AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 4014–4020.
DOI: 10.3934/math.2024195
Received: 30 November 2023
Revised: 08 January 2024
Accepted: 10 January 2024
Published: 11 January 2024

Research article

Covering cross-polytopes with smaller homothetic copies

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Abstract: Let C_n be an n-dimensional cross-polytope and $\Gamma_p(C_n)$ be the smallest positive number γ such that C_n can be covered by p translates of γC_n. We obtain better estimates of $\Gamma_{2^n}(C_n)$ for small n and a universal upper bound of $\Gamma_{2^n}(C_n)$ for all positive integers n.

Keywords: convex body; covering functional; Hadwiger's covering conjecture; homothetic copy
Mathematics Subject Classification: 52A20, 52C17, 52A15

1. Introduction

Let K be a *convex body* in $\mathbb{R}^n$, i.e., a compact convex set having interior points. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$, and the set of convex bodies that are centrally symmetric is denoted by C^n. For each $x \in \mathbb{R}^n$ and each $\lambda > 0$, the set

$$x + \lambda K := \{x + \lambda y \mid y \in K\}$$

is called a *homothetic copy* of K; when $\lambda \in (0, 1)$, it is called a *smaller homothetic copy* of K. For each $K \in \mathcal{K}^n$, we denote by $c(K)$ the least number of translates of $\operatorname{int} K$ needed to cover K. Concerning the least upper bound of $c(K)$ in $\mathcal{K}^n$, there is a long-standing conjecture (see [1–6] for the origin, history, and classical known results concerning this conjecture):

Conjecture 1. *(Hadwiger's covering conjecture [4]) For each $K \in \mathcal{K}^n$, we have*

$$c(K) \leq 2^n,$$

and the equality holds if and only if K is a parallelotope.

Although many people have conducted in-depth research, this conjecture is confirmed completely only for the planar case [7]. In [8], Chuanming Zong proposed a four-step program to attack this conjecture. In this program, it is important to estimate

$$\Gamma_m(K) := \inf\left\{\gamma > 0 \mid \exists\{c_i \mid i \in [m]\} \subseteq \mathbb{R}^n \text{ s.t. } K \subseteq \bigcup_{i\in[m]}(c_i + \gamma K)\right\},$$

i.e., $\Gamma_m(K)$ is the smallest positive number γ such that K can be covered by m translates of γK. The map $\Gamma_m(\cdot)$: $\mathcal{K}^n \to [0,1]$, $K \mapsto \Gamma_m(K)$ is called the covering functional with respect to m, where $[m] := \{i \in \mathbb{Z}^+ \mid 1 \le i \le m\}$. Clearly, $c(K) \le m$ if and only if $\Gamma_m(K) < 1$. For each $m \in \mathbb{Z}^+$, $\Gamma_m(\cdot)$ is an affine invariant. More precisely,

$$\Gamma_m(K) = \Gamma_m(T(K)),\ \forall T \in \mathcal{A}^n,$$

where $\mathcal{A}^n$ is the set of non-degenerate affine transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$.

A compact convex set K is said to be an *d-dimensional cross-polytope* if there exist d linearly independent vectors $v_1, \cdots, v_d$ such that

$$K = \mathrm{conv}\{\pm v_1, \cdots, \pm v_d\}.$$

Clearly, any d-dimensional cross-polytope is the image of

$$C_d = \left\{(\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d \mid \sum_{i\in[d]} |\alpha_i| \le 1\right\}$$

under a non-degenerate affine transformation. Therefore, $\Gamma_m(K) = \Gamma_m(C_d)$ holds for each pair of positive integers m and d. In a recent work [9], Xia Li et al. obtained some estimations of $\Gamma_m(C_d)$ for large d. Moreover, they showed that, if $P \in C^n$ is a convex polytope with $2d$ vertices, then

$$\Gamma_m(P) \le \Gamma_m(C_d), \tag{1.1}$$

which shows the importance of estimating $\Gamma_m(C_d)$.

