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Research article

Solutions and anti-periodic solutions for impulsive differential equations and inclusions containing Atangana-Baleanu fractional derivative of order $\zeta \in (1,2)$ in infinite dimensional Banach spaces

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Abstract: In this paper, we improved recent results on the existence of solutions for nonlinear fractional boundary value problems containing the Atangana-Baleanu fractional derivative of order $\zeta \in (1,2)$. We also derived the exact relations between these fractional boundary value problems and the corresponding fractional integral equations in infinite dimensional Banach spaces. We showed that the continuity assumption on the nonlinear term of these equations is insufficient, give the derived expression for the solution, and present two results about the existence and uniqueness of the solution. We examined the case of impulsive impact and provide some sufficiency conditions for the existence and uniqueness of the solution in these cases. We also demonstrated the existence and uniqueness of anti-periodic solution for the studied problems and considered the problem when the right-hand side was a multivalued function. Examples were given to illustrate the obtained results.

Keywords: AB fractional derivative; fractional differential inclusions; instantaneous impulses; solutions and anti-periodic solutions
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

It has been recognized that the dynamics of complex real-world problems are better described using fractional calculus. Fractional calculus has many applications in engineering [1–7], in environmental, and biological studies [8–12]. As an extension to Newtonian derivatives, researchers have proposed different concepts of fractional derivatives and integrals, each of which generalizes the concept of differentiation and integration of integer order. The best known fractional operators are those of Riemann-Liouville and Caputo. These operators use a singular kernel. The problems arising from the presence of singular kernel were overcome by introducing fractional operators with non singular kernels. Caputo et al. [13] proposed a definition based on the exponential function. Atangana and Baleanu [14] generalized the Caputo fractional operators using kernels based on the Mittag-Leffler function.

Although Atangana and Baleanu's derivative is not the left inverse of the corresponding Atangana and Baleanu's integral (Lemma 2.1 and Remark 2.1 below), there are many applications of Atangana and Baleanu's fractional derivative to differential equations [15–19]. Many researchers obtained results regarding the existence of solutions for fractional differential equations and inclusions involving Atangana and Baleanu derivative in finite dimensional spaces [20–24]. Recently, Al Nuwairan et al. [25] investigated the existence of solutions for non-local impulsive differential equations and inclusions with Atangana and Baleanu derivative of order $\zeta \in (0,1)$ in infinite dimensional spaces.

Impulsive differential equations and impulsive differential inclusions have been an object of interest with wide applications to physics, biology, engineering, medicine, industry, and technology. The impulsive differential equations provide appropriate models for processes that change their state rapidly and cannot be modeled using the ordinary differential equations. An example of such a process is the motion of an elastic ball bouncing vertically on a surface. The moments of the impulses are the times when the ball touches the surface and rapidly changes its velocity. For some applications of impulsive differential equations, we see [26, 27]. Xu et al. [28] studied the exponential stability of stochastic nonlinear delay systems subject to multiple periodic impulses. For further results on the existence of solutions or mild solutions for impulsive differential equations and inclusions, we refer to [29–33].

Kaslik et al. [34] showed that unlike the integer order derivative, the fractional-order derivative of a periodic function cannot be a function with the same period. This implies the non-existence of periodic solutions for a wide class of fractional-order differential systems on bounded intervals. Thus, much attention has been devoted to the study of anti-periodic solutions or S-asymptotically w-periodic solution. Fractional differential equations with anti-periodic conditions have been applied to the study of blood flow, chemical engineering, underground water flow, and population dynamics. The anti-periodic solutions to various fractional differential equations and inclusions are investigated by several authors [35–40] and papers cited therein. Very recently, Abdeljawad et al. [41] proposed a higher-order extension of Atangana–Baleanu fractional operators. For more recent results on fractional differential equations, we refer the reader to [42–44].

Notation 1.1. Throughout this paper, we use the following notation:

- For $b > 0$, let $J = [0, b] \subset \mathbb{R}$. Let m be a natural number, $0 \leq k \leq m, N_k = \{k, k+1, \ldots, m\}, 0 = \iota_0 < \iota_1 \leq < \iota_2 \leq \iota_3 \cdots < \iota_{m+1} = b$ be a partition of J, $J_0 = [0, \iota_1]$, and $J_k = (\iota_k, \iota_{k+1}], k \in N_1$.
- E is a reflexive real Banach space, z_0, z_1 are elements of E.
- $AC(J, E)$ is the Banach space of absolutely continuous functions from J to E.
- $H^1((a,b), E)$ is the Sobolev space $\{z \in L^2((a,b), E) : z' \in L^2((a,b), E)\}$.
- $PC(J, E)$ is the Banach space defined as
  $PC(J, E) = \{z : J \to E, z \in H^1(J_k, E) : z(\iota_k^+)$ and $z(\iota_k^-)$ exist with $z(\iota_k) = z(\iota_k^-)\}, \forall k \in N_1\}$.
  The norm on $PC(J, E)$ is given by $\|z\|_{PC(J,E)} = \sup\{\|z(\iota)\| : \iota \in J\}$.

- $PCH^1(J,E) = \{z \in PC(J,E) : z_{|J_k} \in H^1((\iota_k, \iota_{k+1}), E), \forall k \in N_1\}$.
- $PCH^2(J,E) = \{z \in PC(J,E) : z'_{|J_k} \in H^1((\iota_k, \iota_{k+1}), E), \forall k \in N_1\}$.

  The spaces $PCH^1(J,E)$ and $PCH^2(J,E)$ are Banach spaces endowed with the norms

$$\|z\|_{PCH^t(J,E)} = \max\{\|z_{|J_k}\|_{H^s(J_k,E)} : k \in N_1\}, \ t = 1,2.$$

Recently, it was shown in [20, 21, 23] that the following fractional differential equation:

$$\begin{cases} {}^{ABC}D^{\zeta}_{0,\iota} z(\iota) = w(\iota), \ \iota \in J, \\ z(0) = z_0, z(b) = z_1 \end{cases} \tag{1.1}$$

is equivalent to the fractional integral equation:

$$\begin{aligned} z(\iota) &= z_0 + \frac{\iota(z_1 - z_0)}{b} - \frac{\iota(2-\zeta)}{bM(\zeta-1)} \int_0^b w(s)ds \\ &\quad - \frac{\iota(\zeta-1)}{bM(\zeta-1)\Gamma(\zeta)} \int_0^b (b-s)^{\zeta-1} w(s)ds + \frac{2-\zeta}{M(\zeta-1)} \int_0^\iota w(s)ds \\ &\quad + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota (\iota-s)^{\zeta-1} w(s)ds, \end{aligned} \tag{1.2}$$

where $\zeta \in (1,2)$ and ${}^{ABC}D^{\zeta}_{0,\iota}$ is the Atangana-Baleanu fractional derivative in the Caputo sense of order ζ with lower limit at 0, $w : J \to \mathbb{R}$ is continuous function satisfying $w(0) = 0$ and z_0, z_1 are fixed points. We claim that the assumption of continuity of w is not enough as it does not assure that the function z in Eq (1.2) satisfies $z' \in H^1((0,b))$. Thus, it does not guarantee that z has Atangana-Baleanu fractional derivative of order ζ. Without differentiability, z would not be a solution for Eq (1.1).

In this paper, we provide

(1) A more precise result regarding the relation between the fractional differential equation (1.1) and the fractional integral equation (1.2) in a real Banach space E (Lemma 3.1).

(2) Two results (Theorems 3.1 and 3.2) concerning the existence and uniqueness of solutions for the following boundary value problem containing Atangana-Baleanu fractional derivative

$$\begin{cases} {}^{ABC}D^{\zeta}_{0,\iota} z(\iota) = f(\iota, z(\iota)), \ \iota \in J, 1 < \zeta < 2, \\ z(0) = z_0, z(b) = z_1, \end{cases} \tag{1.3}$$

where $f(0, z(0)) = 0$.

(3) A formula (given in Lemma 4.1) for the relation between the boundary value problem

$$\begin{cases} {}^{ABC}D^{\zeta}_{0,\iota} z(\iota) = w(\iota), \ \iota \in J, \\ z(a) = z_0, z'(a) = z_1 \end{cases}$$

and the integral equation

$$z(\iota) = z_0 + (\iota - a)\left[z_1 - \frac{2-\zeta}{M(\zeta-1)} w(a) - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)} \int_0^a (a-s)^{\zeta-2} w(s)ds\right]$$

$$-\frac{2-\zeta}{M(\zeta-1)}\int_0^a w(s)ds-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^a (a-s)^{\zeta-1}w(s)ds$$
$$+\frac{2-\zeta}{M(\zeta-1)}\int_0^\iota w(s)ds+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1}w(s)ds,\quad \iota\in J.$$

(4) A formula for the solutions to the following impulsive boundary value problem involving Atangana-Baleanu fractional derivative of order $\zeta \in (1,2)$:

$$\begin{cases} {}^{ABC}D_{0,\iota}^{\zeta}u(\iota)=f(\iota,z(\iota)),\iota\in J-\{\iota_1,\iota_2,\ldots,\iota_m\},\\ z(0)=z_0,\ z'(0)=z_1,\\ z(\iota_i^+)=z(\iota_i^-)+I_i(z(\iota_i^-)),\ i\in N_1,\\ z'(\iota_i^+)=z'(\iota_i^-)+\overline{I}_i(z(\iota_i^-)),\ i\in N_1,\end{cases}\tag{1.4}$$

where $f(0,z(0))=0$, and $I_i,\overline{I}_i:E\to E$ are continuous functions (Lemma 4.2). We also establish two results concerning the existence and uniqueness of the solution of (1.4) (Theorems 4.1 and 4.2).

(5) The sufficient conditions for the existence of anti-periodic solution to the following impulsive differential equation involving Atangana-Baleanu fractional derivative of order $\zeta\in(1,2)$

$$\begin{cases} {}^{ABC}D_{0,\iota}^{\zeta}z(\iota)=f(\iota,z(\iota)),\iota\in J-\{\iota_1,\iota_2,\ldots,\iota_m\},\\ z(0)=-z(b),\ z'(0)=-z'(b),\\ z(\iota_i^+)=z(\iota_i^-)+I_i(z(\iota_i^-)),\ i\in N_1,\\ z'(\iota_i^+)=z'(\iota_i^-)+\overline{I}_i(z(\iota_i^-)),\ i\in N_1,\end{cases}\tag{1.5}$$

where $f(0,z(0))=0$, (Theorem 5.1).

(6) The sufficient conditions for the existence of solutions to the impulsive differential inclusion

$$\begin{cases} {}^{ABC}D_{0,\iota}^{\zeta}z(\iota)\in F(\iota,z(\iota)),\iota\in J-\{\iota_1,\iota_2,\ldots,\iota_m\},\\ z(0)=z_0,\ z'(0)=z_1,\\ z(\iota_i^+)=z(\iota_i^-)+I_i(z(\iota_i^-)),\ i\in N_1,\\ z'(\iota_i^+)=z'(\iota_i^-)+\overline{I}_i(z(\iota_i^-)),\ i\in N_1,\end{cases}\tag{1.6}$$

where F is a multi-valued function satisfying $F(0,z_0)=\{0\}$ (Theorem 6.1).

