp_0, \mathfrak{a}_0) \in \mathscr{L} \times \mathscr{L}$ *satisfies that* $\wp_0 \leq \mathcal{U}(\wp_0, \mathfrak{a}_0)$ *and* $\mathfrak{a}_0 \geq \mathcal{U}(\mathfrak{a}_0, \wp_0)$*.*

Proof. Put $\mathfrak{g} = I_{\mathscr{L}}$ in Theorem 14. □

Corollary 16. *Consider a c.m.s* $(\mathscr{L}, \omega, \mathrm{s}, \preceq)$ *where* $\mathrm{s} > 1$ *and* $\mathcal{U} \colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ *be a continuous mapping possessing a mixed monotone property. Suppose the existence of* $\psi \in \Psi$ *such that*

$$\omega(\mathcal{U}(\wp, \mathfrak{a}), \mathcal{U}(\rho, \mathfrak{d})) \leq \frac{1}{\mathrm{s}^k}\mathrm{G}_{\mathfrak{g}}(\wp, \mathfrak{a}, \rho, \mathfrak{d}) - \frac{1}{\mathrm{s}^k}\psi(\mathrm{H}_{\mathfrak{g}}(\wp, \mathfrak{a}, \rho, \mathfrak{d})) \tag{3.50}$$

for all $\wp, \mathfrak{a}, \rho, \mathfrak{d} \in \mathscr{L}$ *with* $\wp \leq \rho$*,* $\mathfrak{a} \geq \mathfrak{d}$ *and* $k > 2$*, where*

$$\begin{aligned}\mathrm{G}_{\mathfrak{g}}(\wp, \mathfrak{a}, \rho, \mathfrak{d}) = \max\{&\frac{\omega(\rho, \mathcal{U}(\rho, \mathfrak{d}))\,[1 + \omega(\wp, \mathcal{U}(\wp, \mathfrak{a}))]}{1 + \omega(\wp, \rho)}, \frac{\omega(\wp, \mathcal{U}(\wp, \mathfrak{a}))\, \omega(\rho, \mathcal{U}(\rho, \mathfrak{d}))}{1 + \omega(\wp, \rho)},\\ &\frac{\omega(\wp, \mathcal{U}(\rho, \mathfrak{d})) + \omega(\rho, \mathcal{U}(\wp, \mathfrak{a}))}{2\mathrm{s}}, \omega(\wp, \rho)\}\end{aligned} \tag{3.51}$$

and

$$\mathrm{H}_{\mathfrak{g}}(\wp, \mathfrak{a}, \rho, \mathfrak{d}) = \max\{\frac{\omega(\rho, \mathcal{U}(\rho, \mathfrak{d}))\,[1 + \omega(\wp, \mathcal{U}(\wp, \mathfrak{a}))]}{1 + \omega(\wp, \rho)}, \omega(\wp, \rho)\}. \tag{3.52}$$

If for some $(\wp_0, \mathfrak{a}_0) \in \mathscr{L} \times \mathscr{L}$ *such that* $\wp_0 \leq \mathcal{U}(\wp_0, \mathfrak{a}_0)$ *and* $\mathfrak{a}_0 \geq \mathcal{U}(\mathfrak{a}_0, \wp_0)$*, then* $\mathcal{U}$ *in* $\mathscr{L}$ *possesses a coupled fixed point.*

Theorem 17. *A coupled common fixed point for* $\mathcal{U}$ *and* $\mathfrak{g}$ *is unique in Theorem 14, if for all* $(\mathfrak{y}, \mathfrak{b}) \in \mathscr{L} \times \mathscr{L}$*,* $(\mathcal{U}(\mathfrak{y}, \mathfrak{b}), \mathcal{U}(\mathfrak{b}, \mathfrak{y}))$ *is comparable to both* $(\mathcal{U}(\wp, \mathfrak{a}), \mathcal{U}(\mathfrak{a}, \wp))$ *and* $(\mathcal{U}(r, s), \mathcal{U}(s, r))$ *for some* $(\mathfrak{y}, \mathfrak{b}) \in \mathscr{L} \times \mathscr{L}$*.*

Proof. By virtue of Theorem 14, a coupled coincidence point in $\mathscr{L}$ for $\mathcal{U}$ and $\mathfrak{g}$ is established. To prove the uniqueness, let us take two coupled coincidence points $(\wp, \mathfrak{a})$ and (r, s), so that

$$\mathcal{U}(\wp,\ \mathfrak{a}) = \mathfrak{g}\wp,\ \ \mathcal{U}(\mathfrak{a}, \wp) = \mathfrak{g}\mathfrak{a},\ \ \mathcal{U}(r, s) = \mathfrak{g}r\ \text{ and }\ \mathcal{U}(s, r) = \mathfrak{g}s.$$

Our objective is to demonstrate that $\mathfrak{g}\wp = \mathfrak{g}r$ and $\mathfrak{g}\mathfrak{a} = \mathfrak{g}s$.

Given the provided conditions, $(\mathcal{U}(\mathfrak{y}, \mathfrak{b}), \mathcal{U}(\mathfrak{b}, \mathfrak{y}))$ is comparable to $(\mathcal{U}(\wp, \mathfrak{a}), \mathcal{U}(\mathfrak{a}, \wp))$ and to $(\mathcal{U}(r, s), \mathcal{U}(s, r))$ for $(\mathfrak{y}, \mathfrak{b}) \in \mathscr{L} \times \mathscr{L}$. Assume that

$$(\mathcal{U}(\wp, \mathfrak{a}), \mathcal{U}(\mathfrak{a}, \wp)) \leq (\mathcal{U}(\mathfrak{y}, \mathfrak{b}), \mathcal{U}(\mathfrak{b}, \mathfrak{y}))\ \text{ and }\ (\mathcal{U}(r, s), \mathcal{U}(s, r)) \leq (\mathcal{U}(\mathfrak{y}, \mathfrak{b}), \mathcal{U}(\mathfrak{b}, \mathfrak{y})).$$

Define $\mathfrak{y}_0 = \mathfrak{y}$ and $\mathfrak{b}_0 = \mathfrak{b}$ and select $(\mathfrak{y}_1, \mathfrak{b}_1) \in \mathscr{L} \times \mathscr{L}$ as follows:

$$\mathfrak{g}\mathfrak{y}_1 = \mathscr{U}(\mathfrak{y}_0, \mathfrak{b}_0), \quad \mathfrak{g}\mathfrak{b}_1 = \mathscr{U}(\mathfrak{b}_0, \mathfrak{y}_0), \quad (n \geq 1).$$

