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Research article

Existence of global solution to 3D density-dependent incompressible Navier-Stokes equations

Jianxia He* **and Ming Li**

Center for Nonlinear Studies, School of Mathematics, Northwest University, Xi'an 710127, China

* **Correspondence:** Email: jxhe2022@163.com.

Abstract: In this article, we are committed to studying the three-dimensional incompressible Navier-Stokes equations, where the viscosity depends on density according to a power law. We investigate the Cauchy problem by constructing an approximation system and bootstrap argument. Finally, we establish the existence of a global strong solution under the conditions of small initial data and the compatibility condition. Meanwhile, the algebraic decay-in-time rates for the solution are also obtained. It is worth pointing out that the degradation of viscosity is allowed.

Keywords: Navier-Stokes equations; density-dependent viscosity; degenerate viscosity; strong solution
Mathematics Subject Classification: 35Q30, 76D05

1. Introduction

As we all know, the Navier-Stokes equations have a profound physical background and play an extremely important role in fluid mechanics. This paper is devoted to the following Navier-Stokes equations in $\mathbb{R}^3$, which can characterize the motion of viscous inhomogeneous incompressible fluids:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu \nabla u) + \nabla P = 0, \\ \operatorname{div} u = 0, \\ (\rho, \rho u)(x, 0) = (\rho_0, \rho_0 u_0)(x). \end{cases} \tag{1.1}$$

Here, $\rho = \rho(x,t),\ u = (u^1, u^2, u^3)(x,t)$ and $P(x,t)$ denote the density, velocity and pressure of the fluid, respectively. μ stands for the viscosity coefficient. In this article, we focus on the Cauchy problem for the system (1.1) with (ρ, u) vanishing at infinity and

$$\mu = \rho^\alpha \ (0 < \alpha < 1). \tag{1.2}$$

The Navier-Stokes equations have always been a hot topic of concern for mathematical researchers. To date, many meaningful research results have been achieved regarding this topic. The mathematical research on incompressible Navier-Stokes equations began with a simple case in which the viscosity μ is a positive constant and the initial density is far from a vacuum. Antontesv and Kazhikov initially proved the global existence of weak solutions and further demonstrated the local existence of strong solutions in [5, 23]. Later, for the 2D and 3D initial-boundary value problems, Ladyzhenskaya and Solonnikov [24] confirmed that the local strong solutions are unique. In fact, this local strong solution is still global, allowing large initial data in two dimensions and requiring small initial data in three dimensions. More results on the well-posedness of solutions can be found in [1, 11, 12] and their references. Under the condition that the initial density contains a vacuum, the global weak solutions were first established by Simon [29]. Based on the vacuum problem, Choe and Kim [9] creatively introduced the compatibility condition given by

$$-\mu\triangle u_0 + \nabla P_0 = \rho_0^{\frac{1}{2}} g, \quad \text{for some } (P_0, g) \in H^1 \times L^2 \tag{1.3}$$

and obtained the local strong solution. Under the condition that $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is small, Craig et al. [10] extended the local solution to the global solution for the whole 3D space.

A more general situation is that the viscosity μ depends on the density, which is more in line with the actual background. At this point, the well-posedness problem of the solution becomes more challenging. Lions [26] established the global weak solutions with the initial density containing a vacuum, which is a big breakthrough. Regarding the 2D Cauchy problem, Gui and Zhang [16] demonstrated the global well-posedness of strong solutions under the assumption that the initial density ρ_0 perturbs near 1. Regarding the initial density allowing for a vacuum, similar to the case of constant viscosity, Cho and Kim [7] obtained a unique local strong solution which required the initial data to satisfy the compatibility condition. For $\mu(\rho) \geq \underline{\mu}$, $\forall\rho \in [0, \infty)$, the unique global strong solution was proved by Huang and Wang [20, 21] under the condition of smallness on $\|\nabla\mu(\rho_0)\|_{L^q}$ $(2 < q < \infty)$ in 2D and on $\|\nabla u_0\|_{L^2}$ in 3D over a bounded domain. The solvability of variable coefficient problem has been studied by many people (see [2, 3, 13, 19, 25, 30]). Later, Lü and Song [27] successfully removed the compatibility condition. On this basis, under the condition that $\|u_0\|_{\dot{H}^\beta}$ $(\frac{1}{2} < \beta \leq 1)$ is small, the global existence and uniqueness of strong solutions to the 3D Cauchy problem were obtained by He et al. [17]. Recently, for the degenerate viscosity case given by $\mu(\rho) = \rho$, He and Guo [18] established the existence of a global strong solution, which required small initial data and a compatibility condition. There are some interesting studies that can be references, see [6, 14, 22, 28].

For a wider range of cases, such as $\mu(\rho) = \rho^\alpha$, the solvability of the Navier-Stokes equations deserves further research. In this case, the problem becomes more complex. Strong degradation brings difficulties to our research, which requires us to fully utilize the structure of the equation. For the case of constant viscosity, the parabolic structure of the momentum equation plays an important role in high-order estimates. However, for the case that we are considering (i.e., $\mu(\rho) = \rho^\alpha$), the parabolicity of the momentum equation may disappear. As a result, high regularity or uniqueness cannot be directly expected from $(1.1)_2$. Therefore, we need some new ideas and to make precise estimates.

Under the assumptions of small initial data and the compatibility condition, we obtain the global existence of a strong solution to the Cauchy problem. The main conclusion is as follows:

Theorem 1.1. *For $3 < k \le 6$ and $2 \le q < 6$, assume that the initial data (ρ_0, u_0) satisfy*

$$0 \le \rho_0 \le \bar{\rho}, \quad \rho_0 \in L^{\frac{3}{2}} \cap H^1 \cap D^{1,k}, \quad \nabla\rho_0^{\frac{\alpha-1}{2}} \in L^{\frac{3}{2}} \cap D^{1,q}, \quad u_0 \in D^1_{0,\sigma} \cap H^3, \tag{1.4}$$

and the compatibility condition is given by

$$-\operatorname{div}(\rho_0^\alpha \nabla u_0) + \nabla P_0 = \rho_0^{\frac{1+\alpha}{2}} g, \quad \textit{for some } (P_0, g) \in H^2 \times H^1. \tag{1.5}$$

There exist two positive constants η_0 and ε_0 that depend on $\bar{\rho}, k, q$ and $\|g\|_{H^1}$ such that if

$$\|\nabla\rho_0^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}} \cap D^{1,q}} \le \eta_0, \quad \|u_0\|_{H^1} \le \varepsilon_0, \tag{1.6}$$

then the system (1.1) *with* (1.2) *has a global strong solution (ρ, u, P) with the following:*

$$\begin{cases} \rho \in C([0,\infty); L^{\frac{3}{2}} \cap H^1 \cap D^{1,k}), \quad \nabla\rho^{\frac{\alpha-1}{2}} \in C([0,\infty); L^{\frac{3}{2}} \cap D^{1,q}), \\ \nabla u \in C([0,\infty); H^1) \cap L^2(0,\infty; H^1), \quad \nabla P \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; L^2), \\ \rho^{\frac{1-\alpha}{2}} u_t \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; L^2), \\ \nabla u_t \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; L^2 \cap L^6). \end{cases} \tag{1.7}$$

Furthermore, it holds that

$$\|\nabla u(\cdot, t)\|_{L^2} \le C_0 t^{-\frac{1}{2}}, \tag{1.8}$$

and

$$\|\rho^{\frac{1-\alpha}{2}} u_t(\cdot, t)\|_{L^2} + \|\nabla u_t(\cdot, t)\|_{L^2} + \|\nabla^2 u(\cdot, t)\|_{L^2} + \|\nabla P(\cdot, t)\|_{L^2} \le C_0 t^{-1}, \tag{1.9}$$

where C_0 is a positive constant that depends on $\bar{\rho}$ and $\|u_0\|_{H^3}$.

Remark 1.1. *For the case that $\mu(\rho) = \rho$, we have previously studied it in detail in [18] and obtained the global strong solution. This article is its promotion and it has a wider scope of application.*

Theorem 1.1 will be proved by constructing an approximation system which has a unique local strong solution and bootstrap argument. We first establish uniform a priori estimates, and the key is to obtain the regularity theory for the Stokes system, which does not depend on the lower bound of viscosity. By combining the local existence of approximated solution with time-uniform a priori estimates, the global approximated solution is obtained. Finally, by using the standard compactness theory, the strong solution of the original system is established.

2. Preliminaries

The results on the existence of solutions to the incompressible Navier-Stokes equations are for non-degenerate viscosity. So, we first construct an approximation system in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\left((\rho^\alpha + \delta)\nabla u\right) + \nabla P = 0, \\ \operatorname{div} u = 0, \\ (\rho, \rho u)(x, 0) = (\rho_{0,\delta}, \rho_{0,\delta} u_0)(x), \\ u(x,t) \to 0, \qquad \text{as } |x| \to \infty, \end{cases} \tag{2.1}$$

where $0 < \delta < 1$ and $\rho_{0,\delta}(x) = \rho_0(x) + \delta$.

The local well-posedness of strong solutions to the system (2.1) is guaranteed by the results of [8].

Lemma 2.1. *Assume that the initial data* $(\rho_{0,\delta}, u_0)$ *satisfy*

$$0 \le \rho_{0,\delta} - \delta \in L^{\frac{3}{2}} \cap H^1 \cap D^{1,k}(3 < k \le 6), \;\; u_0 \in D^1_{0,\sigma} \cap D^2. \tag{2.2}$$

Then, there exist a small time $T_0 > 0$ *and a unique strong solution* $(\rho^\delta, u^\delta, P^\delta)$ *to the system* (2.1) *on* $\mathbb{R}^3 \times (0, T_0]$.

Our high-order a priori estimates will be obtained based on the following regularity theory for Stokes equations, which does not rely on the lower bound of viscosity.

