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Research article

Efficient results on unbounded solutions of fractional Bagley-Torvik system on the half-line

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Abstract: The fractional Bagley-Torvik system (FBTS) is initially created by utilizing fractional calculus to study the demeanor of real materials. It can be described as the dynamics of an inflexible plate dipped in a Newtonian fluid. In the present article, we aim for the first time to discuss the existence and uniqueness (E&U) theories of an unbounded solution for the proposed generalized FBTS involving Riemann-Liouville fractional derivatives in the half-line $(0, \infty)$, by using fixed point theorems (FPTs). Moreover, the Hyers-Ulam stability (HUS), Hyers-Ulam-Rassias stability (HURS), and semi-Hyers-Ulam-Rassias stability (sHURS) are proved. Finally, two numerical examples are given for checking the validity of major findings. By investigating unbounded solutions for the FBTS, engineers gain a deeper understanding of the underlying physics, optimize performance, improve system design, and ensure the stability of the motion of real materials in a Newtonian fluid.

Keywords: fractional derivatives; Bagley-Torvik equation; fixed point theorems; unbounded solutions
Mathematics Subject Classification: 34A08, 34B15

1. Introduction

Fractional calculus is the science of differentials and integrations of arbitrary non-integer orders [1]. In recent decades, fractional differential equations (FDEs) have gained a high interest, which enable the applications of dynamical systems in the field of life sciences. These equations under specific boundary value conditions have numerous applications in various science fields, including physics, engineering, finance, and biology. For example, FDEs can be used to model disease spread and understand a complex physical dynamic systems. As such, FDEs have become an essential tool for scientists, mathematicians, and engineers in many different areas of research; for more details, see these [2–8], and references cited therein. In 1984, Bagley and Torvik are considered the first authors who created a prototype fractional mathematical model to describe the viscoelasticity of real materials [9]. This model is given by

$$\mathcal{M}\, y''(\mathrm{t}) + 2\mathcal{S}\sqrt{pq}\, {}^{\mathcal{RL}}\mathbb{D}^{\frac{3}{2}} y(\mathrm{t}) + k\, y(\mathrm{t}) = \mathfrak{g}(\mathrm{t}), \tag{1.1}$$

where ${}^{\mathcal{RL}}\mathbb{D}^{\frac{3}{2}}$ is the Riemann-Liouville ($\mathcal{RL}$) fractional derivative, and $\mathcal{M}$ is the mass of a plate with surface area $\mathcal{S}$ and displacement y. Moreover, $\mathfrak{g}$ represents the loading force, and k is the stiffness of a spring that connected an immersed plate in a fluid of viscosity p and density q. In fact, the FBTS has attracted the attention of researchers in fields of mathematics and physics. In particular, Syam et al. [10] studied approximate solutions of the FBTS, with boundary conditions $y(0) = A, y(T) = B$, for $\mathrm{t} \in [0, T]$, by using an implicit hybrid method. Saw and Kumar [11] applied the Chebyshev collocation technique for solving the following FBTS:

$$\begin{gathered} k_0 y''(\mathrm{t}) + k_1 {}^{C}\mathbb{D}^{\frac{3}{2}} y(\mathrm{t}) + k_2 y(\mathrm{t}) = \mathfrak{g}(\mathrm{t}), \quad \mathrm{t} \in [0, 1], \\ y(0) = a, \quad y(1) = b, \quad a, b \in \mathbb{R}, \end{gathered}$$

such that ${}^{C}\mathbb{D}^{\frac{3}{2}}$ is a derivative in the Caputo sense. Furthermore, many authors studied the approximate solution of the FBTS under different fractional operators and a variety of boundary conditions by various approaches such as the Gegenbauer wavelet expansion [12], Homotopy analysis method [13], Galerkin approach [14], Legendre-collocation methods [15], Haar wavelet [16], generalized Fibonacci operational [17], and the matrix of Fermat polynomials operational [18]; for more related works see references cited therein.

Indeed, the analytic solution of the FBTS gained the interest of mathematicians. Particularly, Stanek [19] investigated E&U results of negative and positive solutions of the generalized FBTS with two-point boundary conditions. In 2015, the authors of [20] used the Laplace transform in solving the general FBTS without constraints in initial and boundary conditions. Fazli and Nieto [21], employed FPTs to study the existence of a lower and upper solution of the initial FBTS in partially ordered normed linear spaces. Pang et al. [22], studied generalized FBTSs of the form:

$$\begin{gathered} y''(\mathrm{t}) = \mathfrak{g}(\mathrm{t}) - \gamma_1 {}^{C}\mathbb{D}^{\nu}_{0^+} y(\mathrm{t}) - \gamma_2 y(\mathrm{t}) = \mathfrak{F}(\mathrm{t}, y(\mathrm{t}), {}^{C}\mathbb{D}^{\nu}_{0^+} y(\mathrm{t})), \mathrm{t} \in [0, T], \\ y(0) = a, \quad y'(0) = b, \quad a, b \in \mathbb{R}, \end{gathered}$$

where ${}^{C}\mathbb{D}^{\nu}_{0^+}$ is a derivative in the Caputo sense with order $\nu \in (0, 2)$. Additionally, [23] discussed the E&U of the FBTS by a different technique to that used in [21]. Moreover, Zafar et al. [24] used the integral transform technique to investigate solutions of the following general form of FBTS:

$$\begin{gathered} \gamma_0 {}^{C}\mathbb{D}^{\theta}_{0^+} y(\mathrm{t}) + \gamma_1 {}^{C}\mathbb{D}^{\vartheta+1}_{0^+} y(\mathrm{t}) + \gamma_2 y(\mathrm{t}) = \mathfrak{g}(\mathrm{t}), \mathrm{t} > 0, \\ y(0) = a, \quad y'(0) = b, \quad a, b \in \mathbb{R}, \end{gathered}$$

where $\theta \in (1, 2)$ and $\vartheta \in (0, 1)$.

On the other hand, unbounded solutions of dynamic systems often arise where a system exhibits extreme behavior, such as exponential growth or decay. By studying these unbounded solutions for a dynamic system, engineers gain a deeper understanding of how the system behaves beyond the scope of bounded solutions under extreme conditions with its long-term dynamics. Also, unbounded solutions allow us to identify critical points, bifurcation points, and regions of stability or instability in the system; we refer the readers to some related papers [25–28].

Inspired by the above articles, the present work focuses on investigating the E&U theories and some stability kinds such as the HUS, HURS, and sHURS of unbounded solutions for a new class of the generalized FBTS on the half-line $(0, \infty)$ as follows:

$$\begin{cases} {}^{\mathcal{RL}}\mathbb{D}^{\mu}_{0^+}y(\mathrm{t}) + \gamma\, {}^{\mathcal{RL}}\mathbb{D}^{\nu}_{0^+}y(\mathrm{t}) = \mathcal{H}(\mathrm{t}, y(\mathrm{t})), \quad \mathrm{t} \in \zeta = (0, \infty), \\ \lim\limits_{\mathrm{t}\to 0^+} \mathrm{t}^{2-\mu}y(\mathrm{t}) = \varrho_0, \quad \lim\limits_{\mathrm{t}\to\infty} \mathrm{t}^{1-\mu}y(\mathrm{t}) = \varrho_\infty, \end{cases} \tag{1.2}$$

such that ${}^{\mathcal{RL}}\mathbb{D}^{\theta}_{0^+}$ is the $\mathcal{RL}$−fractional derivative of order θ, where $\theta \in \{\mu, \nu\}$ with $\mu, \nu \in (1, 2], \mu > \nu$, and a given function $\mathrm{t}^{2-\mu}\mathcal{H}(\mathrm{t}, y) \in \mathfrak{C}(\zeta \times \varGamma, \varGamma)$, $\gamma, \varrho_0, \varrho_\infty \in \varGamma$, $\gamma \neq 0$, where $\varGamma$ denotes the real Banach space.