Construct the sequences $\{\mathfrak{g}\mathfrak{y}_n\}$ and $\{\mathfrak{g}\mathfrak{b}_n\}$ in $\mathscr{L}$ inductively by implementing the above process using the following equations;

$$\mathfrak{g}\mathfrak{y}_{n+1} = \mathscr{U}(\mathfrak{y}_n, \mathfrak{b}_n), \quad \mathfrak{g}\mathfrak{b}_{n+1} = \mathscr{U}(\mathfrak{b}_n, \mathfrak{y}_n), \quad (n \geq 0).$$

In a similar manner, introduce the sequences $\{\mathfrak{g}\mathscr{o}_n\}$, $\{\mathfrak{g}\mathfrak{a}_n\}$, $\{\mathfrak{g}r_n\}$ and $\{\mathfrak{g}s_n\}$ in $\mathscr{L}$ according to the previous definitions, with the initial values $\mathscr{o}_0 = \mathscr{o}$, $\mathfrak{a}_0 = \mathfrak{a}$, $r_0 = r$, and $s_0 = s$. Additionally, it is apparent that

$$\mathfrak{g}\mathscr{o}_n \to \mathscr{U}(\mathscr{o}, \mathfrak{a}), \ \mathfrak{g}\mathfrak{a}_n \to \mathscr{U}(\mathfrak{a}, \mathscr{o}), \ \mathfrak{g}r_n \to \mathscr{U}(r, s), \ \mathfrak{g}s_n \to \mathscr{U}(s, r), \quad (n \geq 1).$$

Given that

$$(\mathscr{U}(\mathscr{o}, \mathfrak{a}), \mathscr{U}(\mathfrak{a}, \mathscr{o})) = (\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{a}) = (\mathfrak{g}\mathscr{o}_1, \mathfrak{g}\mathfrak{a}_1)$$

is comparable to

$$(\mathscr{U}(\mathfrak{y}, \mathfrak{b}), \mathscr{U}(\mathfrak{b}, \mathfrak{y})) = (\mathfrak{g}\mathfrak{y}, \mathfrak{g}\mathfrak{b}) = (\mathfrak{g}\mathfrak{y}_1, \mathfrak{g}\mathfrak{b}_1),$$

it follows that $(\mathfrak{g}\mathscr{o}_1, \mathfrak{g}\mathfrak{a}_1) \leq (\mathfrak{g}\mathfrak{y}_1, \mathfrak{g}\mathfrak{b}_1)$. Therefore, through induction, we can deduce that

$$(\mathfrak{g}\mathscr{o}_n, \mathfrak{g}\mathfrak{a}_n) \leq (\mathfrak{g}\mathfrak{y}_n, \mathfrak{g}\mathfrak{b}_n), \quad (n \geq 0).$$

Hence, according to (3.38), we obtain

$$\begin{aligned} \phi(\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_{n+1})) \leq \phi(\mathrm{s}^k\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_{n+1})) &= \phi(\mathrm{s}^k\omega(\mathscr{U}(\mathscr{o}, \mathfrak{a}), \mathscr{U}(\mathfrak{y}_n, \mathfrak{b}_n))) \\ &\leq \phi(\mathrm{G}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a}, \mathfrak{y}_n, \mathfrak{b}_n)) - \psi(\mathrm{H}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a}, \mathfrak{y}_n, \mathfrak{b}_n)), \end{aligned} \tag{3.53}$$

where

$$\begin{aligned} \mathrm{G}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a}, \mathfrak{y}_n, \mathfrak{b}_n) = &\max\{\frac{\omega(\mathfrak{g}\mathfrak{y}_n, \mathscr{U}(\mathfrak{y}_n, \mathfrak{b}_n))\,[1 + \omega(\mathfrak{g}\mathscr{o}, \mathscr{U}(\mathscr{o}, \mathfrak{a}))]}{1 + \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)}, \\ &\frac{\omega(\mathfrak{g}\mathscr{o}, \mathscr{U}(\mathscr{o}, \mathfrak{a}))\,\omega(\mathfrak{g}\mathfrak{y}_n, \mathscr{U}(\mathfrak{y}_n, \mathfrak{b}_n))}{1 + \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)}, \\ &\frac{\omega(\mathfrak{g}\mathscr{o}, \mathscr{U}(\mathfrak{y}_n, \mathfrak{b}_n)) + \omega(\mathfrak{g}\mathfrak{y}_n, \mathscr{U}(\mathscr{o}, \mathfrak{a}))}{2\mathrm{s}}, \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)\} \\ = &\max\{0, 0, \frac{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)}{\mathrm{s}}, \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)\} \\ = &\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n) \end{aligned}$$

and

$$\begin{aligned} \mathrm{H}_{\mathfrak{g}}(\mathscr{o}, \mathfrak{a}, \mathfrak{y}_n, \mathfrak{b}_n) &= \max\{\frac{\omega(\mathfrak{g}\mathfrak{y}_n, \mathscr{U}(\mathfrak{y}_n, \mathfrak{b}_n))\,[1 + \omega(\mathfrak{g}\mathscr{o}, \mathscr{U}(\mathscr{o}, \mathfrak{a}))]}{1 + \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)}, \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)\} \\ &= \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n). \end{aligned}$$

Hence, according to (3.38), we obtain

$$\phi(\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_{n+1})) \leq \phi(\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)) - \psi(\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n)). \tag{3.54}$$

By an analogous procedure, we find that

$$\phi(\omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_{n+1})) \leq \phi(\omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)) - \psi(\omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)). \tag{3.55}$$

Combining (3.54) and (3.55), we obtain

$$\begin{aligned}\phi(\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_{n+1})\}) \leq &\phi(\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)\}) \\ &- \psi(\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)\}) \\ &< \phi(\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)\}).\end{aligned} \tag{3.56}$$

Therefore, utilizing the property of ϕ, we deduce the following:

$$\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_{n+1}), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_{n+1})\} < \max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)\},$$

which demonstrates that $\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)\}$ is a decreasing sequence; then,

$$\lim_{n\to+\infty} \max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)\} = \gamma, \quad \gamma \geq 0.$$

By the limiting case in (3.56), we get

$$\phi(\gamma) \leq \phi(\gamma) - \psi(\gamma),$$

this suggests that if $\psi(\gamma) = 0$, then it follows that $\gamma = 0$. As a result,

$$\lim_{n\to+\infty} \max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n)\} = 0.$$

Consequently, we get

$$\lim_{n\to+\infty} \omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{y}_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{b}_n) = 0. \tag{3.57}$$

By a similar argument, we acquire

$$\lim_{n\to+\infty} \omega(\mathfrak{g}r, \mathfrak{g}\mathfrak{y}_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \omega(\mathfrak{g}s, \mathfrak{g}\mathfrak{b}_n) = 0. \tag{3.58}$$