It is well known that $\Gamma_{2^n}([-1,1]^n) = 1/2$, $\forall n \ge 2$. It is interesting to ask whether there exists a universal upper bound for $\Gamma_{2^n}(C_n)$. In this paper, by using elementary yet interesting observations and refining techniques used in the recent works [9, 10], we get better estimates of $\Gamma_{2^n}(C_n)$. Based on this, we present the first nontrivial universal upper bound of $\Gamma_{2^n}(C_n)$ for all positive n. By (1.1), results mentioned above yield also estimates of covering functionals of convex polytopes with few vertices.

Throughout this paper, the dimension n of the underlying space is at least 3.

2. Covering functionals of cross-polytopes

For each $n, k \in \mathbb{Z}^+$, we put

$$M(n,k) = \left\{(\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n \mid \sum_{i\in[n]} |\alpha_i| \le k\right\}.$$

It is known that (cf. [11] or [12])

$$\#M(n,k) = \sum_{i=n-k}^{n} 2^{n-i}\binom{n}{i}\binom{k}{n-i}.$$

Lemma 1. *Let $n, k \in \mathbb{Z}^+$. If $k \le \frac{n}{2}$, then*

$$(n+k)C_n \subseteq nC_n + S_k,$$

where

$$S_k = \left\{(\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n \mid \sum_{i\in[n]} |\alpha_i| = k\right\} \cup \{o\}.$$

Moreover,

$$\#S_k = \sum_{i=1}^{k} 2^i \binom{n}{i}\binom{k-1}{i-1} + 1.$$

Proof. Let $(\alpha_1, \cdots, \alpha_n)$ be an arbitrary point in $(n+k)C_n$. Then

$$\sum_{i\in[n]} |\alpha_i| \le n + k.$$

If $\sum_{i\in[n]} |\alpha_i| \le n$, then $(\alpha_1, \cdots, \alpha_n) \in nC_n \subseteq nC_n + S_k$. Otherwise, there exists $m \in [k]$ such that

$$n + m - 1 < \sum_{i\in[n]} |\alpha_i| \le n + m.$$

On the one hand, since $\sum_{i\in[n]}(|\alpha_i| - \lfloor|\alpha_i|\rfloor) < n$, we have $\sum_{i\in[n]} \lfloor|\alpha_i|\rfloor \ge m$. Then there exist integers $\beta_1, \cdots, \beta_n \ge 0$ such that

$$\beta_i \le |\alpha_i|, \ \forall i \in [n] \ \text{ and } \ \sum_{i\in[n]} \beta_i = m.$$

Clearly, we have

$$\sum_{i\in[n]} \left|\alpha_i - \text{sgn}\alpha_i \cdot \beta_i\right| = \sum_{i\in[n]} (|\alpha_i| - \beta_i) \le n.$$

Therefore,

$$\begin{aligned}(\alpha_1, \cdots, \alpha_n) =& (\alpha_1 - \text{sgn}(\alpha_1)\cdot\beta_1, \cdots, \alpha_n - \text{sgn}(\alpha_n)\cdot\beta_n) \\ &+ (\text{sgn}(\alpha_1)\cdot\beta_1, \cdots, \text{sgn}(\alpha_n)\cdot\beta_n) \\ \in& nC_n + S_m.\end{aligned}$$

On the other hand, set

$$m_i = \begin{cases} \lfloor|\alpha_i|\rfloor, & \text{if } |\alpha_i| - \lfloor|\alpha_i|\rfloor < \frac{1}{2}, \\ \lfloor|\alpha_i|\rfloor + 1, & \text{if } |\alpha_i| - \lfloor|\alpha_i|\rfloor \ge \frac{1}{2}, \end{cases} \quad \forall i \in [n]. \tag{2.1}$$

We have

$$\begin{aligned}(\alpha_1, \cdots, \alpha_n) =& (\alpha_1 - \text{sgn}(\alpha_1)\cdot m_1, \cdots, \alpha_n - \text{sgn}(\alpha_n)\cdot m_n) \\ &+ (\text{sgn}(\alpha_1)\cdot m_1, \cdots, \text{sgn}(\alpha_n)\cdot m_n).\end{aligned}$$