Here, we declare that, to the best of our knowledge, this is the first research work concerning the E&U and some types of HUS of unbounded solutions for FBTS on the half-line $(0, \infty)$, in an applicable space Σ which is defined in Section 3. Further, the FBTS (1.2) covers many existing works in the literature, for instance, it will turn to the original model (1.1), by taking $\mu = 2, \nu = \frac{3}{2}, \gamma = \frac{2\mathcal{S}\sqrt{pq}}{\mathcal{M}}$, and $\mathcal{H}(\mathrm{t}, y(\mathrm{t})) = \frac{1}{\mathcal{M}}(\mathrm{g}(\mathrm{t}) - ky(\mathrm{t}))$.

The remainder of this articles is organized as follows: In Section 2, background materials are provided. In Sections 3 and 4, the qualitative properties of an unbounded solution for the proposed FBTS (1.2) are proved.

2. Background materials

This section presents several important background materials, which are related to this study.

Definition 2.1. *[1] The μ^{th} $\mathcal{RL}$-fractional integral of the integrable function y, with $\mu > 0$, is as follows:*

$$(\mathbb{I}^{\mu}_{0^+}y)(\mathrm{t}) = \int_0^{\mathrm{t}} \frac{(\mathrm{t} - z)^{\mu-1}}{\Gamma(\mu)} y(z)dz, \ \mathrm{t} > 0.$$

Definition 2.2. *[1] The μ^{th} $\mathcal{RL}$-fractional derivative of the integrable function y, with $\mu \in (n-1, n]$, is as follows:*

$$({}^{\mathcal{RL}}\mathbb{D}^{\mu}_{0^+}y)(\mathrm{t}) = (\mathbb{D}^n\, \mathbb{I}^{n-\mu}_{0^+}y)(\mathrm{t}) = \frac{d^n}{d\mathrm{t}^n} \int_0^{\mathrm{t}} \frac{(\mathrm{t} - z)^{n-\mu-1}}{\Gamma(n - \mu)} y(z)dz, \ \mathrm{t} > 0, \mathbb{D} := \frac{d}{d\mathrm{t}}.$$

Lemma 2.3. *[1] Let $n - 1 < \mu, \nu \leq n$, and $y \in L^1([0, b])$, then*

(*i*) $\mathbb{I}^{\mu}_{0^+}\, {}^{\mathcal{RL}}\mathbb{D}^{\mu}_{0^+}y(\mathrm{t}) = y(\mathrm{t}) - \sum_{i=1}^{n} c_i \mathrm{t}^{\mu-i}, \ \ \forall\ \mathrm{t} \in [0, b], c_i \in \mathbb{R}$;

(*ii*) ${}^{\mathcal{RL}}\mathbb{D}^{\mu}_{0^+}\mathbb{I}^{\mu}_{0^+}y(\mathrm{t}) = y(\mathrm{t})$;

(*iii*) $\mathbb{I}^{\mu}_{0^+}\mathbb{I}^{\nu}_{0^+}y(\mathrm{t}) = \mathbb{I}^{\mu+\nu}_{0^+}y(\mathrm{t})$;

(*vi*) ${}^{\mathcal{RL}}\mathbb{D}^{\mu}_{0^+}\mathbb{I}^{\nu}_{0^+}y(\mathrm{t}) = \mathbb{I}^{\mu-\nu}_{0^+}y(\mathrm{t})$.

Lemma 2.4. *[1] For $\eta > 0$ and $\mu > 0$, we have* $\left(\mathbb{I}^{\mu}_{0^+} u^{\eta-1}\right)(\mathrm{t}) = \dfrac{\Gamma(\eta)}{\Gamma(\eta+\mu)}\mathrm{t}^{\eta+\mu-1}$, *and* $\left({}^{\mathcal{RL}}\mathbb{D}^{\mu}_{0^+} u^{\eta-1}\right)(\mathrm{t}) = \dfrac{\Gamma(\eta)}{\Gamma(\eta-\mu)}\mathrm{t}^{\eta-\mu-1}$.

Definition 2.5. *[29] We say that* $\mathrm{d} : Y \times Y \to [0,\infty)$ *is a generalized metric on the nonempty set Y, if the following three properties satisfied:* (i) $\mathrm{d}(\mathrm{t}, z) = 0$, *iff* $\mathrm{t} = z, \forall\, \mathrm{t}, z \in Y$; (ii) $\mathrm{d}(\mathrm{t}, z) = \mathrm{d}(z, \mathrm{t}), \forall\, \mathrm{t}, z \in Y$; (iii) $\mathrm{d}(\mathrm{t}, z) \leq \mathrm{d}(\mathrm{t}, c) + \mathrm{d}(c, z), \forall\, \mathrm{t}, z, c \in Y$.

Theorem 2.6. *[29] Suppose that a mapping $\Psi : Y \to Y$ is contractive with Lipschitz's constant $K < 1$, such that* (Y, d) *admits a generalized complete metric space. Moreover, if* $\mathrm{d}(\Psi^{j+1}\mathrm{t}, \Psi^{j}\mathrm{t}) < \infty$, *for some $j \in \mathbb{N}$ and $\mathrm{t} \in Y$, then the following statements are satisfied:*
(i) *A sequence $\{\Psi^j\}$ tends to a fixed point $\mathrm{t}_0 \in \Psi$;*
(ii) t_0 *admits unique fixed point of Ψ in* $Y^* = \{z \in Y |\ \mathrm{d}(\Psi^j\mathrm{t}, z) < \infty\}$*;*
(iii) *If $z \in Y^*$, then* $\mathrm{d}(z, \mathrm{t}_0) \leq \dfrac{1}{1-K}\mathrm{d}(\Psi z, z)$.

Theorem 2.7. *(Schauder's FPT, [30]). Let a mapping $\Psi : G \to G$ be continuous and compact, such that G is a nonempty, convex, closed, and bounded subset of a Banach space Y. Then, a mapping Ψ admits at least one fixed point in G.*

3. E&U of an unbounded solution

In this part, we start our study by producing the corresponding Volterra integral formula of the FBTS (1.2).

Lemma 3.1. *The FBTS* (1.2) *admits a solution equivalent to the Volterra integral equation*

$$\begin{aligned} y(\mathrm{t}) = & \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{\Gamma(\mu)}\mathcal{H}(z, y(z))dz - \gamma \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{\Gamma(\mu-\nu)} y(z)dz \\ & - \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^{\infty} \mathcal{H}(z, y(z))dz + \varrho_\infty \mathrm{t}^{\mu-1} + \varrho_0 \mathrm{t}^{\mu-2}. \end{aligned} \tag{3.1}$$

Proof. By taking $\mathbb{I}^{\mu}_{0^+}$ on both sides of the FBTS (1.2), and by applying Lemma 2.3, one finds

$$y(\mathrm{t}) = \mathbb{I}^{\mu}_{0^+}\mathcal{H}(\mathrm{t}, y(\mathrm{t})) - \gamma\, \mathbb{I}^{\mu-\nu}_{0^+} y(\mathrm{t}) + c_1 \mathrm{t}^{\mu-1} + c_2 \mathrm{t}^{\mu-2}.$$

Now, applying the boundary condition $\lim\limits_{\mathrm{t}\to 0^+} \mathrm{t}^{2-\mu} y(\mathrm{t}) = \varrho_0$, we find $c_2 = \varrho_0$, then

$$y(\mathrm{t}) = \mathbb{I}^{\mu}_{0^+}\mathcal{H}(\mathrm{t}, y(\mathrm{t})) - \gamma\, \mathbb{I}^{\mu-\nu}_{0^+} y(\mathrm{t}) + c_1 \mathrm{t}^{\mu-1} + \varrho_0 \mathrm{t}^{\mu-2}, \tag{3.2}$$

and by using the condition $\lim\limits_{\mathrm{t}\to\infty} \mathrm{t}^{1-\mu} y(\mathrm{t}) = \varrho_\infty$, we obtain

$$c_1 + \frac{1}{\Gamma(\mu)}\int_0^{\infty} \mathcal{H}(z, y(z))dz = \varrho_\infty,$$

which yields that

$$c_1 = \varrho_\infty - \frac{1}{\Gamma(\mu)}\int_0^{\infty} \mathcal{H}(z, y(z))dz.$$