Thus, utilizing (3.57) and (3.58), we conclude that $\mathfrak{g}\mathscr{o} = \mathfrak{g}r$ and $\mathfrak{g}\mathfrak{a} = \mathfrak{g}s$. Given that $\mathfrak{g}\mathscr{o} = \mathscr{U}(\mathscr{o}, \mathfrak{a})$ and $\mathfrak{g}\mathfrak{a} = \mathscr{U}(\mathfrak{a}, \mathscr{o})$, then the fact that $\mathscr{U}$ and $\mathfrak{g}$ are commutative results in

$$\mathfrak{g}(\mathfrak{g}\mathscr{o}) = \mathfrak{g}(\mathscr{U}(\mathscr{o}, \mathfrak{a})) = \mathscr{U}(\mathfrak{g}\mathscr{o}, \mathfrak{g}\mathfrak{a}) \quad \text{and} \quad \mathfrak{g}(\mathfrak{g}\mathfrak{a}) = \mathfrak{g}(\mathscr{U}(\mathfrak{a}, \mathscr{o})) = \mathscr{U}(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathscr{o}). \tag{3.59}$$

Let $\mathfrak{g}\mathscr{o} = \mathfrak{c}^*$ and $\mathfrak{g}\mathfrak{a} = \mathfrak{e}^*$; then, (3.59) becomes

$$\mathfrak{g}(\mathfrak{c}^*) = \mathscr{U}(\mathfrak{c}^*, \mathfrak{e}^*) \quad \text{and} \quad \mathfrak{g}(\mathfrak{e}^*) = \mathscr{U}(\mathfrak{e}^*, \mathfrak{c}^*). \tag{3.60}$$

This establishes that $\mathscr{U}$, $\mathfrak{g}$ have a coupled coincidence point of $(\mathfrak{c}^*, \mathfrak{e}^*)$. Consequently, $\mathfrak{g}(\mathfrak{c}^*) = \mathfrak{g}r$ and $\mathfrak{g}(\mathfrak{e}^*) = \mathfrak{g}s$, indicating that $\mathfrak{g}(\mathfrak{c}^*) = \mathfrak{c}^*$ and $\mathfrak{g}(\mathfrak{e}^*) = \mathfrak{e}^*$. Therefore, we can deduce from (3.60) that

$$\mathfrak{c}^* = \mathfrak{g}(\mathfrak{c}^*) = \mathscr{U}(\mathfrak{c}^*, \mathfrak{e}^*) \quad \text{and} \quad \mathfrak{e}^* = \mathfrak{g}(\mathfrak{e}^*) = \mathscr{U}(\mathfrak{e}^*, \mathfrak{c}^*).$$

Thus, $\mathscr{U}$, $\mathfrak{g}$ have $(\mathfrak{c}^*, \mathfrak{e}^*)$ which is a coupled common fixed point.

Assuming uniqueness, consider another pair $(\mathfrak{u}^*, \mathfrak{m}^*)$ as a coupled common fixed point for both $\mathscr{U}$ and $\mathfrak{g}$. This implies that

$$\mathfrak{u}^* = \mathfrak{g}\mathfrak{u}^* = \mathscr{U}(\mathfrak{u}^*, \mathfrak{m}^*) \quad \text{and} \quad \mathfrak{m}^* = \mathfrak{g}\mathfrak{m}^* = \mathscr{U}(\mathfrak{m}^*, \mathfrak{u}^*).$$

Given that $(\mathfrak{u}^*, \mathfrak{m}^*)$ is a coupled common fixed point of $\mathscr{U}$ and $\mathfrak{g}$, it follows that $\mathfrak{g}\mathfrak{u}^* = \mathfrak{g}\mathscr{o} = \mathfrak{c}^*$ and, $\mathfrak{g}\mathfrak{m}^* = \mathfrak{g}\mathfrak{a} = \mathfrak{e}^*$. Consequently,

$$\mathfrak{u}^* = \mathfrak{g}\mathfrak{u}^* = \mathfrak{g}\mathfrak{c}^* = \mathfrak{c}^* \quad \text{and} \quad \mathfrak{m}^* = \mathfrak{g}\mathfrak{m}^* = \mathfrak{g}\mathfrak{e}^* = \mathfrak{e}^*.$$

Hence, we have the proof. □

Theorem 18. *Under the additional conditions of Theorem 17, when $\mathfrak{g}\mathscr{o}_0$ and $\mathfrak{g}\mathfrak{a}_0$ are comparable, a common fixed point which is unique exists in $\mathscr{L}$ for $\mathscr{U}$ and $\mathfrak{g}$.*

Proof. Following Theorem 17, the existence of a coupled common fixed point $(\mathscr{o}, \mathfrak{a})$ which is a unique for $\mathscr{U}$ and $\mathfrak{g}$ in $\mathscr{L}$ is established. To prove that $\mathscr{o} = \mathfrak{a}$, consider the hypothesis that $\mathfrak{g}\mathscr{o}_0$ and $\mathfrak{g}\mathfrak{a}_0$ are comparable, assuming, without loss of generality, that $\mathfrak{g}\mathscr{o}_0 \leq \mathfrak{g}\mathfrak{a}_0$. Through induction, it is inferred that $\mathfrak{g}\mathscr{o}_n \leq \mathfrak{g}\mathfrak{a}_n$ holds for all $n \geq 0$, where $\{\mathfrak{g}\mathscr{o}_n\}$ and $\{\mathfrak{g}\mathfrak{a}_n\}$ are sequences derived from Theorem 14.

Now, employing Lemma 6, we obtain

$$\begin{aligned}
\phi(\mathrm{s}^{k-2}\omega(\mathscr{o}, \mathfrak{a})) = \phi(\mathrm{s}^k \frac{1}{\mathrm{s}^2}\omega(\mathscr{o}, \mathfrak{a})) &\leq \lim_{n\to+\infty} \sup \phi(\mathrm{s}^k\omega(\mathscr{o}_{n+1}, \mathfrak{a}_{n+1})) \\
&= \lim_{n\to+\infty} \sup \phi(\mathrm{s}^k\omega(\mathscr{U}(\mathscr{o}_n, \mathfrak{a}_n), \mathscr{U}(\mathfrak{a}_n, \mathscr{o}_n))) \\
&\leq \lim_{n\to+\infty} \sup \phi(\mathrm{G}_{\mathfrak{g}}(\mathscr{o}_n, \mathfrak{a}_n, \mathfrak{a}_n, \mathscr{o}_n)) - \lim_{n\to+\infty} \inf \psi(\mathrm{H}_{\mathfrak{g}}(\mathscr{o}_n, \mathfrak{a}_n, \mathfrak{a}_n, \mathscr{o}_n)) \\
&\leq \phi(\omega(\mathscr{o}, \mathfrak{a})) - \lim_{n\to+\infty} \inf \psi(\mathrm{H}_{\mathfrak{g}}(\mathscr{o}_n, \mathfrak{a}_n, \mathfrak{a}_n, \mathscr{o}_n)) \\
&< \phi(\omega(\mathscr{o}, \mathfrak{a})).
\end{aligned}$$