By the Triangle Inequality, we have

$$n - \sum_{i\in[n]} m_i < \sum_{i\in[n]} |\alpha_i| - \sum_{i\in[n]} m_i \le \sum_{i\in[n]} \left|\alpha_i - \text{sgn}(\alpha_i)\cdot m_i\right| = \sum_{i\in[n]} \left||\alpha_i| - m_i\right| \le \frac{n}{2}.$$

Thus,

$$m_1 + \cdots + m_n > \frac{n}{2} \geq k.$$

Without loss of generality, assume that $\alpha_1, \cdots, \alpha_n \geq 0$, and

$$\alpha_1, \cdots, \alpha_{n_0'} \geq 1,\ \alpha_{n_0'+1}, \cdots, \alpha_{n_0} \in \left[\frac{1}{2}, 1\right),\ \alpha_{n_0+1}, \cdots, \alpha_n \in \left[0, \frac{1}{2}\right).$$

By (2.1), we have

$$\begin{aligned} \beta_i \leq \lfloor \alpha_i \rfloor \leq m_i \leq \lceil \alpha_i \rceil,\ \forall i \in [n_0'], \\ \beta_i = 0 < 1 = m_i,\ \forall i \in [n_0] \setminus [n_0'], \\ \beta_i = m_i = 0,\ \forall i \in [n] \setminus [n_0]. \end{aligned}$$

Then there exist integers $m_1', \cdots, m_n'$ such that

$$\beta_i \leq m_i' \leq m_i,\ \forall i \in [n] \text{ and } \sum_{i \in [n]} m_i' = k.$$

Set, for each $i \in [n]$, $f_i(\lambda) = |\alpha_i - \lambda|$. Then f_i is decreasing on $[\beta_i, \lfloor \alpha_i \rfloor]$. We claim that

$$f_i(\beta_i) \geq f_i(m_i'),\ \forall i \in [n]. \tag{2.2}$$

The case when $m_i' \in [\beta_i, \lfloor \alpha_i \rfloor]$ is clear. If $m_i' > \lfloor \alpha_i \rfloor$, then $m_i' = m_i = \lfloor \alpha_i \rfloor + 1$ and $1/2 \leq \alpha_i - \lfloor \alpha_i \rfloor < 1$. Thus

$$f_i(\beta_i) \geq f_i(\lfloor \alpha_i \rfloor) = \alpha_i - \lfloor \alpha_i \rfloor \geq \frac{1}{2} \geq 1 - (\alpha_i - \lfloor \alpha_i \rfloor) = f_i(\lfloor \alpha_i \rfloor + 1) = f_i(m_i').$$

Hence (2.2) holds as claimed. It follows that

$$\sum_{i \in [n]} \left| \alpha_i - m_i' \right| = \sum_{i \in [n]} f_i(m_i') \leq \sum_{i \in [n]} f_i(\beta_i) \leq n.$$

Therefore,

$$(\alpha_1, \cdots, \alpha_n) = (\alpha_1 - m_1', \cdots, \alpha_n - m_n') + (m_1', \cdots, m_n') \in nC_n + S_k.$$

