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*Research article*

# The relationship between two kinds of structural credit migration models

**Yang Lin**[1,2] **and Jin Liang**[2,*]

[1] Postdoctoral Research Center, Industrial and Commercial Bank of China, Beijing 100140, China
[2] School of Mathematical Sciences, Tongji University, Shanghai 200092, China

* **Correspondence:** Email: liang_jin@tongji.edu.cn.

**Abstract:** In this paper, we presented an asymptotic relationship between the single threshold model and the model with different upgrade and downgrade thresholds for credit migration problems with fixed boundaries. By partial differential equation (PDE) techniques, we proved that the solution of the asymmetric threshold problem converges to that of a single threshold problem uniformly as one of the asymmetric thresholds approaches the other.

**Keywords:** credit migration risk; different upgrade and downgrade thresholds; asymptotic behavior; structural model; corporate bond pricing
**Mathematics Subject Classification:** 35K40, 91G40

## 1. Introduction

Credit rating migration risk is an important kind of credit risk, referring to the possibility of potential losses due to changes in credit rating. The phenomenon of rating movements has long attracted the attention of academia and industry and there have been many studies on the performance and patterns of credit rating transitions [1–6]. As for the valuation of credit migration risk, the used models are mainly categorized into two types: the reduced-form model and the structural model. The former employs a transition density matrix to characterize the process of credit rating migrations, such as [7–9] while the latter utilizes the company's own financial status as the determinant of credit rating changes, such as [10–13]. Each kind of model has its advantages, but the structural model elucidates the mechanism of credit migration by explicitly linking credit ratings to the company's assets and liabilities.

Liang and Zeng ( [10], 2015) constructed the first structural model for assessing credit migration risk by pricing a corporate bond with this risk. By predefining an asset threshold as the boundary for credit migration, the company's asset values are divided into high and low rating regions. In different regions, the asset values follow geometric Brownian motions with different volatilities. The model was derived as an initial value problem of parabolic differential equations which are coupled on a fixed inner boundary. Further, Hu et al. ( [11], 2015) took the asset-liability ratio as a threshold for credit migration and deduced the migration boundary as a free boundary. On these bases, more theoretical analyses are provided, such as the asymptotic traveling wave solution [14], convergence rate of the difference scheme [15], multi-credit rating case [16], steady-state solution [17], etc (see also [18–22]). There are also some empirical results for the single threshold model, for instance, the credit migration boundary was identified by pricing long-term bonds in the U.S. corporate bond market [23]. In those models with the fixed boundaries or free boundaries, the thresholds for upgrades and downgrades are the same. However, this single threshold may lead to infinite frequent changes in credit ratings within a short period, due to the assumption that asset values follow Brownian motions which can cross any level infinitely many times within any time interval. Chen and Liang ( [12], 2021) made an improvement based on the free boundary model by introducing different asset-liability ratio thresholds for upgrades and downgrades, resulting in a pair of migration boundaries. As a buffer zone is formed between these two boundaries, the frequency of credit rating migrations per unit time becomes finite. Liang and Lin ( [13], 2023) made a similar modification, applying a pair of asymmetric asset thresholds to the fixed boundary model, and obtaining a system of partial differential equations coupled with each other on a pair of fixed boundaries. Liang and Lin ( [24], 2023) further explored an asymptotic traveling wave solution with a buffer zone for this kind of model.

These structural credit migration models with thresholds are somewhat similar in form to the credit barrier model considered in Albanese and Chen ( [3], 2006). A stochastic process with state-dependent volatilities was used to model the credit quality process, which directly indicated the dynamic of credit rating, with barrier crossings corresponding to credit migrations and default events, in Albanese and Chen ( [3], 2006). The variable of credit quality might capture the firm's fundamental information but did not articulate specific implications in finance. In the structural models of Liang et al., the credit migrations were delineated by the crossings of the thresholds as well, with the driving factors explicitly identified as the asset value or the asset-liability ratio. Since the asset value was assumed to follow the Brownian motion with different volatilities in different ratings, the distribution of future asset value depends on only the current rating and value of asset, and not on the history of previous states, even in the structural models with buffer zones. Therefore, these models we analyze in this paper cannot capture non-Markov effects in rating transition probabilities, such as those identified in Lando and Skødeberg ( [4], 2002), Nickell et al. ( [6], 2000), etc.