This contradiction suggests that $\mathscr{o} = \mathfrak{a}$. Hence, we have the proof. □

Remark 19. The condition outlined is based on the known fact that a b-metric space transforms into a metric space for $\mathrm{s} = 1$. Consequently, drawing upon Jachymski's [36] findings, the specified condition unfolds;

$$\phi(\omega(\mathscr{U}(\mathscr{o}, \mathfrak{a}), \mathscr{U}(\rho, \mathfrak{d}))) \leq \phi(\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\rho), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{d})\}) - \psi(\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\rho), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{d})\})$$

is equivalent to

$$\omega(\mathscr{U}(\mathscr{o}, \mathfrak{a}), \mathscr{U}(\rho, \mathfrak{d})) \leq \varphi(\max\{\omega(\mathfrak{g}\mathscr{o}, \mathfrak{g}\rho), \omega(\mathfrak{g}\mathfrak{a}, \mathfrak{g}\mathfrak{d})\}).$$

In this scenario, considering that $\phi \in \Phi$, $\psi \in \Psi$ and a self-map φ over $[0, +\infty)$ exists as a continuous function with $\varphi(t) < t, \forall t > 0$ and $\varphi(t) = 0$ under the circumstance that $t = 0$, it can be concluded that our findings provide a broader and more extensive perspective than the results presented in [13, 14, 37], along with other similar outcomes.

Corollary 20. *A non-decreasing self-map $\mathscr{U}$ over a c.m.s. $(\mathscr{L}, \omega, \mathrm{s}, \leq)$ is continuous and $\mathscr{o}_0 \leq \mathscr{U}\mathscr{o}_0$ for $\mathscr{o}_0 \in \mathscr{L}$. Then $\mathscr{U}$ possesses a fixed point in $\mathscr{L}$ if for $(\mathscr{o}, \mathfrak{a}) \in \mathscr{L} \times \mathscr{L}$,*

$$\phi(s\omega(\mathscr{U}\mathscr{o}, \mathscr{U}\mathfrak{a})) \leq \phi(\mathrm{G}(\mathscr{o}, \mathfrak{a})) - \psi(\mathrm{G}(\mathscr{o}, \mathfrak{a})),$$

where $\mathrm{G}(\mathscr{o}, \mathfrak{a})$, ϕ and ψ are the same as in Theorem 7.

Proof. Choose $H(\varpi, \mathfrak{a}) = G(\varpi, \mathfrak{a})$ in the contraction condition (3.3) of Theorem 7. □

Note 1. Additionally, given that the requirements of Theorem 8 are fulfilled by $\mathscr{L}$, it establishes the presence of a fixed point for the mapping $\mathscr{U}$. Also, under the condition (3.15) satisfied by $\mathscr{L}$, the uniqueness of the fixed point is ensured.

Note 2. By adapting the proofs provided in Theorems 11 and 12, one can establish a coincidence point for $\mathscr{U}$ and $\mathfrak{g}$ within $\mathscr{L}$. Similarly, leveraging Theorems 14, 17 and 18, one can derive a coupled coincidence point which is unique, along with a unique common fixed point for $\mathscr{U}$ and $\mathfrak{g}$ within both $\mathscr{L} \times \mathscr{L}$ and $\mathscr{L}$. These results are contingent upon the fulfillment of a generalized contraction condition in Corollary 20, where $G_{\mathfrak{g}}(\varpi, \mathfrak{a})$, $G_{\mathfrak{g}}(\varpi, \mathfrak{a}, \rho, \mathfrak{d})$, and the conditions imposed on ϕ and ψ remain consistent with the previously defined formulations.

Corollary 21. *A non-decreasing self-map $\mathscr{U}$ over a c.m.s. $(\mathscr{L}, \omega, s, \preceq)$ is continuous. Assume that there is a constant* $k \in [0, 1)$*, and for each pair of elements $\varpi, \mathfrak{a} \in \mathscr{L}$ with $\varpi \preceq \mathfrak{a}$,*

$$\omega(\mathscr{U}\varpi, \mathscr{U}\mathfrak{a}) \leq \frac{k}{s} \max\{\frac{\omega(\mathfrak{a}, \mathscr{U}\mathfrak{a})\,[1 + \omega(\varpi, \mathscr{U}\varpi)]}{1 + \omega(\varpi, \mathfrak{a})}, \frac{\omega(\varpi, \mathscr{U}\varpi)\,\omega(\mathfrak{a}, \mathscr{U}\mathfrak{a})}{1 + \omega(\varpi, \mathfrak{a})}, \frac{\omega(\varpi, \mathscr{U}\mathfrak{a}) + \omega(\mathfrak{a}, \mathscr{U}\varpi)}{2s}, \omega(\varpi, \mathfrak{a})\}.$$

If there exists $\varpi_0 \in \mathscr{L}$ with $\varpi_0 \preceq \mathscr{U}\varpi_0$, then $\mathscr{U}$ has a fixed point in $\mathscr{L}$.

Proof. Define $\phi(t) = t$ and $\psi(t) = (1 - k)t$, for all $t \in (0, +\infty)$ in accordance with Corollary 20. □

Note 3. By relaxing the continuity constraints for the mapping $\mathscr{U}$ as described in Corollary 21, it is possible to acquire a fixed point of $\mathscr{U}$ by considering a decreasing sequence $\{\varpi_n\} \subset \mathscr{L}$ in Theorem 8.

We demonstrate the practical relevance of the obtained results for diverse scenarios, including situations in which a metric ω exhibits both continuity and discontinuity within a space $\mathscr{L}$.

Example 22. A metric ω: $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ is defined as below and $\preceq$ is a usual order on $\mathscr{L}$, where $\mathscr{L} = \{1, 2, 3, 4, 5, 6\}$ given the following:

$$\begin{aligned}
&\omega(\varpi, \mathfrak{a}) = \omega(\mathfrak{a}, \varpi) = 0, \text{ if } \varpi, \mathfrak{a} = 1, 2, 3, 4, 5, 6 \text{ and } \varpi = \mathfrak{a},\\
&\omega(\varpi, \mathfrak{a}) = \omega(\mathfrak{a}, \varpi) = 3, \ if \ \varpi, \mathfrak{a} = 1, 2, 3, 4, 5 \text{ and } \varpi \neq \mathfrak{a},\\
&\omega(\varpi, \mathfrak{a}) = \omega(\mathfrak{a}, \varpi) = 12, \ if \ \varpi = 1, 2, 3, 4 \text{ and } \mathfrak{a} = 6,\\
&\omega(\varpi, \mathfrak{a}) = \omega(\mathfrak{a}, \varpi) = 20, \ if \ \varpi = 5 \text{ and } \mathfrak{a} = 6.
\end{aligned}$$

Define a map $\mathscr{U}$: $\mathscr{L} \to \mathscr{L}$ by $\mathscr{U}1 = \mathscr{U}2 = \mathscr{U}3 = \mathscr{U}4 = \mathscr{U}5 = 1, \mathscr{U}6 = 2$ and set $\phi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{4}$, $\forall t \in [0, +\infty)$. Then $\mathscr{U}$ possesses a fixed point in $\mathscr{L}$.