Hence, by putting c_1 into Eq (3.2), one deduces that

$$
\begin{aligned}
y(\mathrm{t}) &= \mathbb{I}^{\mu}_{0^+}\mathcal{H}(\mathrm{t},y(\mathrm{t})) - \gamma\,\mathbb{I}^{\mu-\nu}_{0^+}y(\mathrm{t}) - \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^\infty \mathcal{H}(z,y(z))dz + \varrho_\infty \mathrm{t}^{\mu-1} + \varrho_0 \mathrm{t}^{\mu-2} \\
&= \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{\Gamma(\mu)}\mathcal{H}(z,y(z))dz - \gamma\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{\Gamma(\mu-\nu)}y(z)dz \\
&\quad - \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^\infty \mathcal{H}(z,y(z))dz + \varrho_\infty \mathrm{t}^{\mu-1} + \varrho_0 \mathrm{t}^{\mu-2}.
\end{aligned}
$$

Hence, the proof is finished. □

Now, we assume that $\mathcal{J}$ is a compact interval, and the Banach space of continuous functions is denoted by $\mathfrak{C}(\mathcal{J},\mathit{\Gamma})$, with supremum norm $\|y\|_{\mathit{\Gamma}} = \sup_{\mathrm{t}\in\mathcal{J}}\|y(\mathrm{t})\|$. Towards our aims, we define an applicable Banach space

$$
\Sigma = \left\{ y \,\middle|\, y(\mathrm{t}) \in \mathfrak{C}(\zeta,\mathit{\Gamma}),\ \sup_{\mathrm{t}\in\zeta}\frac{\|y(\mathrm{t})\|}{1+\mathrm{t}^{\mu}} < \infty \right\},
$$

which is gifted with the supremum norm

$$
\|y\|_\Sigma = \sup_{\mathrm{t}\in\zeta}\frac{\|y(\mathrm{t})\|}{1+\mathrm{t}^{\mu}},
$$

where $(\Sigma, \|\cdot\|_\Sigma)$ represents a Banach space, as in the works [31, 32]. Additionally, according to Lemma 3.1, we introduce the operator $\Psi : \Sigma \to \Sigma$, as follows:

$$
\begin{aligned}
(\Psi y)(\mathrm{t}) &= \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{\Gamma(\mu)}\mathcal{H}(z,y(z))dz - \gamma\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{\Gamma(\mu-\nu)}y(z)dz \\
&\quad - \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^\infty \mathcal{H}(z,y(z))dz + \varrho_\infty \mathrm{t}^{\mu-1} + \varrho_0 \mathrm{t}^{\mu-2}, \quad \mathrm{t}\in(0,\infty).
\end{aligned}
\tag{3.3}
$$

For investigating a work analysis, we present the following assumptions:

(AS_1) Let $h_1(\cdot), h_2(\cdot) > 0$, and $y, h_1, h_2, \mathrm{t}^{2-\mu}\mathcal{H}(\mathrm{t},y) : \zeta\times\Sigma \to \Sigma$ are continuous functions, such that

$$
\left\|\mathrm{t}^{2-\mu}\mathcal{H}(\mathrm{t},(1+\mathrm{t}^{\mu})y(\mathrm{t}))\right\| \le h_1(\mathrm{t}) + h_2(\mathrm{t})\left\|y(\mathrm{t})\right\|.
$$

(AS_2) Let $\alpha_1 \in (0,1)$, and $\alpha_2 > 0$ be real constants, such that

$$
\begin{aligned}
&\sup_{\mathrm{t}\in\zeta}\left(\frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}}\frac{h_2(z)}{z^{2-\mu}}dz + \frac{1}{\Gamma(\mu)}\int_0^\infty\frac{h_2(z)}{z^{2-\mu}}dz + \frac{\|\gamma\|}{\Gamma(\mu-\nu+1)}\right) \le \alpha_1 < 1, \\
&\sup_{\mathrm{t}\in\zeta}\left(\frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}}\frac{h_1(z)}{z^{2-\mu}}dz + \frac{1}{\Gamma(\mu)}\int_0^\infty\frac{h_1(z)}{z^{2-\mu}}dz + \|\varrho_\infty\| + \|\varrho_0\|\right) \le \alpha_2 < \infty.
\end{aligned}
$$

(AS_3) Let $\mathfrak{f}(\cdot) > 0$, and $y, \mathfrak{f}, t^{2-\mu}\mathcal{H}(\mathrm{t},y) : \zeta\times\Sigma \to \Sigma$ be continuous functions, such that

$$
\left\|\mathrm{t}^{2-\mu}\left[\mathcal{H}(\mathrm{t},(1+\mathrm{t}^{\mu})y(\mathrm{t})) - \mathcal{H}(\mathrm{t},(1+\mathrm{t}^{\mu})\bar{y}(\mathrm{t}))\right]\right\| \le \mathfrak{f}(\mathrm{t})\left\|y(\mathrm{t}) - \bar{y}(\mathrm{t})\right\|.
$$

(AS_4) Let $\Delta_1 \in (0,1)$, and $\Delta_2 > 0$ be real constants, such that

$$\sup_{\mathrm{t}\in\zeta}\left(\int_0^{\mathrm{t}} \frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz + \int_0^{\infty} \frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz + \frac{\|\gamma\|}{\Gamma(\mu-\nu+1)}\right) \le \Delta_1 < 1,$$

$$\sup_{\mathrm{t}\in\zeta}\left(\int_0^{\mathrm{t}} \frac{1}{\Gamma(\mu)}\left\|\mathcal{H}(z,0)\right\|dz + \int_0^{\infty} \frac{1}{\Gamma(\mu)}\left\|\mathcal{H}(z,0)\right\|dz + \|\varrho_\infty\| + \|\varrho_0\|\right) \le \Delta_2 < \infty.$$

Now, we present the following essential lemma which is needed for our analysis.

Lemma 3.2. *A bounded subset D of Σ is relatively compact in Σ, if*
(i) *A set* $\{\frac{h(\mathrm{t})}{1+\mathrm{t}^\mu}$, *for any* $h \in D, \mathrm{t} \in \mathcal{J}\}$, *is equicontinuous on* $\mathcal{J}$, *such that* $\mathcal{J}$ *is closed, bounded, and a sub-interval of* $(0,\infty)$;
(ii) *For any* $\epsilon > 0$, $\exists\, \delta > 0$, *such that* $\|\frac{h(\mathrm{t}_1)}{1+\mathrm{t}_1^\mu} - \frac{h(\mathrm{t}_2)}{1+\mathrm{t}_2^\mu}\| < \epsilon$, *for any* $\mathrm{t}_1, \mathrm{t}_2 \ge \delta$, *and* $h \in D$.

Proof. The proof can be introduced by the same manner as in [32]. □

Theorem 3.3. *Let* (AS_1) *and* (AS_2) *are hold. Then, the FBTS* (1.2) *possesses at least one solution on the half-line* ζ.

Proof. In order to achieve our goal, let us take the mapping $\Psi : \Sigma \to \Sigma$, as defined in Eq (3.3). Also, we define a bounded closed ball $\mathbb{B}_\rho = \{y \in \Sigma : \|y\|_\Sigma \le \rho\}$, such that $\rho \ge \dfrac{\alpha_2}{1-\alpha_1}$.