Lemma 2.2. *Assume that* ρ *satisfies*

$$0 \le \rho \le \bar{\rho}, \;\; \nabla\rho^{\frac{\alpha-1}{2}} \in L^3 \cap D^{1,q} \, (2 \le q < 6), \;\; \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3} \le \zeta_0 \, (\zeta_0 \, see \, (2.12)). \tag{2.3}$$

Let $(u, P) \in H^1_{0,\sigma} \times L^2$ *be the unique weak solution to the following problem:*

$$\begin{cases} -\operatorname{div}((\rho^\alpha + \delta)\nabla u) + \nabla P = F, & x \in \mathbb{R}^3, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3, \\ u(x) \to 0, & |x| \to \infty. \end{cases} \tag{2.4}$$

Then, the following conclusions are valid:
(1) If $\rho^{-\alpha}F \in L^r$ *with* $r \in [2, 3)$*, then*

$$\|\nabla^2 u\|_{L^r} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^r} \le C\|\rho^{-\alpha}F\|_{L^r}. \tag{2.5}$$

(2) If $\rho^{-\alpha}F \in L^p \cap L^{\frac{6p}{p+6}}$ *with* $p \in [3, 6)$*, then*

$$\|\nabla^2 u\|_{L^p} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^p} \le C\|\rho^{-\alpha}F\|_{L^p} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|\rho^{-\alpha}F\|_{L^{\frac{6p}{p+6}}}. \tag{2.6}$$

(3) If $\rho^{-\alpha}F \in L^2$*,* $\rho^{-\alpha}\nabla F \in L^2$*, and* $F\rho^{-2\alpha}\nabla\rho^\alpha \in L^2$*, then*

$$\begin{aligned} \|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{H^1} \le{}& C\|\rho^{-\alpha}F\|_{L^2} + C\|\rho^{-\alpha}\nabla F\|_{L^2} + C\|F\rho^{-2\alpha}\nabla\rho^\alpha\|_{L^2} \\ &+ C\|(|\nabla u| + |(\rho^\alpha + \delta)^{-1}P|)\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{2q}{q-2}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}. \end{aligned} \tag{2.7}$$

(4) Further, if $F = \operatorname{div} G + H$ *with* $\rho^{-\alpha}G \in L^s \cap L^{\frac{6s}{s+6}}$*, and* $\rho^{-\alpha}H \in L^{\frac{3s}{s+3}}$ *for some* $s \in [\frac{3}{2}, +\infty)$*, then*

$$\|\nabla u\|_{L^s} + \|(\rho^\alpha + \delta)^{-1}P\|_{L^s} \le C\|\rho^{-\alpha}G\|_{L^s} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|\rho^{-\alpha}G\|_{L^{\frac{6s}{s+6}}} + C\|\rho^{-\alpha}H\|_{L^{\frac{3s}{s+3}}}. \tag{2.8}$$

Here, the constant C depends only on $\bar{\rho}$*.*

Proof. Equation $(2.4)_1$ can be rewritten as

$$-\triangle u + \nabla(\frac{P}{\rho^\alpha + \delta}) = \frac{F}{\rho^\alpha + \delta} + \frac{\nabla u \cdot \nabla\rho^\alpha}{\rho^\alpha + \delta} - \frac{P\nabla\rho^\alpha}{(\rho^\alpha + \delta)^2}. \tag{2.9}$$

According to the theory of the Stokes system and Gagliardo-Nirenberg inequality, we have the following for $r \in [2, 3)$:

$$\begin{aligned}
&\|\nabla^2 u\|_{L^r} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^r} \\
\le& C\|(\rho^\alpha + \delta)^{-1}F\|_{L^r} + C\|\nabla u \cdot (\rho^\alpha + \delta)^{-1}\nabla\rho^\alpha\|_{L^r} + C\|(\rho^\alpha + \delta)^{-2}P\nabla\rho^\alpha\|_{L^r} \\
\le& C\|\rho^{-\alpha}F\|_{L^r} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}(\|\nabla u\|_{L^{\frac{3r}{3-r}}} + \|(\rho^\alpha + \delta)^{-1}P\|_{L^{\frac{3r}{3-r}}}) \\
\le& C\|\rho^{-\alpha}F\|_{L^r} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}(\|\nabla^2 u\|_{L^r} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^r}).
\end{aligned}$$

If $\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3} \le \zeta_1 \triangleq \min\{(2C)^{-1}, 1\}$, then (2.5) is valid.

Similarly, for $p \in [3, 6)$,

$$\begin{aligned}
&\|\nabla^2 u\|_{L^p} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^p} \\
\le& C\|(\rho^\alpha + \delta)^{-1}F\|_{L^p} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}(\|\nabla u\|_{L^{\frac{6p}{6-p}}} + \|(\rho^\alpha + \delta)^{-1}P\|_{L^{\frac{6p}{6-p}}}) \\
\le& C\|\rho^{-\alpha}F\|_{L^p} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}(\|\nabla^2 u\|_{L^{\frac{6p}{p+6}}} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^{\frac{6p}{p+6}}}).
\end{aligned} \tag{2.10}$$

Then, $\frac{6p}{p+6} \in [2, 3)$ given that $p \in [3, 6)$. Combining (2.5) and (2.10) yields (2.6).

On the other hand,

$$\begin{aligned}
&\|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{H^1} \\
\le& C\|(\rho^\alpha + \delta)^{-1}F\|_{H^1} + C\|\nabla u \cdot (\rho^\alpha + \delta)^{-1}\nabla\rho^\alpha - (\rho^\alpha + \delta)^{-2}P\nabla\rho^\alpha\|_{H^1} \\
\le& C\|\rho^{-\alpha}F\|_{L^2} + C\|\nabla u \cdot (\rho^\alpha + \delta)^{-1}\nabla\rho^\alpha\|_{L^2} + C\|(\rho^\alpha + \delta)^{-2}P\nabla\rho^\alpha\|_{L^2} + C\|\rho^{-\alpha}\nabla F\|_{L^2} \\
&+ C\|F(\rho^\alpha + \delta)^{-2}\nabla\rho^\alpha\|_{L^2} + C\|\nabla(\nabla u \cdot (\rho^\alpha + \delta)^{-1}\nabla\rho^\alpha - (\rho^\alpha + \delta)^{-2}P\nabla\rho^\alpha)\|_{L^2} \\
\le& \frac{1}{2}(\|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{H^1}) + C\|\rho^{-\alpha}F\|_{L^2} + C\|\rho^{-\alpha}\nabla F\|_{L^2} \\
&+ C\|F\rho^{-2\alpha}\nabla\rho^\alpha\|_{L^2} + C\|(|\nabla u| + |(\rho^\alpha + \delta)^{-1}P|)\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{2q}{q-2}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q},
\end{aligned}$$

where, in the last inequality, one can use the following:

$$\begin{aligned}
&\|\nabla(\nabla u \cdot (\rho^\alpha + \delta)^{-1}\nabla\rho^\alpha - (\rho^\alpha + \delta)^{-2}P\nabla\rho^\alpha)\|_{L^2} \\
=& \|\frac{2\alpha}{\alpha - 1}\nabla(\nabla u \cdot \frac{\rho^\alpha}{\rho^\alpha + \delta}\rho^{\frac{1-\alpha}{2}}\nabla\rho^{\frac{\alpha-1}{2}} - \frac{P}{\rho^\alpha + \delta}\frac{\rho^\alpha}{\rho^\alpha + \delta}\rho^{\frac{1-\alpha}{2}}\nabla\rho^{\frac{\alpha-1}{2}})\|_{L^2} \\
\le& C\|(|\nabla^2 u| + |\nabla(\frac{P}{\rho^\alpha + \delta})|)|\nabla\rho^{\frac{\alpha-1}{2}}|\|_{L^2} + C\|(|\nabla u| + |\frac{P}{\rho^\alpha + \delta}|)(|\frac{\nabla\rho^\alpha}{\rho^\alpha + \delta}||\nabla\rho^{\frac{\alpha-1}{2}}| \\
&+ |\nabla\rho^{\frac{1-\alpha}{2}}||\nabla\rho^{\frac{\alpha-1}{2}}| + |\rho^{\frac{1-\alpha}{2}}||\nabla^2\rho^{\frac{\alpha-1}{2}}| + |\frac{\rho^\alpha\nabla\rho^\alpha}{(\rho^\alpha + \delta)^2}\nabla\rho^{\frac{\alpha-1}{2}}|)\|_{L^2} \\
\le& C(\|\nabla^2 u\|_{L^6} + \|\nabla(\frac{P}{\rho^\alpha + \delta})\|_{L^6})\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3} + C(\|\nabla u\|_{L^6} + \|\frac{P}{\rho^\alpha + \delta}\|_{L^6})\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}^2 \\
&+ C\|(|\nabla u| + |\frac{P}{\rho^\alpha + \delta}|)\rho^{\frac{1-\alpha}{2}}|\nabla^2\rho^{\frac{\alpha-1}{2}}|\|_{L^2} \\
\le& \frac{1}{2}(\|\nabla^3 u\|_{L^2} + \|\nabla^2(\frac{P}{\rho^\alpha + \delta})\|_{L^2}) + C\|\rho^{-\alpha}F\|_{L^2} + C\|(|\nabla u| + |\frac{P}{\rho^\alpha + \delta}|)\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{2q}{q-2}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}.
\end{aligned}$$

Hence, (2.7) holds.