Remark 1.1. Previously, the authors in [29] had investigated problems (1.3) and (1.4) with the Atangana-Belearn derivative replaced with Caputo's. Also in [29, 40] problems (1.5) and (1.6) were studied using Caputo derivative without impulses. Saha et al. [42] established the existence of solutions for problem (1.1) in finite dimensional spaces with the boundary conditions $z(0)=z_0, {}^{ABC}D_{0,\iota}z(b)=z_1$. Indeed, the vast majority of published research on the existence of solutions to differential equations involving Atangana-Baleanu fractional derivative are restricted to finite-dimensional spaces [20–24, 45, 46]. Up to the authors knowledge, there has been no published research on anti-periodic solutions.

The contribution of this paper can be summarized as follows:

(1) In Lemma 3.1, we obtained a precise relationship between the fractional differential Equation (1.1) and the corresponding integral Equation (1.2). We showed in detail that the continuity assumption on the nonlinear term, used earlier, e.g., Theorem 3.6 in [21] and Lemma 2 in [23], is insufficient and should be replaced with the requirement that w lies in the space $H^1((a,b),E)$.

(2) As to our knowledge, Theorem 5.1 showing the existence of an anti-periodic solution for the impulsive fractional differential equation (1.5), with Attange-Baleanu fractional derivative of order $\zeta \in (1,2)$, has not previously appeared in literature.

(3) To our knowledge, there has been no published results on the existence of solutions for impulsive differential equations containing Atangana-Baleanu fractional derivative of order $\zeta \in (1,2)$, or on the existence of anti-periodic solutions for differential equations containing Atangana-Baleanu fractional derivative.

The paper is organized as follows. In the second section, we recall the basic facts and concepts needed for the following sections. In Section 3, we present two existence and uniqueness results for the solution to problem (1.3). Section 4 studies the existence and uniqueness of solutions to problem (1.4), and Section 5 is devoted to showing the existence of solutions to problem (1.5). In Section 6, we prove the existence of solutions for problem (1.6). Three examples are given in the last section to illustrate the obtained results.

2. Preliminaries and notations

Definition 2.1. [14, 19] Let $a < b$ be two real numbers, and $\zeta \in (0,1)$. The Atangana-Baleanu fractional derivative for a function $z \in H^1((a,b),E)$ in the Caputo sense and in the Riemann-Liouville sense of order ζ with lower limit at a are defined by

$$^{ABC}D^{\zeta}_{a,\iota}z(\iota) = \frac{M(\zeta)}{1-\zeta}\int_a^{\iota} z'(x)E_{\zeta}\left(\frac{-\zeta(\iota-x)^{\zeta}}{1-\zeta}\right)dx, \quad \iota \in J,$$

and

$$^{ABR}D^{\zeta}_{a,\iota}z(\iota) = \frac{M(\zeta)}{1-\zeta}\frac{d}{d\iota}\int_a^{\iota} z(x)E_{\zeta}\left(\frac{-\zeta(\iota-x)^{\zeta}}{1-\zeta}\right)dx, \quad \iota \in J,$$

where $M(\zeta) > 0$ is a normalized function satisfying $M(0) = M(1) = 1$, and $E_{\zeta} = E_{\zeta,1}$ is the Mittag-Leffler function given by:

$$E_{\zeta,\beta}(\mu) = \sum_{k=0}^{\infty}\frac{\mu^k}{\Gamma(\zeta k+\beta)}, \ \beta \in \mathbb{R}, \ \mu \in \mathbb{C}.$$

Definition 2.2. [14, 19] Let $a < b$ be two real numbers, and $\zeta \in (0,1)$. The Atangana-Baleanu fractional integral for a function $z \in H^1((a,b),E)$ of order ζ with lower limit at a is given by

$$^{AB}I^{\zeta}_{a,\iota}z(\iota) = \frac{1-\zeta}{M(\zeta)}z(\iota) + \frac{\zeta}{M(\zeta)\Gamma(\zeta)}\int_a^{\iota} z(x)(\iota-x)^{\zeta-1}dx, \ \iota \in J.$$

The following lemma was proved in [14, 19] for $E = \mathbb{R}$. It can be generalized to a Banach space E with little changes in the proof.

Lemma 2.1. Let $z \in H^1((a,b),E)$, $\zeta \in (0,1)$ and $\iota \in J$.

i. ${}^{ABR}D^{\zeta}_{a,\iota}({}^{AB}I^{\zeta}_{a,\iota}z(\iota)) = z(\iota)$ and ${}^{AB}I^{\zeta}_{a,\iota}({}^{ABR}I^{\zeta}_{a,\iota}z(\iota)) = z(\iota)$.

ii. ${}^{ABC}D^{\zeta}_{a,\iota}({}^{AB}I^{\zeta}_{a,\iota}z(\iota)) = z(\iota) - z(a)E_{\zeta}\left(\frac{-\zeta(\iota-a)^{\zeta}}{1-\zeta}\right)$.

iii. ${}^{ABC}I^{\zeta}_{a,\iota}({}^{ABC}I^{\zeta}_{a,\iota}z(\iota)) = z(\iota) - z(a)$.

iv. ${}^{ABR}D^{\zeta}_{a,\iota}z(\iota) = {}^{ABC}D^{\zeta}_{0,\iota}z(\iota) + \frac{M(\zeta)}{1-\zeta}z(a)E_{\zeta}\left(\frac{-\zeta(\iota-a)^{\zeta}}{1-\zeta}\right)$.

v. ${}^{ABR}D^{\zeta}_{a,\iota}c = cE_{\zeta}(\frac{-\zeta}{1-\zeta}(\iota-a)^{\zeta}), {}^{ABC}D^{\zeta}_{a,\iota}c = 0$, for a constant c.

Remark 2.1. Note that the second assertion of Lemma 2.1 implies that ${}^{ABC}D^{\zeta}_{a,\iota}({}^{AB}I^{\zeta}_{a,\iota}z(\iota)) \neq z(\iota)$, unless $z(a) = 0$. Thus, we can not drop the assumption that $f(0,z(0)) = 0$ in problems (1.3)–(1.5) and that $F(0,z_0) = 0$ in problem (1.6).

Definition 2.3. [14, 19] Let $\zeta \in (n, n+1), n \in \mathbb{N}$ and $z : [a,b] \to E$ with $z^{(n)} \in H^1((a,b),E)$. The left Atangana-Baleanu fractional derivative of z, in the Caputo sense and in the Riemann-Liouville sense of order ζ with lower limit at a are defined by

$$\begin{aligned} {}^{ABC}D^{\zeta}_{a,\iota}z(\iota) &= {}^{ABC}D^{\zeta-n}_{a,\iota}z^{(n)}(\iota) \\ &= \frac{M(\zeta-1)}{1-(\zeta-n)}\int_a^{\iota} z^{(n+1)}(x)E_{(\zeta-n)}\left(\frac{-(\zeta-n)(\iota-x)^{(\zeta-n)}}{1-(\zeta-n)}\right)dx, \end{aligned}$$

and

$$\begin{aligned} {}^{ABR}D^{\zeta}_{a,\iota}z(\iota) &= {}^{ABR}D^{\zeta-n}_{a,\iota}z^{(n)}(\iota) \\ &= \frac{M(\zeta-n)}{1-(\zeta-n)}\frac{d}{d\iota}\int_a^{\iota} z^{(n)}(x)E_{(\zeta-n)}\left(\frac{-(\zeta-n)(\iota-x)^{(\zeta-n)}}{1-(\zeta-n)}\right)dx. \end{aligned}$$

Definition 2.4. [14, 19] Let $\zeta \in (n, n+1), n \in \mathbb{N}$ and $z : [a,b] \to E$ with $z^{(n)} \in H^1((a,b),E)$. The left Atangana-Baleanu fractional integral for z, of order ζ with lower limit at a, is defined by

$${}^{AB}I^{\zeta}_{a,\iota}z(\iota) = I^{n}_{a,\iota}\,{}^{AB}I^{\zeta-n}_{a,\iota}z(\iota).$$

As in [20, 22], one can prove the following lemma.

Lemma 2.2. Let $\zeta \in (1,2)$ and $z : J \to E$ with $z' \in H^1((a,b),E)$. For any $\iota \in [a,b]$,

(1) ${}^{ABR}D^{\zeta}_{a,\iota}\left({}^{AB}I^{\zeta}_{a,\iota}z(\iota)\right) = z(\iota)$.

(2) ${}^{ABC}D^{\zeta}_{a,\iota}\left({}^{AB}I^{\zeta}_{a,\iota}z(\iota)\right) = {}^{ABC}D^{\zeta-1}_{a,\iota}\left(\frac{d}{d\iota}\left(I\left({}^{AB}I^{\zeta-1}_{0,\iota}z(\iota)\right)\right)\right) = {}^{ABC}D^{\zeta-1}_{a,\iota}\left({}^{AB}I^{\zeta-1}_{0,\iota}z(\iota)\right)$
$= z(\iota) - z(a)E_{\zeta-1}\left(\frac{-(\zeta-1)(\iota-a)^{\zeta-1}}{2-\zeta}\right)$.

(3) ${}^{AB}I^{\zeta}_{a,\iota}\left({}^{ABC}D^{\zeta}_{a,\iota}z(\iota)\right) = z(\iota) - c_0 - c_1(\iota - a)$.

We end this section by listing some assumptions that are used later.

Assumptions 2.1. Let $f : J \times E \to E$ be a function, we assume the following:

- (A_1) : For any $\delta > 0$ there is $L_\delta > 0$ such that for any $x, y \in E$ with $\|x\| \leq \delta$, $\|y\| \leq \delta$ and any $s, \iota \in J$, we have
$$\|f(\iota, x) - f(s, y)\| \leq |s - \iota| + L_\delta \|x - y\|.$$
- (A_2) : There is $\sigma > 0$ such that for any $x, y \in E$, we have
$$\|f(\iota, x) - f(\iota, y)\| \leq \sigma \|x - y\|, \forall \iota \in J.$$
- (A_3) : For every $i \in N_1$, the functions $I_i, \overline{I_i} : E \to E$ are continuous, compact and there exist positive constants h_i , $\overline{h_i}$ $(i = 1, 2, .., m)$ such that
$$\|I_i(x)\| \leq h_i \|x\|, \forall x \in E. \text{ and } \left\|\overline{I_i}(x)\right\| \leq \overline{h_i} \|x\|, \forall\, x \in E. \tag{2.1}$$
- (A_4) : For every $i \in N_1$, there exists positive constants δ_i , η_i, such that
$$\|I_i(x) - I_i(y)\| \leq \delta_i \|x - y\|, \forall\, x \in E, \tag{2.2}$$
and
$$\left\|\overline{I_i}(x) - \overline{I_i}(y)\right\| \leq \eta_i \|x - y\|, \forall\, x \in E. \tag{2.3}$$

3. Existence solutions of problem (1.3)

In this section, we stat and prove the relationship between the fractional differential Equation (1.1) and the fractional integral Equation (1.2) in a reflexive Banach space E .

Lemma 3.1. Let $\zeta \in (1, 2)$.

(1) If $w : J \to E$ is continuous and $z : J \to E$ is a solution to Eq (1.1), then z satisfies the integral equation (1.2).

(2) If $w \in H^1((0, b), E)$ with $w(0) = 0$ and z satisfies Eq (1.2), then $z' \in H^1((0, b), E)$ and z is a solution to Eq (1.1).

Proof.