In fact, our analysis will be done according to Schauder's technique. Thus, first we show that $\Psi : \mathbb{B}_\rho \to \mathbb{B}_\rho$. For $y \in \mathbb{B}_\rho$, and $\mathrm{t} \in \zeta$, we have

$$\begin{aligned}
\left\|\frac{(\Psi y)(\mathrm{t})}{1+\mathrm{t}^\mu}\right\| \le\ & \frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}(1+\mathrm{t}^\mu)}\left\|z^{2-\mu}\mathcal{H}(z,y(z))\right\|dz + \frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)}\|y(z)\|\,dz \\
& + \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^{\infty} \frac{\left\|z^{2-\mu}\mathcal{H}(z,y(z))\right\|}{z^{2-\mu}(1+\mathrm{t}^\mu)}dz + \frac{\|\varrho_\infty\|\,\mathrm{t}^{\mu-1}}{(1+\mathrm{t}^\mu)} + \frac{\|\varrho_0\|\,\mathrm{t}^{\mu-2}}{(1+\mathrm{t}^\mu)} \\
\le\ & \frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}(1+\mathrm{t}^\mu)}\left(h_1(z) + h_2(z)\frac{\left\|y(z)\right\|}{1+z^\mu}\right)dz \\
& + \frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)}\|y(z)\|\,dz \\
& + \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)(1+\mathrm{t}^\mu)}\int_0^{\infty} \frac{1}{z^{2-\mu}}\left(h_1(z) + h_2(z)\frac{\left\|y(z)\right\|}{1+z^\mu}\right)dz + \frac{\|\varrho_\infty\|\,\mathrm{t}^{\mu-1}}{(1+\mathrm{t}^\mu)} + \frac{\|\varrho_0\|\,\mathrm{t}^{\mu-2}}{(1+\mathrm{t}^\mu)} \\
\le\ & \frac{1}{\Gamma(\mu)(1+\mathrm{t}^\mu)}\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}}h_1(z)dz + \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)(1+\mathrm{t}^\mu)}\int_0^{\infty} \frac{1}{z^{2-\mu}}h_1(z)dz \\
& + \frac{\left\|y\right\|_\Sigma}{\Gamma(\mu)(1+\mathrm{t}^\mu)}\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}}h_2(z)dz + \frac{\mathrm{t}^{\mu-1}\left\|y\right\|_\Sigma}{\Gamma(\mu)(1+\mathrm{t}^\mu)}\int_0^{\infty} \frac{1}{z^{2-\mu}}h_2(z)dz \\
& + \frac{\|\gamma\|\,\|y\|_\Sigma}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)}dz + \frac{\|\varrho_\infty\|\,\mathrm{t}^{\mu-1}}{(1+\mathrm{t}^\mu)} + \frac{\|\varrho_0\|\,\mathrm{t}^{\mu-2}}{(1+\mathrm{t}^\mu)} \\
\le\ & \frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}} \frac{h_1(z)}{z^{2-\mu}}dz + \frac{1}{\Gamma(\mu)}\int_0^{\infty} \frac{h_1(z)}{z^{2-\mu}}dz
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\left\|y\right\|_{\Sigma}}{\Gamma(\mu)} \int_0^{\mathrm{t}} \frac{h_2(z)}{z^{2-\mu}} dz + \frac{\left\|y\right\|_{\Sigma}}{\Gamma(\mu)} \int_0^{\infty} \frac{h_2(z)}{z^{2-\mu}} dz \\
&+ \frac{\|\gamma\|\,\|y\|_{\Sigma}}{\Gamma(\mu-\nu+1)} + \|\varrho_\infty\| + \|\varrho_0\| \\
\leq\ & \alpha_2 + \alpha_1\rho \leq \rho.
\end{aligned}
$$

Thus, $\|\Psi y\|_{\Sigma} \leq \rho$, which means $\Psi : \mathbb{B}_\rho \to \mathbb{B}_\rho$.

Now, it is easy to show that Ψ is continuous mapping due to the continuity of the functions y and $\mathcal{H}$, along with the Lebesgue dominated convergence approach, as follows:

Let $\{y_n\}_{n\in\mathbb{N}}$ be a convergence sequence in $\mathbb{B}_\rho$ that converges to y as n tends to ∞. Then,

$$
\mathrm{t}^{2-\mu}\mathcal{H}(\mathrm{t}, y_n(\mathrm{t})) \to \mathrm{t}^{2-\mu}\mathcal{H}(\mathrm{t}, y(\mathrm{t})), \text{ as } n \to \infty,
$$

and so

$$
\begin{aligned}
\lim_{n\to\infty} \frac{(\Psi y_n)(\mathrm{t})}{1+\mathrm{t}^\mu} &= \frac{1}{\Gamma(\mu)} \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}(1+\mathrm{t}^\mu)} \lim_{n\to\infty} z^{2-\mu}\mathcal{H}(z, y_n(z))dz \\
&- \frac{\|\gamma\|}{\Gamma(\mu-\nu)} \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)} \lim_{n\to\infty} y_n(z)dz \\
&- \frac{1}{\Gamma(\mu)} \int_0^{\infty} \frac{\mathrm{t}^{\mu-1}}{z^{2-\mu}(1+\mathrm{t}^\mu)} \lim_{n\to\infty} z^{2-\mu}\mathcal{H}(z, y_n(z))dz + \frac{\|\varrho_\infty\|\,\mathrm{t}^{\mu-1}}{(1+\mathrm{t}^\mu)} + \frac{\|\varrho_0\|\,\mathrm{t}^{\mu-2}}{(1+\mathrm{t}^\mu)} \\
&= \frac{(\Psi y)(\mathrm{t})}{1+\mathrm{t}^\mu}.
\end{aligned}
$$

Next, we prove that Ψ is an equicontinuous mapping on any compact interval $\mathcal{J} \subset \zeta$.

Consider B to be a bounded subset of $\mathbb{B}_\rho$, and $\mathcal{J} \subset \zeta$ a compact interval. Thus, for any $y \in B$, and $\mathrm{t}_1, \mathrm{t}_2 \in \mathcal{J}$ with $\mathrm{t}_1 \leq \mathrm{t}_2$, one finds

$$
\begin{aligned}
\left\| \frac{(\Psi y)(\mathrm{t}_2)}{1+\mathrm{t}_2^\mu} - \frac{(\Psi y)(\mathrm{t}_1)}{1+\mathrm{t}_1^\mu} \right\| &\leq \left\| \frac{1}{\Gamma(\mu)} \int_0^{\mathrm{t}_2} \frac{(\mathrm{t}_2-z)^{\mu-1}}{(1+\mathrm{t}_2^\mu)} \mathcal{H}(z, y(z))dz - \frac{1}{\Gamma(\mu)} \int_0^{\mathrm{t}_1} \frac{(\mathrm{t}_1-z)^{\mu-1}}{(1+\mathrm{t}_1^\mu)} \mathcal{H}(z, y(z))dz \right\| \\
&+ \left\| \frac{\gamma}{\Gamma(\mu-\nu)} \int_0^{\mathrm{t}_2} \frac{(\mathrm{t}_2-z)^{\mu-\nu-1}}{(1+\mathrm{t}_2^\mu)} y(z)dz - \frac{\gamma}{\Gamma(\mu-\nu)} \int_0^{\mathrm{t}_1} \frac{(\mathrm{t}_1-z)^{\mu-\nu-1}}{(1+\mathrm{t}_1^\mu)} y(z)dz \right\| \\
&+ \frac{\left|\mathrm{t}_2^{\mu-1} - \mathrm{t}_1^{\mu-1}\right|}{(1+\mathrm{t}_2^\mu)(1+\mathrm{t}_1^\mu)} \int_0^{\infty} \frac{\left\|\mathcal{H}(z, y(z))\right\|}{\Gamma(\mu)} dz \\
&+ \frac{\|\varrho_\infty\| \left|(\mathrm{t}_2^{\mu-1}(1+\mathrm{t}_1^\mu) - \mathrm{t}_1^{\mu-1}(1+\mathrm{t}_2^\mu))\right|}{(1+\mathrm{t}_2^\mu)(1+\mathrm{t}_1^\mu)} + \frac{\|\varrho_0\| \left|\mathrm{t}_2^{\mu-2}(1+\mathrm{t}_1^\mu) - \mathrm{t}_1^{\mu-2}(1+\mathrm{t}_2^\mu)\right|}{(1+\mathrm{t}_2^\mu)(1+\mathrm{t}_1^\mu)} \\
&\leq \frac{1}{\Gamma(\mu)} \int_0^{\mathrm{t}_1} \left| \frac{(\mathrm{t}_2-z)^{\mu-1}}{(1+\mathrm{t}_2^\mu)} - \frac{(\mathrm{t}_1-z)^{\mu-1}}{(1+\mathrm{t}_1^\mu)} \right| \left\|\mathcal{H}(z, y(z))\right\| dz \\
&+ \frac{\|\gamma\|}{\Gamma(\mu-\nu)} \int_0^{\mathrm{t}_1} \left| \frac{(\mathrm{t}_2-z)^{\mu-\nu-1}}{(1+\mathrm{t}_2^\mu)} - \frac{(\mathrm{t}_1-z)^{\mu-\nu-1}}{(1+\mathrm{t}_1^\mu)} \right| \|y(z)\|\, dz \\
&+ \left\| \frac{1}{\Gamma(\mu)} \int_{\mathrm{t}_1}^{\mathrm{t}_2} \frac{(\mathrm{t}_2-z)^{\mu-1}}{(1+\mathrm{t}_2^\mu)} \mathcal{H}(z, y(z))dz \right\| + \left\| \frac{\gamma}{\Gamma(\mu-\nu)} \int_{\mathrm{t}_1}^{\mathrm{t}_2} \frac{(\mathrm{t}_2-z)^{\mu-\nu-1}}{(1+\mathrm{t}_2^\mu)} y(z)dz \right\|
\end{aligned}
$$