Furthermore, if $F = \operatorname{div} G + H$, we rewrite (2.9) as

$$-\triangle u + \nabla(\frac{P}{\rho^\alpha + \delta}) = \operatorname{div}(\frac{G}{\rho^\alpha + \delta}) + \tilde{H}, \tag{2.11}$$

where

$$\tilde{H} \triangleq \frac{G \cdot \nabla \rho^\alpha}{(\rho^\alpha + \delta)^2} + \frac{H}{\rho^\alpha + \delta} + \frac{\nabla u \cdot \nabla \rho^\alpha}{\rho^\alpha + \delta} - \frac{P \nabla \rho^\alpha}{(\rho^\alpha + \delta)^2}.$$

It follows from $(2.4)_{2,3}$, (2.11) and the Sobolev inequality that, for $s \in [\frac{3}{2}, +\infty)$,

$$\begin{aligned} \|(\rho^\alpha + \delta)^{-1} P\|_{L^s} &\le C\|\nabla u\|_{L^s} + C\|(\rho^\alpha + \delta)^{-1} G\|_{L^s} + C\|(-\Delta)^{-1} \operatorname{div} \tilde{H}\|_{L^s} \\ &\le C\|\nabla u\|_{L^s} + C\|\rho^{-\alpha} G\|_{L^s} + C\|\tilde{H}\|_{L^{\frac{3s}{s+3}}}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla u\|_{L^s} &\le C\|\nabla \times u\|_{L^s} \\ &\le C\|(-\Delta)^{-1} \nabla \times \operatorname{div}\left((\rho^\alpha + \delta)^{-1} G\right)\|_{L^s} + C\|(-\Delta)^{-1} \nabla \times \tilde{H}\|_{L^s} \\ &\le C\|\rho^{-\alpha} G\|_{L^s} + C\|\tilde{H}\|_{L^{\frac{3s}{s+3}}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\nabla u\|_{L^s} + \|(\rho^\alpha + \delta)^{-1} P\|_{L^s} &\le C\|\rho^{-\alpha} G\|_{L^s} + C\|\tilde{H}\|_{L^{\frac{3s}{s+3}}} \\ &\le \|\rho^{-\alpha} G\|_{L^s} + C\|(\rho^\alpha + \delta)^{-1} G \cdot \rho^{-1} \nabla \rho\|_{L^{\frac{3s}{s+3}}} + C\|\rho^{-\alpha} H\|_{L^{\frac{3s}{s+3}}} \\ &\quad + C\|\nabla u \cdot \rho^{-1} \nabla \rho\|_{L^{\frac{3s}{s+3}}} + C\|(\rho^\alpha + \delta)^{-1} P \rho^{-1} \nabla \rho\|_{L^{\frac{3s}{s+3}}} \\ &\le C\|\rho^{-\alpha} G\|_{L^s} + C\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^6} \|\rho^{-\alpha} G\|_{L^{\frac{6s}{s+6}}} + C\|\rho^{-\alpha} H\|_{L^{\frac{3s}{s+3}}} \\ &\quad + C\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^3} (\|\nabla u\|_{L^s} + \|(\rho^\alpha + \delta)^{-1} P\|_{L^s}). \end{aligned}$$

If

$$\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^3} \le \zeta_0 \triangleq \min\{(2C)^{-1}, \zeta_1\}, \tag{2.12}$$

it is clear that (2.8) holds. The proof is finished.

3. A priori estimates

This section mainly aims to obtain uniform a priori estimates of the local strong solution to the system (2.1), which is necessary to obtain the global existence of approximate solution. We abbreviate the approximate solution $(\rho^\delta, u^\delta, P^\delta)$ as (ρ, u, P). The C in this section represents some positive constants that depend on $\bar{\rho}$, k and q but are independent of δ and T.

Theorem 3.1. *For $3 < k \le 6$ and $2 \le q < 6$, assume that the initial data $(\rho_{0,\delta}, u_0)$ satisfy*

$$0 \le \rho_{0,\delta} - \delta \le \bar{\rho}, \;\; \rho_{0,\delta} - \delta \in L^{\frac{3}{2}} \cap H^1 \cap D^{1,k}, \;\; \nabla \rho_0^{\frac{\alpha-1}{2}} \in L^{\frac{3}{2}} \cap D^{1,q}, \;\; u_0 \in D^1_{0,\sigma} \cap H^3, \tag{3.1}$$

as well as the condition (1.5). *Then there exist two positive constants* η_0 *and* ε_0 *that depend on* $\bar{\rho}, k, q$ *and* $\|g\|_{H^1}$ *such that if*

$$\|\nabla\rho_0^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}\cap D^{1,q}} \leq \eta_0, \quad \|u_0\|_{H^1} \leq \varepsilon_0, \tag{3.2}$$

then the system (2.1) *has a unique global strong solution* (ρ, u, P) *with the following:*

$$\begin{cases} \rho - \delta \in C([0,\infty); L^{\frac{3}{2}} \cap H^1 \cap D^{1,k}), \ \ \nabla\rho^{\frac{\alpha-1}{2}} \in C([0,\infty); L^{\frac{3}{2}} \cap D^{1,q}), \\ \nabla u \in C([0,\infty); H^1) \cap L^2(0,\infty; H^1), \ \ \nabla P \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; L^2), \\ \rho^{\frac{1-\alpha}{2}} u_t \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; L^2), \ \ \rho^{\frac{1-\alpha}{2}} u_{tt} \in L^2(0,\infty; L^2), \\ \nabla u_t \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; L^2 \cap L^6), \ \ \rho^{-\alpha} P_t \in L^2(0,\infty; L^6). \end{cases} \tag{3.3}$$

In order to prove Theorem 3.1, whose proof is placed after Proposition 3.2, we need to establish the global a priori estimates. Denote

$$M \triangleq \|\nabla\rho_0^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}\cap D^{1,q}}.$$

Proposition 3.1. *There exist two positive constants* η_0 *and* ε_0 *that depend on* $\bar{\rho}$, k *and* q *such that if* (ρ, u, P) *is a strong solution of* (2.1) *satisfying*

$$\sup_{t\in[0,T]} \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}\cap D^{1,q}} \leq 4M, \qquad \int_0^T \|\nabla u\|_{L^2}^4 dt \leq 2\|u_0\|_{H^1}^2, \tag{3.4}$$

then

$$\sup_{t\in[0,T]} \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}\cap D^{1,q}} \leq 3M, \qquad \int_0^T \|\nabla u\|_{L^2}^4 dt \leq \|u_0\|_{H^1}^2, \tag{3.5}$$

provided that

$$M \leq \eta_0, \quad \|u_0\|_{H^1} \leq \varepsilon_0. \tag{3.6}$$

Before proving Proposition 3.1, we first establish time-weighted energy estimates, as shown in Lemmas 3.1–3.4. By the a priori hypotheses $(3.4)_1$ and Gagliardo-Nirenberg inequality (see [4, 15]), we have

$$\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3} \leq C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}^{\frac{2q-3}{3(q-1)}} \|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}^{\frac{q}{3(q-1)}} \leq CM, \tag{3.7}$$

$$\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6} \leq C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}^{\frac{q-2}{2(q-1)}} \|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}^{\frac{q}{2(q-1)}} \leq CM. \tag{3.8}$$

Lemma 3.1. *Suppose that* (ρ, u, P) *is the local strong solution of* (2.1) *that satisfies* (3.4). *There exists a positive constant* η_3 *such that if* $M \leq \eta_3$*, then*

$$\sup_{t\in[0,T]} \|\rho - \delta\|_{L^{\frac{3}{2}}\cap L^2} \leq \|\rho_0\|_{L^{\frac{3}{2}}\cap L^2}, \tag{3.9}$$

$$\sup_{t\in[0,T]} (\|\rho^{\frac{1-\alpha}{2}} u\|_{L^2}^2 + \sigma_1(t)\|\nabla u\|_{L^2}^2) + \int_0^T [\|\nabla u\|_{L^2}^2 + \sigma_1(t)(\|(\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2)]dt \leq C\|u_0\|_{H^1}^2, \tag{3.10}$$

where $\sigma_1(t) = \max\{1, t\}$.

Proof. The transport equation $(2.1)_1$ and divergence-free $(2.1)_3$ imply that

$$\delta \leq \rho(x,t) \leq \bar{\rho} + 1, \ \sup_{t\in[0,T]} \|\rho - \delta\|_{L^{\frac{3}{2}}\cap L^2} \leq \|\rho_{0,\delta} - \delta\|_{L^{\frac{3}{2}}\cap L^2} = \|\rho_0\|_{L^{\frac{3}{2}}\cap L^2}, \tag{3.11}$$

which provides the following:

$$\frac{1}{2}\rho^{1-\alpha} \leq \frac{\rho}{\rho^\alpha + \delta} \leq \rho^{1-\alpha}. \tag{3.12}$$

Multiplying $(2.1)_2$ by $(\rho^\alpha + \delta)^{-1}u$, integrating the result with respect to x over $\mathbb{R}^3$, and then using the Gagliardo-Nirenberg inequality and $(3.4)_1$, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\rho}{\rho^\alpha + \delta}|u|^2 dx + \int |\nabla u|^2 dx \\
=&\int \nabla u \cdot \frac{\nabla\rho^\alpha}{\rho^\alpha + \delta} \cdot u dx - \int (\rho^\alpha + \delta)^{-1} P \frac{\nabla\rho^\alpha}{\rho^\alpha + \delta} \cdot u dx \\
\leq&\|\frac{\nabla\rho^\alpha}{\rho^\alpha + \delta}\|_{L^{\frac{3}{2}}} \|u\|_{L^6}(\|\nabla u\|_{L^6} + \|(\rho^\alpha + \delta)^{-1}P\|_{L^6}) \\
\leq&C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}\|\nabla u\|_{L^2}(\|\nabla^2 u\|_{L^2} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^2}) \\
\leq&\frac{1}{4}\|\nabla u\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^2}^2,
\end{aligned} \tag{3.13}$$

where one can utilize the simple fact that

$$\|\frac{\nabla\rho^\alpha}{\rho^\alpha + \delta}\|_{L^{\frac{3}{2}}} = \|\frac{\alpha\rho^{\alpha-1}\nabla\rho}{\rho^\alpha + \delta}\|_{L^{\frac{3}{2}}} = \|\frac{2\alpha}{\alpha-1}\frac{\rho^\alpha}{\rho^\alpha + \delta}\rho^{\frac{1-\alpha}{2}}\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}} \leq C\|\rho\|_{L^\infty}^{\frac{1-\alpha}{2}}\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}} \leq C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}.$$