(1) By applying ${}^{AB}I^{\zeta}_{0,\iota}$ to both sides of Eq (1.1) and using the definition of ${}^{AB}I^{\zeta}_{0,\iota}$, the third assertion of Lemma 2.2, and Definition (2.4), we obtain that for any $\iota \in [0, b]$

$$\begin{aligned} z(\iota) &= c_0 + \iota c_1 + {}^{AB} I^{\zeta}_{0,\iota} w(\iota) \\ &= c_0 + \iota c_1 + I_{0,\iota}({}^{AB}I^{\zeta-1}_{0,\iota} w(\iota)) \\ &= c_0 + \iota c_1 + I_{0,\iota}[\frac{1 - (\zeta - 1)}{M(\zeta - 1)} w(\iota) + \frac{\zeta - 1}{M(\zeta - 1)} I^{\zeta-1}_{0,\iota} w(\iota)] \\ &= c_0 + \iota c_1 + \int_0^{\iota} \frac{2 - \zeta}{M(\zeta - 1)} w(s) ds + \frac{\zeta - 1}{M(\zeta - 1)} I^{\zeta}_{0,\iota} w(\iota) \\ &= c_0 + \iota c_1 + \int_0^{\iota} \frac{2 - \zeta}{M(\zeta - 1)} w(s) ds + \frac{\zeta - 1}{M(\zeta - 1)\Gamma(\zeta)} \int_0^{\iota} w(s)(\iota - s)^{\zeta-1} ds. \end{aligned} \tag{3.1}$$

From the boundary conditions $z(0) = z_0$ and $z(b) = z_1$, it follows that $c_0 = z_0$ and

$$z_1 = z_0 + bc_1 + \int_0^b \frac{2-\zeta}{M(\zeta-1)} w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)} \int_0^b w(s)(b-s)^{\zeta-1} ds,$$

i.e.,

$$c_1 = \frac{z_1}{b} \quad - \quad \frac{z_0}{b} - \int_0^b \frac{2-\zeta}{bM(\zeta-1)} w(s)ds - \frac{\zeta-1}{bM(\zeta-1)\Gamma(\zeta)} \int_0^b w(s)(b-s)^{\zeta-1} ds. \qquad (3.2)$$

Substituting the values of c_0 and c_1 into (3.1), we obtain

$$\begin{aligned} z(\iota) \;\; = \;\; & \frac{\iota z_1 + z_0(b-\iota)}{b} - \frac{\iota(2-\zeta)}{bM(\zeta-1)} \int_0^b w(s)ds - \frac{\iota(\zeta-1)}{bM(\zeta-1)\Gamma(\zeta)} \int_0^b w(s)(b-s)^{\zeta-1} ds. \\ & + \frac{2-\zeta}{M(\zeta-1)} \int_0^\iota w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota w(s)(\iota-s)^{\zeta-1} ds. \end{aligned}$$

(2) Assume that $w \in H^1((0,b),E)$ with $w(0) = 0$, and that Eq (1.2) holds. Clearly $z(0) = z_0$ and $z(b) = z_1$. Moreover,

$$z(\iota) = c_0 + \iota c_1 + \frac{2-\zeta}{M(\zeta-1)} \int_0^\iota w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota w(s)(\iota-s)^{\zeta-1} ds, \qquad (3.3)$$

where $c_0 = z_0$ and c_1 is given by (3.2). Since $\zeta > 1$, Eq (3.3) gives us that

$$z'(\iota) = c_1 + \frac{2-\zeta}{M(\zeta-1)} w(\iota) + \frac{\zeta-1}{M(\zeta-1)} I_{0,\iota}^{\zeta-1} w(\iota), \text{ for } \; a.e. \; \iota \in J, \qquad (3.4)$$

where $I_{0,\iota}^{\zeta-1}$ is the Riemann-Liouville fractional integral of order $\zeta - 1$. Since $w \in H^1(J,E)$, $\zeta - 1 \in (0,1)$, and E is reflexive, w has a Bochner integrable derivative w' almost everywhere, and

$$w(s) = w(0) + \int_0^s w'(x)dx, \quad \forall s \in w.$$

This implies that

$$I_{0,\iota}^{\zeta-1} w(\iota) = \frac{1}{\Gamma(\zeta-1)} \int_0^\iota (\iota-s)^{\zeta-2} w(s)ds = \frac{1}{\Gamma(\zeta-1)} \int_0^\iota (\iota-s)^{\zeta-2} \left[\int_0^s w'(x)dx \right] ds,$$

i.e., $I_{0,\iota}^{\zeta-1} w(\iota)$ is the primitive of a Bochner integrable function, hence is absolutely continuous. Thus Eq (3.4) is valid for every $\iota \in J$. Moreover,

$$z^{(2)}(\iota) = \frac{2-\zeta}{M(\zeta-1)} w'(\iota) + \frac{\zeta-1}{M(\zeta-1)} \frac{d}{d\iota} (I^{\,\zeta-1} w(\iota)),$$

giving us that $z' \in H^1((0,b),E)$. Equation (3.2) implies

$$z(\iota) = c_0 + \iota c_1 + {}^{AB} I_{0,\iota}^{\zeta} w(\iota), \; \iota \in J.$$

Finally, by the second assertion of Lemma 2.2,
${}^{ABC}D_{0,\iota}^{\zeta} z(\iota) = {}^{ABC} D_{0,\iota}^{\zeta} \, {}^{AB}I_{0,\iota}^{\zeta} w(\iota) = w(\iota) - w(0) E_{\zeta-1}(\frac{-(\zeta-1)\iota^{\zeta-1}}{2-\zeta}) = w(\iota), \; \iota \in J.$

□

Remark 3.1. Note that

(1) The first assertion of Lemma 3.1 has been proved in Lemma 2 in [23] for the case where $E = \mathbb{R}$.

(2) The solution formula of problem (1.1) does not follow from the first assertion of Lemma 3.1, nor from Lemma 2 in [23].

(3) The assumption $w(0) = 0$ cannot be omitted in the second assertion of Lemma 3.1 since

$$^{ABC}D^{\zeta}_{0,\iota}\, {}^{AB}I^{\zeta}_{0,\iota} w(\iota) = w(\iota) - w(0)E_{\zeta-1}\left(\frac{-(\zeta-1)\iota^{\zeta-1}}{2-\zeta}\right) \neq w(\iota).$$

(4) If w is continuous and not in $H^1((0,b),E)$, then Eq (3.4) does not imply the existence of $z^{(2)}$. Therefore, without the assumption $w \in H^1((0,b),E)$, there is no guarantee that ${}^{ABC}D^{\zeta}_{0,\iota}z(\iota)$ exists.

(5) Lemma 3.1 gives a more accurate statement of Lemma 2 in [23] and generalizes it to the infinite dimensional case.

The results in Lemma 3.1 can be summarized as follows.

Lemma 3.2. Let $w \in H^1((0,b),E)$ with $w(0) = 0$. A function $z : J \to E$ is a solution of problem (1.1) if and only if

$$\begin{aligned} z(\iota) &= z_0 + \frac{\iota(z_1 - z_0)}{b} - \frac{\iota(2-\zeta)}{bM(\zeta-1)}\int_0^b w(s)ds - \frac{\iota(\zeta-1)}{bM(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}w(s)ds \\ &\quad + \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}w(s)ds. \end{aligned} \tag{3.5}$$

Theorem 3.1. Let $f : J \times E \to E$ be a function. If (A_1) holds, then problem (1.3) has a unique solution provided that $f(0, z_0) = 0$ and there is $r > 0$ such that

$$\|z_0\| + \|z_1\| + 2\left(b + rL_r + \|f(0,0)\|\right)\left[\frac{b(2-\zeta)}{M(\zeta-1)} + \frac{(\zeta-1)b^{\zeta}}{M(\zeta-1)\Gamma(\zeta)}\right] < r. \tag{3.6}$$

Proof. Define $T : C(J,E) \to C(J,E)$ by

$$\begin{aligned} T(z)(\iota) &= z_0 + \frac{\iota(z_1 - z_0)}{b} - \frac{\iota(2-\zeta)}{bM(\zeta-1)}\int_0^b f(s,z(s))ds \\ &\quad - \frac{\iota(\zeta-1)}{bM(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}f(s,z(s))ds + \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} f(s,z(s))ds \\ &\quad + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}f(s,z(s))ds. \end{aligned} \tag{3.7}$$

Using the Schauder fixed point theorem, we will show that T has a unique fixed point. Set $B_0 = \{z \in C(J,E) : \|z\|_{C(J,E)} \leq r\}$.

- Step 1: $T(B_0) \subseteq B_0$. Let $z \in B_0$. It follows that from (A_1)

$$\|f(\iota, z(\iota))\| \leq \|f(\iota, z(\iota)) - f(0,0)\| + \|f(0,0)\|$$

$$\begin{aligned} &\leq \; b + L_r\|z(\iota)\| + \|f(0,0)\| \\ &\leq \; b + rL_r + \|f(0,0)\|, \; \forall\, \iota \in J. \end{aligned} \tag{3.8}$$

From (3.6)–(3.8), one has

$$\begin{aligned} \|T(z)(\iota)\| \;&\leq\; \|z_0(1-\frac{\iota}{b}) + \frac{\iota}{b}z_1\| \\ &\quad +2\left(b + rL_r + \|f(0,0)\|\right)\left[\frac{b(2-\zeta)}{M(\zeta-1)} + \frac{(\zeta-1)b^{\zeta}}{M(\zeta-1)\Gamma(\zeta)}\right] \\ &\leq\; \|z_0\| + \|z_1\| \\ &\quad +2\left(b + rL_r + \|f(0,0)\|\right)\left[\frac{b(2-\zeta)}{M(\zeta-1)} + \frac{(\zeta-1)b^{\zeta}}{M(\zeta-1)\Gamma(\zeta)}\right] \\ &<\; r, \end{aligned}$$

showing that $T(B_0) \subseteq B_0$.

- Step 2: $T(B_0)$ is equicontinuous. Let $z \in B_0$ and $\iota, \iota + \lambda \in J$. Using (3.7), we obtain

$$\begin{aligned} \|T(z)(\iota+\lambda) - T(z)(\iota)\| \;&\leq\; \left\|\frac{\lambda(z_1 - z_0)}{b}\right\| + \frac{\lambda(2-\zeta)\left(b + rL_r + \|f(0,0)\|\right)}{M(\zeta-1)} \\ &\quad + \frac{\lambda(\zeta-1)(b + rL_r + \|f(0,0)\|)b^{\zeta}}{\zeta M(\zeta-1)\Gamma(\zeta)} + \frac{\lambda(2-\zeta)(b + rL_r + \|f(0,0)\|)}{M(\zeta-1)} \\ &\quad + \frac{(\zeta-1)(b + rL_r + \|f(0,0)\|)}{M(\zeta-1)\Gamma(\zeta)}\left[\int_0^{\iota+\lambda} (\iota+\lambda-s)^{\zeta-1} - (\iota-s)^{\zeta-1})ds\right]. \end{aligned}$$

Since $\zeta - 1 > 0, \|T(z)(\iota+\lambda) - T(z)(\iota)\| \to 0$ when $\lambda \to 0$, independently of z, proving the assertion.

- Step 3: For $n \geq 1$, let $B_n = \overline{conv}T(B_{n-1})$, and $B = \cap_{n\geq 0}B_n$.

  Let B, B_n be as defined above, then the set B is a non empty compact subset of $C(J,E)$. It follows from Step 1, that $B_n \subseteq B_{n-1}$, $n \geq 1$. By Cantor intersection property [47], it is enough to show that

$$\lim_{n\to\infty} \chi_{C(J,E)}(B_n) = 0, \tag{3.9}$$

  where $\chi_{C(J,E)}$ is the Hausdorff measure of noncompactness on $C(J,E)$ [48].

