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Research article

Fixed point results for P-contractive mappings on M-metric space and application

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Abstract: In this paper, we elucidate a pivotal fixed point theorem for P-contraction mappings defined on M-metric spaces, offering a novel perspective on the interplay between mappings and the underlying space structure. This theorem's significance becomes evident when compared with earlier results, underscoring its potential to enhance our understanding of fixed point theory in M-metric spaces and its broader applications.

Keywords: fixed point; P-contractive mapping; M-metric space; Langevin equation
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1. Introduction

A fundamental concept in the realm of mathematical analysis and topology is the study of metric spaces, which provide a framework for understanding the notion of distance between elements of a set. Metric spaces are an essential part of modern mathematics and have found numerous applications in various branches of science and engineering. Among the intriguing properties of metric spaces is their ability to reveal deep insights into the behavior of mappings.

In this context, the exploration of fixed point results within the context of metric spaces has become a significant area of interest for mathematicians. Fixed point theory deals with the study of self-maps on metric spaces that leave certain points unchanged. These fixed points have far-reaching implications in various fields, including analysis, functional analysis, and differential equations, making them a critical subject of investigation.

In this paper, we will focus on fixed point theory on M-metric spaces. An M-metric space is a generalization of the ordinary metric space, designed to capture more complex notions of distance and convergence. The study of fixed point results in M-metric spaces offers a rich and challenging landscape for mathematicians, leading to the development of powerful theorems and tools for understanding the behavior of mappings within these spaces. Through this exploration, we will present a fundamental fixed point theorem for P-contraction mappings on M-metric spaces and compare it with earlier results.

Asadi et al. [8] introduced the idea of the M-metric. The M-metric form of the Banach contraction principle was then demonstrated. First, we recall the definition and some properties of it.

We will use the following notations in the rest of this paper:

$$\begin{aligned} \mu_{x,y} &= \min\{\mu(x,x), \mu(y,y)\} \\ M_{x,y} &= \max\{\mu(x,x), \mu(y,y)\}. \end{aligned}$$

Definition 1. *[8] Consider a nonempty set $\mathcal{M}$ and a mapping $\mu : \mathcal{M} \times \mathcal{M} \to [0, \infty)$. Then μ is called M-metric on $\mathcal{M}$ if, for all $x, y, z \in \mathcal{M}$,*

m1) $\mu(x,y) = \mu(x,x) = \mu(y,y) \Leftrightarrow x = y$,

m2) $\mu_{x,y} \leq \mu(x,y)$,

m3) $\mu(x,y) = \mu(y,x)$,

m4) $\mu(x,y) - \mu_{x,y} \leq (\mu(x,z) - \mu_{x,z}) + \left(\mu(z,y) - \mu_{z,y}\right)$.

Then the couple $(\mathcal{M}, \mu)$ is called M-metric space.

It is clear that every ordinary metric space and every partial metric space in the sense of Matthews [14] is an M-metric space. The converse, however, may not be true as seen in the following examples:

Example 1. *Let $\mathcal{M} = [0, \infty)$. Define two functions $\mu_1, \mu_2 : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by $\mu_1(x,y) = \frac{x+y}{2}$ and $\mu_2(x,y) = \min\{x,y\}$. Then, both μ_1 and μ_2 are M-metrics on $\mathcal{M}$, however, they are neither ordinary metric nor partial metric on $\mathcal{M}$.*

We can find further examples of M-metric space in [1,23,24]. Let $(\mathcal{M}, \mu)$ be an M-metric space and $x \in \mathcal{M}$. Then the open ball centered at $x \in \mathcal{M}$ and radius $\varepsilon > 0$ in the M-metric space is defined as

$$B(x, \varepsilon) = \{y \in \mathcal{M} : \mu(x,y) < \mu_{xy} + \varepsilon\}.$$

We call a subset U of $\mathcal{M}$ open if and only if there is a real number $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ for every $x \in U$. Then, the family τ_μ of all open subsets of $\mathcal{M}$ is a topology on $\mathcal{M}$, which is a T_0 topology.

Definition 2. *[8] Let $(\mathcal{M}, \mu)$ be an M-metric space, $\{x_n\} \subset \mathcal{M}$ be a sequence and $x \in \mathcal{M}$. Then,*

(1) $\{x_n\}$ is said to be M-convergent to x, denoting $x_n \to x$, if and only if

$$\lim_{n\to\infty} (\mu(x_n, x) - \mu_{x_n x}) = 0.$$

(2) $\{x_n\}$ is called M-Cauchy sequence if

$$\lim_{n,k\to\infty} (\mu(x_n, x_k) - \mu_{x_n, x_k}) \text{ and } \lim_{n,k\to\infty} (M_{x_n, x_k} - \mu_{x_n, x_k})$$

exist and are finite.

(3) $(\mathcal{M}, \mu)$ is said to be M-complete if and only if every M-Cauchy sequence on this space M-converges to a point $x \in \mathcal{M}$ such that

$$\lim_{n\to\infty}(\mu(x_n, x) - \mu_{x_n x}) = 0 \text{ and } \lim_{n\to\infty}(M_{x_n,x} - \mu_{x_n,x}) = 0.$$

Remark 1. *Let $(\mathcal{M}, \mu)$ be an M-metric space. The function ν defined by*

$$\nu(x, y) = \mu(x, y) - 2\mu_{x,y} + M_{x,y}$$

is an ordinary metric on $\mathcal{M}$. Further, we know that a sequence $\{x_n\}$ is an M-Cauchy sequence in $(\mathcal{M}, \mu)$ if and only if it is a Cauchy sequence in the metric space $(\mathcal{M}, \nu)$. Also, $(\mathcal{M}, \mu)$ is M-complete if and only if $(\mathcal{M}, \nu)$ is complete.