According to whether or not the upgrade threshold and downgrade threshold are the same, the structural models are classified into the single threshold model and the model with different upgrade and downgrade thresholds. We already know that substituting a pair of asymmetric thresholds for a single threshold avoids high-frequency fluctuations in credit ratings. However a natural question arises: What is the relationship between those two kinds of models? Intuitively, when upgrade threshold and downgrade threshold are very close, the asymmetric threshold model's behavior should be similar to that of a single threshold model. This article attempts to answer this question rigorously within the scope of the credit migration problem with fixed boundaries. We consider two fixed boundary models: one model from [10] with $X(X > 0)$ as a critical threshold for credit migration, and the other model from [13] with $X$ as a threshold for downgrades, and $X_\varepsilon := X + \varepsilon$ (depending on a small parameter $\varepsilon > 0$) as a threshold for upgrades. By techniques of partial differential equations, we prove that the solution of the asymmetric threshold problem converges to that of the single threshold problem when the upgrade threshold $X_\varepsilon$ approaches the downgrade threshold $X$ as $\varepsilon \to 0$. To our knowledge, it is the first time that the asymptotic relationship between the single threshold and a pair of asymmetric thresholds has been shown in credit migration problems. Therefore, when $\varepsilon$ is small enough, the different upgrade and downgrade threshold model can be approximated by a single threshold model, which already has more theoretical and empirical results.

The remainder of the paper is organized as follows. In Section 2, the single threshold and the different downgrade and upgrade threshold model for credit migration problems with fixed boundaries are reviewed. In Section 3, we prove a series of lemmas to establish $\varepsilon$-independent estimates for the solution of the asymmetric threshold problem. In Section 4, a key step is shown that the first-order derivatives of the asymmetric threshold problem's solution on both sides of $x = X$ tend to be equal, as $\varepsilon$ approaches zero. It follows that the solution of this model converges to that of a single threshold model by compactness. Section 5 is a summary of this paper.

## 2. Model review

The structural models with fixed migration boundaries assess credit rating migration risk by pricing a zero-coupon corporate bond.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Assuming that the company only issues one bond with a face value of 1 and a maturity of $T$, the bond is considered as a contingent claim of the company's asset value on the space $(\Omega, \mathcal{F}, P)$. At maturity, the bond will default and pay out the remaining assets if the company's asset value is less than the face value of the bond. Let $S_t$ denote the company's asset value in the risk-neutral world. It satisfies

$$dS_t = \begin{cases} rS_t dt + \sigma_H S_t dW_t, & \text{in high rating region,} \\ rS_t dt + \sigma_L S_t dW_t, & \text{in low rating region,} \end{cases}$$

where $r$ is the risk-free interest rate, and

$$\sigma_H < \sigma_L \tag{2.1}$$

represent volatilities (positive constants) of the company under the high and low credit grades respectively. $W_t$ is the standard Brownian motion which generates the filtration $\{\mathcal{F}_t\}$. High and low rating regions are determined by the company's asset value. Inequality (2.1) captures the characteristics of the high and low credit ratings, with the volatility of asset return in the lower rating region being greater than the volatility in the higher rating region.

### 2.1. Single threshold model

Liang and Zeng ( [10], 2015) (referred to as LZ's model hereafter) gave a single predetermined threshold to divide asset value into high and low rating regions. Through a standard variable transformation $x = \log S$ and renaming $T - t$ as $t$, we set the migration boundary as $X (X > 0)$ and represented the values of low and high-rated bonds as $v^L(x, t)$ and $v^H(x, t)$, respectively. The model is derived as the following partial differential equation problem:

$$\begin{cases} \frac{\partial v^H}{\partial t} - \frac{1}{2}\sigma_H^2 \frac{\partial^2 v^H}{\partial x^2} - (r - \frac{1}{2}\sigma_H^2)\frac{\partial v^H}{\partial x} + rv^H = 0, & \text{for } x > X, t > 0, \\ \frac{\partial v^L}{\partial t} - \frac{1}{2}\sigma_L^2 \frac{\partial^2 v^L}{\partial x^2} - (r - \frac{1}{2}\sigma_L^2)\frac{\partial v^L}{\partial x} + rv^L = 0, & \text{for } x < X, t > 0, \\ v^H(x,0) = \min\{e^x, 1\}, & \text{for } x > X, \\ v^L(x,0) = \min\{e^x, 1\}, & \text{for } x < X, \\ v^H(X,t) = v^L(X,t), & \text{for } t > 0, \\ \frac{\partial v^H}{\partial x}(X,t) = \frac{\partial v^L}{\partial x}(X,t), & \text{for } t > 0. \end{cases} \tag{2.2}$$