  Let $n \geq 1$ be a fixed natural number and $\varepsilon > 0$. By Lemma 3 in [49], there exists a sequence (z_k), $k \geq 1$ in B_{n-1} such that

$$\chi_{C(J,E)}(B_n) = \chi_{C(J,E)}T(B_{n-1}) \leq 2\chi_{C(J,E)}\{T(z_k) : k \geq 1\} + \varepsilon. \tag{3.10}$$

  Since B_n is equicontinuous, inequality (3.10) becomes

$$\chi_{C(J,E)}(B_n) \leq 2\max_{\iota\in J}\chi_E\{T(z_k)(\iota) : k \geq 1\} + \varepsilon. \tag{3.11}$$

Let $\iota \in J$ be fixed. In view of (3.8)

$$\|f(\iota, z_m(\iota)) - f(\iota, z_n(\iota))\| \leq L_r \|z_m(\iota) - z_n(\iota)\|, \forall n, m \in \mathbb{N}.$$

It follows that

$$\chi_E\{f(\iota, z_k(\iota)) : k \geq 1\} \leq L_r \chi_E\{z_k(\iota) : k \geq 1\}. \tag{3.12}$$

We also have that

$$\begin{aligned} T(z_k)(\iota) &= z_0 + \frac{\iota(z_1 - z_0)}{b} - \frac{\iota(2-\zeta)}{bM(\zeta-1)}\int_0^b f(s, z_k(s))ds \\ &-\frac{(\zeta-1)}{bM(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1} f(s, z_k(s))ds + \frac{2-\zeta}{M(\zeta-1)}\int_0^\iota f(s, z_k(s))ds \\ &+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1} f(s, z_k(s))ds. \end{aligned} \tag{3.13}$$

Since $\zeta > 1$, Eqs (3.12) and (3.13) give

$$\begin{aligned} \chi\{T(z_k)(\iota) : k \geq 1\} &\leq \int_0^b \chi_E\{z_k(s) : k \geq 1\}ds\left[\frac{2(2-\zeta)L_r}{M(\zeta-1)} + \frac{2b^{\zeta-1}(\zeta-1)L_r}{M(\zeta-1)\Gamma(\zeta)}\right] \\ &\leq b\left[\frac{2(2-\zeta)L_r}{M(\zeta-1)} + \frac{2b^{\zeta-1}(\zeta-1)L_r}{M(\zeta-1)\Gamma(\zeta)}\right]\chi_{C(J,E)}(B_{n-1}). \end{aligned} \tag{3.14}$$

Using (3.11) and (3.14), we obtain that

$$\chi_{C(J,E)}(B_n) \leq 4bL_r\left[\frac{2-\zeta}{M(\zeta-1)} + \frac{b^{\zeta-1}(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)}\right]\chi_{C(J,E)}(B_{n-1}), \forall n \in \mathbb{N}.$$

This inequality yields that

$$\chi_{C(J,E)}(B_n) \leq \chi_{C(T,E)}(B_0) 4bL_r\left[\frac{2-\zeta}{M(\zeta-1)} + \frac{b^{\zeta-1}(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)}\right]^{n-1}. \tag{3.15}$$

The inequality in (3.6) implies that $4bL_r\left[\frac{(2-\zeta)}{M(\zeta-1)} + \frac{b^{\zeta}(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)}\right] < 1$, and thus, (3.15) implies (3.9).

- Step 4: The function $T_{|B} : B \to B$ is continuous. Assume that $z_n \to z$ in B. Note that for $n \geq 1$ and $\iota \in J$, we have

$$\begin{aligned} T(z_n)(\iota) &= z_0 + \frac{\iota(z_1 - z_0)}{b} - \frac{\iota(2-\zeta)}{bM(\zeta-1)}\int_0^b f(s, z_n(s))ds, \\ &-\frac{\iota(\zeta-1)}{bM(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1} f(s, z_n(s))ds + \frac{2-\zeta}{M(\zeta-1)}\int_0^\iota f(s, z_n(s))ds \\ &+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1} f(s, z_n(s))ds. \end{aligned}$$

Using $(A_1), \zeta > 1$, the inequality (3.8), and the Lebesgue dominated convergence theorem, we obtain that $T(z_n) \to T(z)$.

It follows from Steps (1) to (4) and Schauder's fixed point theorem that there is $z \in B$ such that $z = T(z)$. That is,

$$\begin{aligned} z(\iota) &= z_0 + \frac{\iota(z_1 - z_0)}{b} - \frac{\iota(2-\zeta)}{bM(\zeta-1)} \int_0^b w(s)ds \\ &\quad - \frac{\iota(\zeta-1)}{bM(\zeta-1)\Gamma(\zeta)} \int_0^b (b-s)^{\zeta-1} w(s)Bs + \frac{2-\zeta}{M(\zeta-1)} \int_0^\iota w(s)ds \\ &\quad + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota (\iota-s)^{\zeta-1} w(s)ds, \end{aligned}$$

where $w(\iota) = f(\iota, z(\iota)),\ \iota \in J.$

Next, we show that this function z is a solution for problem (1.3). By Lemma 3.1, it is sufficient to show that $w \in H^1(J, E)$. Since $\zeta > 1$, then

$$z'(t) = \frac{2-\zeta}{M(\zeta-1)} w(t) + \frac{\zeta-1}{M(\zeta-1)} I_{0,t}^{\zeta-1} w(t), t \in J.$$

From (A_1), w is absolutely continuous, and since E is reflexive, the function $t \to I_{0,t}^{\zeta-1} w(t)$ is absolutely continuous. Hence $w \in H^1(J, E)$.

To show the uniqueness of the solution, let $z, v \in C(J, E)$ be two solutions for problem (1.3) and $\iota \in J$. Since z, v are solutions, it follows from (A_1) that

$$\begin{aligned} \|T(z)(\iota) - T(v)(\iota)\| &\leq \frac{(2-\zeta)}{M(\zeta-1)} \int_0^b \|f(s, z(s)) - f(s, v(s)))\| \\ &\quad + \frac{b^{\zeta-1}(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \int_0^b \|f(s, z(s)) - f(s, v(s))\|ds \\ &\quad + \frac{2-\zeta}{M(\zeta-1)} \int_0^\iota \|f(s, z(s)) - f(s, v(s))\|ds \\ &\quad + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota (\iota-s)^{\zeta-1} \|f(s, z(s)) - f(s, v(s))\|ds \\ &\leq \frac{L_r(2-\zeta)}{M(\zeta-1)} \int_0^b \|z(s)) - v(s)\|ds + \frac{L_r b^{\zeta-1}(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \int_0^b \|z(s)) - v(s)\|ds \\ &\quad + \frac{L_r(2-\zeta)}{M(\zeta-1)} \int_0^\iota \|z(s)) - v(s)\|ds + \frac{L_r(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota (\iota-s)^{\zeta-1} \|z(s)) - v(s)\|ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|z(\iota) - v(\iota)\| &\leq \frac{L_r(2-\zeta)b}{M(\zeta-1)} \|z-v\| + \frac{L_r b^\zeta(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \|z-v\| + \frac{L_r(2-\zeta)b}{M(\zeta-1)} \|z-v\| + \frac{L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} \|z-v\| \\ &\leq \|z-v\| \left[\frac{2L_r(2-\zeta)b}{M(\zeta-1)} + \frac{2L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} \right]. \end{aligned}$$

Since ι is arbitrary, it follows that

$$\|z-v\|_{C(J,E)} \leq \|z-v\|_{C(J,E)} \left[\frac{2L_r(2-\zeta)b}{M(\zeta-1)} + \frac{2L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} \right].$$

Inequality (3.6) implies $\frac{2L_r(2-\zeta)b}{M(\zeta-1)} + \frac{2L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} < 1$, consequently, $\|z-v\|_{C(J,E)} = 0$, and $z = v$. □

In the following, another existence and uniqueness result for solutions of problem (1.3) is obtained. Replacing the assumption (A_1) by (A_2) simplifies the inequality (3.6) enabling us to use the Banach fixed point theorem for contraction mappings instead of the Schauder fixed point.

Theorem 3.2. Let $f : J\times E \to E$. If (A_2) is satisfied, then problem (1.3) has a unique solution provided that $f(0, z_0) = 0$ and

$$\frac{2b\sigma(2-\zeta)}{M(\zeta-1)} + \frac{2b^{\zeta}(\zeta-1)\sigma}{M(\zeta-1)\Gamma(\zeta)} < 1. \tag{3.16}$$

Proof. Consider the function $T : C(J, E) \to C(J, E)$ defined by (3.7). Let $z, v \in C(J, E)$. For any $\iota \in J$,

$$\begin{aligned}\|T(z)(\iota) - T(v)(\iota)\| \le\ & \frac{(2-\zeta)}{M(\zeta-1)} \int_0^b \|f(s, z(s)) - f(s, z(s))\|ds \\ & + \frac{b^{\zeta-1}(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \int_0^b \|f(s, z(s)) - f(s, z(s))\|ds \\ & + \frac{2-\zeta}{M(\zeta-1)} \int_0^\iota \|f(s, z(s)) - f(s, z(s))\|ds \\ & + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota (\iota-s)^{\zeta-1}\|f(s, z(s)) - f(s, z(s))\|ds.\end{aligned}$$

Since $\zeta > 1$, this inequality together with $(A2)$ imply that

$$\begin{aligned}\|T(z)(\iota) - T(v)(\iota)\| &\le \frac{2\sigma(2-\zeta)}{M(\zeta-1)} \int_0^b \|z(s) - v(s))\|ds + \frac{2b^{\zeta-1}(\zeta-1)\sigma}{M(\zeta-1)\Gamma(\zeta)} \int_0^b \|z(s) - v(s))\|ds \\ &\le \left[\frac{2b\sigma(2-\zeta)}{M(\zeta-1)} + \frac{2b^{\zeta}(\zeta-1)\sigma}{M(\zeta-1)\Gamma(\zeta)}\right]\|z - v\|ds.\end{aligned}$$

Thus,

$$\|T(z) - T(v)\| \le \left[\frac{2b\sigma(2-\zeta)}{M(\zeta-1)} + \frac{2b^{\zeta}(\zeta-1)\sigma}{M(\zeta-1)\Gamma(\zeta)}\right]\|z - v\|.$$

Using (3.16), we obtain that T is contraction, and hence has a unique fixed point. □

4. Existence of solutions for problem (1.4)

The following lemmas will be used for deriving an existence result for solutions of problem (1.4).

Lemma 4.1.

(1) If $w : J \to E$ is continuous , $a \in [0, b), z : J \to E$ be such that $z' \in H^1((0, b), E)$ and

$$\begin{cases} {}^{ABC}D^{\zeta}_{0,\iota} z(\iota) = w(\iota),\ \iota \in J, \\ z(a) = z_0,\ z'(a) = z_1, \end{cases} \tag{4.1}$$

then for any $\iota \in J$,

$$z(\iota) = z_0 + (\iota - a)\left[z_1 - \frac{2-\zeta}{M(\zeta-1)}w(a) - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)} \int_0^a (a-s)^{\zeta-2}w(s)ds\right]$$

$$-\frac{2-\zeta}{M(\zeta-1)}\int_0^a w(s)ds-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^a (a-s)^{\zeta-1}w(s)ds$$
$$+\frac{2-\zeta}{M(\zeta-1)}\int_0^\iota w(s)ds+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1}w(s)ds. \quad (4.2)$$

(2) If $a\in[0,b), w:J\to E$ be continuous with $w(0)=0$ and $z:J\to E$ are such that (4.2) holds, then $z'\in H^1((0,b),E)$ and z is a solution for (4.1).