Lemma 1. *[8] Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M-metric space $(\mathcal{M}, \mu)$. Then,*

$$\lim_{n\to\infty}\left(\mu(x_n, y_n) - \mu_{x_n y_n}\right) = \mu(x, y) - \mu_{xy}.$$

Asadi et al. [8] proved the following fixed point theorem.

Theorem 1. *Let $(\mathcal{M}, \mu)$ be an M-complete M-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a contraction mapping with respect to μ, that is, there exists $k \in [0, 1)$ such that*

$$\mu(\mathcal{T}x, \mathcal{T}y) \le k\mu(x, y) \tag{1.1}$$

for all $x, y \in \mathcal{M}$. Then, $\mathcal{T}$ has a unique fixed point.

Following Asadi et al., several investigations on fixed point theory in this space have been conducted, as well as publications outlining the topological structure and some basic aspects of the M-metric space [6, 7, 15, 17].

In this paper, we will introduce the concepts of P-contraction and P-contractivity in M-metric space in order to gain a new perspective on fixed point theory in this space. It will be obvious that every contraction mapping is a P-contraction, but since an example is needed to show that the converse is not true, we will provide a proper example. Then, using the two concepts we introduced, we will present some fixed point theorems that will generalize some theorems obtained previously in the literature. Finally, since there are not many applications of fixed point theorems obtained in M-metric space in the literature, we will present an existence and uniqueness theorem for a second-order (p, q)-difference Langevin equation in order to demonstrate the applicability of these theorems. For more information about P-contraction in the literature, we suggest to readers the papers [3–5, 16].

2. Main results

The existence theorem of fixed point for P-contractive mappings on M-metric space will be presented in this section.

Definition 3. *Let $(\mathcal{M}, \mu)$ be an M-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a mapping. Then, $\mathcal{T}$ is said to be a P-contraction mapping, if there exists $L \in [0, 1)$ such that*

$$\mu(\mathcal{T}x, \mathcal{T}y) \le L\left[\mu(x, y) + |\mu(x, \mathcal{T}x) - \mu(y, \mathcal{T}y)|\right] \tag{2.1}$$

for all $x, y \in \mathcal{M}$. $\mathcal{T}$ is said to be a P-contractive mapping, if it satisfies

$$\mu(\mathcal{T}x, \mathcal{T}y) < \mu(x, y) + |\mu(x, \mathcal{T}x) - \mu(y, \mathcal{T}y)| \tag{2.2}$$

for all $x, y \in \mathcal{M}$ with $\mu(x, y) > 0$. $\mathcal{T}$ is said to have 0-property if

$$\mu(x, x) = 0 \Leftrightarrow \mu(\mathcal{T}x, \mathcal{T}x) = 0$$

holds for all $x \in \mathcal{M}$.

Remark 2. *It is clear that every contraction mapping is also a P-contraction on an M-metric space. But the converse may not be true as shown in the following example.*

Example 2. *Let $\mathcal{M} = [0, 1] \cup [2, 3]$ with*

$$\mu(x, y) = \begin{cases} 1 + |x - y| & , \quad x \neq y \\ 0 & , \quad x = y \end{cases},$$

then $(\mathcal{M}, \mu)$ is an M-metric space. Define $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{T}x = \begin{cases} 1 & , \quad x \in [0, 1] \\ 0 & , \quad x \in [2, 3] \end{cases}.$$

Then, $\mathcal{T}$ is not contraction, since

$$\mu(\mathcal{T}1, \mathcal{T}2) = \mu(1, 0) = 2 = \mu(1, 2).$$

However it is a P-contraction with $L = \frac{1}{2}$. In order to show this, we will consider the following cases:
Case 1. If $x, y \in [0, 1]$, then

$$\mu(\mathcal{T}x, \mathcal{T}y) = \mu(1, 1) = 0 \leq \frac{1}{2}\left[\mu(x, y) + |\mu(x, \mathcal{T}x) - \mu(y, \mathcal{T}y)|\right].$$

Case 2. If $x, y \in [2, 3]$, then

$$\mu(\mathcal{T}x, \mathcal{T}y) = \mu(0, 0) = 0 \leq \frac{1}{2}\left[\mu(x, y) + |\mu(x, \mathcal{T}x) - \mu(y, \mathcal{T}y)|\right].$$

Case 3. If $x \in [0, 1]$ and $y \in [2, 3]$, then we have

$$\mu(\mathcal{T}x, \mathcal{T}y) = \mu(1, 0) = 2$$

and

$$\begin{aligned} \mu(x, y) + |\mu(x, \mathcal{T}x) - \mu(y, \mathcal{T}y)| &= 1 + |x - y| + |\mu(x, 1) - \mu(y, 0)| \\ &= \begin{cases} 1 + y - x + |1 - x - y| & , \quad x \neq 1 \\ 2y + 1 & , \quad x = 1 \end{cases} \end{aligned}$$

$$= \begin{cases} 2y & , \quad x \neq 1 \\ 2y+1 & , \quad x = 1 \end{cases}$$

and so

$$\mu(\mathcal{T}x, \mathcal{T}y) = 2 = \frac{1}{2}4 \leq \frac{1}{2}2y \leq \frac{1}{2}\left[\mu(x,y) + |\mu(x,\mathcal{T}x) - \mu(y,\mathcal{T}y)|\right].$$

Here we will provide a further example to support our definiton. This time, we will present an example showing that P-contractive mappings may not be P-contraction mapping in an M-metric space that is not an ordinary metric space.