We define a solution $v$ over the entire region $(-\infty, \infty) \times [0, \infty)$ as follows:

$$v = \begin{cases} v^H(x,t), \text{ for } x \geq X, t \geq 0, \\ v^L(x,t), \text{ for } x < X, t \geq 0. \end{cases} \tag{2.3}$$

*2.2. Different upgrade and downgrade threshold model*

Liang and Lin ( [13], 2023 ) (referred to as LL's model hereafter) proposed a pair of asymmetric thresholds for credit migrations: one threshold for downgrades and the other slightly higher threshold for upgrades. After the same change of variables as in Section 2.1 , we have set the downgrade threshold as $X(X > 0)$ and the upgrade threshold as $X_\varepsilon := X + \varepsilon, \varepsilon > 0$. Denote by $u_\varepsilon^L(x,t)$ and $u_\varepsilon^H(x,t)$ the bond values in low rating and high rating, respectively. They are captured by the following PDE problem:

$$\begin{cases} \frac{\partial u_\varepsilon^H}{\partial t} - \frac{1}{2}\sigma_H^2 \frac{\partial^2 u_\varepsilon^H}{\partial x^2} - (r - \frac{1}{2}\sigma_H^2)\frac{\partial u_\varepsilon^H}{\partial x} + ru_\varepsilon^H = 0, & \text{for } x > X, t > 0, \\ \frac{\partial u_\varepsilon^L}{\partial t} - \frac{1}{2}\sigma_L^2 \frac{\partial^2 u_\varepsilon^L}{\partial x^2} - (r - \frac{1}{2}\sigma_L^2)\frac{\partial u_\varepsilon^L}{\partial x} + ru_\varepsilon^L = 0, & \text{for } x < X_\varepsilon, t > 0, \\ u_\varepsilon^H(x,0) = \min\{e^x, 1\}, & \text{for } x > X, \\ u_\varepsilon^L(x,0) = \min\{e^x, 1\}, & \text{for } x < X_\varepsilon, \\ u_\varepsilon^H(X,t) = u_\varepsilon^L(X,t), & \text{for } t > 0, \\ u_\varepsilon^L(X_\varepsilon,t) = u_\varepsilon^H(X_\varepsilon,t), & \text{for } t > 0. \end{cases} \tag{2.4}$$

Note that $v_\varepsilon^L$ and $v_\varepsilon^H$ overlap in $[X, X_\varepsilon] \times [0, \infty)$. To be comparable with $v$, we rearrange $u_\varepsilon$ over the entire region as the following formula:

$$u_\varepsilon = \begin{cases} u_\varepsilon^H(x,t), \text{ for } x \geq X, t \geq 0, \\ u_\varepsilon^L(x,t), \text{ for } x < X, t \geq 0. \end{cases} \tag{2.5}$$

*2.3. Asymptotic relationship*

Letting the upgrade threshold $X_\varepsilon$ approach the downgrade threshold $X$, we find an asymptotic behavior of LL's model, i.e., as $\varepsilon \to 0$, $u_\varepsilon$ converges to $v$. This actuallly constructs an asymptotic relationship between LL's model and LZ's model. To prove this, we first establish some $\varepsilon$-independent estimates for the solution $(u_\varepsilon^L, u_\varepsilon^H)$ . Based on this, $|u_{\varepsilon x}^L(X-,t) - u_{\varepsilon x}^H(X+,t)|$ approaching 0 as $\varepsilon \to 0$ is verified as a key step. Further, we suggest that any convergent subsequence of $u_\varepsilon$ tends to the solution $v$ of LZ's model as $\varepsilon \to 0$.

## 3. Estimates for the solution of LL's model

Define $\mathcal{L}^i = \frac{\partial}{\partial t} - \frac{1}{2}\sigma_i{}^2\frac{\partial^2}{\partial x^2} - (r - \frac{1}{2}\sigma_i{}^2)\frac{\partial}{\partial x} + r\cdot, i = H, L$; $Q_\varepsilon^L = (-\infty, X_\varepsilon)\times(0,\infty)$, and $Q^H = (X,\infty)\times(0,\infty)$.