Moreover,

$$\begin{aligned} \#S_k =& \#M_2(n,k) - \#M_2(n,k-1) + 1 \\ =& \sum_{i=n-k}^{n} 2^{n-i} \binom{n}{i} \binom{k}{n-i} - \sum_{i=n-k+1}^{n} 2^{n-i} \binom{n}{i} \binom{k-1}{n-i} + 1 \\ =& 2^k \binom{n}{n-k} \binom{k}{k} + 2^{k-1} \binom{n}{n-k+1} \binom{k}{k-1} + \cdots + \binom{n}{n} \binom{k}{0} \\ & - 2^{k-1} \binom{n}{n-k+1} \binom{k-1}{k-1} - \cdots - \binom{n}{n} \binom{k-1}{0} + 1 \\ =& 2^k \binom{n}{n-k} + \sum_{i=1}^{k-1} 2^i \binom{n}{n-i} \left[\binom{k}{i} - \binom{k-1}{i} \right] + 1 \\ =& 2^k \binom{n}{n-k} \binom{k-1}{k-1} + \sum_{i=1}^{k-1} 2^i \binom{n}{n-i} \binom{k-1}{i-1} + 1 \\ =& \sum_{i=1}^{k} 2^i \binom{n}{n-i} \binom{k-1}{i-1} + 1 = \sum_{i=1}^{k} 2^i \binom{n}{i} \binom{k-1}{i-1} + 1. \end{aligned}$$

□

For each $n \in \mathbb{Z}^+$, let $k_1(n)$ be the nonnegative integer satisfying

$$\sum_{i=1}^{k_1(n)} 2^i \binom{n}{i}\binom{k_1(n)-1}{i-1} + 1 \le 2^n < \sum_{i=1}^{k_1(n)+1} 2^i \binom{n}{i}\binom{k_1(n)}{i-1} + 1.$$

It is easy to prove that $k_1(n) \le \frac{n}{2}$.

Corollary 2. *For each $n \in \mathbb{Z}^+$, we have*

$$\Gamma_{2^n}(C_n) \le \frac{n}{n + k_1(n)}.$$

Remark 3. It can be verified that

$$\begin{aligned}\sum_{i=1}^{k} 2^i \binom{n}{i}\binom{k-1}{i-1} + 1 \le & 2^k \sum_{i=1}^{k} \binom{n}{i}\binom{k-1}{i-1} = 2^k \sum_{i=1}^{k} \binom{n}{n-i}\binom{k-1}{i-1} \\ =& 2^k\left[\binom{n}{n-1}\binom{k-1}{0} + \cdots + \binom{n}{n-k}\binom{k-1}{k-1}\right] \\ =& 2^k\binom{n+k-1}{n-1} \le 2^k\binom{n+k}{n}.\end{aligned}$$

For $x \in (0, +\infty)$, we define

$$g(x) = \frac{2^x(1+x)^{(1+x)}}{x^x}.$$

Clearly, g is strictly increasing on $(0, +\infty)$, and $\lim_{x\to 0^+} g(x) = 1$. For each $t \in (1, +\infty)$, let $b(t)$ be the solution to the equation $g(x) = t$. Numerical calculation shows that $b(2) \approx 0.205597$. We can easily prove that, if $k_2(n)$ is the integer satisfying

$$2^{k_2(n)}\binom{n + k_2(n)}{n} \le 2^n < 2^{k_2(n)+1}\binom{n + k_2(n) + 1}{n},$$

then we have $\lim_{n\to\infty} \frac{k_2(n)}{n} = b(2)$ [9, 10]. It can be verified that

$$\lim_{n\to\infty} \frac{k_1(n)}{n} > b(2).$$

Therefore, the estimate in Corollary 2 is slightly better than that given by [9, Proposition 5] in the asymptotical sense, and it is much better for particular choices of small n. For example, we have $k_1(7) = 2$ and $k_2(7) = 1$. It follows that

$$\Gamma_{128}(C_7) \le \frac{n}{n + k_1(n)} = \frac{7}{7+2} \approx 0.78,$$

which is better than $\Gamma_{128}(C_7) \le \frac{n}{n+k_2(n)} = \frac{7}{7+1} \le 0.875$ [9]. See Table 1 for more examples.