If we apply the following:

$$M \leq \eta_1 \triangleq \min\{C^{-1}\zeta_0, 1\}, \tag{3.14}$$

combined with (3.7), we will have that $\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3} \leq \zeta_0$. Applying Lemma 2.2 with $F = -\rho u_t - \rho u \cdot \nabla u$, we deduce from (2.5) and the Sobolev inequality that

$$\|\nabla^2 u\|_{L^2} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^2} \leq C\|\rho^{\frac{1-\alpha}{2}}u_t + u\cdot\nabla u\|_{L^2} \leq C\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}},$$

which, together with Young's inequality, can lead to the following:

$$\|\nabla^2 u\|_{L^2} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{L^2} \leq C\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3. \tag{3.15}$$

Given that

$$\nabla P = \nabla(\frac{P}{\rho^\alpha + \delta})(\rho^\alpha + \delta) + P\frac{\nabla\rho^\alpha}{\rho^\alpha + \delta},$$

there is

$$\|\nabla P\|_{L^2} \leq \|\rho^\alpha + \delta\|_{L^\infty}\|\nabla(\frac{P}{\rho^\alpha + \delta})\|_{L^2} + \|P\|_{L^6}\|\frac{\nabla\rho^\alpha}{\rho^\alpha + \delta}\|_{L^3} \leq C\|\nabla(\frac{P}{\rho^\alpha + \delta})\|_{L^2} + C\|\nabla P\|_{L^2}\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}.$$

If $M \leq \eta_2 \triangleq \min\{(2C)^{-1},\ \eta_1\}$, one can obtain

$$\|\nabla P\|_{L^2} \leq C\|\nabla(\frac{P}{\rho^\alpha+\delta})\|_{L^2}. \tag{3.16}$$

Putting (3.15) into (3.13), we obtain

$$\frac{d}{dt}\int \frac{\rho}{\rho^\alpha+\delta}|u|^2dx + \int |\nabla u|^2dx \leq C\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6. \tag{3.17}$$

Next, multiplying $(2.1)_2$ by $(\rho^\alpha+\delta)^{-1}u_t$ and integrating over $\mathbb{R}^3$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2dx + \int \frac{\rho}{\rho^\alpha+\delta}|u_t|^2dx\\
=&-\int \frac{\rho}{\rho^\alpha+\delta}u\cdot\nabla u\cdot u_tdx + \int \nabla u\cdot\frac{\nabla\rho^\alpha}{\rho^\alpha+\delta}\cdot u_tdx - \int(\rho^\alpha+\delta)^{-1}P\frac{\nabla\rho^\alpha}{\rho^\alpha+\delta}\cdot u_tdx\\
\leq& C\|u\|_{L^6}\|\nabla u\|_{L^3}\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}(\|\nabla u\|_{L^6} + \|(\rho^\alpha+\delta)^{-1}P\|_{L^6})\\
\leq&(\frac{1}{16}+CM)\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6.
\end{aligned}$$

If

$$M \leq \eta_3 \triangleq \min\{(4C)^{-1}, \eta_2\}, \tag{3.18}$$

then we have

$$\frac{d}{dt}\int |\nabla u|^2dx + \int \rho^{1-\alpha}|u_t|^2dx \leq C\|\nabla u\|_{L^2}^6. \tag{3.19}$$

Using Grönwall's inequality and (3.4), we can obtain

$$\sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2 + \int_0^T \|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2dt \leq C\|\nabla u_0\|_{L^2}^2, \tag{3.20}$$

$$\sup_{t\in[0,T]}\|\rho^{\frac{1-\alpha}{2}}u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2dt \leq C\|u_0\|_{H^1}^2. \tag{3.21}$$

Furthermore, by taking $\sigma_1(t) = \max\{1,t\}$, multiplying (3.19) by $\sigma_1(t)$, and using Grönwall's inequality, (3.4), (3.15), (3.16) and (3.21), one can obtain

$$\sup_{t\in[0,T]}\sigma_1(t)\|\nabla u\|_{L^2}^2 + \int_0^T \sigma_1(t)(\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2)dt \leq C\|u_0\|_{H^1}^2.$$

Lemma 3.2. *Suppose that* (ρ,u,P) *is the local strong solution of* (2.1) *that satisfies* (3.4)*, with the initial data* $(\rho_{0,\delta},\ u_0)$ *satisfying the condition* (1.5)*. If* $M \leq \eta_3$*, then*

$$\begin{aligned}
&\sup_{t\in[0,T]}\sigma_1^2(t)[\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2 + \|\nabla^2u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2] + \int_0^T \sigma_1^2(t)\|\nabla u_t\|_{L^2}^2dt\\
\leq& C\|u_0\|_{H^2}^2 + C\|g\|_{L^2}^2 + C\int_0^T \sigma_1^2(t)(\|\nabla u_t\|_{L^6} + \|\rho^{-\alpha}P_t\|_{L^6})^2dt.
\end{aligned} \tag{3.22}$$

Proof. Differentiating $(2.1)_2$ with respect to t, one has

$$\rho u_{tt} + \rho u \cdot \nabla u_t - \operatorname{div}((\rho^\alpha + \delta)\nabla u_t) + \nabla P_t = -\rho_t u_t - (\rho u)_t \cdot \nabla u + \operatorname{div}(\partial_t \rho^\alpha \nabla u). \tag{3.23}$$

Multiplying (3.23) by $\rho^{-\alpha} u_t$ and integrating it over $\mathbb{R}^3$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^{1-\alpha}|u_t|^2 dx + \int \frac{\rho^\alpha + \delta}{\rho^\alpha}|\nabla u_t|^2 dx \\
=& - \int \rho^{-\alpha}\rho_t |u_t|^2 dx - \int (\rho u)_t \cdot \nabla u \cdot \rho^{-\alpha} u_t dx + \int \operatorname{div}(\partial_t \rho^\alpha \nabla u) \cdot \rho^{-\alpha} u_t dx \\
&- \alpha \int \rho^{-\alpha-1} P_t \nabla \rho \cdot u_t dx + \alpha \int \frac{\rho^\alpha + \delta}{\rho^\alpha} \nabla u_t \cdot \rho^{-1}\nabla\rho \cdot u_t dx \\
\triangleq& \sum_{i=1}^{5} J_i.
\end{aligned} \tag{3.24}$$

Now, we estimate the right-hand terms of (3.24) one by one. Thanks to the Gagliardo-Nirenberg inequality and Young's inequality, we arrive at the following formulas:

$$\begin{aligned}
|J_1| + |J_2| \le& C\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u\|_{L^6}\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^4}^2 + C\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|u_t\|_{L^6} + \|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^4}^2\|\nabla u\|_{L^2} \\
\le& \frac{1}{8}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{10},
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
J_3 =& \alpha \int \rho^{-1}\partial_k \rho u^k \partial_i u^j \partial_i u_t^j dx - \alpha^2 \int \rho^{-1}\partial_k \rho u^k \partial_i u^j \rho^{-1}\partial_i \rho u_t^j dx \\
\le& C\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} + C\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^3}^2\|u\|_{L^\infty}\|\nabla u\|_{L^6}\|u_t\|_{L^6} \\
\le& \frac{1}{8}\|\nabla u_t\|_{L^2}^2 + C\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^3\|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}^{10},
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
|J_4| + |J_5| \le& 2\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}\|u_t\|_{L^6}(\|\rho^{-\alpha}P_t\|_{L^6} + \|\nabla u_t\|_{L^6}) \\
\le& \frac{1}{8}\|\nabla u_t\|_{L^2}^2 + C(\|\rho^{-\alpha}P_t\|_{L^6} + \|\nabla u_t\|_{L^6})^2.
\end{aligned} \tag{3.27}$$

Based on the above estimates, it can be concluded that

$$\begin{aligned}
&\frac{d}{dt}\int \rho^{1-\alpha}|u_t|^2 dx + \int |\nabla u_t|^2 dx \\
\le& C\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^2(\|\nabla u\|_{L^2}^4 + \|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}\|\nabla u\|_{L^2}) + C\|\nabla u\|_{L^2}^{10} + C(\|\rho^{-\alpha}P_t\|_{L^6} + \|\nabla u_t\|_{L^6})^2.
\end{aligned} \tag{3.28}$$

Multiplying (3.28) by $\sigma_1^2(t)$ and taking advantage of Grönwall's inequality, the condition (1.5), and (3.10), one has

$$\begin{aligned}
&\sup_{t\in[0,T]} \sigma_1^2(t)\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^2 + \int_0^T \sigma_1^2(t)\|\nabla u_t\|_{L^2}^2 dt \\
\le& C\|u_0\|_{H^2}^2 + C\|g\|_{L^2}^2 + C\int_0^T \sigma_1^2(t)(\|\rho^{-\alpha}P_t\|_{L^6} + \|\nabla u_t\|_{L^6})^2 dt.
\end{aligned} \tag{3.29}$$

With the help of (3.15), (3.16) and (3.29), we can deduce (3.22). This completes the proof of the lemma.