For any given $\varepsilon$, we iteratively define a sequence $\{u_{\varepsilon k}^L, u_{\varepsilon k}^H\}_{k=0}^{\infty}$, which is proved to decrease with $k$ and converge to $(u_\varepsilon^L, u_\varepsilon^H)$ as $k \to 0$ in Liang and Lin ( [13], 2023). In detail, the sequence satisfies equations

$$\mathcal{L}^L[u_{\varepsilon k}^L] = 0 \text{ in } Q_\varepsilon^L,\ \mathcal{L}^H[u_{\varepsilon k}^H] = 0 \text{ in } Q^H,$$

with the initial value $\min\{e^x, 1\}$. Starting from $u_{\varepsilon 0}^L(X_\varepsilon, t) \equiv 1$ and by the induction assumption $u_{\varepsilon k}^H(X,t) = u_{\varepsilon k}^L(X,t)$ and $u_{\varepsilon(k+1)}^L(X_\varepsilon, t) = u_{\varepsilon k}^H(X_\varepsilon, t)$, we have completed the definition of the sequence.

The estimations of the maximum norm of $(u_\varepsilon^L, u_\varepsilon^H)$ and its first-order derivative with respect to time $t$ are carried out by induction on the sequence $\{u_{\varepsilon k}^L, u_{\varepsilon k}^H\}_{k=0}^{\infty}$. As for the boundary estimate of $(u_{\varepsilon x}^L, u_{\varepsilon x}^H)$, we introduce barrier functions.

**Lemma 3.1.**

$$0 \le u_\varepsilon^L \le \min\{e^x, 1\} \textit{ in } Q_\varepsilon^L,\ 0 \le u_\varepsilon^H \le \min\{e^x, 1\} \textit{ in } Q^H. \tag{3.1}$$

*Proof.* By the induction and maximum principle, for each $k \ge 0$, $0 \le u_{\varepsilon k}^H \le \min\{e^x, 1\}$ in $Q^H$, and $0 \le u_{\varepsilon k}^L \le \min\{e^x, 1\}$ in $Q_\varepsilon^L$. □

**Lemma 3.2.**

$$-C_1 \le \frac{\partial u_\varepsilon^L}{\partial t} \le 0 \textit{ in } Q_\varepsilon^L \backslash \bar{Q}_\rho,\ -C_1 \le \frac{\partial u_\varepsilon^H}{\partial t} \le 0 \textit{ in } Q^H, \tag{3.2}$$

*where* $Q_\rho = (-\rho/2, \rho/2) \times (0, \rho^2/4)$ *and* $0 < \rho < X$.

*Proof.* We claim that $-C_1 \le u_{\varepsilon kt}^L, u_{\varepsilon kt}^H \le 0$ for any $k \in \mathbb{N}$ and $C_1$ is independent of $\varepsilon$ and $k$.

Differentiating $u_{\varepsilon k}^L$ and $u_{\varepsilon k}^H$ with respect to $t$, they satisfy

$$\mathcal{L}^L[u_{\varepsilon kt}^L] = 0 \text{ in } Q_\varepsilon^L,\ \mathcal{L}^H[u_{\varepsilon kt}^L] = 0 \text{ in } Q^H.$$

On the migration boundaries, $u_{\varepsilon kt}^L(X_\varepsilon, t) = u_{\varepsilon(k-1)t}^H(X_\varepsilon, t)$ and $u_{\varepsilon kt}^H(X,t) = u_{\varepsilon kt}^L(X,t)$.

It is also clear that initially

$$u_{\varepsilon kt}^L(x,0) = 0 \text{ for } x < 0, \quad u_{\varepsilon kt}^L(x,0) = -r \text{ for } 0 < x < X_\varepsilon, \tag{3.3}$$

$$u_{\varepsilon kt}^H(x,0) = -r \text{ for } x > X. \tag{3.4}$$

At $x = 0, u_{\varepsilon kxx}^L(x,0)$ produces a Dirac measure of density $-1$. Thus, in the distribution sense,

$$u_{\varepsilon kt}^L(x,0) \le 0 \text{ for } x < X_\varepsilon. \tag{3.5}$$

From the standard parabolic estimates (see e.g. [25]), there exists constants $c_1, c_2$ independent of $\varepsilon$ and $k$, such that

$$u_{\varepsilon kt}^L \ge -c_2 - \frac{c_2}{\sqrt{t}} exp(-c_1\frac{x^2}{t}) \text{ for } |x| < \frac{\rho}{2}, 0 < t \le \frac{\rho^2}{4},$$

where $0 < \rho < X$. Take $C_1 \ge r$ such that

$$C_1 \ge c_2 + \frac{c_2}{\sqrt{t}} exp(-c_1\frac{x^2}{t}) \text{ on } \{|x| = \frac{\rho}{2}, 0 < t \le \frac{\rho^2}{4}\} \cup \{|x| < \frac{\rho}{2}, t = \frac{\rho^2}{4}\}.$$