$$
+\frac{\left|\mathrm{t}_2^{\mu-1}-\mathrm{t}_1^{\mu-1}\right|}{(1+\mathrm{t}_2^{\mu})(1+\mathrm{t}_1^{\mu})}\int_0^{\infty}\frac{\left\|\mathcal{H}(z,y(z))\right\|}{\Gamma(\mu)}dz
$$
$$
+\frac{\|\varrho_\infty\|\left|(\mathrm{t}_2^{\mu-1}(1+\mathrm{t}_1^{\mu})-\mathrm{t}_1^{\mu-1}(1+\mathrm{t}_2^{\mu}))\right|}{(1+\mathrm{t}_2^{\mu})(1+\mathrm{t}_1^{\mu})}+\frac{\|\varrho_0\|\left|\mathrm{t}_2^{\mu-2}(1+\mathrm{t}_1^{\mu})-\mathrm{t}_1^{\mu-2}(1+\mathrm{t}_2^{\mu})\right|}{(1+\mathrm{t}_2^{\mu})(1+\mathrm{t}_1^{\mu})},
$$

which yields that $\left|\frac{(\Psi y)(\mathrm{t}_2)}{1+\mathrm{t}_2^{\mu}}-\frac{(\Psi y)(\mathrm{t}_1)}{1+\mathrm{t}_1^{\mu}}\right|\to 0$, when t_1 tends to t_2, which means Ψ is an equicontinuous mapping on $\mathcal{J}$.

Next, we investigate that Ψ is equiconvergent at ∞. For achieving this goal, we know that $\lim\limits_{\mathrm{t}\to\infty}\frac{\mathrm{t}^{\mu-1}}{1+\mathrm{t}^{\mu}}=0$, and then for any $\epsilon>0,\exists\ \delta_1>0,\forall\,\mathrm{t}>\delta_1$, which implies that $\left|\frac{\mathrm{t}^{\mu-1}}{1+\mathrm{t}^{\mu}}\right|<\frac{\epsilon}{2}$. So, for each $\mathrm{t}_1,\mathrm{t}_2>\delta_1$, one has

$$
\left|\frac{\mathrm{t}_2^{\mu-1}}{1+\mathrm{t}_2^{\mu}}-\frac{\mathrm{t}_1^{\mu-1}}{1+\mathrm{t}_1^{\mu}}\right|\le\left|\frac{\mathrm{t}_2^{\mu-1}}{1+\mathrm{t}_2^{\mu}}\right|+\left|\frac{\mathrm{t}_1^{\mu-1}}{1+\mathrm{t}_1^{\mu}}\right|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
$$

Similarly, $\lim\limits_{\mathrm{t}\to\infty}\frac{(\mathrm{t}-z)^{\mu-1}}{1+\mathrm{t}^{\mu}}=0$, that is, for any $\epsilon>0,\exists\,\delta_2>0,\forall\,\mathrm{t}>\delta_2$, we have $\left|\frac{(\mathrm{t}-z)^{\mu-1}}{1+\mathrm{t}^{\mu}}\right|<\frac{\epsilon}{2}$. Thus, for each $\mathrm{t}_1,\mathrm{t}_2>\delta_2$, we get $\left|\frac{(\mathrm{t}_2-z)^{\mu-1}}{1+\mathrm{t}_2^{\mu}}-\frac{(\mathrm{t}_1-z)^{\mu-1}}{1+\mathrm{t}_1^{\mu}}\right|<\epsilon$. Moreover, $\lim\limits_{\mathrm{t}\to\infty}\frac{\mathrm{t}^{\mu-2}}{1+\mathrm{t}^{\mu}}=0$, and then, for any $\epsilon>0,\exists\,\delta_3>0,\forall\,\mathrm{t}>\delta_3$, yields that $\left|\frac{\mathrm{t}^{\mu-2}}{1+\mathrm{t}^{\mu}}\right|<\frac{\epsilon}{2}$. Hence, for each $\mathrm{t}_1,\mathrm{t}_2>\delta_3$, one obtains $\left|\frac{\mathrm{t}_2^{\mu-2}}{1+\mathrm{t}_2^{\mu}}-\frac{\mathrm{t}_1^{\mu-2}}{1+\mathrm{t}_1^{\mu}}\right|<\epsilon$. In the same manner, since $\lim\limits_{\mathrm{t}\to\infty}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{1+\mathrm{t}^{\mu}}=0$, for any $\epsilon>0,\exists\,\delta_4>0,\forall\,\mathrm{t}>\delta_4$, we have $\left|\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{1+\mathrm{t}^{\mu}}\right|<\frac{\epsilon}{2}$. Thus, for each $\mathrm{t}_1,\mathrm{t}_2>\delta_4$, we find $\left|\frac{(\mathrm{t}_2-z)^{\mu-\nu-1}}{1+\mathrm{t}_2^{\mu}}-\frac{(\mathrm{t}_1-z)^{\mu-\nu-1}}{1+\mathrm{t}_1^{\mu}}\right|<\epsilon$.

Therefore, for any $\epsilon>0$, by choosing $\delta\ge\max\{\delta_1,\delta_2,\delta_3,\delta_4\}$, for all $\mathrm{t}_1,\mathrm{t}_2>\delta$ and for any $y\in B$, one finds

$$
\begin{aligned}
\left\|\frac{(\Psi y)(\mathrm{t}_2)}{1+\mathrm{t}_2^{\mu}}-\frac{(\Psi y)(\mathrm{t}_1)}{1+\mathrm{t}_1^{\mu}}\right\|\le&\frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}_1}\frac{1}{z^{2-\mu}}\left|\frac{(\mathrm{t}_2-z)^{\mu-1}}{(1+\mathrm{t}_2^{\mu})}-\frac{(\mathrm{t}_1-z)^{\mu-1}}{(1+\mathrm{t}_1^{\mu})}\right|\left\|z^{2-\mu}\mathcal{H}(z,y(z))\right\|dz\\
&+\frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}_1}\left|\frac{(\mathrm{t}_2-z)^{\mu-\nu-1}}{(1+\mathrm{t}_2^{\mu})}-\frac{(\mathrm{t}_1-z)^{\mu-\nu-1}}{(1+\mathrm{t}_1^{\mu})}\right|\|y(z)\|\,dz\\
&+\frac{1}{\Gamma(\mu)}\int_{\mathrm{t}_1}^{\mathrm{t}_2}\frac{1}{z^{2-\mu}}\left|\frac{(\mathrm{t}_2-z)^{\mu-1}}{(1+\mathrm{t}_2^{\mu})}\right|\left\|z^{2-\mu}\mathcal{H}(z,y(z))\right\|dz\\
&+\frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_{\mathrm{t}_1}^{\mathrm{t}_2}\left|\frac{(\mathrm{t}_2-z)^{\mu-\nu-1}}{(1+\mathrm{t}_2^{\mu})}\right|\|y(z)\|\,dz\\
&+\left|\frac{\mathrm{t}_2^{\mu-1}}{(1+\mathrm{t}_2^{\mu})}-\frac{\mathrm{t}_1^{\mu-1}}{(1+\mathrm{t}_1^{\mu})}\right|\int_0^{\infty}\frac{\left\|z^{2-\mu}\mathcal{H}(z,y(z))\right\|}{z^{2-\mu}\Gamma(\mu)}dz\\
&+\|\varrho_\infty\|\left|\frac{\mathrm{t}_2^{\mu-1}}{(1+\mathrm{t}_2^{\mu})}-\frac{\mathrm{t}_1^{\mu-1}}{(1+\mathrm{t}_1^{\mu})}\right|+\|\varrho_0\|\left|\frac{\mathrm{t}_2^{\mu-2}}{(1+\mathrm{t}_2^{\mu})}-\frac{\mathrm{t}_1^{\mu-2}}{(1+\mathrm{t}_1^{\mu})}\right|\\
\le&\frac{\epsilon}{\Gamma(\mu)}\int_0^{\mathrm{t}_1}\frac{1}{z^{2-\mu}}\left(h_1(\mathrm{t})+h_2(\mathrm{t})\frac{\left\|y(\mathrm{t})\right\|}{1+z^{\mu}}\right)dz+\frac{\epsilon\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}_1}\|y(z)\|\,dz\\
&+\frac{\frac{\epsilon}{2}}{\Gamma(\mu)}\int_{\mathrm{t}_1}^{\mathrm{t}_2}\frac{1}{z^{2-\mu}}\left(h_1(\mathrm{t})+h_2(\mathrm{t})\frac{\left\|y(\mathrm{t})\right\|}{1+z^{\mu}}\right)dz+\frac{\frac{\epsilon}{2}\|\gamma\|}{\Gamma(\mu-\nu)}\int_{\mathrm{t}_1}^{\mathrm{t}_2}\|y(z)\|\,dz
\end{aligned}
$$