Lemma 3.3. *Suppose that* (ρ, u, P) *is the local strong solution of* (2.1)*, and that it satisfies* (3.4)*. There exist two positive constants* η_4 *and* ε_1 *such that if* $M \le \eta_4$ *and* $\|u_0\|_{H^1} \le \varepsilon_1$*, then*

$$\begin{aligned}
&\sup_{t\in[0,T]} [\sigma_2(t)\|\nabla u_t\|_{L^2}^2 + \sigma_1^2(t)(\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2)] \\
&+ \int_0^T [\sigma_2(t)\|\rho^{\frac{1-\alpha}{2}} u_{tt}\|_{L^2}^2 + \sigma_1^2(t)(\|\nabla u_t\|_{L^2\cap L^6}^2 + \|\rho^{-\alpha} P_t\|_{L^6}^2)]dt \\
\le& C\|u_0\|_{H^3}^2 + C\|g\|_{H^1}^2,
\end{aligned} \tag{3.30}$$

with $\sigma_2(t) = \max\{\lambda, t^2\}$ *(*λ *see* (3.42)*).*

Proof. According to Lemma 2.2, the Sobolev inequality and (3.23), we can deduce the following for $p \in [3, 6)$:

$$\begin{aligned}
&\|\nabla^2 u\|_{L^p} + \|\nabla((\rho^\alpha + \delta)^{-1} P)\|_{L^p} \\
\le& C\|\rho^{-\alpha}(\rho u_t + \rho u\cdot\nabla u)\|_{L^p} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|\rho^{-\alpha}(\rho u_t + \rho u\cdot\nabla u)\|_{L^{\frac{6p}{p+6}}} \\
\le& C\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^{\frac{6-p}{2p}}\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^6}^{\frac{3p-6}{2p}} + C\|u\|_{L^\infty}\|\nabla u\|_{L^p} + C\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^{\frac{3}{p}}\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^6}^{\frac{p-3}{p}} + C\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{6p}{p+6}}} \\
\le& C\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}} + C\|\rho^{\frac{1-\alpha}{2}} u_t\|_{L^2}^{\frac{3}{p}}\|\nabla u_t\|_{L^2}^{\frac{p-3}{p}} + C\|\nabla u\|_{L^2}^{\frac{3}{p}}\|\nabla^2 u\|_{L^2}^{\frac{2p-3}{p}} + C\|\nabla u\|_{L^2}^{\frac{p+6}{2p}}\|\nabla^2 u\|_{L^2}^{\frac{3p-6}{2p}},
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
&\|\nabla u_t\|_{L^6} + \|(\rho^\alpha + \delta)^{-1} P_t\|_{L^6} \\
\le& C\|\rho^{-\alpha}\partial_t\rho^\alpha\nabla u\|_{L^6} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|\rho^{-\alpha}\partial_t\rho^\alpha\nabla u\|_{L^3} + C\|\rho^{-\alpha}(\rho u_{tt} + \rho u\cdot\nabla u_t + \rho_t u_t + (\rho u)_t\cdot\nabla u)\|_{L^2} \\
\le& C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|u\|_{L^\infty}\|\nabla u\|_{L^\infty} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}^2\|u\|_{L^\infty}\|\nabla u\|_{L^6} + C\|\rho^{\frac{1-\alpha}{2}} u_{tt}\|_{L^2} + C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2} \\
&+ C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u\|_{L^\infty}\|u_t\|_{L^6} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u\|_{L^\infty}^2\|\nabla u\|_{L^6} + C\|\nabla u\|_{L^3}\|u_t\|_{L^6} \\
\le& C\|\rho^{\frac{1-\alpha}{2}} u_{tt}\|_{L^2} + C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{3}{2}} \\
&+ C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^\infty}.
\end{aligned} \tag{3.32}$$

Multiplying (3.23) by $(\rho^\alpha + \delta)^{-1} u_{tt}$, integrating the resulting equality over $\mathbb{R}^3$, and using integration by parts, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla u_t|^2 dx + \int \frac{\rho}{\rho^\alpha + \delta}|u_{tt}|^2 dx \\
=& -\int \frac{\rho}{\rho^\alpha + \delta}(u_t\cdot\nabla u + u\cdot\nabla u_t)\cdot u_{tt} dx - \int \rho_t u_t\cdot(\rho^\alpha + \delta)^{-1} u_{tt} dx - \int \rho_t u\cdot\nabla u\cdot(\rho^\alpha + \delta)^{-1} u_{tt} dx \\
&+ \int \nabla u_t\cdot\frac{\nabla\rho^\alpha}{\rho^\alpha + \delta}\cdot u_{tt} dx - \int (\rho^\alpha + \delta)^{-1} P_t \frac{\nabla\rho^\alpha}{\rho^\alpha + \delta}\cdot u_{tt} dx - \int \partial_i(u^k\partial_k\rho^\alpha\partial_i u^j)(\rho^\alpha + \delta)^{-1} u_{tt}^j dx \\
\triangleq& \sum_{i=6}^{11} J_i.
\end{aligned} \tag{3.33}$$

Owing to the Sobolev inequality and (3.32), one can obtain

$$\begin{aligned}|J_6| &\le C\|u_t\|_{L^6}\|\nabla u\|_{L^3}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2} + C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}\\ &\le \frac{1}{40}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2},\end{aligned} \tag{3.34}$$

$$\begin{aligned}|J_7+J_8| &\le \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u\|_{L^\infty}\|u_t\|_{L^6}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2} + \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u\|_{L^\infty}^2\|\nabla u\|_{L^6}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}\\ &\le \frac{1}{40}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^4,\end{aligned} \tag{3.35}$$

$$\begin{aligned}|J_9|+|J_{10}| &\le \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}(\|\nabla u_t\|_{L^6} + \|(\rho^\alpha+\delta)^{-1}P_t\|_{L^6})\\ &\le CM\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}(\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12} + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2\\ &\quad + \|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac32} + \|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^\infty})\\ &\le \frac{11}{40}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^4\\ &\quad + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^\infty}^2,\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}J_{11} &= -\alpha\int \partial_i u^k\rho^{\frac{\alpha-3}{2}}\partial_k\rho\partial_i u^j\frac{\rho^{\frac{\alpha+1}{2}}}{\rho^\alpha+\delta}u_{tt}^j dx - \alpha\int u^k\partial_i(\rho^{\frac{\alpha-3}{2}}\partial_k\rho)\partial_i u^j\frac{\rho^{\frac{\alpha+1}{2}}}{\rho^\alpha+\delta}u_{tt}^j dx\\ &\quad - \frac{\alpha(\alpha+1)}{2}\int u^k\rho^{\frac{\alpha-3}{2}}\partial_i\rho\rho^{\frac{\alpha-3}{2}}\partial_k\rho\partial_i u^j\frac{\rho}{\rho^\alpha+\delta}u_{tt}^j dx - \alpha\int u^k\rho^{\frac{\alpha-3}{2}}\partial_k\rho\partial_i\partial_i u^j\frac{\rho^{\frac{\alpha+1}{2}}}{\rho^\alpha+\delta}u_{tt}^j dx\\ &\le \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|\nabla u\|_{L^6}^2\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2} + \|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}\\ &\quad + \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}^2\|u\|_{L^\infty}\|\nabla u\|_{L^6}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2} + \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|u\|_{L^\infty}\|\nabla^2 u\|_{L^3}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}\\ &\le \frac{1}{40}\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^4 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^{\frac{2q}{q-2}}}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3\\ &\quad + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2 u\|_{L^3}^2,\end{aligned} \tag{3.37}$$

provided that

$$M \le \eta_4 \triangleq \min\{(4C)^{-1}, \eta_3\}. \tag{3.38}$$

Substituting the above estimates into (3.33) yields

$$\begin{aligned}&\frac12\frac{d}{dt}\int|\nabla u_t|^2dx + \frac12\int\rho^{1-\alpha}|u_{tt}|^2dx\\ &\le C\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + C\|\nabla^2 u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^4 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3\\ &\quad + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2 u\|_{L^3}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^\infty}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^{\frac{2q}{q-2}}}^2\\ &\triangleq C\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + \sum_{i=12}^{17}J_i.\end{aligned} \tag{3.39}$$

Multiply (3.39) by $\sigma_2(t) \triangleq \max\{\lambda, t^2\}$ (λ is a positive number to be determined) and integrate the resulting equation over $[0,t]$. By virtue of Lemma 3.1 and Lemma 3.2, we arrive at the following:

$$\int_0^t \sigma_2(\tau)J_{12}d\tau \leq C\sup_{\tau\in[0,t]}(\sigma_1^2(\tau)\|\nabla^2 u\|_{L^2}^2)\sup_{\tau\in[0,t]}\frac{\sigma_2(\tau)}{\sigma_1^3(\tau)}\int_0^t \sigma_1(\tau)\|\nabla^2 u\|_{L^2}^2 d\tau \leq C\lambda\|u_0\|_{H^1}^2 E, \tag{3.40}$$

where $E \triangleq \|u_0\|_{H^2}^2 + \|g\|_{L^2}^2 + \int_0^t \sigma_1^2(\tau)(\|\nabla u_t\|_{L^6} + \|\rho^{-\alpha}P_t\|_{L^6})^2 d\tau$ and we have used the fact that $\sup_{\tau\in[0,t]}\frac{\sigma_2(\tau)}{\sigma_1^j(\tau)} \leq \lambda,\ \forall j \geq 2$.

Similarly,

$$\begin{aligned}
&\int_0^t \sigma_2(\tau)J_{13}d\tau \leq C\lambda\|u_0\|_{H^1}^2 E,\\
&\int_0^t \sigma_2(\tau)J_{14}d\tau + \int_0^t \sigma_2(\tau)J_{15}d\tau \leq C\lambda\|u_0\|_{H^1}^2 E + C\lambda\|u_0\|_{H^1}^2,\\
&\int_0^t \sigma_2(\tau)J_{16}d\tau \leq C\int_0^t \sigma_2(\tau)\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{2(2p-6)}{5p-6}}\|\nabla^2 u\|_{L^p}^{\frac{6p}{5p-6}}d\tau \leq C\lambda\|u_0\|_{H^1}^2 E + C\lambda\|u_0\|_{H^1}^2.
\end{aligned}$$

For $2 \leq q \leq 3$, it holds that

$$\int_0^t \sigma_2(\tau)J_{17}d\tau \leq C\int_0^t \sigma_2(\tau)\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^6}^{\frac{2(3q-6)}{q}}\|\nabla u\|_{L^\infty}^{\frac{2(6-2q)}{q}}d\tau \leq C\lambda\|u_0\|_{H^1}^2 E + C\lambda\|u_0\|_{H^1}^2.$$

For $3 < q < 6$, we have

$$\int_0^t \sigma_2(\tau)J_{17}d\tau \leq C\int_0^t \sigma_2(\tau)\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{2(q-3)}{q}}\|\nabla u\|_{L^6}^{\frac{6}{q}}d\tau \leq C\lambda\|u_0\|_{H^1}^2 E + C\lambda\|u_0\|_{H^1}^2.$$