It follows that on $\{x \le -\frac{\rho}{2}, t = 0\} \cup \{|x| = \frac{\rho}{2}, 0 < t \le \frac{\rho^2}{4}\} \cup \{|x| < \frac{\rho}{2}, t = \frac{\rho^2}{4}\} \cup \{\frac{\rho}{2} \le x \le X, t = 0\}$,

$$u^L_{\varepsilon kt} \ge -C_1. \tag{3.6}$$

When $k = 0$, $u_{\varepsilon 0}(X_\varepsilon, t) = 1$ and we have $u_{\varepsilon 0t}(X_\varepsilon, t) = 0$ on $x = X_\varepsilon$ for $t > 0$.

By further approximating the initial data with smooth functions if necessary, we conclude by (3.5) and maximum principle that

$$u^L_{\varepsilon 0t} \le 0 \text{ in } Q^L_\varepsilon.$$

We conclude by (3.6) and minimum principle that

$$u^L_{\varepsilon 0t} \ge -C_1 \text{ in } Q^L_\varepsilon \backslash \bar{Q}_\rho,$$

where $Q_\rho = (-\rho/2, \rho/2) \times (0, \rho^2/4)$.

We assume that $-C_1 \le u^L_{\varepsilon kt} \le 0$ in $Q^L_\varepsilon \backslash \bar{Q}_\rho$ holds for $k \ge 1$. Since $u^H_{\varepsilon kt}(X, t) = u^L_{\varepsilon kt}(X, t)$ for $t > 0$ and (3.4), we conclude by extremum principle

$$-C_1 \le u^H_{\varepsilon kt} \le 0 \text{ in } Q^H.$$

From $u^L_{\varepsilon(k+1)t}(X_\varepsilon, t) = u^H_{\varepsilon kt}(X_\varepsilon, t)$ and (3.5),(3.6), applying the extremum principle gives

$$-C_1 \le u^L_{\varepsilon(k+1)t} \le 0 \text{ in } Q^L_\varepsilon \backslash \bar{Q}_\rho.$$

Thus, by induction, we derive that $-C_1 \le u^L_{\varepsilon kt} \le 0$ in $Q^L_\varepsilon \backslash \bar{Q}_\rho$ and $-C_1 \le u^H_{\varepsilon kt} \le 0$ in $Q^H$ for any $k \ge 0$ and $C_1$ is independent of $\varepsilon$ and $k$. The lemma's results can be obtained by taking limits with $k$. □

**Lemma 3.3.**

$$\left|\frac{\partial u^L_\varepsilon}{\partial x}\right| \le C_2 \textit{ in } Q^L_\varepsilon, \ \left|\frac{\partial u^H_\varepsilon}{\partial x}\right| \le C_3 \textit{ in } Q^H. \tag{3.7}$$

*Proof.* We estimate $u^L_{\varepsilon x}$ as an example and the same process works for $u^H_{\varepsilon x}$.

We fixed $\varepsilon$ and first deal with the $u^L_{\varepsilon x}(X_\varepsilon, t)$. Let $K_1 (\rho < K_1 < X)$ be a constant and $\eta = X - K_1$. Define $w^L_\varepsilon = u^L_\varepsilon - \frac{X_\varepsilon - x}{X_\varepsilon - K_1} u^L_\varepsilon(K_1, t) - \frac{x - K_1}{X_\varepsilon - K_1} u^L_\varepsilon(X_\varepsilon, t)$. Thus, $w^L_\varepsilon(K_1, t) = w^L_\varepsilon(X_\varepsilon, t) = 0$ for $t > 0$ and $w^L_\varepsilon(x, t) = 0$ for $K_1 \le x \le X_\varepsilon$. In $Q^L_{\varepsilon K_1} = \{K_1 < x < X_\varepsilon, t > 0\}$,