$$+\frac{\epsilon}{\Gamma(\mu)}\int_0^\infty \frac{1}{z^{2-\mu}}\left(h_1(\mathrm{t})+h_2(\mathrm{t})\frac{\|y(\mathrm{t})\|}{1+z^\mu}\right)dz+(\|\varrho_\infty\|+\|\varrho_0\|)\,\epsilon.$$

Hence, for any $\epsilon>0$, there exists $\delta>0$ such that, for all $\mathrm{t}_1,\mathrm{t}_2>\delta$, we deduce that $\Psi: B\to B$ is equiconvergent at ∞. According to Lemma 3.2, we conclude that $\Psi:\mathbb{B}_\rho\to\mathbb{B}_\rho$ is completely continuous. Hence, in view of Schauder's technique 2.7, we infer that Ψ possesses at least one fixed point, which means the FBTS (1.2) possesses at least one solution on the half-line ζ. □

Theorem 3.4. *Let* (AS_3) *and* (AS_4) *are hold. Then, the FBTS* (1.2) *admits an exactly one solution on the half-line* ζ.

Proof. To prove this theorem, we define the mapping $\Psi:\Sigma\to\Sigma$, as given in Eq (3.3). Hence, Ψ maps Σ into itself, due to using (AS_3) and (AS_4), as follows:

$$\begin{aligned}
\left\|\frac{(\Psi y)(\mathrm{t})}{1+\mathrm{t}^\mu}\right\| &\le \frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}(1+\mathrm{t}^\mu)}\left\|z^{2-\mu}\mathcal{H}(z,y(z))\right\|dz+\frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)}\|y(z)\|\,dz\\
&\quad+\frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^\infty\frac{\left\|z^{2-\mu}\mathcal{H}(z,y(z))\right\|}{z^{2-\mu}(1+\mathrm{t}^\mu)}dz+\frac{\|\varrho_\infty\|\,\mathrm{t}^{\mu-1}}{(1+\mathrm{t}^\mu)}+\frac{\|\varrho_0\|\,\mathrm{t}^{\mu-2}}{(1+\mathrm{t}^\mu)}\\
&\le \frac{1}{\Gamma(\mu)}\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}(1+\mathrm{t}^\mu)}\left(\mathfrak{f}(z)\frac{\|y(z)\|}{(1+z^\mu)}+\left\|z^{2-\mu}\mathcal{H}(z,0)\right\|\right)dz\\
&\quad+\frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)}\|y(z)\|\,dz\\
&\quad+\frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^\infty\frac{1}{z^{2-\mu}(1+\mathrm{t}^\mu)}\left(\mathfrak{f}(z)\frac{\|y(z)\|}{(1+z^\mu)}+\left\|z^{2-\mu}\mathcal{H}(z,0)\right\|\right)dz\\
&\quad+\frac{\|\varrho_\infty\|\,\mathrm{t}^{\mu-1}}{(1+\mathrm{t}^\mu)}+\frac{\|\varrho_0\|\,\mathrm{t}^{\mu-2}}{(1+\mathrm{t}^\mu)}\\
&\le\left(\int_0^{\mathrm{t}}\frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz+\int_0^\infty\frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz+\frac{\|\gamma\|}{\Gamma(\mu-\nu+1)}\right)\|y\|_\Sigma\\
&\quad+\int_0^{\mathrm{t}}\frac{1}{\Gamma(\mu)}\left\|\mathcal{H}(z,0)\right\|dz+\int_0^\infty\frac{1}{\Gamma(\mu)}\left\|\mathcal{H}(z,0)\right\|dz+\|\varrho_\infty\|+\|\varrho_0\|\\
&\le\Delta_1\|y\|_\Sigma+\Delta_2<\infty.
\end{aligned}$$

Next, we prove that Ψ is a contractive operator on Σ. Then, for any $y,\bar{y}\in\Sigma$, and by applying (AS_3) and (AS_4), one has

$$\begin{aligned}
\frac{\|(\Psi y)(\mathrm{t})-(\Psi\bar{y})(\mathrm{t})\|}{1+\mathrm{t}^\mu} &\le \int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-1}}{z^{2-\mu}\Gamma(\mu)(1+\mathrm{t}^\mu)}\left\|z^{2-\mu}\mathcal{H}(z,y(z))-z^{2-\mu}\mathcal{H}(z,\bar{y}(z))\right\|dz\\
&\quad+\frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)}\|y(z)-\bar{y}(z)\|dz\\
&\quad+\int_0^\infty\frac{\mathrm{t}^{\mu-1}}{z^{2-\mu}\Gamma(\mu)(1+\mathrm{t}^\mu)}\left\|z^{2-\mu}\mathcal{H}(z,y(z))-z^{2-\mu}\mathcal{H}(z,\bar{y}(z))\right\|dz\\
&\le\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-1}\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)(1+\mathrm{t}^\mu)}\frac{\|y(z)-\bar{y}(z)\|}{(1+z^\mu)}dz+\frac{\|\gamma\|}{\Gamma(\mu-\nu)}\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{(1+\mathrm{t}^\mu)}\|y(z)-\bar{y}(z)\|dz
\end{aligned}$$

$$+\int_0^{\infty}\frac{\mathrm{t}^{\mu-1}\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)(1+\mathrm{t}^{\mu})}\frac{\|y(z)-\bar{y}(z)\|}{(1+z^{\mu})}dz$$
$$\leq\left(\int_0^{\mathrm{t}}\frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz+\int_0^{\infty}\frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz+\frac{\|\gamma\|}{\Gamma(\mu-\nu+1)}\right)\|y-\bar{y}\|_{\Sigma}.$$

Hence, we infer that $\|\Psi y-\Psi\bar{y}\|_{\Sigma}\leq\Delta_1\|y-\bar{y}\|_{\Sigma}$, due to $\Delta_1\in(0,1)$. This implies that Ψ is a contractive mapping, and based on the Banach contraction technique, we deduce that Ψ admits exactly one fixed point y_0 in Σ. Therefore, the FBTS (1.2) possesses one solution on the half-line $(0,\infty)$. □

4. Stability of an unbounded solution

In this section, we are concerned with HURS, HUS, and sHURS. To this end, we define suitable metrics $\mathrm{d}_1(\cdot)$ and $\mathrm{d}_2(\cdot)$ on the space Σ, as follows:

$$\mathrm{d}_1(y,\bar{y})=\inf_{\mathrm{t}\in\zeta}\left\{\Lambda\in\zeta\Big|\ \frac{\|y(\mathrm{t})-\bar{y}(\mathrm{t})\|}{1+\mathrm{t}^{\mu}}\leq\Lambda\chi(\mathrm{t})\right\},$$

such that $\chi(\mathrm{t})>0$, is a continuous and non-decreasing function on the half-line ζ, and

$$\mathrm{d}_2(y,\bar{y})=\sup_{\mathrm{t}\in\zeta}\left\{\Lambda\in\zeta\Big|\ \frac{\|y(\mathrm{t})-\bar{y}(\mathrm{t})\|}{\chi(\mathrm{t})(1+\mathrm{t}^{\mu})}\leq\Lambda\right\},$$

such that $\chi(\mathrm{t})>0$, is a continuous and non-increasing function on the half-line ζ. Similar to the work [33], and references therein, we can show that $\mathrm{d}_1(\cdot)$ and $\mathrm{d}_2(\cdot)$ represent metrics on the Banach space Σ.