Table 1. Comparison of estimates of $\Gamma_{2^n}(C_n)$.

n	$k_2(n)$	$\frac{n}{n+k_2(n)}$	$k_1(n)$	$\frac{n}{n+k_1(n)}$
7	1	0.875	2	0.778
11	2	0.846	3	0.786
16	3	0.842	4	0.8
20	4	0.833	5	0.8
25	5	0.833	6	0.806

Theorem 4. *For each $n \geq 3$, we have*

$$\Gamma_{2^n}(C_n) \leq \frac{6}{7}.$$

Proof. By numerical calculations, for each $3 \leq n \leq 49$, we have $\Gamma_{2^n}(C_n) \leq \frac{6}{7}$. Set $c = b(2) - 0.02$. Then for each $n \geq 50$, we have $cn \leq b(2)n - 1$, which shows that $(1 + c)n \leq n + \lfloor b(2)n \rfloor$. Therefore,

$$\Gamma_{2^n}(C_n) \leq \frac{n}{n + \lfloor b(2)n \rfloor} \leq \frac{n}{(1 + c)n} \approx 0.8435.$$

Thus, for each $n \geq 3$, we have

$$\Gamma_{2^n}(C_n) \leq \max\left\{\frac{6}{7}, 0.8435\right\} = \frac{6}{7}.$$ □

3. Conclusions

By refining techniques used in the recent works [9, 10], we get better estimates of $\Gamma_{2^n}(C_n)$ and the first nontrivial universal upper bound of $\Gamma_{2^n}(C_n)$. It is natural to find universal bounds of $\Gamma_{2^n}(B_p^n)$ for fixed $p \in (1, \infty)$, where B_p^n is the closed unit ball of $(\mathbb{R}^n, \|\cdot\|_p)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are supported by the [National Natural Science Foundation of China] grant numbers [12071444] and the [Fundamental Research Program of Shanxi Province of China] grant numbers [202103021223191].

Conflict of interest

There is no conflicts of interest between all authors.

References

1. K. Bezdek, Hadwiger's covering conjecture and its relatives, *Am. Math. Mon.*, **99** (1992), 954–956. https://doi.org/10.1080/00029890.1992.11995963

2. K. Bezdek, *Classical topics in discrete geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, New York: Springer, 2010. https://doi.org/10.1007/978-1-4419-0600-7
3. K. Bezdek, M. A. Khan, *The geometry of homothetic covering and illumination*, In: Discrete Geometry and Symmetry, Cham: Springer, 2018. https://doi.org/10.1007/978-3-319-78434-2
4. V. Boltyanski, H. Martini, P. S. Soltan, *Excursions into combinatorial geometry*, Universitext, Berlin: Springer-Verlag, 1997. https://doi.org/10.1007/978-3-642-59237-9
5. P. Brass, W. Moser, J. Pach, *Research problems in discrete geometry*, New York: Springer, 2005. https://doi.org/10.1007/0-387-29929-7
6. H. Martini, V. Soltan, Combinatorial problems on the illumination of convex bodies, *Aequationes Math.*, **57** (1999), 121–152. https://doi.org/10.1007/s000100050074
7. F. W. Levi, Überdeckung eines Eibereiches durch Parallelverschiebungen seines offenen Kerns, *Arch. Math.*, **6** (1955), 369–370. https://doi.org/10.1007/BF01900507
8. C. Zong, A quantitative program for Hadwiger's covering conjecture, *Sci. China Math.*, **53** (2010), 2551–2560. https://doi.org/10.1007/s11425-010-4087-3
9. X. Li, L. Meng, S. Wu, Covering functionals of convex polytopes with few vertices, *Arch. Math.*, **119** (2022), 135–146. https://doi.org/10.1007/s00013-022-01727-z
10. F. Xue, Y. Lian, Y. Zhang, *On Hadwiger's covering functional for the simplexand the cross-polytope*, arXiv Preprint, 2021. https://doi.org/10.48550/arXiv.2108.13277
11. U. Betke, M. Henk, Intrinsic volumes and lattice points of crosspolytopes, *Monatsh. Math.*, **115** (1993), 27–33. https://doi.org/10.1007/BF01311208
12. G. Pólya, G. Szegő, *Problems and theorems in analysis: Series, integral calculus, theory of functions*, Classics in Mathematics, Berlin: Springer-Verlag, 1998. https://doi.org/10.1007/978-3-642-61905-2

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