Example 3. *Let* $\mathcal{M} = C^{+}[0,1]$ *be the family of all continuous and non-negative real vaued functions defined on* $[0,1]$*, and define*

$$\mu(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\} + \min\{f(0), g(0)\},$$

for $f, g \in \mathcal{M}$*. Then* $(\mathcal{M}, \mu)$ *is an* M*-metric space, however it is not an ordinary metric space. Consider the subset*

$$\mathcal{M}^{*} = \{f \in \mathcal{M} : 0 = f(0) \leq f(t) \leq f(1) = 1\}$$

and define a mapping $\mathcal{T} : \mathcal{M}^{*} \to \mathcal{M}^{*}$*, by* $\mathcal{T}f(t) = tf(t)$ *for* $t \in [0,1]$*. Then* $\mathcal{T}$ *is not* P*-contraction. Indeed, consider the sequences of functions* $\{f_n\}$ *and* $\{g_n\}$ *in* $\mathcal{M}^{*}$ *defined as* $f_n(t) = t^{2n}$ *and* $g_n(t) = t^{n}$*. In this case, we have*

$$\begin{aligned} \mu(\mathcal{T}f_n, \mathcal{T}g_n) &= \sup\{|\mathcal{T}f_n(t) - \mathcal{T}g_n(t)| : t \in [0,1]\} + \min\{\mathcal{T}f_n(0), \mathcal{T}g_n(0)\} \\ &= \sup\left\{\left|t^{2n+1} - t^{n+1}\right| : t \in [0,1]\right\} \\ &= \left(\frac{n+1}{2n+1}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right) \to \frac{1}{4} \text{ as } n \to \infty, \end{aligned}$$

$$\begin{aligned} \mu(f_n, g_n) &= \sup\{|f_n(t) - g_n(t)| : t \in [0,1]\} + \min\{f_n(0), g_n(0)\} \\ &= \sup\left\{\left|t^{2n} - t^{n}\right| : t \in [0,1]\right\} \\ &= \frac{1}{4}, \end{aligned}$$

$$\begin{aligned} \mu(f_n, \mathcal{T}f_n) &= \sup\{|f_n(t) - \mathcal{T}f_n(t)| : t \in [0,1]\} + \min\{f_n(0), \mathcal{T}f_n(0)\} \\ &= \sup\left\{\left|t^{2n} - t^{2n+1}\right| : t \in [0,1]\right\} \\ &= \left(\frac{2n}{2n+1}\right)^{2n}\left(\frac{1}{2n+1}\right) \to 0 \text{ as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} \mu(g_n, \mathcal{T}g_n) &= \sup\{|g_n(t) - \mathcal{T}g_n(t)| : t \in [0,1]\} + \min\{g_n(0), \mathcal{T}g_n(0)\} \\ &= \sup\left\{\left|t^{n} - t^{n+1}\right| : t \in [0,1]\right\} \end{aligned}$$

$$= \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) \to 0 \; as \; n \to \infty.$$

Therefore, we have

$$\lim_{n\to\infty} \frac{\mu(\mathcal{T} f_n, \mathcal{T} g_n)}{\mu(f_n, g_n) + |\mu(f_n, \mathcal{T} f_n) - \mu(g_n, \mathcal{T} g_n)|} = 1.$$

This shows that we cannot find a constant $L \in [0, 1)$ *satisfying inequality (2.1). On the other hand, we have, for all* $f, g \in \mathcal{M}^*$,

$$\begin{aligned}
\mu(\mathcal{T} f, \mathcal{T} g) &= \sup \{|\mathcal{T} f(t) - \mathcal{T} g(t)| : t \in [0, 1]\} + \min \{\mathcal{T} f(0), \mathcal{T} g(0)\} \\
&= \sup \{|tf(t) - tg(t)| : t \in [0, 1]\} \\
&< \sup \{|f(t) - g(t)| : t \in [0, 1]\} \\
&= \sup \{|f(t) - g(t)| : t \in [0, 1]\} + \min \{f(0), g(0)\} \\
&= \mu(f, g) \\
&\leq \mu(f, g) + |\mu(f, \mathcal{T} f) - \mu(g, \mathcal{T} g)| \, .
\end{aligned}$$

Hence $\mathcal{T}$ *is P-contractive.*

Now we are ready to present our main result.