Proof.

(1) Apply ${}^{AB}I^{\zeta}_{0,\iota}$ on both side of the equation ${}^{ABC}D^{\zeta}_{0,\iota}z(\iota)=w(\iota);\iota\in[0,b]$. As in the proof of first assertion of Lemma 3.1, we obtain for any $\iota\in[0,b]$

$$z(\iota)=c_0+\iota c_1+\frac{2-\zeta}{M(\zeta-1)}\int_0^\iota w(s)ds+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1}w(s)ds.$$

Using the boundary conditions $z(a)=z_0, z'(a)=z_1$, we obtain

$$c_0=z_0-ac_1-\frac{2-\zeta}{M(\zeta-1)}\int_0^a w(s)ds-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^a (a-s)^{\zeta-1}w(s)ds, \quad (4.3)$$

and

$$z_1=c_1+\frac{2-\zeta}{M(\zeta-1)}w(a)+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^a (a-s)^{\zeta-2}w(s)ds.$$

This gives that

$$c_1=z_1-\frac{2-\zeta}{M(\zeta-1)}w(a)-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^a (a-s)^{\zeta-2}w(s)ds. \quad (4.4)$$

From (4.3) and (4.4), we obtain

$$\begin{aligned} c_0 &= z_0-a[z_1-\frac{2-\zeta}{M(\zeta-1)}w(a)-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^a (a-s)^{\zeta-2}w(s)ds] \\ &\quad -\frac{2-\zeta}{M(\zeta-1)}\int_0^a w(s)ds-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^a (a-s)^{\zeta-1}w(s)ds. \end{aligned} \quad (4.5)$$

Substituting the values of c_0 and $c_1$into $z(\iota)$, we obtain

$$\begin{aligned} z(\iota) &= z_0-a[z_1-\frac{2-\zeta}{M(\zeta-1)}w(a)-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^a (a-s)^{\zeta-2}w(s)ds] \\ &\quad -\frac{2-\zeta}{M(\zeta-1)}\int_0^a w(s)ds-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^a (a-s)^{\zeta-1}w(s)ds \\ &\quad +\iota[z_1-\frac{2-\zeta}{M(\zeta-1)}w(a)-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^a (a-s)^{\zeta-2}w(s)ds] \\ &\quad +\frac{2-\zeta}{M(\zeta-1)}\int_0^\iota w(s)ds+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1}w(s)ds \end{aligned}$$

$$= z_0 + (\iota - a)[z_1 - \frac{2-\zeta}{M(\zeta-1)}w(a) - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^a (a-s)^{\zeta-2}w(s)ds]$$
$$-\frac{2-\zeta}{M(\zeta-1)}\int_0^a w(s)ds - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^a (a-s)^{\zeta-1}w(s)ds$$
$$+\frac{2-\zeta}{M(\zeta-1)}\int_0^\iota w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1}w(s)ds.$$

Hence, Eq (4.2) is verified.

(2) Suppose that $w : J \to E$ be continuous function with $w(a) = 0$ and Eq (4.2) holds. Clearly $z(a) = z_0$ and $z'(a) = z_1$. As in the proof of second assertion of Lemma 3.1, we can show that $z' \in H^1((0,b),E)$. For any $\iota \in [a,b]$, we have

$$\begin{aligned}
{}^{ABC}D_{0,\iota}^{\zeta}z(\iota) &= {}^{ABC}D_{0,\iota}^{\zeta-1}z'(\iota)\\
&= {}^{ABC}D_{0,\iota}^{\zeta-1}(z_1 - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^a (a-s)^{\zeta-2}w(s)ds\\
&\quad + \frac{2-\zeta}{M(\zeta-1)}w(\iota) + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^\iota (\iota-s)^{\zeta-2}w(s)ds)(\iota)\\
&= {}^{ABC}D_{0,\iota}^{\zeta-1}\left(\frac{2-\zeta}{M(\zeta-1)}w(\iota) + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^\iota (\iota-s)^{\zeta-2}w(s)ds\right)(\iota)\\
&= {}^{ABC}D_{0,\iota}^{\zeta-1}\left({}^{AB}I_{0,\iota}^{1-\zeta}w(\iota)\right) = w(\iota) - w(0)E_{1-\zeta}\left(\frac{-(1-\zeta)\iota^{(1-\zeta)}}{2-\zeta}\right)\\
&= w(\iota).
\end{aligned}$$

□

Remark 4.1. Following the same method, used in the above proof, a generalization of Theorem 4 in [21] can be derived for any Banach space.

Lemma 4.2. If $w \in PCH^1(J,E)$ with $w(0) = 0$ and $z : J \to E$ be a function satisfying

$$\begin{aligned}
z(\iota) &= z_0 + \iota z_1 + \sum_{i=1}^{k} I_i(z(\iota_i^-)) + \sum_{i=1}^{k}(\iota - \iota_i)\overline{I}_i(z(\iota_i^-)) + \frac{2-\zeta}{M(\zeta-1)}\int_0^\iota w(s)ds\\
&\quad + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1}w(s)ds, \text{ where } \iota \in J_k, k \in N_0, \qquad (4.6)
\end{aligned}$$

then $z \in PCH^2(J,\ E)$ and satisfies the impulsive fractional differential equation:

$$\begin{cases}
{}^{ABC}D_{0,\iota}^{\zeta}z(\iota) = w(\iota),\ \iota \in J - \{\iota_1, \iota_2, \dots, \iota_m\},\\
z(0) = z_0,\ z'(0) = z_1,\\
z(\iota_i^+) = z(\iota_i^-) + I_i(z(\iota_i)), \quad i \in N_1,\\
z'(\iota_i^+) = z'(\iota_i^-) + \overline{I}_i(z(\iota_i)), \quad i \in N_1.
\end{cases} \qquad (4.7)$$

Note that for $k = 0$, in Eq (4.6), the sum $\sum_{i=1}^{k}$ is an empty sum and conventionally, equals zero.

Proof. For any $\iota \in J_0$,

$$z(\iota) = z_0 + \iota z_1 + \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota} w(s)(\iota-s)^{\zeta-1}ds. \tag{4.8}$$

Clearly, $z(0) = z_0$, $z'(0) = z_1$. Since $w \in PCH^1(J, E)$ and $w(0) = 0$, it follows by the second statement of Lemma 4.1, that z is a solution for the fractional differential equation:

$$\begin{cases} {}^{ABC}D_{0,\iota}^{\zeta} z(\iota) = w(\iota),\ \iota \in J_0, \\ z(0) = z_0,\ z'(0) = z_1. \end{cases}$$

Let us define a function v on $J_1 = (\iota_1, \iota_2]$ by:

$$\begin{aligned} v(\iota) &= z(\iota_1^-) + I_1(z(\iota_1^-)) \\ &\quad + (\iota-\iota_1)\left[z'(\iota_1^-) + \overline{I}_1(z(\iota_1)) - \frac{2-\zeta}{M(\zeta-1)}w(\iota_1) - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^{\iota_1}(\iota_1-s)^{\zeta-2}w(s)ds\right] \\ &\quad - \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota_1} w(s)ds - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota_1}(\iota_1-s)^{\zeta-1}w(s)ds \\ &\quad + \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}w(s)ds. \end{aligned} \tag{4.9}$$

From the second assertion of Lemma 4.1, v is a solution for the fractional differential equation:

$$\begin{cases} {}^{ABC}D_{0,\iota}^{\zeta} z(\iota) = w(\iota),\ \iota \in J_1, \\ z(\iota_1^+) = z(\iota_1^-) + I_1(z(\iota_1^-)), \\ z'(\iota_1^+) = z^{(1)}(\iota_1^-) + \overline{I}_1(z(\iota_1^-)). \end{cases} \tag{4.10}$$

Let $\iota \in J_1$. We show that, $v(\iota) = z(\iota)$. From Eq (4.8), it follows that

$$z(\iota_1^-) = z_0 + \iota_1 z_1 + \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota_1} w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota_1}(\iota_1-s)^{\zeta-1}w(s)ds$$

and

$$z'(\iota_1^-) = z_1 + \frac{2-\zeta}{M(\zeta-1)}w(\iota_1) + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^{\iota_1}(\iota_1-s)^{\zeta-2}w(s)ds.$$

By substituting the values of $z(\iota_1^-)$ and $z'(\iota_1^-)$ into Eq (4.9), we obtain

$$\begin{aligned} v(\iota) &= z_0 + \iota_1 z_1 + \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota_1} w(s)ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota_1} w(s)(\iota_1-s)^{\zeta-1}ds \\ &\quad + I_1(z(\iota_1^-)) + (\iota-\iota_1)[z_1 + \frac{2-\zeta}{M(\zeta-1)}w(\iota_1) \\ &\quad + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^{\iota_1}(\iota_1-s)^{\zeta-2}w(s)ds + \overline{I}_1(z(\iota_1^-)) - \frac{2-\zeta}{M(\zeta-1)}w(\iota_1) \\ &\quad - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)}\int_0^{\iota_1}(a-s)^{\zeta-2}w(s)ds] \end{aligned}$$

$$
\begin{aligned}
&-\frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota_1} w(s)ds-\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota_1}(a-s)^{\zeta-1}w(s)ds\\
&+\frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} w(s)ds+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}w(s)ds\\
=\;& z_0+\iota z_1+I_1(z(\iota_1^-))+(\iota-\iota_1)\overline{I_1}(z(\iota_1^-))+\frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} w(s)ds\\
&+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}w(s)ds. \qquad (4.11)
\end{aligned}
$$

Therefore, $v(\iota)=z(\iota), \forall\iota\in J_1$. Since

$$
z(\iota_1^+)-z(\iota_1^-)=I_1(z(\iota_1^-)),\ \text{ and }\ z'(\iota_1^+)-z'(\iota_1^-)=\overline{I_1}(z(\iota_1^-)),
$$

then z is a solution for the fractional differential equation (4.10). By repeating the above steps for $J_k; k\in N_2$, the proof follows. □

Definition 4.1. A function $z\in PCH^2(J,E)$ is said to be a solution for problem (1.4) if it has left Atangana-Baleanu fractional derivative of order ζ on each $J_k,\ k\in N_1$ and satisfies Eq (4.6).

In the following theorem, we provide an existence result for problem (1.4).

Theorem 4.1. Let $f: J\times E\to E$ with $f(0,z_0)=0$ and $I_i,\overline{I_i}: E\to E$ $(i\in N_1)$ be functions. If both Assumptions (A_1) and (A_3) are satisfied, then problem (1.4) has a unique solution provided that $f(0,z_0)=0$, and there is $r>0$ such that

$$
\|z_0\|+b\left(\|z_1\|+rhm(1+b)+(b+rL_r\ +\|f(0,0)\|\right)\left[\frac{(2-\zeta)b}{M(\zeta-1)}+\frac{(\zeta-1)b^{\zeta}}{M(\zeta-1)\Gamma(\zeta+1)}\right]<r, \qquad (4.12)
$$

where $h=\max\{\sum_{i=1}^m h_i\,,\sum_{i=1}^m \overline{h_i}\}$.