Proof. It is clear that when $s = 2$, $(\mathscr{L}, \omega, s, \preceq)$ forms a c.m.s. Let us examine the various scenarios for elements ϖ and $\mathfrak{a}$ within $\mathscr{L}$.
Case 1. Assume that ϖ and $\mathfrak{a}$ are elements of the set $\{1, 2, 3, 4, 5\}$, with $\varpi < \mathfrak{a}$. Consequently,

$$\omega(\mathscr{U}\varpi, \mathscr{U}\mathfrak{a}) = \omega(1, 1) = 0.$$

Thus,

$$\phi(2\omega(\mathscr{U}\varpi, \mathscr{U}\mathfrak{a})) = 0 \leq \phi(G(\varpi, \mathfrak{a})) - \psi(G(\varpi, \mathfrak{a})).$$

Case 2. Assume that ϱ is any one of $\{1,2,3,4,5\}$ and $\mathfrak{a} = 6$. In this scenario, $\omega(\mathcal{U}\varrho, \mathcal{U}\mathfrak{a}) = \omega(1,2) = 3$, $G(6,5) = 20$ and $G(\varrho, 6) = 12$, for $\varrho \in \{1,2,3,4\}$. As a result, we have

$$\phi(2\omega(\mathcal{U}\varrho, \mathcal{U}\mathfrak{a})) \leq \frac{G(\varrho, \mathfrak{a})}{4} = \phi(G(\varrho, \mathfrak{a})) - \psi(G(\varrho, \mathfrak{a})).$$

Therefore, the condition established in Corollary 20 is satisfied. Additionally, the other prerequisites outlined in Corollary 20 are met. Consequently, $\mathcal{U}$ possesses a fixed point in $\mathscr{L}$ since Corollary 20 is applicable for $\mathcal{U}$, ϕ, ψ and the space $(\mathscr{L}, \omega, s, \preceq)$. □

Example 23. A metric ω: $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ is specified on the set

$$\mathscr{L} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\}$$

with the standard order $\leq$ in the following manner:

$$\omega(\varrho, \mathfrak{a}) = \begin{cases} 0, & for\ \varrho = \mathfrak{a}, \\ 1, & for\ \varrho \neq \mathfrak{a} \in \{0,1\}, \\ |\varrho - \mathfrak{a}|, & for\ \varrho, \mathfrak{a} \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\}, \\ 2, & otherwise. \end{cases}$$

Define $\mathcal{U}$: $\mathscr{L} \to \mathscr{L}$ under the conditions that $\mathcal{U}0 = 0$, $\mathcal{U}\frac{1}{n} = \frac{1}{12n}, \forall n \geq 1$ and $\phi(t) = t$, $\psi(t) = \frac{4t}{5}, 0 \leq t < \infty$. Then $\mathcal{U}$ possesses a fixed point in $\mathscr{L}$.

Proof. Clearly, when $s = \frac{12}{5}$, the space $(\mathscr{L}, \omega, s, \preceq)$ forms a c.m.s. Additionally, as defined, the metric ω is a b-metric space which is discontinuous. When examining $\varrho, \mathfrak{a} \in \mathscr{L}$ where $\varrho < \mathfrak{a}$, the following situations arise:
Case 1. If $\varrho = 0$ and $\mathfrak{a} = \frac{1}{n}, \forall n \geq 1$, then,

$$\omega(\mathcal{U}\varrho, \mathcal{U}\mathfrak{a}) = \omega(0, \frac{1}{12n}) = \frac{1}{12n} \quad \text{and} \quad G(\varrho, \mathfrak{a}) = \frac{1}{n} \quad \text{or} \quad G(\varrho, \mathfrak{a}) = \{1, 2\}.$$

Therefore, we have

$$\phi\left(\frac{12}{5}\omega(\mathcal{U}\varrho, \mathcal{U}\mathfrak{a})\right) \leq \frac{G(\varrho, \mathfrak{a})}{5} = \phi(G(\varrho, \mathfrak{a})) - \psi(G(\varrho, \mathfrak{a})).$$

Case 2. If $\varrho = \frac{1}{m}$ and $\mathfrak{a} = \frac{1}{n}$ with $m > n \geq 1$, then,

$$\omega(\mathcal{U}\varrho, \mathcal{U}\mathfrak{a}) = \omega(\frac{1}{12m}, \frac{1}{12n}) \quad \text{and} \quad G(\varrho, \mathfrak{a}) \geq \frac{1}{n} - \frac{1}{m} \quad \text{or} \quad G(\varrho, \mathfrak{a}) = 2.$$

Therefore,

$$\phi\left(\frac{12}{5}\omega(\mathcal{U}\varrho, \mathcal{U}\mathfrak{a})\right) \leq \frac{G(\varrho, \mathfrak{a})}{5} = \phi(G(\varrho, \mathfrak{a})) - \psi(G(\varrho, \mathfrak{a})).$$

Thus, the conditions specified in Corollary 20, including condition the defined condition, along with the fulfillment of the remaining assumptions, are met. Consequently, there exists a fixed point for $\mathcal{U}$ within the space $\mathscr{L}$. □

Example 24. Consider $\mathscr{L}$ such that it consists of all continuous functions on [a, b], denoted by $\mathscr{L} = C[a, b]$. A b-metric ω on $\mathscr{L}$ defined as

$$\omega(\theta_1, \theta_2) = \sup_{t \in [a,b]} \{|\theta_1(t) - \theta_2(t)|^2\}$$

for any $\theta_1, \theta_2 \in \mathscr{L}$ with the partial order $\preceq$ defined with $\theta_1 \preceq \theta_2$, if $a \le \theta_1(t) \le \theta_2(t) \le b$, where $0 \le a < b$ and $t \in [a, b]$. Define $\mathscr{U}\colon \mathscr{L} \to \mathscr{L}$ under the conditions that $\mathscr{U}\theta = \frac{\theta}{5}, \theta \in \mathscr{L}$ and the distance functions $\phi(t) = t$, $\psi(t) = \frac{t}{3}$, $0 \le t < \infty$. Thus, $\mathscr{U}$ possesses a unique fixed point in $\mathscr{L}$.