Theorem 2. *Let* $(\mathcal{M}, \mu)$ *be an M-complete M-metric space and* $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ *be a P-contraction mapping. Then,* $\mathcal{T}$ *has a unique fixed point.*

Proof. Assume that $x_0 \in \mathcal{M}$ and define the sequence $\{x_n\}$ by $x_n = \mathcal{T}^n x_0$. Then we have

$$\begin{aligned}
\mu(x_n, x_n) &= \mu(\mathcal{T} x_{n-1}, \mathcal{T} x_{n-1}) \\
&\leq L \left[\mu(x_{n-1}, x_{n-1}) + |\mu(x_{n-1}, \mathcal{T} x_{n-1}) - \mu(x_{n-1}, \mathcal{T} x_{n-1})|\right] \\
&= L\mu(x_{n-1}, x_{n-1})
\end{aligned}$$

for all $n \in \mathbb{N}$, and so we have

$$\mu(x_n, x_n) \leq L^n \mu(x_0, x_0).$$

This shows that

$$\lim \mu(x_n, x_n) = 0. \tag{2.3}$$

On the other hand we have

$$\begin{aligned}
\mu(x_{n+1}, x_n) &= \mu(\mathcal{T} x_n, \mathcal{T} x_{n-1}) \\
&\leq L \left[\mu(x_n, x_{n-1}) + |\mu(x_n, \mathcal{T} x_n) - \mu(x_{n-1}, \mathcal{T} x_{n-1})|\right] \\
&= L \left[\mu(x_n, x_{n-1}) + |\mu(x_n, x_{n+1}) - \mu(x_{n-1}, x_n)|\right]
\end{aligned} \tag{2.4}$$

for all $n \in \mathbb{N}$. Now, we will consider two cases:
Case 1. Assume there exists $k \in \mathbb{N}$ such that $\mu(x_k, x_{k+1}) \geq \mu(x_{k-1}, x_k)$. Then by (2.4) we have

$$\mu(x_k, x_{k+1}) \leq L\mu(x_k, x_{k+1}).$$

This shows that $\mu(x_k, x_{k+1}) = 0$ (otherwise the last inequality is a contradiction) and also $\mu(x_{k-1}, x_k) = 0$. Hence by triangular inequality of μ, we have

$$\begin{aligned} \mu(x_{k-1}, x_{k+1}) - \mu_{x_{k-1},x_{k+1}} &\leq (\mu(x_{k-1}, x_k) - \mu_{x_{k-1},x_k}) \\ &\quad + (\mu(x_k, x_{k+1}) - \mu_{x_k,x_{k+1}}) \\ &= -[\mu_{x_{k-1},x_k} + \mu_{x_k,x_{k+1}}]. \end{aligned} \tag{2.5}$$

The last inequality is possible only if $\mu_{x_{k-1},x_k} = 0 = \mu_{x_k,x_{k+1}}$ or equivalently

$$\min\{\mu(x_{k-1}, x_{k-1}), \mu(x_k, x_k)\} = 0 = \min\{\mu(x_k, x_k), \mu(x_{k+1}, x_{k+1})\}. \tag{2.6}$$

Now if $\mu(x_{k-1}, x_{k-1}) = 0$, then by (2.1) we have

$$\begin{aligned} \mu(x_k, x_k) &= \mu(\mathcal{T}x_{k-1}, \mathcal{T}x_{k-1}) \\ &\leq L[\mu(x_{k-1}, x_{k-1}) + |\mu(x_{k-1}, \mathcal{T}x_{k-1}) - \mu(x_{k-1}, \mathcal{T}x_{k-1})|] \\ &= 0 \end{aligned}$$

and so $x_k = x_{k-1}$, that is, x_{k-1} is a fixed point of $\mathcal{T}$.

If $\mu(x_{k+1}, x_{k+1}) = 0$, then again by (2.1) we have

$$\begin{aligned} \mu(x_k, x_k) &= \mu(\mathcal{T}x_{k-1}, \mathcal{T}x_{k-1}) \\ &\leq L[\mu(x_{k-1}, x_{k-1}) + |\mu(x_{k-1}, \mathcal{T}x_{k-1}) - \mu(x_{k-1}, \mathcal{T}x_{k-1})|] \\ &= L\mu(x_{k-1}, x_{k-1}) \\ &\leq \mu(x_{k-1}, x_{k-1}) \end{aligned}$$

and so by (2.6) we have $\mu(x_k, x_k) = 0$. Therefore, $x_k = x_{k+1}$, that is, x_k is a fixed point of $\mathcal{T}$.

If $\mu(x_k, x_k) = 0$, then again by (2.1) we have

$$\begin{aligned} \mu(x_{k+1}, x_{k+1}) &= \mu(\mathcal{T}x_k, \mathcal{T}x_k) \\ &\leq L[\mu(x_k, x_k) + |\mu(x_k, \mathcal{T}x_k) - \mu(x_k, \mathcal{T}x_k)|] \\ &= 0 \end{aligned}$$

and so $x_k = x_{k+1}$, that is, x_k is a fixed point of $\mathcal{T}$.

Case 2. Assume that $\mu(x_n, x_{n+1}) < \mu(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Then from (2.4), we have

$$\mu(x_n, x_{n+1}) \leq \frac{2L}{1+L}\mu(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Since $\frac{2L}{1+L} < 1$, this shows that

$$\lim \mu(x_n, x_{n+1}) = 0. \tag{2.7}$$

Now, we show that $\{x_n\}$ is M-Cauchy sequence. For this, it is enough to show that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{M}, \nu)$. First note that from (2.3) and (2.7) we have

$$\lim \mu(x_n, x_n) = 0 = \lim \mu(x_n, x_{n+1})$$

and so we have

$$\lim_{n,\mu\to\infty} \mu_{x_n,x_\mu} = 0 = \lim_{n,\mu\to\infty} M_{x_n,x_\mu}.$$