Proof. Using Schauder's fixed point theorem, we show that the function $R: PC(J,E)\to PC(J,E)$ given by

$$
\begin{aligned}
R(z)(\iota)\;=\;& z_0+\iota z_1+\sum_{i=1}^{k} I_i(z(\iota_i^-))+\sum_{i=1}^{k}(\iota-\iota_i)\overline{I_i}(z(\iota_i^-))+\frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} f(s,z(s))ds\\
&+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}f(s,z(s))ds,\ \text{ where }\ \iota\in J_k, k\in N_0, \qquad (4.13)
\end{aligned}
$$

has a fixed point. Set $B_0=\{z\in PC(J,E):\|z\|_{PC(J,E)}\leq r\}$. The remainder of the proof is similar to the steps used in proving Theorem (3.1), so we give it in outline.

- Step 1: Let $z\in B_0$ and $\iota\in J_k,\ k=0,1,2,..,m$. Using (2.1), (3.8) , (4.12), and (4.13), we obtain that for $\iota\in J_k, k=1,2,..,m$,

$$
\begin{aligned}
\|R(z)(\iota)\| \;\leq\;& \|z_0\|+b\|z_1\|+rmh(1+b)+(b+rL_r\ +\|f(0,0)\|)\left[\frac{(2-\zeta)b}{M(\zeta-1)}+\frac{(\zeta-1)b^{\zeta}}{M(\zeta-1)\Gamma(\zeta+1)}\right]\\
<\;& r
\end{aligned}
$$

  from which we deduce that $R(B_0)\subseteq B_0$.

- Step 2: Let $Z = R(B_0)$. We claim that Z is equicontinuous on every $J_k, k \in N_0 = \{0, 1, 2, .., m\}$. Let $k \in N_0$ be fixed, $z \in B_0$ and $\iota, \iota + \lambda \in J_k$. Using (4.13), we get

$$\begin{aligned} \|R(z)(\iota+\lambda) - R(z)(\iota)\| &\leq \lambda\left(\|z_1\| + \frac{2-\zeta}{M(\zeta-1)}\|f(0,z_0)\|\right) + \frac{\lambda(2-\zeta)}{M(\zeta-1)}(b + rL_r + \|f(0,0)\|) \\ &\quad + \frac{(\zeta-1)(b + rL_r + \|f(0,0)\|)}{M(\zeta-1)\Gamma(\zeta)} \int_0^\iota \left[(\iota+\lambda-s)^{\zeta-1} - (\iota-s)^{\zeta-1}\right] ds. \end{aligned}$$

Since $\zeta - 1 > 0$, we have $\|R(z)(\iota+\lambda) - R(z)(\iota)\| \to 0$ as $\lambda \to 0$, independently of z.

- Step 3: We show that $B = \cap_{n\geq 1} B_n$ is non-empty and compact in $PC(J, E)$, where $B_n = \overline{conv}(R(B_{n-1})), \forall n \geq 1$. By Step 1, it follows that $B_n, n \geq 1$ is a decreasing sequence of non-empty, closed convex and bounded subsets of $PC(J, E)$, and hence it is sufficient to show that

$$\lim_{n\to\infty} \chi_{PC(J,E)}(B_n) = 0, \tag{4.14}$$

where $\chi_{PC(J,E)}$ is the Hausdorff measure of noncompactness on $PC(J, E)$.

Let $n \geq 1$ be a fixed natural number and $\varepsilon > 0$. In view of Lemma 3 in [49], there exists a sequence $(z_k), k \geq 1$ in B_{n-1} such that

$$\begin{aligned} \chi_{PC(J,E)}(B_n) &= \chi_{PC(J,E)} R(B_{n-1}) \leq 2\chi_{PC(J,E)}\{R(z_k) : k \geq 1\} + \varepsilon \\ &= 2 \max_{i=0,1,\ldots,m} \chi_i\{R(z_k)_{|\overline{J_i}} : k \geq 1\} + \varepsilon, \end{aligned}$$

where χ_i is the Hausdorff measure of noncompactness on $C(\overline{J_i}, E)$. Since $R(B_{n-1})$ is equicontinuous, the above inequality becomes

$$\chi_{PC(J,E)}(B_n) \leq 2 \max_{i=0,1,\ldots,m} \sup_{\iota\in\overline{J_i}} \chi\{R(z_k)(\iota) : k \geq 1\} + \varepsilon, \tag{4.15}$$

where χ is the the Hausdorff measure of noncompactness on E. Since $I_i, \overline{I_i}, i = 0, .., m$, are compact, we have

$$\sum_{i=1}^{m} \chi\{I_i(z_k(\iota_i^-)) : k \geq 1\} = \sum_{i=1}^{m} \chi\{(\iota - \iota_i)(\overline{I_i}(z_k(\iota_i^-)) : k \geq 1\} = 0.$$

Thus, as in (3.12)

$$\chi\{R(z_k)(\iota) : k \geq 1\} \leq \chi_{C(J,E)}(B_{n-1}) L_r \left(\frac{2-\zeta}{M(\zeta-1)} + \frac{(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)}\right),$$

from which,

$$\chi_{PC(J,E)}(B_n) \leq \chi_{PC(J,E)}(B_0) \left[L_r \left(\frac{2-\zeta}{M(\zeta-1)} + \frac{(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)}\right)\right]^{n-1}.$$

Inequality (4.12) insures that $L_r \left(\frac{2-\zeta}{M(\zeta-1)} + \frac{(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)}\right) < 1$, and Eq (4.14) follows.

- Step 4: The function $R_{|B} : B \to B$ is continuous. Let $z_n \to z$ in B and $y_n = R(z_n)$. The proof follows from the continuity of both $I_i, \overline{I_i}$; $i = 0, 1, 2, .., m$, by following the same arguments in Step 4 of the proof of Theorem (3.1).

As a result of Steps (1) to (4) and Schauder's fixed point theorem, there is $z \in B \subseteq PC(J, E)$ such that $z = R(z)$.

To show the uniqueness of the solution, let z and v be two solutions for problem (1.4). For $\iota \in J_0$, we have

$$\begin{aligned} \|z(\iota) - v(\iota)\| &\leq \frac{L_r(2-\zeta)b}{M(\zeta-1)} \sup_{s\in J_0} \|z(s) - v(s)\| + \frac{L_r b^\zeta(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \sup_{s\in J_0} \|z(s) - v(s)\| \\ &\leq \sup_{s\in J_0} \|z(s) - v(s)\| \left[\frac{L_r(2-\zeta)b}{M(\zeta-1)} + \frac{L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} \right]. \end{aligned}$$

Thus,

$$\sup_{s\in J_0} \|z(s) - v(s)\| \leq \sup_{s\in J_0} \|z(s) - v(s)\| \left[\frac{L_r(2-\zeta)b}{M(\zeta-1)} + \frac{L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} \right].$$

The inequality (4.12) gives

$$\frac{L_r(2-\zeta)b}{M(\zeta-1)} + \frac{L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} < 1.$$

Thus, $\sup_{s\in J_0} \|z(s) - v(s)\| = 0$, and hence $z(s) = v(s), \forall s \in J_0$.

Assume that $\iota \in J_1$. Because $z(\iota_1^-) = v(\iota_1^-)$, it yields that,

$$\begin{aligned} \|z(\iota) - v(\iota)\| &\leq \|I_1(z(\iota_1^-)) - I_1(v(\iota_1^-))\| + (\iota - \iota_1)\|\overline{I_1}(z(\iota_1^-)) - \overline{I_1}(v(\iota_1^-))\| \\ &\quad + \frac{L_r(2-\zeta)b}{M(\zeta-1)} \sup_{s\in J_0} \|z(s) - v(s)\| + \frac{L_r b^\zeta(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \sup_{s\in J_0} \|z(s) - v(s)\| \\ &= \frac{L_r(2-\zeta)b}{M(\zeta-1)} \sup_{s\in J_0} \|z(s) - v(s)\| + \frac{L_r b^\zeta(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)} \sup_{s\in J_0} \|z(s) - v(s)\| \\ &\leq \sup_{s\in J_0} \|z(s) - v(s)\| \left[\frac{L_r(2-\zeta)b}{M(\zeta-1)} + \frac{L_r(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta)} \right]. \end{aligned}$$

As above, we obtain that $z(\iota) = v(\iota), \forall\ \iota \in J_1$. By continuing in the same manner, we show that $z = v$. □

Next, we show that replacing the Assumptions (A_1), and (A_3) in Theorem 4.1 by (A_2), and (A_4) simplifies (4.12). In fact this enable us to apply Banach fixed point theorem for contraction mappings instead of Schauder fixed point.

Theorem 4.2. Let $f : J \times E \to E$ such that $f(0, z_0) = 0$ and $I_i, \overline{I_i} : E \to E$ $(i \in N_1)$ be functions. If Assumptions (A_2) and (A_4) are satisfied, then problem (1.4) has a unique solution provided that

$$\sum_{i=1}^{m} (\delta_i + b\eta_i) + \frac{\sigma(2-\zeta)b}{M(\zeta-1)} + \frac{\sigma(\zeta-1)b^\zeta}{M(\zeta-1)\Gamma(\zeta+1)} < 1. \tag{4.16}$$

Proof. Let $R : PC(J, E) \to PC(J, E)$, be function given by Eq (4.13), and $z, v \in PC(J, E)$. For each $\iota \in J_k; k \in N_1$, we have

$$\|R(z)(\iota) - R(v)(\iota)\| \leq \left[\sum_{i=1}^{m} \delta_i + b\eta_i) + \frac{\sigma(2-\zeta)b}{M(\zeta-1)} + \frac{\sigma(\zeta-1)b^{\zeta}}{M(\zeta-1)\Gamma(\zeta+1)}\right] \|z - v\|_{PC(J,E)}.$$

Thus, R is contraction. By applying Banach fixed point theorem, we obtain that R has a unique fixed point, and such point is a solution for problem (1.4). □

5. Existence of solutions for problem (1.5)

To obtain the sufficient conditions for the existence of anti-periodic solution for problem (1.5), we consider the following problem:

$$\begin{cases} {}^{ABC}D_{0,\iota}^{\zeta} z(\iota) = w(\iota),\ \iota \in J - \{\iota_1, \iota_2, \ldots, \iota_m\}, \\ z(0) = -z(b),\ z'(0) = -z'(b), \\ z(\iota_i^+) = z(\iota_i^-) + I_i(z(\iota_i)),\ i \in N_1, \\ z'(\iota_i^+) = z'(\iota_i^-) + \overline{I}_i(z(\iota_i)),\ i \in N_1. \end{cases} \tag{5.1}$$

Note that problem (5.1) can be obtained from (4.7) by setting $z_0 = -z(b)$ and $z_1 = -z'(b)$. Therefore, the solution of (5.1) is given by Eq (4.6) after substituting the values of z_0 and z_1.