Proof. Given the specified assumptions, it is evident that $(\mathscr{L}, \omega, s, \preceq)$ forms a c.m.s. with $s = 2$, satisfying the conditions stated in Corollary 20 and Note 1. Moreover, for all $\theta_1, \theta_2 \in \mathscr{L}$, the function

$$\min(\theta_1, \theta_2)(t) = \min\{\theta_1(t), \theta_2(t)\}$$

is continuous, satisfying the assumptions of Corollary 20 and Note 1. Thus in $\mathscr{L}$, $\mathscr{U}$ possesses a unique fixed point $\theta = 0$. □

4. Application

We examine the presence of a unique solution to a nonlinear quadratic integral [15], considering its applicability in light of the coupled fixed point defined in Corollary 16 outlined in this paper.

Let us examine the following integral equation of a quadratic and nonlinear nature:

$$x(t) = \delta(t) + \alpha \int_0^1 k_1(t, q) g_1(q, x(q)) dq \int_0^1 k_2(t, q) g_2(q, x(q)) dq, \quad t \in I = [0, 1], \ \alpha \ge 0. \tag{4.1}$$

Denote Γ that consists of self-maps γ over $[0, +\infty)$ to satisfy the following conditions:

(i) $(\gamma(t))^q \le \gamma(t^q)$, $\forall$ $q \ge 1$ and γ is non-decreasing.

(ii) $\exists$ $\phi \in \Phi$ with $\gamma(t) = t - \phi(t)$, $\forall t \in [0, \infty)$.

For example, $\gamma_1(t) = rt$, $0 \le r < 1$ and $\gamma_2(t) = \frac{t}{t+1}$ are in Γ [15].

We will examine the equation given by (4.1) as subject to the following:

(c_1) The functions g_i: $I \times \mathbb{R} \to \mathbb{R}$, with $g_i(t, \sigma) \ge 0$ are continuous and $\xi_i \in L^1(I)$ are two functions such that $g_i(t, \sigma) \le \xi_i(t)$ for $i = 1, 2$.

(c_2) $g_1(t, \sigma)$ exhibits monotonic non-decreasing behavior in σ, while $g_2(t, \mathfrak{a})$ demonstrates monotonic descending characteristics in $\mathfrak{a}$ for every $\sigma, \mathfrak{a} \in \mathbb{R}$ and $t \in I$.

(c_3) The function δ: $I \to \mathbb{R}$ is continuous.

(c_4) The functions denoted by k_i: $I \times I \to \mathbb{R}$, $i = 1, 2$ exhibit continuity in I at every t for each $q \in I$ and measurability in $q \in I$ for each $t \in I$ under the given conditions

$$\int_0^1 k_i(t, q) \xi_i(q) dq \le K, i = 1, 2 \quad \text{and} \quad k_i(t, \sigma) \ge 0.$$

(c_5) For any $\sigma, \mathfrak{a} \in \mathbb{R}$ with $\sigma \ge \mathfrak{a}$ a function $\gamma \in \Gamma$ and the constants $0 \le L_i < 1, (i = 1, 2)$ exist such that

$$|g_i(t, \sigma) - g_i(t, \mathfrak{a})| \le L_i \gamma(\sigma - \mathfrak{a}), (i = 1, 2).$$

(c_6) There are functions $\mathrm{l}_1, \mathrm{l}_2 \in \mathrm{C(I)}$ with

$$\begin{aligned}\mathrm{l}_1(\mathrm{t}) &\leq \delta(\mathrm{t}) + \alpha \int_0^1 \mathrm{k}_1(\mathrm{t},\mathrm{q})\mathrm{g}_1(\mathrm{q},\mathrm{l}_1(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t},\mathrm{q})\mathrm{g}_2(\mathrm{q},\mathrm{l}_2(\mathrm{q}))d\mathrm{q} \\ &\leq \delta(\mathrm{t}) + \lambda \int_0^1 \mathrm{k}_1(\mathrm{t},\mathrm{q})\mathrm{g}_1(\mathrm{q},\mathrm{l}_2(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t},\mathrm{q})\mathrm{g}_2(\mathrm{q},\mathrm{l}_1(\mathrm{q}))d\mathrm{q} \\ &\leq \mathrm{l}_2(\mathrm{t}).\end{aligned}$$

(c_7) $\max\{\mathrm{L}_1{}^{\mathrm{q}}, \mathrm{L}_2{}^{\mathrm{q}}\}\alpha^{\mathrm{q}}\mathrm{K}^{2\mathrm{q}} \leq \frac{1}{2^{4\mathrm{q}-3}}$.

Let us consider the space $\mathscr{L} = \mathrm{C(I)}$ with $\mathrm{I} = [0,1]$, which denotes the set of continuous maps equipped with a metric.

$$\mathrm{d}(\varpi, \mathfrak{a}) = \sup_{\mathrm{t}\in\mathrm{I}} |\varpi(\mathrm{t}) - \mathfrak{a}(\mathrm{t})| \quad \text{for } \varpi, \mathfrak{a} \in \mathrm{C(I)}.$$

It is evident that the space can be endowed with a partial order defined by

$$\varpi, \mathfrak{a} \in \mathrm{C(I)}, \varpi \leq \mathfrak{a} \iff \varpi(\mathrm{t}) \leq \mathfrak{a}(\mathrm{t}), \ \forall \mathrm{t} \in \mathrm{I}.$$

For $\mathrm{q} \geq 1$, a metric ω defined as below,

$$\omega(\varpi, \mathfrak{a}) = (\mathrm{d}(\varpi, \mathfrak{a})^q) = (\sup_{\mathrm{t}\in\mathrm{I}} |\varpi(\mathrm{t}) - \mathfrak{a}(\mathrm{t})|)^q = \sup_{\mathrm{t}\in\mathrm{I}} |\varpi(\mathrm{t}) - \mathfrak{a}(\mathrm{t})|^q \quad \text{for } \varpi, \mathfrak{a} \in \mathrm{C(I)},$$

forms a complete b-metric space for $\mathrm{s} = 2^{\mathrm{q}-1}$ according to [16].

Additionally, $\mathscr{L} \times \mathscr{L} = \mathrm{C(I)} \times \mathrm{C(I)}$ forms a p.o.s with the specified order relation, as follows

$$(\varpi, \mathfrak{a}), (\rho, \mathfrak{d}) \in \mathscr{L} \times \mathscr{L}, \quad (\varpi, \mathfrak{a}) \leq (\rho, \mathfrak{d}) \iff \varpi \leq \rho \ \text{ and } \ \mathfrak{a} \geq \mathfrak{d}.$$

Moreover, for all $\varpi, \mathfrak{a} \in \mathscr{L}$ and any $\mathrm{t} \in \mathrm{I}$, $max\{\varpi(\mathrm{t}), \mathfrak{a}(\mathrm{t})\}$ serves as both an upper and lower bound for $\varpi, \mathfrak{a}$ in $\mathscr{L}$. Consequently, for any

$$(\varpi, \mathfrak{a}), (\rho, \mathfrak{d}) \in \mathscr{L} \times \mathscr{L}, (max\{\varpi, \rho\}, min\{\mathfrak{a}, \mathfrak{d}\}) \in \mathscr{L} \times \mathscr{L}$$

is comparable to both $(\varpi, \mathfrak{a})$ and $(\rho, \mathfrak{d})$.