Assume that $\{x_n\}$ is not Cauchy sequence in $(\mathcal{M}, \nu)$. Then there exist $\varepsilon > 0$ and subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $n_k > m_k > k$ and $\nu(x_{n_k}, x_{m_k}) \geq \varepsilon$. Now, for m_k we can choose the smallest n_k satisfying $n_k > m_k$ and $\nu(x_{n_k}, x_{m_k}) \geq \varepsilon$. Hence $\nu(x_{n_k-1}, x_{m_k}) < \varepsilon$. Hence we have

$$\varepsilon \leq \nu(x_{n_k}, x_{m_k}) \leq \nu(x_{n_k}, x_{n_k-1}) + \nu(x_{n_k-1}, x_{m_k}) < \nu(x_{n_k}, x_{n_k-1}) + \varepsilon.$$

Since $\nu(x_{n_k}, x_{n_k-1}) \to 0$, we have

$$\lim_{k\to\infty} \nu(x_{n_k}, x_{m_k}) = \varepsilon.$$

Again, since

$$\nu(x_{n_k}, x_{m_k}) \leq \nu(x_{n_k}, x_{n_k-1}) + \nu(x_{n_k-1}, x_{m_k-1}) + \nu(x_{m_k-1}, x_{m_k})$$

and

$$\nu(x_{n_k-1}, x_{m_k-1}) \leq \nu(x_{n_k-1}, x_{n_k}) + \nu(x_{n_k}, x_{m_k}) + \nu(x_{m_k}, x_{m_k-1})$$

we have by letting $k \to \infty$ in these inequalities

$$\lim_{k\to\infty} \nu(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Therefore we have

$$\begin{aligned} \lim_{k\to\infty} \mu(x_{n_k}, x_{m_k}) &= \lim_{k\to\infty} \left(\mu(x_{n_k}, x_{m_k}) - 2\mu_{x_{n_k},x_{m_k}} + M_{x_{n_k},x_{m_k}}\right) \\ &= \lim_{k\to\infty} \nu(x_{n_k}, x_{m_k}) = \varepsilon \end{aligned}$$

and in a similar way we can get

$$\lim_{k\to\infty} \mu(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Now by (2.1) and (2.7) we have

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} \mu(x_{n_k}, x_{m_k}) \\ &\leq \lim_{k\to\infty} L\left[\mu(x_{n_k-1}, x_{m_k-1}) + \left|\nu(x_{n_k-1}, x_{n_k}) - \nu(x_{m_k-1}, x_{m_k})\right|\right] \\ &= L\varepsilon, \end{aligned}$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in $(\mathcal{M}, \nu)$ and so it an M-Cauchy sequence in the M-complete M-metric space $(\mathcal{M}, \mu)$. Therefore, there exists $z \in \mathcal{M}$ such that

$$\lim_{n\to\infty} (\mu(x_n, z) - \mu_{x_n z}) = 0$$

and

$$\lim_{n\to\infty} (M_{x_n,z} - \mu_{x_n,z}) = 0.$$

Since $\lim_{n\to\infty} \mu_{x_n z} = 0$ we have $\lim_{n\to\infty} \mu(x_n, z) = 0$ and $\lim_{n\to\infty} M_{x_n,z} = 0$. Therefore, we have $\mu(z, z) = 0$. Now, we want to show that z is a fixed point of $\mathcal{T}$. From (2.1), we have

$$\mu(\mathcal{T}z, \mathcal{T}z) \leq L\left[\mu(z, z) + |\mu(z, \mathcal{T}z) - \mu(z, \mathcal{T}z)|\right] = 0$$

and so $\mu(\mathcal{T}z, \mathcal{T}z) = 0$. On the other hand, from (2.1), we have

$$\mu(x_n, \mathcal{T}z) \leq L\left[\mu(x_{n-1}, z) + |\mu(x_{n-1}, x_n) - \mu(z, \mathcal{T}z)|\right]$$

and letting $n \to \infty$ and using Lemma 1, we get (note that $\mu(z, z) = 0$ and $\lim \mu(x_n, x_n) = 0$)

$$\begin{aligned}
\mu(z, \mathcal{T}z) &= \mu(z, \mathcal{T}z) - \mu_{z,\mathcal{T}z} \\
&= \lim_{n\to\infty} \left(\mu(x_n, \mathcal{T}z) - \mu_{x_n,\mathcal{T}z}\right) \\
&= \lim_{n\to\infty} \mu(x_n, \mathcal{T}z) - \lim_{n\to\infty} \mu_{x_n,\mathcal{T}z} \\
&= \lim_{n\to\infty} \mu(x_n, \mathcal{T}z) \\
&\leq \lim_{n\to\infty} L\left[\mu(x_{n-1}, z) + |\mu(x_{n-1}, x_n) - \mu(z, \mathcal{T}z)|\right] \\
&= L\mu(z, \mathcal{T}z).
\end{aligned}$$

This is possible only if $\mu(z, \mathcal{T}z) = 0$. Therefore we have

$$\mu(z, \mathcal{T}z) = \mu(\mathcal{T}z, \mathcal{T}z) = \mu(z, z)$$

and so $z = \mathcal{T}z$.