Lemma 5.1. Let $w \in PCH^1(J, E)$ with $w(0) = 0$. The solution function of problem (5.1) is given by Eq (4.6), where z_0, z_1 are given as follows:

$$\begin{aligned} z_0 &= \frac{b(2-\zeta)}{4M(\zeta-1)} w(b) + \frac{b(\zeta-1)}{4M(\zeta-1)\Gamma(\zeta-1)} \int_0^b (b-s)^{\zeta-2} w(s)ds. \\ &\quad -\frac{1}{2}\sum_{i=1}^{m} I_i(z(\iota_i^-)) - \frac{1}{4}\sum_{i=1}^{m}(b - 2\iota_i)\overline{I}_i(z(\iota_i^-)) \\ &\quad -\frac{1}{2}\frac{2-\zeta}{M(\zeta-1)}\int_0^b w(s)ds - \frac{1}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1} w(s)ds, \end{aligned} \tag{5.2}$$

and

$$z_1 = \frac{-1}{2}\left[\sum_{i=1}^{m} \overline{I}_i(z(\iota_i^-)) + \frac{2-\zeta}{M(\zeta-1)} w(b) + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)} \int_0^b (b-s)^{\zeta-2} w(s)ds\right]. \tag{5.3}$$

Proof. Using Eq (4.6) and the boundary conditions $z(0) = -z(b)$, $z'(0) = -z'(b)$, we obtain

$$z_1 = -z_1 - \sum_{i=1}^{m} \overline{I}_i(z(\iota_i^-)) - \frac{2-\zeta}{M(\zeta-1)} w(b) - \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta-1)} \int_0^b (b-s)^{\zeta-2} w(s)ds.$$

So, Eq (5.3) is verified. Moreover,

$$z_0 = -[z_0 + bz_1 + \sum_{i=1}^{m} I_i(z(\iota_i^-)) + \sum_{i=1}^{m}(b - \iota_i)\overline{I}_i(z(\iota_i^-))$$

$$+\frac{2-\zeta}{M(\zeta-1)}\int_0^b w(s)ds+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}w(s)ds],$$

i.e.,

$$\begin{aligned} z_0 &= \frac{-1}{2}[bz_1+\sum_{i=1}^m I_i(z(\iota_i^-))+\sum_{i=1}^m (b-\iota_i)\overline{I}_i(z(\iota_i^-)) \\ &\quad +\frac{2-\zeta}{M(\zeta-1)}\int_0^b w(s)ds+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}w(s)ds]. \end{aligned}$$

This equation along with Eq (5.3) lead to

$$\begin{aligned} z_0 &= \frac{b(2-\zeta)}{4M(\zeta-1)}w(b)+\frac{b(\zeta-1)}{4M(\zeta-1)\Gamma(\zeta-1)}\int_0^b (b-s)^{\zeta-2}w(s)ds. \\ &\quad -\frac{1}{2}\sum_{i=1}^m I_i(z(\iota_i^-))-\frac{1}{4}\sum_{i=1}^m (b-2\iota_i)\overline{I}_i(z(\iota_i^-)) \\ &\quad -\frac{1}{2}\frac{2-\zeta}{M(\zeta-1)}\int_0^b w(s)ds-\frac{1}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}w(s)ds]. \end{aligned}$$

□

By substituting the values of z_0 and z_1 into Eq (4.6), we obtain the following

Corollary 5.1. Let $w \in PCH^1(J,E)$ with $w(0)=0$. The solution of system in (5.1) is given by:

$$\begin{aligned} z(\iota) &= \left(\frac{b}{4}-\frac{\iota}{2}\right)\frac{2-\zeta}{M(\zeta-1)}w(b)+\left(\frac{b}{4}-\frac{\iota}{2}\right)\int_0^b (b-s)^{\zeta-2}w(s)ds-\frac{1}{2}\frac{2-\zeta}{M(\zeta-1)}\int_0^b w(s)ds \\ &\quad -\frac{1}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}w(s)ds-\frac{1}{2}\sum_{i=1}^m I_i(z(\iota_i^-))-\frac{b}{4}\sum_{i=1}^m \overline{I}_i(z(\iota_i^-))-\frac{1}{2}\sum_{i=1}^m (\iota-\iota_i)\overline{I}_i(z(\iota_i^-)) \\ &\quad +\sum_{i=1}^k I_i(z(\iota_i^-))+\sum_{i=1}^k (\iota-\iota_i)\overline{I}_i(z(\iota_i^-))+\frac{2-\zeta}{M(\zeta-1)}\int_0^\iota w(s)ds \\ &\quad +\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^\iota (\iota-s)^{\zeta-1}w(s)ds, \quad \text{where } \iota \in J_k, k \in N_0. \end{aligned} \tag{5.4}$$

As a result of Corollary 5.1, we state the following definition.

Definition 5.1. A function $z \in PCH^2(J,E)$ is said to be a solution for problem (1.5) if it has left Atangana-Baleanu fractional derivative of order ζ on each J_k, where $k \in N_0$, and satisfies the integral equation:

$$\begin{aligned} z(\iota) &= \left(\frac{b}{4}-\frac{\iota}{2}\right)\frac{2-\zeta}{M(\zeta-1)}f(b,z(b))+\left(\frac{b}{4}-\frac{\iota}{2}\right)\int_0^b (b-s)^{\zeta-2}f(s,z(s))ds-\frac{1}{2}\frac{2-\zeta}{M(\zeta-1)}\int_0^b f(s,z(s))ds \\ &\quad -\frac{1}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}f(s,z(s))ds-\frac{1}{2}\sum_{i=1}^m I_i(z(\iota_i^-)) \end{aligned}$$

$$-\frac{b}{4}\sum_{i=1}^{m}\overline{I}_i(z(\iota_i^-)) - \frac{1}{2}\sum_{i=1}^{m}(\iota-\iota_i)\overline{I}_i(z(\iota_i^-))$$
$$+\sum_{i=1}^{k}I_i(z(\iota_i^-)) + \sum_{i=1}^{k}(\iota-\iota_i)\overline{I}_i(z(\iota_i^-)) + \frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} f(s,z(s))ds$$
$$+\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}f(s,z(s))ds, \text{ where } \iota\in J_k, k\in N_0. \quad (5.5)$$

Theorem 5.1. Under the assumptions of Theorem 4.2, problem (1.5) has a unique solution provided that

$$\left[\frac{\sigma b}{2} + \frac{\sigma b^{\zeta}}{2(\zeta-1)} + \frac{2\sigma b(2-\zeta)}{M(\zeta-1)} + \frac{3\sigma b^{\zeta}}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta+1)}\frac{3}{2}\sum_{i=1}^{m}\delta_i + \frac{7b}{4}\sum_{i=1}^{m}\eta_i\right] < 1. \quad (5.6)$$

Proof. Consider the function $\mathcal{R} : PC(J,E) \to PC(J,E)$ defined as:

$$\begin{aligned}
\mathcal{R}(z)(\iota) \;=\; & (\frac{b}{4}-\frac{\iota}{2})\frac{2-\zeta}{M(\zeta-1)}f(b,z(b)) + (\frac{b}{4}-\frac{\iota}{2})\int_0^b (b-s)^{\zeta-2}f(s,z(s))ds \\
& -\frac{1}{2}\frac{2-\zeta}{M(\zeta-1)}\int_0^b f(s,z(s))ds - \frac{1}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^b (b-s)^{\zeta-1}f(s,z(s))ds \\
& -\frac{1}{2}\sum_{i=1}^{m}I_i(z(\iota_i^-)) - \frac{b}{4}\sum_{i=1}^{m}\overline{I}_i(z(\iota_i^-)) - \frac{1}{2}\sum_{i=1}^{m}(\iota-\iota_i)\overline{I}_i(z(\iota_i^-)) \\
& +\sum_{i=1}^{k}I_i(z(\iota_i^-)) + \sum_{i=1}^{k}(\iota-\iota_i)\overline{I}_i(z(\iota_i^-)) \\
& +\frac{2-\zeta}{M(\zeta-1)}\int_0^{\iota} f(s,z(s))ds + \frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta)}\int_0^{\iota}(\iota-s)^{\zeta-1}f(s,z(s))ds,
\end{aligned}$$

where $\iota\in J_k, k\in N_0$. (5.7)

Let $z, v\in PC(J,E)$, $\iota\in J_k, k\in N_1$. By the assumptions (A_2), (A_4), and above equality, we have

$$\begin{aligned}
\|\mathcal{R}(z)(\iota)-\mathcal{R}(v)(\iota)\| \;\le\; & [\frac{\sigma b}{2} + \frac{\sigma b^{\zeta}}{2(\zeta-1)} + \frac{\sigma b}{2}\frac{2-\zeta}{M(\zeta-1)} + \frac{\sigma b^{\zeta}}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta+1)} \\
& +\frac{3}{2}\sum_{i=1}^{m}\delta_i + \frac{b}{4}\sum_{i=1}^{m}\eta_i + \frac{3b}{2}\sum_{i=1}^{m}\eta_i + \frac{\sigma b(2-\zeta)}{M(\zeta-1)} + \frac{\sigma b^{\zeta}(\zeta-1)}{M(\zeta-1)\Gamma(\zeta+1)}]\|z-v\| \\
=\; & [\frac{\sigma b}{2} + \frac{\sigma b^{\zeta}}{2(\zeta-1)} + \frac{2\sigma b(2-\zeta)}{M(\zeta-1)} + \frac{3\sigma b^{\zeta}}{2}\frac{\zeta-1}{M(\zeta-1)\Gamma(\zeta+1)} \\
& +\frac{3}{2}\sum_{i=1}^{m}\delta_i + \frac{7b}{4}\sum_{i=1}^{m}\eta_i]\|z-v\|.
\end{aligned}$$

This equation along with (5.6) shows that $\mathcal{R}$ is a contraction. Therefore, problem (1.5) has a unique solution. □

6. Existence of solutions for problem(1.6)

Definition 6.1. Let Λ and Δ be two normed spaces. A multi-valued function $G : \Lambda \to 2^E$ with non-empty closed, bounded and convex values is called ρ-Lipschitz if

$$h(G(x) - G(y)) \leq \rho \, \|x - y\|_\Lambda, \forall x, y \in \Lambda,$$

where h is the Hausdorff distance.

For information about the multi-valued functions, we refer the reader to [50].

Lemma 6.1. ([51], Theorem 7) Let $(\Omega, \sum, \mu)$ be a $\sigma-$ finite measure space and (T, d) be a metric space. If $G : T \to 2^{L^p(\Omega,\mathbb{R}^n)}, p \in [1, \infty)$ is a ρ-Lipschitz multi-valued function with non-empty, closed, convex, bounded and decomposable values, then there is single-valued function $f : T \to 2^{L^p(\Omega,\mathbb{R}^n)}$ such that $f(t) \in G(t), a.e$, and $\|f(z) - f(v)\| \leq \xi_G \; \rho \|z - v\|_{L^p(\Omega,\mathbb{R}^n)}$, $\forall z, v \in T$, where ξ_G is a positive real number.

In the following theorem, we provide sufficient conditions for the existence of solutions of problem (1.6).