It is easy to deduce from the definitions of $\sigma_1(t)$ and $\sigma_2(t)$ that

$$\int_0^t \sigma_2'(\tau)\|\nabla u_t\|_{L^2}^2 d\tau \leq \sup_{\tau\in[0,t]}\frac{\sigma_2'(\tau)}{\sigma_1^2(\tau)}\int_0^t \sigma_1^2(\tau)\|\nabla u_t\|_{L^2}^2 d\tau \leq \frac{C}{\sqrt{\lambda}}E. \tag{3.41}$$

By combining these formulas with (3.32) and applying the following:

$$\lambda \triangleq (64C^2)^{-1}, \quad \|u_0\|_{H^1} \leq \varepsilon_1 \triangleq \min\{512C^3, 1\}, \tag{3.42}$$

we obtain

$$\begin{aligned}
&\sum_{i=12}^{17}\int_0^t \sigma_2(\tau)J_i d\tau + \int_0^t \sigma_2'(\tau)\|\nabla u_t\|_{L^2}^2 d\tau\\
&\leq (C\lambda\|u_0\|_{H^1}^2 + \frac{C}{\sqrt{\lambda}})E + C\lambda\|u_0\|_{H^1}^2\\
&\leq (C\lambda\|u_0\|_{H^1}^2 + \frac{C}{\sqrt{\lambda}})\Big(\lambda\int_0^t \sigma_2(\tau)(\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^4\\
&\quad + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^3 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^\infty}^2)d\tau + \|u_0\|_{H^2}^2 + \|g\|_{L^2}^2\Big) + C\lambda\|u_0\|_{H^1}^2
\end{aligned}$$

$$\leq \frac{1}{4}\int_0^t \sigma_2(\tau)(\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2})d\tau + \frac{1}{4}\int_0^t \sigma_2(\tau)(J_{13}+J_{14}+J_{16})d\tau$$
$$+ C\|u_0\|_{H^2}^2 + C\|g\|_{L^2}^2. \tag{3.43}$$

Considering (3.39) and (3.43) and taking advantage of the condition (1.5) and Grönwall's inequality, it can be inferred that

$$\sup_{t\in[0,T]}\sigma_2(t)\|\nabla u_t\|_{L^2}^2 + \int_0^T \sigma_2(t)\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 dt \leq C(\|u_0\|_{H^3}^2 + \|g\|_{H^1}^2). \tag{3.44}$$

Meanwhile, by virtue of Lemma 3.2, (3.43) and (3.44), we have

$$\sup_{t\in[0,T]}\sigma_1^2(t)(\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla(\frac{P}{\rho^\alpha+\delta})\|_{L^2}^2) + \int_0^T \sigma_1^2(t)\|\nabla u_t\|_{L^2}^2 dt$$
$$\leq C\|u_0\|_{H^2}^2 + C\|g\|_{L^2}^2 + C\int_0^T \sigma_1^2(t)(\|\nabla u_t\|_{L^6} + \|\rho^{-\alpha}P_t\|_{L^6})^2 dt$$
$$\leq C\|u_0\|_{H^2}^2 + C\|g\|_{L^2}^2 + C\int_0^T \sigma_2(t)\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 dt + C\int_0^T \sigma_2(t)\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} dt$$
$$\leq C\|u_0\|_{H^3}^2 + C\|g\|_{H^1}^2.$$

Hence, we have completed the proof of Lemma 3.3.

Lemma 3.4. *Suppose that* (ρ, u, P) *is the local strong solution of* (2.1)*, and that it satisfies* (3.4). *Then,*

$$\int_0^T \|\nabla u\|_{L^\infty} dt \leq C\|u_0\|_{H^1}^{\frac{6-p}{2p}}(1 + \|u_0\|_{H^3} + \|g\|_{H^1}), \quad \forall p \in (3,6), \tag{3.45}$$

provided that $M \leq \eta_4$ *and* $\|u_0\|_{H^1} \leq \varepsilon_1$.

Proof. Taking into account Lemma 2.2, the Gagliardo-Nirenberg inequality, (3.10), (3.30) and (3.31), one has

$$\int_0^T \|\nabla u\|_{L^\infty} dt \leq C\int_0^T \|\nabla u\|_{L^{\frac{3r}{3-r}}}^{\alpha}\|\nabla^2 u\|_{L^p}^{1-\alpha} dt$$
$$\leq C\int_0^T \|\nabla^2 u\|_{L^r} dt + C\int_0^T \|\nabla^2 u\|_{L^p} dt$$
$$\leq C\int_0^T \|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} dt + C\int_0^T \|\nabla u\|_{L^2}^{\frac{3}{r}}\|\nabla^2 u\|_{L^2}^{\frac{2r-3}{r}} dt$$
$$+ C\int_0^T \|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u_t\|_{L^2}^{\frac{3p-6}{2p}} dt + C\int_0^T \|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^{\frac{3}{p}}\|\nabla u_t\|_{L^2}^{\frac{p-3}{p}} dt$$
$$+ C\int_0^T \|\nabla u\|_{L^2}^{\frac{3}{p}}\|\nabla^2 u\|_{L^2}^{\frac{2p-3}{p}} dt + C\int_0^T \|\nabla u\|_{L^2}^{\frac{p+6}{2p}}\|\nabla^2 u\|_{L^2}^{\frac{3p-6}{2p}} dt$$
$$\leq C\|u_0\|_{H^1}^{\frac{6-r}{2r}}(\|u_0\|_{H^3} + \|g\|_{H^1})^{\frac{3r-6}{2r}} + C\|u_0\|_{H^1}^{\frac{6-p}{2p}}(\|u_0\|_{H^3} + \|g\|_{H^1})^{\frac{3p-6}{2p}}$$
$$+ C\|u_0\|_{H^1}^{\frac{3}{p}}(\|u_0\|_{H^3} + \|g\|_{H^1})^{\frac{p-3}{p}} + C\|u_0\|_{H^1}$$
$$\leq C\|u_0\|_{H^1}^{\frac{6-p}{2p}}(1 + \|u_0\|_{H^3} + \|g\|_{H^1}),$$

where $\alpha = \frac{r(p-3)}{3(p-r)}$, $2 < r < 3$, $3 < p < 6$.

Now, we can use Lemmas 3.1–3.4 to prove Proposition 3.1.

Proof of Proposition 3.1. First, based on $(2.1)_1$ and $(2.1)_3$, we find that

$$(\partial_i \rho^{\frac{\alpha-1}{2}})_t + (\partial_i u \cdot \nabla)\rho^{\frac{\alpha-1}{2}} + u \cdot \nabla \partial_i \rho^{\frac{\alpha-1}{2}} = 0. \tag{3.46}$$

Multiplying this formula by $\frac{3}{2}|\nabla \rho^{\frac{\alpha-1}{2}}|^{-\frac{1}{2}} \partial_i \rho^{\frac{\alpha-1}{2}}$ and then integrating over $\mathbb{R}^3$, we have

$$\frac{d}{dt}\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}} \leq \|\nabla u\|_{L^\infty}\|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}. \tag{3.47}$$

Choose some small positive constant ε_2 satisfying

$$C\varepsilon_2^{\frac{6-p}{2p}}(1 + \|u_0\|_{H^3} + \|g\|_{H^1}) \leq \ln \frac{3}{2}. \tag{3.48}$$

If $\|u_0\|_{H^1} \leq \min\{\varepsilon_1, \ \varepsilon_2\}$, combined with Grönwall's inequality, (3.45) and (3.47), we have

$$\sup_{t\in[0,T]} \|\nabla \rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}} \leq \|\nabla \rho_{0,\delta}^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}} \exp\{\int_0^T \|\nabla u\|_{L^\infty} dt\} \leq \frac{3}{2}\|\nabla \rho_0^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}. \tag{3.49}$$

Similarly, the following formula can also be obtained:

$$\sup_{t\in[0,T]} \|\nabla \rho\|_{L^2\cap L^k} \leq 2\|\nabla \rho_0\|_{L^2\cap L^k}, \tag{3.50}$$

which shows that

$$\|\rho_t\|_{L^{\frac{3}{2}}} \leq \|u\|_{L^6}\|\nabla \rho\|_{L^2} \leq C\|\nabla \rho_0\|_{L^2}. \tag{3.51}$$

On the other hand, taking the derivative of (3.46) with respect to x_k $(k = 1, 2, 3)$, we get

$$(\partial_k\partial_i \rho^{\frac{\alpha-1}{2}})_t = -(\partial_k\partial_i u \cdot \nabla)\rho^{\frac{\alpha-1}{2}} - (\partial_i u \cdot \nabla)\partial_k \rho^{\frac{\alpha-1}{2}} - \partial_k u \cdot \nabla \partial_i \rho^{\frac{\alpha-1}{2}} - u \cdot \nabla \partial_k \partial_i \rho^{\frac{\alpha-1}{2}}.$$

After standard calculations, we can obtain

$$\frac{d}{dt}\|\nabla^2 \rho^{\frac{\alpha-1}{2}}\|_{L^q} \leq 2\|\nabla u\|_{L^\infty}\|\nabla^2 \rho^{\frac{\alpha-1}{2}}\|_{L^q} + \||\nabla^2 u||\nabla \rho^{\frac{\alpha-1}{2}}|\|_{L^q}. \tag{3.52}$$