Now, let w also be a fixed point of $\mathcal{T}$. Then from (2.1), we have

$$\begin{aligned}
\mu(w, w) &= \mu(\mathcal{T}w, \mathcal{T}w) \\
&\leq L\left[\mu(w, w) + |\mu(w, \mathcal{T}w) - \mu(w, \mathcal{T}w)|\right] \\
&= L\mu(w, w),
\end{aligned}$$

which shows that $\mu(w, w) = 0$. Hence we have

$$\begin{aligned}
\mu(z, w) &= \mu(\mathcal{T}z, \mathcal{T}w) \\
&\leq L\left[\mu(z, w) + |\mu(z, \mathcal{T}z) - \mu(w, \mathcal{T}w)|\right] \\
&= L\left[\mu(z, w) + |\mu(z, z) - \mu(w, w)|\right] \\
&= L\mu(z, w).
\end{aligned}$$

Therefore, we have $\mu(z, w) = 0 = \mu(z, z) = \mu(w, w)$, hence $z = w$. □

Remark 2 and Example 2 show that Theorem 2 is a proper generalization of Theorem 1. Here we provide a further example.

Example 4. *Let* $\mathcal{M} = \mathbb{N} \times \mathbb{N} \subseteq \mathbb{R}^2$ *with*

$$\mu(m, n) = |m_1 - n_1| + \min\{m_2, n_2\},$$

where $m = (m_1, m_2)$*,* $n = (n_1, n_2)$*, then* $(\mathcal{M}, \mu)$ *is an M-complete M-metric space. Define* $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ *by*

$$\mathcal{T}m = \mathcal{T}(m_1, m_2) = \begin{cases} (1, 0) & , \quad m_1 \in \{0, 1, 2\} \\ (0, 0) & , \quad \textit{otherwise} \end{cases}.$$

Now, we claim that $\mathcal{T}$ is P-contraction mapping with $L = \frac{1}{2}$, that is, we have

$$\mu(\mathcal{T}m, \mathcal{T}n) \leq \frac{1}{2} \left[\mu(m, n) + |\mu(m, \mathcal{T}m) - \mu(n, \mathcal{T}n)| \right] \tag{2.8}$$

for all $m = (m_1, m_2)$, $n = (n_1, n_2) \in \mathcal{M}$. It is clear that (2.8) holds for both $m_1, n_1 \in \{0, 1, 2\}$ and $m_1, n_1 \notin \{0, 1, 2\}$, which in both cases the left-hand side of (2.8) is zero. Now assume $m_1 \in \{0, 1, 2\}$ and $n_1 \notin \{0, 1, 2\}$, then we have

$$\begin{aligned}
\mu(\mathcal{T}m, \mathcal{T}n) &= \mu((1, 0), (0, 0)) = 1, \\
\mu(m, n) &= |m_1 - n_1| + \min\{m_2, n_2\} = n_1 - m_1 + \min\{m_2, n_2\}, \\
\mu(m, \mathcal{T}m) &= \mu((m_1, m_2), (1, 0)) = |m_1 - 1|, \\
\mu(n, \mathcal{T}n) &= \mu((n_1, n_2), (0, 0)) = n_1
\end{aligned}$$

and hence

$$\begin{aligned}
\mu(m, n) + |\mu(m, \mathcal{T}m) - \mu(n, \mathcal{T}n)| &= n_1 - m_1 + \min\{m_2, n_2\} + ||m_1 - 1| - n_1| \\
&= 2n_1 - m_1 - |m_1 - 1| + \min\{m_2, n_2\} \\
&\geq 2n_1 - 3.
\end{aligned}$$

Therefore we have

$$\mu(\mathcal{T}m, \mathcal{T}n) = 1 \leq n_1 - \frac{3}{2} = \frac{1}{2}(2n_1 - 3) \leq \frac{1}{2} \left[\mu(m, n) + |\mu(m, \mathcal{T}m) - \mu(n, \mathcal{T}n)| \right].$$

This shows that $\mathcal{T}$ is a P-contraction mapping, and hence by Theorem 2 it has a unique fixed point. On the other hand, for $m = (2, 0)$ and $n = (3, 1)$ we have

$$\mu(\mathcal{T}m, \mathcal{T}n) = 1 = \mu(m, n),$$

and hence T is not a contraction mapping. Therefore Theorem 1 cannot be applied to this example.

Now, we present a fixed point theorem for P-contractive mappings.

Theorem 3. *Let $(\mathcal{M}, \mu)$ be an M-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a P-contractive mapping having 0-property. Consider the function $f : \mathcal{M} \to \mathbb{R}$ defined by $f(x) = \mu(x, \mathcal{T}x)$. If there exists $x_0 \in \mathcal{M}$ such that $f(x_0) \leq f(\mathcal{T}x_0)$, then $\mathcal{T}$ has a unique fixed point.*

Proof. Let x_0 be point as mentioned. In this case, if $\mu(x_0, \mathcal{T}x_0) > 0$, then by the P-contractivity of $\mathcal{T}$, we have

$$\begin{aligned}
f(\mathcal{T}x_0) &= \mu(\mathcal{T}x_0, \mathcal{T}\mathcal{T}x_0) \\
&< \mu(x_0, \mathcal{T}x_0) + |\mu(x_0, \mathcal{T}x_0) - \mu(\mathcal{T}x_0, \mathcal{T}\mathcal{T}x_0)| \\
&= f(x_0) + |f(x_0) - f(\mathcal{T}x_0)| \\
&= f(x_0) + f(\mathcal{T}x_0) - f(x_0) \\
&= f(\mathcal{T}x_0),
\end{aligned}$$

which is a contradiction. Hence, we have $\mu(x_0, \mathcal{T}x_0) = 0$. Therefore we get $\mu_{x_0, \mathcal{T}x_0} = 0$, and so by the 0-property of $\mathcal{T}$, we have $x_0 = \mathcal{T}x_0$. The uniqueness of fixed point is easily seen by the P-contractivity of $\mathcal{T}$. $\square$