$$\begin{aligned}\mathcal{L}^L_0 w^L_\varepsilon = &- r u^L_\varepsilon - \frac{X_\varepsilon - x}{X_\varepsilon - K_1} u^L_{\varepsilon t}(K_1, t) - \frac{x - K_1}{X_\varepsilon - K_1} u^L_{\varepsilon t}(X_\varepsilon, t) \\ &+ (r - \frac{1}{2}\sigma_L^2) \frac{u^L_\varepsilon(X_\varepsilon, t) - u^L_\varepsilon(K_1, t)}{X_\varepsilon - K_1},\end{aligned}$$

where $\mathcal{L}^i_0 = \frac{\partial}{\partial t} - \frac{1}{2}{\sigma_i}^2 \frac{\partial^2}{\partial x^2} - (r - \frac{1}{2}{\sigma_i}^2)\frac{\partial}{\partial x}$. Actually, $\mathcal{L}^L_0 w^L_\varepsilon$ can be bounded by a number $G$ independent of $\varepsilon$,

$$\begin{aligned}\sup_{Q^L_{\varepsilon K_1}} |\mathcal{L}^L_0 w^L_\varepsilon| &\le r \sup_{Q^L_{\varepsilon K_1}} |u^L_\varepsilon| + \sup_{Q^L_{\varepsilon K_1}} |u^L_{\varepsilon t}| + \frac{2r + \sigma_L^2}{\eta} \sup_{Q^L_{\varepsilon K_1}} |u^L_\varepsilon| \\ &\le r + C_1 + \frac{2r + \sigma_L^2}{\eta} \triangleq G.\end{aligned}$$

In the case $\sigma_L^2 \neq 2r$, we introduce the function

$$z_\varepsilon^L(x,t) = \frac{2C}{\sigma_L^2 - 2r}(1 - \exp((1 - \frac{2r}{\sigma_L^2})(x - X_\varepsilon))) + \frac{2G}{\sigma_L^2 - 2r}(x - X_\varepsilon),$$

where $C$ is a constant to be determined later. Obviously, $\mathcal{L}_0^L z_\varepsilon^L = G \geq \mathcal{L}_0^L w_\varepsilon^L$ in $Q_{\varepsilon K_1}^L$ and $z_\varepsilon^L(X_\varepsilon, t) = 0$ for $t > 0$. To make

$$z_{\varepsilon x}^L = -\frac{2C}{\sigma_L^2}\exp((1 - \frac{2r}{\sigma_L^2})(x - X_\varepsilon)) + \frac{2G}{\sigma_L^2 - 2r} \leq 0$$

hold for $K_1 < x < X_\varepsilon, t > 0$, we choose $C = \frac{\sigma_L^2 G}{\sigma_L^2 - 2r}\exp((1 - \frac{2r}{\sigma_L^2})(\eta + \varepsilon))$. Thus, on the parabolic boundary of $Q_{\varepsilon K_1}^L$ we clearly have $z_\varepsilon^L \geq \pm w_\varepsilon^L$. It follows by comparison principle that

$$z_\varepsilon^L \geq \pm w_\varepsilon^L \text{ in } Q_{\varepsilon K_1}^L.$$

Since $z_\varepsilon^L(X_\varepsilon, t) = \pm w_\varepsilon^L(X_\varepsilon, t) = 0$,

$$\frac{\pm(w_\varepsilon^L(x,t) - w_\varepsilon^L(X_\varepsilon,t))}{X_\varepsilon - x} \leq \frac{z_\varepsilon^L(x,t) - z_\varepsilon^L(X_\varepsilon,t)}{X_\varepsilon - x}$$

for $x < X_\varepsilon$. Letting $x \to X_\varepsilon$, we have

$$|w_{\varepsilon x}^L(X_\varepsilon,t)| \leq -z_{\varepsilon x}^L(X_\varepsilon,t) = -\frac{2G}{\sigma_L^2 - 2r}(1 - \exp((1 - \frac{2r}{\sigma_L^2})(\eta + \varepsilon))).$$

Then, by $u_{\varepsilon x}^L(X_\varepsilon,t) = w_{\varepsilon x}^L(X_\varepsilon,t) + [u_\varepsilon^L(X_\varepsilon,t) - u_\varepsilon^L(K_1,t)]/(X_\varepsilon - K_1)$ there exists a constant $C_2 > 1$ independent of $\varepsilon$ such that

$$|u_{\varepsilon x}^L(X_\varepsilon,t)| \leq -\frac{2G}{\sigma_L^2 - 2r}(1 - \exp((1 - \frac{2r}{\sigma_L^2})(\eta + 1))) + 2/\eta \leq C_2,$$

when $\varepsilon < 1$.