It indicates that

$$\begin{aligned}
\sup_{t\in[0,T]} \|\nabla^2 \rho^{\frac{\alpha-1}{2}}\|_{L^q} &\leq (\|\nabla^2 \rho_{0,\delta}^{\frac{\alpha-1}{2}}\|_{L^q} + \int_0^T \||\nabla^2 u||\nabla \rho^{\frac{\alpha-1}{2}}|\|_{L^q} dt) \exp\{2\int_0^T \|\nabla u\|_{L^\infty} dt\} \\
&\leq (M + CM^2 + CM\|u_0\|_{H^1}^{\frac{6-p}{2p}}(1 + \|u_0\|_{H^3} + \|g\|_{H^1})) \\
&\quad \times \exp\{C\|u_0\|_{H^1}^{\frac{6-p}{2p}}(1 + \|u_0\|_{H^3} + \|g\|_{H^1})\} \\
&\leq (\frac{9}{8}M + CM^2) \exp\{C\|u_0\|_{H^1}^{\frac{6-p}{2p}}(1 + \|u_0\|_{H^3} + \|g\|_{H^1})\},
\end{aligned} \tag{3.53}$$

where we have used the following facts:

$$\begin{aligned}
\|\nabla^2\rho_{0,\delta}^{\frac{\alpha-1}{2}}\|_{L^q} &= \|\frac{\alpha-1}{2}\frac{\partial_i\partial_j\rho_0}{(\rho_0+\delta)^{\frac{3-\alpha}{2}}}+\frac{(\alpha-1)(\alpha-3)}{4}\frac{\partial_i\rho_0\partial_j\rho_0}{(\rho_0+\delta)^{\frac{5-\alpha}{2}}}\|_{L^q}\\
&\le \|\frac{\alpha-1}{2}(\frac{\partial_i\partial_j\rho_0}{\rho_0^{\frac{3-\alpha}{2}}}+\frac{\alpha-3}{2}\frac{\partial_i\rho_0\partial_j\rho_0}{\rho_0^{\frac{5-\alpha}{2}}})(\frac{\rho_0}{\rho_0+\delta})^{\frac{3-\alpha}{2}}\|_{L^q}\\
&\quad+\|\frac{(\alpha-1)(\alpha-3)}{4}(-\frac{\partial_i\rho_0\partial_j\rho_0}{\rho_0(\rho_0+\delta)^{\frac{3-\alpha}{2}}}+\frac{\partial_i\rho_0\partial_j\rho_0}{(\rho_0+\delta)^{\frac{5-\alpha}{2}}})\|_{L^q}\\
&\le \|\frac{\alpha-1}{2}(\frac{\partial_i\partial_j\rho_0}{\rho_0^{\frac{3-\alpha}{2}}}+\frac{\alpha-3}{2}\frac{\partial_i\rho_0\partial_j\rho_0}{\rho_0^{\frac{5-\alpha}{2}}})\|_{L^q}+\|\frac{(\alpha-1)(\alpha-3)}{4}\frac{\partial_i\rho_0\partial_j\rho_0}{\rho_0^{3-\alpha}}\frac{\delta\rho_0^{2-\alpha}}{(\rho_0+\delta)^{\frac{5-\alpha}{2}}}\|_{L^q}\\
&\le \|\nabla^2\rho_0^{\frac{\alpha-1}{2}}\|_{L^q}+C\|\nabla\rho_0^{\frac{\alpha-1}{2}}\|_{L^{2q}}^2\le\|\nabla^2\rho_0^{\frac{\alpha-1}{2}}\|_{L^q}+C\|\nabla\rho_0^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}^{\frac{2q-3}{3(q-1)}}\|\nabla^2\rho_0^{\frac{\alpha-1}{2}}\|_{L^q}^{\frac{4q-3}{3(q-1)}}\\
&\le M+CM^2,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \||\nabla^2 u||\nabla\rho^{\frac{\alpha-1}{2}}|\|_{L^q}dt &\le \int_0^T\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{pq}{p-q}}}\|\nabla^2u\|_{L^p}dt\\
&\le C\int_0^T\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}}^{\frac{2pq+3q-3p}{3p(q-1)}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}^{\frac{pq-3q}{3p(q-1)}}\|\nabla^2u\|_{L^p}dt\\
&\le CM\|u_0\|_{H^1}^{\frac{6-p}{2p}}(1+\|u_0\|_{H^3}+\|g\|_{H^1}),\ \ \forall\ \max\{q,3\}<p<6.
\end{aligned}$$

Then, if

$$M\le\eta_0\triangleq\min\{(8C)^{-1},\ \eta_4\},\qquad \|u_0\|_{H^1}\le\min\{\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\},$$

where ε_3 satisfying $C\varepsilon_3^{\frac{6-p}{2p}}(1+\|u_0\|_{H^3}+\|g\|_{H^1})\le\ln\frac{6}{5}$, it can be inferred from (3.53) that

$$\sup_{t\in[0,T]}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}\le\frac{3}{2}M. \tag{3.54}$$

Meanwhile, (3.46) yields the following:

$$\begin{aligned}
\|\partial_t\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{6q}{q+6}}} &\le \||\nabla u||\nabla\rho^{\frac{\alpha-1}{2}}|\|_{L^{\frac{6q}{q+6}}}+\||u||\nabla^2\rho^{\frac{\alpha-1}{2}}|\|_{L^{\frac{6q}{q+6}}}\\
&\le\|\nabla u\|_{L^6}\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^q}+\|u\|_{L^6}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}\\
&\le C.
\end{aligned}\tag{3.55}$$

Furthermore, it can be easily derived from (3.10) that

$$\int_0^T\|\nabla u\|_{L^2}^4dt\le\sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2\int_0^T\|\nabla u\|_{L^2}^2dt\le C\|u_0\|_{H^1}^4\le\|u_0\|_{H^1}^2, \tag{3.56}$$

provided that

$$\|u_0\|_{H^1}\le\varepsilon_0\triangleq\min\{C^{-\frac{1}{2}},\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\}.$$

It is now clear that if (3.6) holds, then (3.5) is valid. The proof of Proposition 3.1 is completed.

Taking all of the a priori estimates (see Proposition 3.1, Lemma 3.1 and Lemma 3.3, (3.31), (3.50), (3.51) and (3.55)) together, the following proposition can be obtained.

Proposition 3.2. *It holds that*

$$\sup_{t\in[0,T]} \|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}\cap D^{1,q}} \le 3\|\nabla\rho_0^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}\cap D^{1,q}}, \tag{3.57}$$

$$\begin{aligned}
&\sup_{t\in[0,T]} [\|\rho-\delta\|_{L^{\frac{3}{2}}\cap H^1\cap D^{1,k}} + \|\rho_t\|_{L^{\frac{3}{2}}} + \|\partial_t\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac{6q}{q+6}}} + \|\rho^{\frac{1-\alpha}{2}}u\|_{L^2}^2 + \|\nabla^2 u\|_{L^p} + \sigma_1(t)\|\nabla u\|_{L^2}^2 + \sigma_2(t)\|\nabla u_t\|_{L^2}^2 \\
&\quad + \sigma_1^2(t)(\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2)] + \|\nabla^3 u\|_{L^2} + \int_0^T [\|\nabla u\|_{L^2}^2 + \sigma_1(t)(\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \\
&\quad + \|\nabla P\|_{L^2}^2)]dt + \int_0^T [\sigma_2(t)\|\rho^{\frac{1-\alpha}{2}}u_{tt}\|_{L^2}^2 + \sigma_1^2(t)(\|\nabla u_t\|_{L^2\cap L^6}^2 + \|\rho^{-\alpha}P_t\|_{L^6}^2)]dt \le C,
\end{aligned} \tag{3.58}$$

provided that

$$\|\nabla\rho_0^{\frac{\alpha-1}{2}}\|_{L^{\frac{3}{2}}\cap D^{1,q}} \le \eta_0, \quad \|u_0\|_{H^1} \le \varepsilon_0.$$

Proof. All estimates have been obtained except for $\|\nabla^3 u\|_{L^2}$. We now estimate $\|\nabla^3 u\|_{L^2}$. According to Lemma 2.2, one has

$$\begin{aligned}
&\|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha+\delta)^{-1}P)\|_{H^1} \\
&\le C\|\rho^{-\alpha}(\rho u_t + \rho u\cdot\nabla u)\|_{L^2} + C\|\rho^{-\alpha}\nabla(\rho u_t + \rho u\cdot\nabla u)\|_{L^2} + C\|(\rho u_t + \rho u\cdot\nabla u)\rho^{-2\alpha}\nabla\rho^\alpha\|_{L^2} \\
&\quad + C\|(|\nabla u| + |\frac{P}{\rho^\alpha+\delta}|)\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{2q}{q-2}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q} \\
&\le C\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\rho^{-\alpha}(|\nabla\rho||u_t| + \rho|\nabla u_t| + |\nabla\rho||u||\nabla u| + \rho|\nabla u|^2 + \rho|u||\nabla^2 u|)\|_{L^2} \\
&\quad + C\|(u_t + u\cdot\nabla u)\rho^{-\alpha}\nabla\rho^\alpha\|_{L^2} + C\|(|\nabla u| + |(\rho^\alpha+\delta)^{-1}P|)\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{2q}{q-2}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q} \\
&\le C\|\rho^{\frac{1-\alpha}{2}}u_t\|_{L^2} + C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|u_t\|_{L^6} + C\|\nabla u_t\|_{L^2} + C\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^6} \\
&\quad + C\|\nabla u\|_{L^4}^2 + C\|u\|_{L^\infty}\|\nabla^2 u\|_{L^2} + \frac{1}{4}(\|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha+\delta)^{-1}P)\|_{H^1}) + C \\
&\le \frac{1}{4}(\|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha+\delta)^{-1}P)\|_{H^1}) + C,
\end{aligned}$$

owing to, for $2 \le q \le 3$,

$$\begin{aligned}
&\|(|\nabla u| + |(\rho^\alpha+\delta)^{-1}P|)\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{2q}{q-2}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q} \\
&\le C(\|\nabla u\|_{L^{\frac{2q}{q-2}}} + \|(\rho^\alpha+\delta)^{-1}P\|_{L^{\frac{2q}{q-2}}})\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q} \\
&\le C(\|\nabla u\|_{L^6}^{\frac{2q-3}{q}}\|\nabla^2 u\|_{L^6}^{\frac{3-q}{q}} + \|(\rho^\alpha+\delta)^{-1}P\|_{L^6}^{\frac{2q-3}{q}}\|\nabla((\rho^\alpha+\delta)^{-1}P)\|_{L^6}^{\frac{3-q}{q}})\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q} \\
&\le \frac{1}{4}(\|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha+\delta)^{-1}P)\|_{H^1}) + C,
\end{aligned}$$

and for $3 < q < 6$,

$$
\begin{aligned}
&\|(|\nabla u| + |(\rho^\alpha + \delta)^{-1}P|)\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{2q}{q-2}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q} \\
&\le C(\|\nabla u\|_{L^6} + \|(\rho^\alpha + \delta)^{-1}P\|_{L^6})\|\rho^{\frac{1-\alpha}{2}}\|_{L^{\frac{3q}{q-3}}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q} \\
&\le C.
\end{aligned}
$$