Theorem 25. *Given the conditions* (c_1)–(c_7)*, the integral Eq* (4.1) *admits a unique solution within the set* $\mathrm{C(I)}$.

Proof. Define a mapping $\mathscr{U}\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ as follows:

$$\mathscr{U}(\varpi, \mathfrak{a})(\mathrm{t}) = \delta(\mathrm{t}) + \alpha \int_0^1 \mathrm{k}_1(\mathrm{t},\mathrm{q})\mathrm{g}_1(\mathrm{q},\varpi(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t},\mathrm{q})\mathrm{g}_2(\mathrm{q},\mathfrak{a}(\mathrm{q}))d\mathrm{q} \quad \text{for } \mathrm{t} \in \mathrm{I}.$$

Therefore, the well-definiteness of $\mathscr{U}$ is ensured by the given hypotheses. To demonstrate the mixed monotone property of $\mathscr{U}$, consider the case in which $\varpi_1 \leq \varpi_2$ and $\mathrm{t} \in \mathrm{I}$

$$\begin{aligned}\mathscr{U}(\varpi_1, \mathfrak{a})(\mathrm{t}) - \mathscr{U}(\varpi_2, \mathfrak{a})(\mathrm{t}) =&\delta(\mathrm{t}) + \alpha \int_0^1 \mathrm{k}_1(\mathrm{t},\mathrm{q})\mathrm{g}_1(\mathrm{q},\varpi_1(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t},\mathrm{q})\mathrm{g}_2(\mathrm{q},\mathfrak{a}(\mathrm{q}))d\mathrm{q} \\ &- \delta(\mathrm{t}) - \alpha \int_0^1 \mathrm{k}_1(\mathrm{t},\mathrm{q})\mathrm{g}_1(\mathrm{q},\varpi_2(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t},\mathrm{q})\mathrm{g}_2(\mathrm{q},\mathfrak{a}(\mathrm{q}))d\mathrm{q} \\ =&\alpha \int_0^1 \mathrm{k}_1(\mathrm{t},\mathrm{q})[\mathrm{g}_1(\mathrm{q},\varpi_1(\mathrm{q})) - \mathrm{g}_1(\mathrm{q},\varpi_2(\mathrm{q}))]d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t},\mathrm{q})\mathrm{g}_2(\mathrm{q},\mathfrak{a}(\mathrm{q}))ds \\ \leq&0.\end{aligned}$$

Similarly, we establish that

$$\mathscr{U}(\varpi, \mathfrak{a}_1)(\mathrm{t}) \leq \mathscr{U}(\varpi, \mathfrak{a}_2)(\mathrm{t})$$

holds true if $\mathfrak{a}_1 \leq \mathfrak{a}_2$ and $\mathrm{t} \in \mathrm{I}$. Thus, $\mathscr{U}$ possesses the mixed monotone property. Furthermore, for $(\varpi, \mathfrak{a}) \leq (\rho, \mathfrak{d})$, which implies that $\varpi \leq \rho$ and $\mathfrak{a} \geq \mathfrak{d}$, we obtain

$$\begin{aligned}
\mathfrak{d}|\mathscr{U}(\varpi, \mathfrak{a})(\mathrm{t}) - \mathscr{U}(\rho, \mathfrak{d})(\mathrm{t})| \leq & |\alpha \int_0^1 \mathrm{k}_1(\mathrm{t}, \mathrm{q})\mathrm{g}_1(\mathrm{q}, \varpi(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t}, \mathrm{q})[\mathrm{g}_2(\mathrm{q}, \mathfrak{a}(\mathrm{q})) - \mathrm{g}_2(\mathrm{q}, \mathfrak{d}(\mathrm{q}))]d\mathrm{q} \\
& + \alpha \int_0^1 \mathrm{k}_2(\mathrm{t}, \mathrm{q})\mathrm{g}_2(\mathrm{q}, \mathfrak{d}(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_1(\mathrm{t}, \mathrm{q})[\mathrm{g}_1(\mathrm{q}, \varpi(\mathrm{q})) - \mathrm{g}_1(\mathrm{q}, \rho(\mathrm{q}))]d\mathrm{q}| \\
\leq & \alpha \int_0^1 \mathrm{k}_1(\mathrm{t}, \mathrm{q})\mathrm{g}_1(\mathrm{q}, \varpi(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_2(\mathrm{t}, \mathrm{q})|\mathrm{g}_2(\mathrm{q}, \mathfrak{a}(\mathrm{q})) - \mathrm{g}_2(\mathrm{q}, \mathfrak{d}(\mathrm{q}))|d\mathrm{q} \\
& + \alpha \int_0^1 \mathrm{k}_2(\mathrm{t}, \mathrm{q})\mathrm{g}_2(\mathrm{q}, \mathfrak{d}(\mathrm{q}))d\mathrm{q} \int_0^1 \mathrm{k}_1(\mathrm{t}, \mathrm{q})|\mathrm{g}_1(\mathrm{q}, \varpi(\mathrm{q})) - \mathrm{g}_1(\mathrm{q}, \rho(\mathrm{q}))|d\mathrm{q} \\
\leq & \alpha \int_0^1 \mathrm{k}_1(\mathrm{t}, \mathrm{q})\mathrm{l}_1(\mathrm{q})d\mathrm{q} \int_0^1 \Lambda_2(\mathrm{t}, \mathrm{q})L_2\gamma[\mathfrak{a}(\mathrm{q}) - \mathfrak{d}(\mathrm{q})]d\mathrm{q} \\
& + \alpha \int_0^1 \mathrm{k}_2(\mathrm{t}, \mathrm{q})\mathrm{l}_2(\mathrm{q})d\mathrm{q} \int_0^1 \mathrm{k}_1(\mathrm{t}, \mathrm{q})L_1\gamma[\rho(\mathrm{q}) - \varpi(\mathrm{q})]d\mathrm{q}.
\end{aligned}$$

Given that γ is an increasing function and under the conditions $\varpi \leq \rho$, $\mathfrak{a} \geq \mathfrak{d}$, we can conclude that

$$\gamma(\rho(\mathrm{q}) - \varpi(\mathrm{q})) \leq \gamma(\sup_{\mathrm{t}\in\mathrm{I}} |\varpi(\mathrm{q}) - \rho(\mathrm{q})|) = \gamma(d(\varpi, \rho))$$

and

$$\gamma(\mathfrak{a}(\mathrm{q}) - \mathfrak{d}(\mathrm{q})) \leq \gamma(\sup_{\mathrm{t}\in\mathrm{I}} |\mathfrak{a}(\mathrm{q}) - \mathfrak{d}(\mathrm{q})|) = \gamma(d(\mathfrak{a}, \mathfrak{d})).$$