Definition 4.1. *[34] The solution of the FBTS* (1.2) *is HURS, if for each continuous function $y:\zeta=(0,\infty)\to\Sigma$, satisfying*

$$\Big\|y(\mathrm{t})-\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-1}}{\Gamma(\mu)}\mathcal{H}(z,y(z))dz+\gamma\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{\Gamma(\mu-\nu)}y(z)dz$$
$$+\frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^{\infty}\mathcal{H}(z,y(z))dz-\varrho_{\infty}\mathrm{t}^{\mu-1}-\varrho_0\mathrm{t}^{\mu-2}\Big\|\leq\mathbb{I}_{0^+}^{\mu}\chi(\mathrm{t}),\quad \mathrm{t}\in\zeta,$$

where $\chi(\mathrm{t})>0$, is a continuous and increasing function on the half-line ζ, then there is exactly one solution y_0 for the FBTS (1.2)*, with*

$$\frac{\|y(\mathrm{t})-y_0(\mathrm{t})\|}{1+\mathrm{t}^{\mu}}\leq\Lambda\chi(\mathrm{t}),\quad\forall\,\mathrm{t}\in\zeta,$$

where $\Lambda>0$ is a constant independent of y,y_0. Additionally, by taking $\xi\geq0$ instead of $\chi(\mathrm{t})$, then the solution of the FBTS (1.2) *is HUS.*

Theorem 4.2. *Let assumptions* (AS_3) *and* (AS_4) *be fulfilled, and $\chi(\mathrm{t})>0$ be a continuous increasing function on the half-line ζ, and a function $y:\zeta=(0,\infty)\to\Sigma$ is continuous satisfying*

$$\Big\|y(\mathrm{t})-\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-1}}{\Gamma(\mu)}\mathcal{H}(z,y(z))dz+\gamma\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{\Gamma(\mu-\nu)}y(z)dz$$

$$+\frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^\infty \mathcal{H}(z,y(z))dz-\varrho_\infty \mathrm{t}^{\mu-1}-\varrho_0\mathrm{t}^{\mu-2}\Big\|\le \mathbb{I}^\mu_{0^+}\chi(\mathrm{t}),\quad \mathrm{t}\in\zeta. \tag{4.1}$$

Then, there is exactly one solution $y_0\in\Sigma$, such that

$$\frac{\|y(\mathrm{t})-y_0(\mathrm{t})\|}{1+\mathrm{t}^\mu}\le\frac{\Pi}{1-\Delta_1}\chi(\mathrm{t}),\quad \forall\,\mathrm{t}\in\zeta, 0<\Delta_1<1, \tag{4.2}$$

where $\sup_{\mathrm{t}\in\zeta}\frac{\mathrm{t}^\mu}{\Gamma(\mu+1)(1+\mathrm{t}^\mu)}\le\Pi<\infty$, *which implies that the solution of the FBTS* (1.2)*, is HURS, and it follows that it HUS.*

Proof. Let us recall the contractive operator $\Psi:\Sigma\to\Sigma$ as defined in Eq (3.3). By the metric $\mathrm{d}_1(\cdot)$, and (AS_3), (AS_4), for $y,\bar{y}\in\Sigma$, we find

$$\begin{aligned}\frac{\|(\Psi y)(\mathrm{t})-(\Psi\bar{y})(\mathrm{t})\|}{1+\mathrm{t}^\mu}&\le\Lambda\chi(\mathrm{t})\left(\int_0^{\mathrm{t}}\frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz+\int_0^\infty\frac{\mathfrak{f}(z)}{z^{2-\mu}\Gamma(\mu)}dz+\frac{\|\gamma\|}{\Gamma(\mu-\nu+1)}\right)\\&\le\Delta_1\,\Lambda\chi(\mathrm{t}),\quad\forall\,\mathrm{t}\in\zeta, 0<\Delta_1<1.\end{aligned}$$

Thus, one has

$$\mathrm{d}_1(\Psi y,\Psi\bar{y})\le\Delta_1\,\Lambda=\Delta_1\,\mathrm{d}_1(y,\bar{y}),\quad 0<\Delta_1<1.$$

Due to inequality (4.1), we obtain

$$\frac{\|(y)(\mathrm{t})-(\Psi y)(\mathrm{t})\|}{1+\mathrm{t}^\mu}\le\sup_{\mathrm{t}\in\zeta}\frac{\mathrm{t}^\mu}{\Gamma(\mu+1)(1+\mathrm{t}^\mu)}\chi(\mathrm{t})=\Pi\,\chi(\mathrm{t}),\quad \mathrm{t}\in\zeta. \tag{4.3}$$

According to inequality (4.3), we have

$$\mathrm{d}_1(y,\Psi y)\le\Pi<\infty.$$

Hence, in view of Theorem 2.6, there is exactly one fixed point y_0, and

$$\mathrm{d}_1(y,y_0)\le\frac{1}{1-\Delta_1}\mathrm{d}_1(\Psi y,y)\le\frac{\Pi}{1-\Delta_1},\quad 0<\Pi<1.$$

Consequently, the solution of the FBTS (1.2) is HURS, and, for $\chi(\mathrm{t})=1$, it follows that the solution of the FBTS (1.2) is HUS. □

Definition 4.3. *[34] The solution of problem* (1.2) *is sHURS, if for each continuous function $y:\zeta=(0,\infty)\to\Gamma$, satisfying*

$$\begin{aligned}&\Big\|y(\mathrm{t})-\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-1}}{\Gamma(\mu)}\mathcal{H}(z,y(z))dz+\gamma\int_0^{\mathrm{t}}\frac{(\mathrm{t}-z)^{\mu-\nu-1}}{\Gamma(\mu-\nu)}y(z)dz\\&+\frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)}\int_0^\infty\mathcal{H}(z,y(z))dz-\varrho_\infty\mathrm{t}^{\mu-1}-\varrho_0\mathrm{t}^{\mu-2}\Big\|\le\mathbb{I}^\mu_{0^+}\xi,\quad\mathrm{t}\in\zeta,\end{aligned}$$

where $\xi\ge0$, there is exactly one solution y_0 of the FBTS (1.2)*, and a constant $\Lambda>0$ independent of y, y_0 for some continuous decreasing function $\chi(\mathrm{t})>0$ on the half-line ζ, where*

$$\frac{\|y(\mathrm{t})-y_0(\mathrm{t})\|}{1+\mathrm{t}^\mu}\le\Lambda\chi(\mathrm{t}),\quad\forall\,\mathrm{t}\in\zeta.$$

Theorem 4.4. *Let* (AS_3) *and* (AS_4) *hold, and* $\chi(\mathrm{t}) > 0$ *be a continuous decreasing function on the half-line* ζ*, and a function* $y : (0,\infty) \to \Sigma$ *be continuous, satisfying*

$$\Big\| y(\mathrm{t}) - \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-1}}{\Gamma(\mu)} \mathcal{H}(z, y(z))dz + \gamma \int_0^{\mathrm{t}} \frac{(\mathrm{t}-z)^{\mu-\nu-1}}{\Gamma(\mu-\nu)} y(z)dz + \frac{\mathrm{t}^{\mu-1}}{\Gamma(\mu)} \int_0^{\infty} \mathcal{H}(z, y(z))dz - \varrho_\infty \mathrm{t}^{\mu-1} - \varrho_0 \mathrm{t}^{\mu-2} \Big\| \leq \mathbb{I}^{\mu}_{0^+}\xi, \quad \mathrm{t} \in \zeta, \tag{4.4}$$

where $\xi > 0$*. Then, there is exactly one solution* $y_0 \in \Sigma$*, and a constant* $\Xi > 0$*, where*

$$\frac{\|y(\mathrm{t}) - y_0(\mathrm{t})\|}{1+\mathrm{t}^{\mu}} \leq \frac{\xi \Pi \Xi}{1-\Delta_1}\chi(\mathrm{t}), \quad \forall\, \mathrm{t} \in \zeta, 0 < \Delta_1 < 1, \tag{4.5}$$

where $\sup\limits_{\mathrm{t}\in\zeta} \dfrac{\mathrm{t}^{\mu}}{\Gamma(\mu+1)(1+\mathrm{t}^{\mu})} \leq \Pi < \infty$*, which implies that the FBTS* (1.2) *has a solution of sHURS.*