In the case $\sigma_L^2 = 2r$, we can introduce the function

$$z_\varepsilon^L(x,t) = -\frac{G}{\sigma_L^2}(x - X_\varepsilon)^2 - \frac{2G}{\sigma_L^2}(\eta + \varepsilon)(x - X_\varepsilon),$$

then by a similar analysis as above, we can conclude that there exists a number $C_2 > 1$ independent of $\varepsilon$ such that

$$|u_{\varepsilon x}^L(X_\varepsilon,t)| \leq \frac{2G}{\sigma_L^2}(\eta + 1) + 2/\eta \leq C_2,$$

when $\varepsilon < 1$.

It is known that $u_{\varepsilon x}^L(x,0) = e^x$ for $x < 0$ and $u_{\varepsilon x}^L(x,0) = 0$ for $0 < x < X_\varepsilon$. Thus it follows by maximum principle that $|u_{\varepsilon x}^L(x,t)| \leq C_2$ in $Q_\varepsilon^L$ when $\varepsilon < 1$. □

**Lemma 3.4.**

$$\left|\frac{\partial^2 u_\varepsilon^L}{\partial x^2}\right| \leq C_4 \text{ in } Q_\varepsilon^L \backslash \bar{Q}_\rho, \quad \left|\frac{\partial^2 u_\varepsilon^H}{\partial x^2}\right| \leq C_5 \text{ in } Q^H. \tag{3.8}$$

*Proof.* The corollary of Lemmas 3.1–3.3. □

## 4. Asymptotic relationship

In this section, we establish the asymptotic relationship between LL's model and LZ's model, when $X_\varepsilon$ approaches $X$. A key lemma is proved to show $|u^L_{\varepsilon x} - u^H_{\varepsilon x}| \to 0$ on $x = X$ along $\varepsilon \to 0$. Then, by compact embedding theorem, we deduce that there exists a subsequence $u_{\varepsilon_j}$ converging a function $u$ which is examined as a solution of the single threshold problem (2.2). By the uniqueness of the solution, we obtain $v \equiv u$ and $u_\varepsilon \to v$ uniformly as $\varepsilon \to 0$.

**Lemma 4.1.** *As* $\varepsilon \to 0$,

$$\left|\frac{\partial u^L_\varepsilon}{\partial x}(X-,t) - \frac{\partial u^H_\varepsilon}{\partial x}(X+,t)\right| \to 0 \tag{4.1}$$

*uniformly for* $0 \le t \le T$.

*Proof.* Let

$$h_\varepsilon(t) = \frac{u^L_\varepsilon(X+\varepsilon,t) - u^L_\varepsilon(X,t)}{\varepsilon} = \frac{u^H_\varepsilon(X+\varepsilon,t) - u^H_\varepsilon(X,t)}{\varepsilon},$$

for $0 \le t \le T$. By the mean value theorem, there is $\mu_1(t) \in [0,\varepsilon]$ such that $h_\varepsilon(t) = u^L_{\varepsilon x}(X+\mu_1(t),t)$, then,

$$\begin{aligned}|u^L_{\varepsilon x}(X+,t) - h_\varepsilon(t)| &= |u^L_{\varepsilon x}(X,t) - u^L_{\varepsilon x}(X+\mu_1(t),t)| \\ &= |u^L_{\varepsilon xx}(X+\mu_2(t),t)|\mu_1(t) \\ &\le \varepsilon \sup_{Q^L_\varepsilon \setminus \bar{Q}_\rho} |u^L_{\varepsilon xx}|,\end{aligned}$$

where $0 \le \mu_2(t) \le \mu_1(t)$. The second equal sign holds by the mean value theorem. Similarly, we can derive

$$|u^H_{\varepsilon x}(X+,t) - h_\varepsilon(t)| \le \varepsilon \sup_{Q^H} |u^H_{\varepsilon xx}|.$$