Thus,

$$\begin{aligned}
|\mathscr{U}(\varpi, \mathfrak{a}(\mathrm{t})) - \mathscr{U}(\rho, \mathfrak{d})(\mathrm{t})| &\leq \alpha K \int_0^1 \mathrm{k}_2(\mathrm{t}, \mathrm{q})\mathrm{L}_2\gamma(\mathrm{d}(\mathfrak{a}, \mathfrak{d}))\mathrm{dq} + \alpha \mathrm{K} \int_0^1 \mathrm{k}_1(\mathrm{t}, \mathrm{q})\mathrm{L}_1\gamma(\mathrm{d}(\rho, \varpi))d\mathrm{q} \\
&\leq \alpha \mathrm{K}^2 \max\{\mathrm{L}_1, \mathrm{L}_2\}[\gamma(\mathrm{d}(\rho, \varpi)) + \gamma(\mathrm{d}(\mathfrak{a}, \mathfrak{d}))].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\omega(\mathscr{U}(\varpi, \mathfrak{a}), \mathscr{U}(\rho, \mathfrak{d})) &= \sup_{\mathrm{t}\in\mathrm{I}} |\mathscr{U}(\varpi, \mathfrak{a})(\mathrm{t}) - \mathscr{U}(\rho, \mathfrak{d})(\mathrm{t})|^q \\
&\leq \{\alpha \mathrm{K}^2 \max\{\mathrm{L}_1, \mathrm{L}_2\}[\gamma(d(\rho, \varpi)) + \gamma(d(\mathfrak{a}, \mathfrak{d}))]\}^q \\
&= \alpha^q K^{2q} \max\{\mathrm{L}_1{}^q, \mathrm{L}_2{}^q\}[\gamma(d(\rho, \varpi)) + \gamma(d(\mathfrak{a}, \mathfrak{d}))]^q,
\end{aligned}$$

and for $\mathrm{q} > 1$, we have that $(\mathrm{m} + \mathrm{n})^{\mathrm{q}} \leq 2^{\mathrm{q}-1}(\mathrm{m}^{\mathrm{q}} + \mathrm{n}^{\mathrm{q}})$ for $\mathrm{m}, \mathrm{n} \in (0, \infty)$; then,

$$\begin{aligned}
\omega(\mathscr{U}(\varpi, \mathfrak{a}), \mathscr{U}(\rho, \mathfrak{d})) &\leq 2^{\mathrm{q}-1}\alpha^{\mathrm{q}}\mathrm{K}^{2\mathrm{q}} max\{\mathrm{L}_1{}^{\mathrm{q}}, \mathrm{L}_2{}^{\mathrm{q}}\}[(\gamma(\mathrm{d}(\rho, \varpi)))^{\mathrm{q}} + (\gamma(\mathrm{d}(\mathfrak{a}, \mathfrak{d})))^{\mathrm{q}}] \\
&\leq 2^{\mathrm{q}-1}\alpha^{\mathrm{q}}\mathrm{K}^{2\mathrm{q}} max\{\mathrm{L}_1{}^{\mathrm{q}}, \mathrm{L}_2{}^{\mathrm{q}}\}[\gamma(\mathrm{d}(\rho, \varpi)) + \gamma(\mathrm{d}(\mathfrak{a}, \mathfrak{d}))] \\
&\leq 2^{\mathrm{q}}\alpha^{\mathrm{q}}\mathrm{K}^{2\mathrm{q}} max\{\mathrm{L}_1{}^{\mathrm{q}}, \mathrm{L}_2{}^{\mathrm{q}}\}[\gamma \mathrm{G}_{\mathrm{g}}(\varpi, \mathfrak{a}, \rho, \mathfrak{d})] \\
&\leq 2^{\mathrm{q}}\alpha^{\mathrm{q}}\mathrm{K}^{2\mathrm{q}} max\{\mathrm{L}_1{}^{\mathrm{q}}, \mathrm{L}_2{}^{\mathrm{q}}\}[\mathrm{G}_{\mathrm{g}}(\varpi, \mathfrak{a}, \rho, \mathfrak{d}) - \psi(\mathrm{H}_{\mathrm{g}}(\varpi, \mathfrak{a}, \rho, \mathfrak{d}))] \\
&\leq \frac{1}{2^{3\mathrm{q}-\mathrm{k}}}\mathrm{G}_{\mathrm{g}}(\varpi, \mathfrak{a}, \rho, \mathfrak{d}) - \frac{1}{2^{3\mathrm{q}-\mathrm{k}}}\psi(\mathrm{H}_{\mathrm{g}}(\varpi, \mathfrak{a}, \rho, \mathfrak{d})),
\end{aligned}$$

this implies that the mapping $\mathscr{U}$ fulfills the contraction condition (3.50) specified in Corollary 16.

Lastly, consider the functions $\mathfrak{l}_1, \mathfrak{l}_2$ as indicated in assumption (c_6). By (c_6), we obtain

$$\mathfrak{l}_1 \leq \mathscr{U}(\mathfrak{l}_1, \mathfrak{l}_2) \leq \mathscr{U}(\mathfrak{l}_2, \mathfrak{l}_1) \leq \mathfrak{l}_2.$$

Therefore, based on Theorem 17, the mapping $\mathscr{U}$ possesses a unique coupled fixed point $(\mathfrak{c}^*, \mathfrak{e}^*) \in \mathscr{L} \times \mathscr{L}$. Given $\mathfrak{l}_1 \leq \mathfrak{l}_2$, according to Theorem 18, we conclude that $\mathfrak{c}^* = \mathfrak{e}^*$, suggesting that $\mathfrak{c}^* = \mathscr{U}(\mathfrak{c}^*, \mathfrak{c}^*)$. Consequently, $\mathfrak{c}^* \in \mathrm{C}(\mathrm{I})$ stands as the unique solution to (4.1). □

5. Conclusions

This study has established results related to the fixed point of a mapping that adheres to the conditions of a rational contraction with auxiliary functions, demonstrating its uniqueness. Furthermore, we extend these findings to obtain coupled fixed points for two mappings within an ordered metric space. The results presented here generalize and extend existing comparable findings in the literature. Numerical examples have been provided to illustrate and support the outcomes. Moreover, these results have the potential for further generalization and extension in various ordered metric spaces, given the necessary topological conditions.

Additionally, the aforementioned findings can be extended and generalized by incorporating the idea of *wt*-distance in a metric type space [47], cone *b*-metric spaces over Banach algebras [48] and (ϕ, ψ)-weak contractions as presented in [49].

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article. The authors declare that they have only used the Grammarly free AI tools to improve the manuscript's grammar.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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