Remark 3. *If the function $f(x) = \mu(x, \mathcal{T}x)$ attains its minimum, then there exists $x_0 \in \mathcal{M}$ such that $f(x_0) = \inf f(\mathcal{M})$. In this case we have $f(x_0) \leq f(\mathcal{T}x_0)$ and so by Theorem 3, x_0 is the unique fixed point of $\mathcal{T}$.*

Theorem 4. *Let $(\mathcal{M}, \mu)$ be a compact M-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a P-contractive mapping having 0-property. If the function $f(x) = \mu(x, \mathcal{T}x)$ is lower semicontinuous, then $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$.*

Proof. By Theorem 2.5.4 of [2], f attains its minimum. Then, from Theorem 3 and Remark 3, $\mathcal{T}$ has a unique fixed point. □

3. Application

In this section, to show the applicability of both Theorem 1 and also Theorem 2 we deal with an existence and uniqueness theorem for a second-order (p, q)-difference Langevin equation with boundary conditions of the form

$$\begin{cases} D_{p,q}\left(D_{p,q} + \gamma\right) x(t) = f\left(t, x(t)\right), \ t \in [0, 1], \\ x(0) = \alpha, \ D_{p,q}x(0) = \beta, \end{cases} \tag{3.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $0 < q < p \leq 1$, and γ, α, β are given constants. First, let us review basic definitions and theorems about (p, q)-calculus which can be found in [22]. The (p, q)-derivative and (p, q)-integral of a function g are defined by the formulas, for constants $0 < q < p \leq 1$,

$$D_{p,q}g(t) = \begin{cases} \dfrac{g(pt) - g(qt)}{(p - q)t}, & t \neq 0, \\ \lim\limits_{t \to 0} D_{p,q}g(t), & t = 0, \end{cases}$$

and

$$\int_0^t g(s)d_{p,q}s = (p - q)t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} g\left(\frac{q^n}{p^{n+1}}t\right)$$

provided the right hand side converges.

The (p, q)-integration by parts is given by

$$\int_a^b g(pt)D_{p,q}h(t)d_{p,q}t = g(t)h(t)|_a^b - \int_a^b h(qt)D_{p,q}g(t)d_{p,q}t, \tag{3.2}$$

$$D_{p,q}\left(\int_0^t g(s)d_{p,q}s\right) = g(t), \tag{3.3}$$

$$\int_0^t D_{p,q}g(s)d_{p,q}s = g(t) - g(0), \tag{3.4}$$

$$\int_a^t D_{p,q}g(s)d_{p,q}s = g(t) - g(a) \text{ for } a \in (0, t). \tag{3.5}$$

We can find more informations about q-calculus, (p,q)-calculus and their relations between fixed point theory in [9–13, 18–21].

Assuming $f(t,x(t)) = 0$ for each $t \in [p,1]$, it can be seen that (3.1) is equivalent to the integral equation defined by

$$x(t) = \alpha + (\beta + \gamma\alpha)\, t - \gamma \int_0^t x(s) d_{p,q}s + \int_0^{\frac{t}{p}} (t - pqs) f(s, x(s)) d_{p,q}s. \tag{3.6}$$

Assume that $C[0,1]$ is the space of all real valued continuous functions defined on $[0,1]$. Define an operator $\mathcal{T} : C[0,1] \to C[0,1]$ by

$$\mathcal{T}u(t) = \alpha + (\beta + \gamma\alpha)\, t - \gamma \int_0^t u(s) d_{p,q}s + \int_0^{\frac{t}{p}} (t - pqs) f(s, u(s)) d_{p,q}s.$$

Hence, if u is a fixed point of $\mathcal{T}$, then it is a solution of the integral equation (3.6) and so identically, we can say that it is a solution of (p,q)-difference Langevin equation (3.1).

Here, we will consider the space $\mathcal{X} = C[0,1]$. Define an M-metric on $\mathcal{X}$ as

$$\mu(u,v) = \max\left\{ \sup_{t\in[0,1]} |u(t)|, \sup_{t\in[0,1]} |v(t)| \right\} + \sup_{t\in[0,1]} \{|u(t) - v(t)|\}$$

for $u, v \in \mathcal{X}$. In this case, $(\mathcal{X}, \mu)$ is an M-complete M-metric space.

Now consider the following assumptions:

(A1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ continuous and $f(t,x) = 0$ for each $t \in [p,1]$,
(A2) $\alpha(1+\gamma) + \beta = 0$,
(A3) there exists $L_1 \geq 0$ such that $f(t,x) \leq L_1 x$ for all $x \in [0,\infty)$,
(A4) there exists $L_2 \geq 0$ such that $|f(t,x) - f(t,y)| \leq L_2 |x-y|$ for all $x, y \in [0,\infty)$.

Theorem 5. *In addition to (A1)–(A4), suppose that*

$$|\gamma| + \max\{L_1, L_2\} \frac{p + q - qp^2}{p^2(p+q)} < 1,$$

then the (p,q)-difference Langevin equation (3.1) has a unique solution.