Proof. In the same way as Theorem 4.2, let $\Psi : \Sigma \to \Sigma$ is a contractive mapping, as defined in (3.3). By metric $\mathrm{d}_2(\cdot)$ and (AS_3), (AS_4), one gets

$$\frac{\|(\Psi y)(\mathrm{t}) - (\Psi \bar{y})(\mathrm{t})\|}{\chi(\mathrm{t})(1+\mathrm{t}^{\mu})} \leq \Delta_1 \Lambda, \quad \forall\, \mathrm{t} \in \zeta, 0 < \Delta_1 < 1.$$

So,

$$\mathrm{d}_2(\Psi y, \Psi \bar{y}) \leq \Delta_1 \Lambda = \Delta_1 \mathrm{d}_2(y, \bar{y}), \quad 0 < \Delta_1 < 1.$$

According to continuity, positiveness, and the decreasing of the function $\chi(\mathrm{t})$, $\forall\, \mathrm{t} \in \zeta$, there is $\Xi > 0$, such that $\frac{1}{\chi(\mathrm{t})} \leq \Xi$. Thus, in view of inequality (4.4), we have

$$\frac{\|(y)(\mathrm{t}) - (\Psi y)(\mathrm{t})\|}{\chi(\mathrm{t})(1+\mathrm{t}^{\mu})} \leq \sup_{\mathrm{t}\in\zeta} \frac{\xi \mathrm{t}^{\mu}}{\chi(\mathrm{t})\Gamma(\mu+1)(1+\mathrm{t}^{\mu})} = \Pi \Xi \xi, \quad \mathrm{t} \in \zeta. \tag{4.6}$$

Further, by the inequality (4.6), we find

$$\mathrm{d}_2(y, \Psi y) \leq \Pi \Xi \xi < \infty.$$

Hence, in view of the Theorem 2.6, there is an exactly one fixed point y_0, and

$$\mathrm{d}_2(y, y_0) \leq \frac{1}{1-\Delta_1} \mathrm{d}_2(\Psi y, y) \leq \frac{\Pi \Xi \xi}{1-\Delta_1}, \quad 0 < \Delta_1 < 1.$$

Therefore, the FBTS (1.2) has a solution with sHURS, and the desired proof is completed. □

5. Examples

In this part, we illustrate the obtained findings with the following examples.

Example 5.1. *Consider the FBTS given by*

$${}^{\mathcal{RL}}\mathbb{D}^{2}_{0^+} y(\mathrm{t}) + \frac{1}{5} {}^{\mathcal{RL}}\mathbb{D}^{\frac{3}{2}}_{0^+} y(\mathrm{t}) = \frac{1}{9\, e^{\mathrm{t}}} + \frac{y(\mathrm{t})}{9(1+\mathrm{t}^2)^2}, \quad \mathrm{t} \in \zeta = (0,\infty), \tag{5.1}$$

subjected to the boundary conditions

$$\lim_{t\to 0^+} y(t) = 2, \quad \lim_{t\to\infty} t^{-1} y(t) = 5. \tag{5.2}$$

Here, $\mu = 2, \nu = \frac{3}{2}, \varrho_0 = 2, \varrho_\infty = 5, \gamma = \frac{1}{5}$, *and*

$$\mathcal{H}(t, y(t)) = \frac{1}{9\, e^t} + \frac{y(t)}{9(1+t^2)^2}.$$

Thus, $\left|\mathcal{H}(t,(1+t^2)y(t))\right| \le \frac{1}{9\,e^t} + \frac{1}{9(1+t^2)}|y(t)|$, *which implies* $h_1(t) = \frac{1}{9\,e^t}$, *and* $h_2(t) = \frac{1}{9(1+t^2)}$. *Moreover, we have*

$$\sup_{t\in\zeta}\left(\frac{1}{\Gamma(2)}\int_0^t \frac{1}{9\,(1+z^2)}dz + \frac{1}{\Gamma(2)}\int_0^\infty \frac{1}{9\,(1+z^2)}dz + \frac{1}{5\,\Gamma(1.5)}\right) \le \alpha_1 \approx 0.574742 < 1,$$
$$\sup_{t\in\zeta}\left(\frac{1}{\Gamma(2)}\int_0^t \frac{1}{9\,e^z}dz + \frac{1}{\Gamma(2)}\int_0^\infty \frac{1}{9\,e^z}dz + 5 + 2\right) \le \alpha_2 \approx 7.22222 < \infty.$$

Hence, the hypotheses (AS_1) *and* (AS_2) *are satisfied, and then in view of Theorem* 3.3*, the FBTS (5.1 and 5.2) possesses at least one solution on the half-line* ζ.

Example 5.2. *Assume that the generalized FBTS given by*

$${}^{\mathcal{RL}}\mathbb{D}_{0^+}^{\frac{3}{2}} y(t) - \frac{1}{12}\, {}^{\mathcal{RL}}\mathbb{D}_{0^+}^{\frac{5}{4}} y(t) = \frac{\sin(t) y(t)}{15\ e^t(1+t^2)}, \quad t \in \zeta = (0,\infty), \tag{5.3}$$

subjected to the boundary conditions

$$\lim_{t\to 0^+} y(t) = \frac{1}{2}, \quad \lim_{t\to\infty} t^{-1} y(t) = \frac{1}{3}. \tag{5.4}$$

Here, $\mu = \frac{3}{2}, \nu = \frac{5}{4}, \varrho_0 = \frac{1}{2}, \varrho_\infty = \frac{1}{3}, \gamma = \frac{-1}{12}$, *and*

$$\mathcal{H}(t, y(t)) = \frac{\sin(t) y(t)}{15\ e^t(1+t^2)} \quad \mathcal{H}(t,0) = 0.$$

Thus,

$$t^{\frac{1}{2}}\left\|\mathcal{H}(t,(1+t^2)y(t)) - \mathcal{H}(t,(1+t^2)\bar{y}(t))\right\| \le \frac{t^{\frac{1}{2}}}{15\,e^t}\left\|y(t)-\bar{y}(t)\right\|,$$

which yields that $\mathfrak{f}(t) = \frac{t^{\frac{1}{2}}}{15\,e^t}$, *and hence we get* $\Delta_1 \approx 0.242389 < 1$, *and* $\Delta_2 \approx 0.83333 < \infty$.

Therefore, hypotheses (AS_3) *and* (AS_4) *are satisfied, and based on Theorem* 3.4*, the FBTS (5.3 and 5.4) admits exactly one solution on the half-line* ζ.

6. Conclusions

This paper was concerned with the study of the generalized FBTS (1.2), which is considered one of the most important dynamic systems in the mechanics field. It describes the motion of real materials in a Newtonian fluid. The qualitative properties such as the E&U, UHS, HURS, and sHURS of unbounded solutions for the proposed dynamic system (1.2) were discussed by utilizing Banach and Schauder FPTs, along with nonlinear analysis subjects on the half-line $(0, \infty)$. Finally, we support our work with two numerical examples for checking the validity of outcomes. This study is constrained by $\mathcal{RL}$ fractional derivative properties, details of the proposed system given in (1.2), and the space of analysis Σ. Unbounded solutions results in this study give us a deeper understanding of the FBTS under extreme conditions for improving system design, optimizing performance, and testing the stability of real materials in a Newtonian fluid. In the future, our focus will be on studying sufficient conditions of positive solutions for the FBTS involving the Hilfer fractional derivative, which is connected to the $\mathcal{RL}$ and Caputo fractional derivatives.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

The authors declare that they have no competing interest.

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