Considering the continuity of $u^L_{\varepsilon x}$ across $x = X$, we have for $0 \le t \le T$,

$$\begin{aligned}&|u^L_{\varepsilon x}(X-,t) - u^H_{\varepsilon x}(X+,t)| \\ =&|u^L_{\varepsilon x}(X+,t) - u^H_{\varepsilon x}(X+,t)| \\ =&|u^L_{\varepsilon x}(X+,t) - h_\varepsilon(t) + h_\varepsilon(t) - u^H_{\varepsilon x}(X+,t)| \\ \le&\varepsilon(C_4 + C_5).\end{aligned}$$

As $C_4, C_5$ are independent of the small parameter $\varepsilon$, the result is seen by letting $\varepsilon \to 0$. □

**Theorem 4.1.** *As* $\varepsilon \to 0$, $u_\varepsilon(x,t)$ *tends to* $v(x,t)$ *uniformly in* $(-\infty,\infty) \times [0,T]$.

*Proof.* Treat $u^L_\varepsilon\big|_{x\le X}$ as $u^L_\varepsilon$ restricted on $(-\infty, X] \times [0,T]$. From Lemmas 3.1–3.4 , $u^L_\varepsilon\big|_{x\le X}$ is bounded in $W^{2,1}_\infty((-\infty,X) \times [0,T] \setminus \bar{Q}_\rho)$. By the compact embedding theorem, there exists $u^L$ and a subsequence $\varepsilon_j$ of $\varepsilon$ such that, as $\varepsilon_j \to 0$,

$$u^L_{\varepsilon_j}\Big|_{x\le X} \to u^L \text{ in } C^{1+\alpha,\frac{1+\alpha}{2}}(([-A,X]\times[0,T])\setminus Q_\rho),\ 0<\alpha<1, \tag{4.2}$$

for any $A > 1$ and $\rho < X$. Similarly, $u_\varepsilon^H$ is bounded in $W_\infty^{2,1}((X,\infty)\times[0,T])$ and there exists $u^H$ such that

$$u_{\varepsilon_j}^H \to u^H \text{ in } C^{1+\alpha,\frac{1+\alpha}{2}}([X,A]\times[0,T]), \tag{4.3}$$

along a subsequence of $\varepsilon_j$ if necessary.

Thus, it is clear that

$$u_{\varepsilon_j x}^L(X-,t) \to u_x^L(X-,t),\ u_{\varepsilon_j x}^H(X+,t) \to u_x^H(X+,t), \tag{4.4}$$

uniformly for $0 \le t \le T$. By Lemma 4.1, we conclude that

$$u_x^L(X-,t) = u_x^H(X+,t). \tag{4.5}$$

It then can be verified that $u = (u^L, u^H)$ is a solution of the problem (2.2). By the uniqueness of the solution to this problem, we have $v \equiv u$. This implies that any convergent subsequence of $u_\varepsilon^L\big|_{x\le X}$ or $u_\varepsilon^H$ has the same limit and $u_\varepsilon^L\big|_{x\le X}$ or $u_\varepsilon^H$ converges as $\varepsilon \to 0$.

It follows that as $\varepsilon \to 0$,

$$u_\varepsilon^L(X-,t) \to v^L(X-,t),\ u_\varepsilon^H(X+,t) \to v^H(X+,t), \tag{4.6}$$

uniformly for $0 \le t \le T$. Since $u_\varepsilon^L(x,0)\big|_{x\le X} = v^L(x,0)$ and $u_\varepsilon^H(x,0) = v^H(x,0)$, implying the maximum norm estimation in their regions, respectively, gives that

$$u_\varepsilon^L\big|_{x\le X} \to v^L \text{ uniformly in } (-\infty, X]\times[0,T], \tag{4.7}$$

$$u_\varepsilon^H \to v^H \text{ uniformly in } [X,\infty)\times[0,T], \tag{4.8}$$

as $\varepsilon \to 0$. □

## 5. Conclusions

By PDE techniques, we showed an asymptotic relationship between two kinds of structural models for credit migration problems with fixed boundaries. When the downgrade threshold was locked and the upgrade threshold approached to it, the solution of the model with a pair of asymmetric thresholds converged to that of the single threshold model in which the downgrade threshold serves as a unique migration threshold. Symmetrically, by fixing the upgrade threshold and moving the downgrade threshold, a similar conclusion will be obtained. As far as we know, it was the first time that the relationship between the single threshold and a pair of asymmetric thresholds was studied in credit migration problems. This may contribute to generalizing existing research results by using a single threshold model to approximate models with different upgrade and downgrade thresholds.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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