Proof. Consider the M-metric space $(\mathcal{X}, \mu)$ mentioned above. Then $\mathcal{T}$ is a self mapping of $\mathcal{X}$ because of (A1) and (A2). Also from (A2)–(A4) we have, for all $u, v \in \mathcal{X}$

$$\begin{aligned}
\mu_1 \quad : &= \max\left\{ \sup_{t\in[0,1]} \mathcal{T}u(t), \sup_{t\in[0,1]} \mathcal{T}v(t) \right\} \\
&= \max\left\{ \begin{array}{l} \sup_{t\in[0,1]} \left\{ \alpha + (\beta+\gamma\alpha)\, t - \gamma \int_0^t u(s) d_{p,q}s + \int_0^{\frac{t}{p}} (t - pqs) f(s,u(s)) d_{p,q}s \right\}, \\ \sup_{t\in[0,1]} \left\{ \alpha + (\beta+\gamma\alpha)\, t - \gamma \int_0^t v(s) d_{p,q}s + \int_0^{\frac{t}{p}} (t - pqs) f(s,v(s)) d_{p,q}s \right\} \end{array} \right\} \\
&\leq \max\left\{ \begin{array}{l} \alpha + (\beta+\gamma\alpha) + |\gamma| \sup_{t\in[0,1]} u(t) + \sup_{t\in[0,1]} \int_0^{\frac{t}{p}} (t - pqs) f(s,u(s)) d_{p,q}s, \\ \alpha + (\beta+\gamma\alpha) + |\gamma| \sup_{t\in[0,1]} v(t) + \sup_{t\in[0,1]} \int_0^{\frac{t}{p}} (t - pqs) f(s,v(s)) d_{p,q}s \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max\left\{\begin{array}{l} |\gamma| \sup_{t\in[0,1]} u(t) + L_1 \sup_{t\in[0,1]} u(t) \sup_{t\in[0,1]} \int_0^{\frac{t}{p}} (t-pqs) d_{p,q}s, \\ |\gamma| \sup_{t\in[0,1]} v(t) + L_1 \sup_{t\in[0,1]} v(t) \sup_{t\in[0,1]} \int_0^{\frac{t}{p}} (t-pqs) d_{p,q}s \end{array}\right\} \\
&= \left(|\gamma| + L_1 \frac{p+q-qp^2}{p^2(p+q)}\right) \max\left\{\sup_{t\in[0,1]} u(t), \sup_{t\in[0,1]} v(t)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_2 \quad : &= \sup_{t\in[0,1]} \{|\mathcal{T}u(t) - \mathcal{T}v(t)|\} \\
&= \sup_{t\in[0,1]} \left\{\left| \begin{array}{l} \gamma \int_0^t u(s) d_{p,q}s - \int_0^{\frac{t}{p}} (t-pqs) f(s,u(s)) d_{p,q}s \\ -\gamma \int_0^t v(s) d_{p,q}s + \int_0^{\frac{t}{p}} (t-pqs) f(s,v(s)) d_{p,q}s \end{array}\right|\right\} \\
&= \sup_{t\in[0,1]} \left\{\left|\gamma \int_0^t [u(s)-v(s)]\, d_{p,q}s + \int_0^{\frac{t}{p}} (t-pqs) \left[f(s,v(s)) - f(s,u(s))\right] d_{p,q}s\right|\right\} \\
&\leq \sup_{s\in[0,1]} \{|u(s)-v(s)|\} \sup_{t\in[0,1]} \left\{\left|\gamma \int_0^t d_{p,q}s + L_2 \int_0^{\frac{t}{p}} (t-pqs) d_{p,q}s\right|\right\} \\
&\leq \left(|\gamma| + L_2 \frac{p+q-qp^2}{p^2(p+q)}\right) \sup_{s\in[0,1]} \{|u(s)-v(s)|\}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\mu(\mathcal{T}u, \mathcal{T}v) &= \mu_1 + \mu_2 \\
&\leq \left(|\gamma| + L_1 \frac{p+q-qp^2}{p^2(p+q)}\right) \max\left\{\sup_{t\in[0,1]} u(t), \sup_{t\in[0,1]} v(t)\right\} \\
&\quad + \left(|\gamma| + L_2 \frac{p+q-qp^2}{p^2(p+q)}\right) \sup_{s\in[0,1]} \{|u(s)-v(s)|\} \\
&\leq \lambda \left\{\max\left\{\sup_{t\in[0,1]} u(t), \sup_{t\in[0,1]} v(t)\right\} + \sup_{t\in[0,1]} \{|u(t)-v(t)|\}\right\} \\
&= \lambda \mu(u,v),
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda &= \max\left\{|\gamma| + L_1 \frac{p+q-qp^2}{p^2(p+q)}, |\gamma| + L_2 \frac{p+q-qp^2}{p^2(p+q)}\right\} \\
&= |\gamma| + \max\{L_1, L_2\} \frac{p+q-qp^2}{p^2(p+q)} < 1.
\end{aligned}
$$

Therefore, by Theorem 1 (and also Theorem 2) $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$. That is, the (p,q)-difference Langevin equation (3.1) has a unique solution. □

4. Conclusions

We introduced the notion of P-contractive mappings on M-metric spaces. Next, we presented a few fixed-point outcomes for these mappings. We also provided a few examples to support our theoretical findings. Finally, we give an existence and uniqueness theorem for a second-order (p, q)-difference Langevin equation. This study yields novel results that advance fixed-point theory and applications. The introduced P-contractiveness can be used to prove new theoretical conclusions for future research, and the theoretical results can be applied to produce the existence and uniqueness results for specific types of equations, including fractional order integral and differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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