Thus, we have

$$
\|\nabla^2 u\|_{H^1} + \|\nabla((\rho^\alpha + \delta)^{-1}P)\|_{H^1} \le C.
$$

With all of the a priori estimates in this section in hand, we are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Lemma 2.1 illustrates the fact that there exists a $T_0 > 0$ such that the system (2.1) has a unique local strong solution (ρ, u, P) on $\mathbb{R}^3 \times (0, T_0]$. We now extend this local solution to a global one. It follows from (3.1) that there exists a $T_1 \in (0, T_0]$ such that (3.4) holds for $T = T_1$. Denote

$$
T^* \triangleq \sup\{T | (\rho, u, P) \text{ is a strong solution on } \mathbb{R}^3 \times (0, T] \text{ and (3.4) holds}\}. \tag{3.59}
$$

Then, $T^* \ge T_1 > 0$. From Proposition 3.1, it can be seen that $T^* = T_1$. For any $0 < T \le T_1$ with T finite, we can deduce from (3.57) and (3.58) that

$$
\rho - \delta \in C([0, T]; L^{\frac{3}{2}} \cap H^1 \cap D^{1,k}), \quad \nabla\rho^{\frac{\alpha-1}{2}} \in C([0, T]; L^{\frac{3}{2}} \cap D^{1,q}). \tag{3.60}
$$

By virtue of Proposition 3.2, it can be inferred that

$$
\nabla u \in C([0, T]; H^1), \tag{3.61}
$$

where we have used the following standard embedding:

$$
L^\infty([0, T]; H^2 \cap W^{1,p}) \cap H^1([0, T]; L^2) \hookrightarrow C([0, T]; H^1) \cap C(\mathbb{R}^3 \times [0, T]).
$$

We assert that $T^* = \infty$. Otherwise, we assume that $T^* < \infty$. According to Proposition 3.1, (3.5) holds at $T = T^*$. It follows from (3.60) and (3.61) that

$$
(\rho^*, u^*)(x) \triangleq (\rho, u)(x, T^*) = \lim_{t \to T^*} (\rho, u)(x, t)
$$

satisfies

$$
0 \le \rho^* - \delta \in L^{\frac{3}{2}} \cap H^1 \cap D^{1,k}, \quad \nabla\rho^{*\frac{\alpha-1}{2}} \in L^{\frac{3}{2}} \cap D^{1,q}, \quad u^* \in D^1_{0,\sigma} \cap D^2.
$$

Then, with (ρ^*, u^*) as the initial data, we can use Proposition 3.1 to extend the local strong solution of (2.1) beyond T^*, which contradicts the definition of T^*. So $T^* = \infty$. The proof of Theorem 3.1 is finished.

4. Proof of Theorem 1.1

With the global existence of the approximate solutions (see Theorem 3.1) and a priori estimates independent of δ, we can take the limit on the approximation system and obtain the solution of the original problem described by (1.1) with (1.2).

Proof. The definition of $\rho_{0,\delta}$ can indicate that when $\delta \to 0$, we have the following for any $R > 1$:

$$\rho_{0,\delta} \to \rho_0 \quad \text{in } L^{\frac{3}{2}}(B_R) \cap H^1(B_R) \cap D^{1,k}(B_R).$$

Taking advantage of Proposition 3.2, we know that when $\delta \to 0$, the approximated solution sequence $(\rho^\delta, u^\delta, P^\delta)$ converges to (ρ, u, P) in the weak sense, and if necessary, we select its subsequence. As $\delta \to 0$, the standard compactness theory can provide the following:

$$\begin{cases} \rho^\delta - \delta \rightharpoonup \rho \text{ weak} - * \text{ in } L^\infty(0,\infty; L^{\frac{3}{2}}(\mathbb{R}^3) \cap D^{1,k}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)), \\ \rho_t^\delta \rightharpoonup \rho_t \text{ weak} - * \text{ in } L^\infty(0,\infty; L^{\frac{3}{2}}(\mathbb{R}^3)), \\ \rho^\delta \to \rho \text{ in } C([0,\infty); L^s(B_R)), \text{ for any } 2 \le s < 6, \\ u^\delta \rightharpoonup u \text{ weak} - * \text{ in } L^\infty([0,\infty); L^6(\mathbb{R}^3)), \\ \nabla u^\delta \to \nabla u \text{ in } C([0,\infty); L^2(\mathbb{R}^3)), \\ \rho^\delta u^\delta \to \rho u \text{ in } C([0,\infty); L^2(B_R)), \\ \nabla^2 u^\delta \rightharpoonup \nabla^2 u \text{ weak} - * \text{ in } L^\infty([0,\infty); L^2(\mathbb{R}^3)), \\ \nabla P^\delta \rightharpoonup \nabla P \text{ weak} - * \text{ in } L^\infty([0,\infty); L^2(\mathbb{R}^3)), \\ \nabla u_t^\delta \rightharpoonup \nabla u_t \text{ weak} - * \text{ in } L^\infty([0,\infty); L^2(\mathbb{R}^3)). \end{cases} \tag{4.1}$$

Since $(\rho^\delta, u^\delta, P^\delta)$ is the strong solution to the system (2.1), as $\delta \to 0$, we have that $\forall \varphi \in C_c^\infty(\mathbb{R}^3 \times [0,T])$,

$$\operatorname{div} u = 0,$$
$$\int_0^T \int [\rho_t \varphi + \rho u \cdot \nabla \varphi] dxdt = 0,$$
$$\int_0^T \int [\rho u \varphi_t + \rho u \otimes u \cdot \nabla \varphi - \rho^\alpha \nabla u \cdot \nabla \varphi + \nabla P \varphi] dxdt = 0.$$

Thanks to the regularity of ρ and u, using Holder's inequality and the Sobolev inequality, we have

$$\|\operatorname{div}(\rho u)\|_{L^{\frac{3}{2}}} = \|u \cdot \nabla \rho\|_{L^{\frac{3}{2}}} \le \|u\|_{L^6} \|\nabla \rho\|_{L^2} \le C,$$

$$\begin{aligned} \|(\rho u)_t\|_{L^2} &= \|\rho_t u + \rho u_t\|_{L^2} \\ &= \|u \cdot \nabla \rho u\|_{L^2} + \|\rho u_t\|_{L^2} \\ &\le C \|\nabla \rho\|_{L^2} \|u\|_{L^\infty}^2 + \|\rho\|_{L^3} \|u_t\|_{L^6} \\ &\le C \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + \|\rho\|_{L^3} \|\nabla u_t\|_{L^2} \\ &\le C, \end{aligned}$$

$$\begin{aligned}
\|\operatorname{div}(\rho u\otimes u)\|_{L^2} &= \|\operatorname{div}(\rho u)u+\rho u\cdot\nabla u\|_{L^2}\\
&\le \|u\cdot\nabla\rho u\|_{L^2}+\|\rho u\cdot\nabla u\|_{L^2}\\
&\le C\|\nabla\rho\|_{L^3}\|u\|_{L^\infty}\|u\|_{L^6}+\|\rho\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^6}\\
&\le C\|\nabla\rho\|_{L^2}^{\frac{2(k-3)}{3(k-2)}}\|\nabla\rho\|_{L^k}^{\frac{k}{3(k-2)}}\|\nabla^2u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac32}+\|\nabla\rho\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^2}\\
&\le C,
\end{aligned}$$

and

$$\begin{aligned}
\|\operatorname{div}(\rho^\alpha\nabla u)\|_{L^2} &= \|\rho^\alpha\Delta u+\nabla\rho^\alpha\cdot\nabla u\|_{L^2}\\
&\le C\|\rho\|_{L^\infty}^\alpha\|\nabla^2u\|_{L^2}+\|\nabla\rho^\alpha\|_{L^3}\|\nabla u\|_{L^6}\\
&\le C\|\rho\|_{L^\infty}^\alpha\|\nabla^2u\|_{L^2}++\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^3}\|\nabla^2u\|_{L^2}\\
&\le C\|\rho\|_{L^\infty}^\alpha\|\nabla^2u\|_{L^2}+\|\nabla\rho^{\frac{\alpha-1}{2}}\|_{L^{\frac32}}^{\frac{2q-3}{3(q-1)}}\|\nabla^2\rho^{\frac{\alpha-1}{2}}\|_{L^q}^{\frac{q}{3(q-1)}}\|\nabla^2u\|_{L^2}\\
&\le C.
\end{aligned}$$

Hence,

$$\int_0^T\int[\rho_t+\operatorname{div}(\rho u)]\varphi dxdt=0,$$

$$\int_0^T\int[(\rho u)_t+\operatorname{div}(\rho u\otimes u)-\operatorname{div}(\rho^\alpha\nabla u)+\nabla P]\varphi dxdt=0.$$

Due to the arbitrariness of φ, it can be inferred that for almost everywhere $(x,t)\in\mathbb{R}^3\times(0,\infty)$, we have the following:

$$\begin{cases}
\rho_t+\operatorname{div}(\rho u)=0,\\
(\rho u)_t+\operatorname{div}(\rho u\otimes u)-\operatorname{div}(\rho^\alpha\nabla u)+\nabla P=0,\\
\operatorname{div}u=0.
\end{cases}$$

Therefore, (ρ,u,P) is a strong solution to the Navier-Stokes system given by (1.1) with (1.2) on $\mathbb{R}^3\times(0,\infty)$ satisfying (1.7). Hence, Theorem 1.1 is proved.

5. Conclusions

We studied the nonhomogeneous incompressible Navier-Stokes equations with variable viscous coefficient and established a global strong solution. This result can help researchers better understand the behavior of fluids obtained by mixing two incompressible and immiscible fluids with different densities, and also contribute to the study of fluid motion containing molten materials. It has certain reference value in some applications of fluid mechanics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest that may influence the publication of this paper.

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