Theorem 6.1. Let $F : J \times L^2(J, \mathbb{R}) \to 2^{L^2(J,\mathbb{R})}$ be an ρ-Lipschitz multi-valued function with non-empty, closed, convex, bounded and decomposable values, and $I_i, \overline{I_i} : L^2(J, \mathbb{R}) \to L^2(J, \mathbb{R}), i \in N_1$ be functions such that

$$\|I_i(x) - I_i(y)\| \leq \delta_i \, \|x - y\| \, , \forall \; x \in L^2(J, \mathbb{R}), \forall i \in N_1,$$

and

$$\left\|\overline{I_i}(x) - \overline{I_i}(y)\right\| \leq \eta_i \, \|x - y\| \, , \forall \; x \in L^2(J, \mathbb{R}), \forall i \in N_1,$$

where δ_i , η_i, are positive real numbers, then problem (1.6) has a solution provided that $F(0, z_0) = \{0\}$ and

$$\sum_{i=1}^{k}(\delta_i + \eta_i) + \frac{\xi_F \; \rho(2 - \zeta)b}{M(\zeta - 1)} + \frac{\sigma(\zeta - 1)b^\zeta}{M(\zeta - 1)\Gamma(\zeta + 1)} < 1. \tag{6.1}$$

Proof. Let $E = L^2(J, \mathbb{R})$. The set $T = J \times E$ is a complete metric space, where $d((\iota_1, x_1), (\iota_2, x_2)) = |\iota_1 - \iota_2| + \|x_1 - x_2\|_E$. By Lemma 6.1, there exists $f : J \times E \to E$ satisfying $f(\iota, x) \in F(\iota, x), a.e.$, and

$$\|f(\iota, x) - f(s, y)\| \leq \xi_F \; \rho(|\iota - s| + \|x - y\|_E), \; \forall(\iota, x), (s, y) \in T.$$

By applying Theorem (4.2), the following fractional boundary problem

$$\begin{cases} {}^{ABC}D^{\zeta}_{0,\iota} z(\iota) = f(\iota, z(\iota)), \iota \in J - \{\iota_1, \iota_2, \ldots, \iota_m\}, \\ z(0) = z_0, \; z'(0) = z_1, \\ z(\iota_i^+) = z(\iota_i^-) + K_i(z(\iota_i^-)), \; i = 1, 2, \ldots, m, \\ z'(\iota_i^+) = z'(\iota_i^-) + \overline{K_i}(z(\iota_i^-)), \; i = 1, 2, \ldots, m. \end{cases}$$

has a solution. Since $f(\iota, x) \in F(\iota, x) \; a.e$, we have ${}^{ABC}D^{\zeta}_{0,\iota} z(\iota) \in F(\iota, z(\iota)) \; a.e.$ for $\iota \in J - \{\iota_1, \iota_2, \ldots, \iota_m\}$ which completes the proof. □

Example 6.1. Let $E = L^2[0,\pi]$, $\zeta \in (1,2)$, $J = [0,1]$ and $z_0 : [0,\pi] \to \mathbb{R}$ be the zero function. Define $f : J \times E \to E$ by

$$f(w,\varsigma)(\eta) = \frac{\sin w}{\sqrt{\pi}} + \lambda\varsigma^2(\eta);\ w \in J,\ \varsigma \in E,\ \eta \in [0,\pi], \tag{6.2}$$

where $\lambda > 0$. Thus, $f(0,z_0)(\eta) = 0; \forall \eta \in [0,\pi]$. For any $\varsigma_1, \varsigma_2 \in E = L^2[0,\pi]$ and any $w_1, w_2 \in J$, we have

$$\begin{aligned}
\|f(w_1,\varsigma_1) - f(w_2,\varsigma_2)\|_{L^2[0,\pi]} &= \left(\int_0^\pi \frac{1}{\sqrt{\pi}}|(\sin w_1 - \sin w_2) + \lambda^2(\varsigma_1^2(\eta) - \varsigma_2^2(\eta))|^2 d\eta\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\pi \frac{1}{\sqrt{\pi}}|\sin w_1 - \sin w_2|^2 d\eta\right)^{\frac{1}{2}} + \lambda\left(\int_0^\pi |\varsigma_1^2(\eta) - \varsigma_2^2(\eta)|^2 d\eta\right)^{\frac{1}{2}} \\
&\le |\sin w_1 - \sin w_2| + \lambda\left(\int_0^\pi |(\varsigma_1(\eta)) + \varsigma_2(\eta))\,(\varsigma_1(\eta)) - \varsigma_2(\eta))|^2 d\eta\right)^{\frac{1}{2}} \\
&\le |w_1 - w_2| + \lambda\,|< \varsigma_1 + \varsigma_2, \varsigma_1 - \varsigma_2 >| \\
&\le |w_1 - w_2| + \lambda\|\varsigma_1 + \varsigma_2\|\,\|\varsigma_1 - \varsigma_2\| \\
&\le |w_1 - w_2| + \lambda(\|\varsigma_1\| + \|\varsigma_2\|)\,\|\varsigma_1 - \varsigma_2\|.
\end{aligned}$$

Thus, (A_1) is satisfied with $L_\delta = 2\lambda\delta$. By applying Theorem 3.1, we have that if there is $r > 0$ such that

$$\|z_1\| + 2(1 + 2\lambda r^2)\left[\frac{(2-\zeta)}{M(\zeta-1)} + \frac{(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)}\right] < r, \tag{6.3}$$

then there is a unique function $z : [0,1] \to L^2[0,\pi]$ satisfying the boundary value problem

$$\begin{cases} {}^{ABC}D_{0,\iota}^{\zeta} z(\iota)(s) = \frac{\sin\iota}{\sqrt{\pi}} + \varsigma^2(s), & \iota \in J,\ \varsigma \in E,\ s \in [0,\pi], \\ z(0) = z_0, z(1) = z_1, \end{cases} \tag{6.4}$$

(see relation (3.6)).

The inequality (6.3) is equivalent to

$$4\lambda\omega r^2 - r + \|z_1\| + 2\omega < 0, \tag{6.5}$$

where $\omega = \frac{(2-\zeta)}{M(\zeta-1)} + \frac{(\zeta-1)}{M(\zeta-1)\Gamma(\zeta)}$.

If $16\,\lambda\,\omega(\|z_1\| + 2\omega) < 1$, then the equation $4\omega\lambda r^2 - r + \|z_1\| + 2\omega = 0$ has two positive solutions, namely

$$r_1 = \frac{1 - \sqrt{1 - 16\lambda\omega(\|z_1\| + 2\omega)}}{8\omega} \text{ and } r_2 = \frac{1 + \sqrt{1 - 16\lambda\omega(\|z_1\| + 2\omega)}}{8\omega}.$$

Therefore, (6.5) will be satisfied for $r \in (r_1, r_2)$. Thus, inequality (6.3) has a solution provided the following inequality

$$16\,\lambda\,\omega(\|z_1\| + 2\omega) < 1 \tag{6.6}$$

holds. The last inequality will hold if we choose λ sufficiently small.

Example 6.2. Let $E = L^2[0,\pi]$, $\zeta \in (1,2)$, $J = [0,1]$. Define $f : J \times E \to E$ by:

$$f(w,\varsigma)(\eta) := \frac{\sin w}{\sqrt{\pi}} + \sigma \sin\varsigma(\eta);\ w \in J,\ \varsigma \in E,\ \eta \in [0,\pi], \tag{6.7}$$

where, $\sigma > 0$. Let $z_0 : [0,\pi] \to \mathbb{R}$, be the zero function and $z_1 \in L^2[0,\pi]$ be a fixed function. Note that $f(0,z_0)(\eta) = 0$, $\forall \eta \in [0,\pi]$. For every $\varsigma_1, \varsigma_2 \in E = L^2[0,\pi]$ and every $w_1, w_2 \in J$, we have

$$\begin{aligned} \|f(w,\varsigma_1) - f(w,\varsigma_2)\|_{L^2[0,\pi]} &= \sigma\left(\int_0^\pi |\sin\varsigma_1(\eta) - \sin\varsigma_2(\eta)|^2 d\eta\right)^{\frac{1}{2}} \\ &\le \sigma\left(\int_0^\pi |\varsigma_1(\eta) - \varsigma_2(\eta)|^2 d\eta\right)^{\frac{1}{2}} = \sigma\|\varsigma_1 - \varsigma_2\|_{L^2[0,\pi]} \\ &\le \sigma\|\varsigma_1 - \varsigma_2\|_{L^2[0,\pi]}. \end{aligned}$$

Thus, (A_2) is satisfied. By applying Theorem 3.2, there is a unique $z : [0,1] \to L^2[0,\pi]$ such that $z' \in H^1((0,1), L^2[0,\pi])$ and z satisfies the boundary value problem:

$$\begin{cases} {}^{ABC}D^{\zeta}_{0,\iota} z(\iota)(\eta) = \frac{\sin\iota}{\sqrt{\pi}} + \upsilon \sin(z(\iota)\eta),\ \iota \in [0,1], \eta \in [0,\pi], \\ z(0) = z_0, z(1) = z_1, \end{cases} \tag{6.8}$$

provided the following condition is satisfied

$$\frac{2\sigma(2-\zeta)}{M(\zeta-1)} + \frac{2(\zeta-1)\sigma}{M(\zeta-1)\Gamma(\zeta)} < 1. \tag{6.9}$$

The last inequality holds by choosing σ sufficiently small.

Example 6.3. Let E, ζ, J, f as be in Example 6.2, $\iota_0 = 0, \iota_1 = \frac{1}{2}$ and $\iota_2 = 1$. Define $I_1(x) = \lambda\, Proj_K(x)$, $\overline{I}_1(x) = \lambda\, Proj_Z(x), \forall x \in E$, where $\lambda > 0$, K and Z are convex and compact subset of $L^2[0,\pi]$ and $\lambda > 0$. Then, (A_2) is satisfied. Also, (2.2) and (2.3) are verified with $\delta_1 = \eta_1 = \lambda$. By applying Theorem 4.2, the problem

$$\begin{cases} ({}^{ABC}D^{\zeta}_{0,\iota} z(\iota))(\eta) = \frac{\sin\iota}{\sqrt{\pi}} + \upsilon \sin(z(\iota)\eta),\ \iota \in [0,1] - \{0, \frac{1}{2}\}, \eta \in [0,\pi], \\ z(0) = z_0,\ z'(0) = z_1, \\ z(\iota_1^+) = z(\iota_1^-) + I_1(z(\iota_1^-)), \\ z'(\iota_1^+) = z'(\iota_1^-) + \overline{I_1}(z(\iota_1^-)), \end{cases} \tag{6.10}$$

has a solution provided that

$$2\lambda + \frac{\sigma(2-\zeta)}{M(\zeta-1)} + \frac{\sigma(\zeta-1)}{M(\zeta-1)\Gamma(\zeta+1)} < 1. \tag{6.11}$$

This inequality will hold by choosing σ and λ sufficiently small.

In a similar manner, many examples of the application of Theorems 5.1 and 6.1 can be provided.

7. Discussion and conclusions

The relationships between some boundary value problems involving the Atangana-Baleanu fractional derivative (AB) of order $\zeta \in (1,2)$ and the corresponding fractional integral equations were obtained in infinite dimensional Banach spaces with or without impulses. We showed that the continuity assumption on the nonlinear term is insufficient and must be replaced by membership in the space $H^1((a,b),E)$. The sufficient conditions for the existence of solutions for differential equations and inclusions involving AB fractional derivative in the presences of instantaneous impulses in infinite dimensional Banach spaces were established. Additionally, the sufficient conditions for the existence and uniqueness of solutions and anti-periodic solutions for differential equations and inclusions containing AB fractional derivative of order $\zeta \in (1,2)$ in the presences of instantaneous impulses in infinite dimensional Banach spaces were obtained.

The major contributions of this work can be summarized as follows:

(1) A modified formula for relationship between the solution of problem (1.1) and the corresponding integral equation (1.2) is derived.

(2) A new class of boundary value for differential equations and inclusions containing AB derivative with instantaneous impulses in infinite dimensional Banach spaces were formulated with and without impulsive effects.

(3) The existence/uniqueness of solutions and anti-periodic solutions for the considered problems and the corresponding inclusions were proved.

We provide methods to deal with differential equations and differential inclusions in infinite dimensional Banach spaces. i.e., to extend the results in [19–24, 45, 46] to infinite dimensional spaces. The methods used in this paper can help researchers to generalize many of the results cited above in the presence of impulsive effects, infinite dimensional Banach spaces, and when the right hand side of the equation is a multi-valued function. We suggest the following topics for future research

- Study the existence of S-asymptotically w-periodic solutions for problems (1.3) and (1.4).
- Extend the recent work by Saha et al. [42] to infinite dimensional Branch spaces.
- Generalize the present work to the case where the Atangana and Baleanu's derivative is replaced by the Atangana and Baleanu's derivative with respect to another function.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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