AIMS Mathematics

AIMS Mathematics, 9(2): 3798–3809.
DOI: 10.3934/math.2024187
Received: 01 November 2023
Revised: 20 December 2023
Accepted: 22 December 2023
Published: 11 January 2024

http://www.aimspress.com/journal/Math

Research article

The Rishi Transform method for solving multi-high order fractional differential equations with constant coefficients

Ali Turab[1], Hozan Hilmi[2], Juan L.G. Guirao[3], Shabaz Jalil[2], Nejmeddine Chorfi[4] and Pshtiwan Othman Mohammed[5,*]

[1] School of Software, Northwestern Polytechnical University, 127 West Youyi Road, Beilin District, Xi'an 710072, China
[2] Department of Mathematics, College of Science, University of Sulaimani, Sulaymani 46001, Kurdistan Region, Iraq
[3] Department of Applied Mathematics and Statistics, Technical University of Cartagena, Hospital de Marina, Cartagena 30203, Spain
[4] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[5] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Kurdistan Region, Iraq

* **Correspondence:** Email: pshtiwansangawi@gmail.com.

Abstract: In this paper, we suggest the Rishi transform, which may be used to find the analytic (exact) solution to multi-high-order linear fractional differential equations, where the Riemann-Liouville and Caputo fractional derivatives are used. We first developed the Rishi transform of foundational mathematical functions for this purpose and then described the important characteristics of the Rishi transform, which may be applied to solve ordinary differential equations and fractional differential equations. Following that, we found an exact solution to a particular example of fractional differential equations. We looked at four numerical problems and solved them all step by step to demonstrate the value of the Rishi transform. The results show that the suggested novel transform, "Rishi Transform," yields exact solutions to multi-higher-order fractional differential equations without doing complicated calculation work.

Keywords: linear fractional differential equations; Riemann-Liouville's fractional derivative; Liouville-Caputo's fractional derivative; Rishi transform; inverse Rishi transform

Mathematics Subject Classification: 30C45, 30C50

1. Introduction

A strong tool for explaining the memory and inherited characteristics of diverse components and methods is fractional calculus [1–4]. It may be used in a variety of scientific and technical domains, including biology, chemistry, acoustics, fluid mechanics, anomalous diffusion, viscoelasticity and others. In these applications, a family of integro-differential equations with singularities involved fractional differential equations [5,6]. Several analytic or numerical approaches to solving fractional differential equations have already been published, including [7–10].

Due to their three key characteristics of simplicity, accuracy and providing results without the need for time-consuming calculation work, integral transforms are currently researchers' first choice among other mathematical techniques to determine the answers to problems in science, social science and engineering [11]. Fractional calculus is also useful in engineering and mathematical modeling [12]. Moreover, the fractional approach is useful in chemistry and using a meshless technique, a numerical solution of fractional reaction-convection-diffusion is shown for modeling PEM fuel cells [13,14]. Some new integral transformations are developed to solve differential equations and fractional differential equations, such as Kamal transformation [15], Aboodh transformation [16], Sumudu transformation [17] and Rishi transformation [18], which are a few examples of recent developments in integral transforms and fractional differential equations. The renowned issues were examined by Aggarwal and other scholars [19] using a variety of integral transforms.

Our goal of this article is to create the "Rishi Transform", a novel integral transform with essential qualities for fractional and fractional differential equations, and to ascertain the solution of the fractional differential equations. The "Rishi transform" that has been suggested is superior to the other transforms that have already been developed since it offers the precise solutions to the issues without requiring time-consuming computations. Laplace transform, a well-known and often used integral transform, and the Rishi transform are dualistic.

We want to generate fractional formulas for another transformation. Thus, just as most transformations have them, we have derived the fractional formula of the transformation and use it to analyze some problems. Other transformations for fractional integrals and derivatives were derived, and we studied the Rishi transform and derived fractional formulas and it is applications.

2. Basic definitions and properties

In this section, provides some fundamental definitions and properties of fractional calculus theory, which will be used in this study.

Definition 2.1 (see [7,20]). The Riemann-Liouville integral operator of fractional order $\alpha > 0$ is defined as:

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad -\infty \leq a < t < \infty, \tag{1}$$

where $\Gamma(\alpha)$ is Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$.

Definition 2.2 (see [7,20]). The Riemann-Liouville differential operator of fractional order $\alpha > 0$, and $m = \lceil\alpha\rceil$ is defined as:

$$ {}^{RL}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s) ds.. \tag{2}$$

Definition 2.3 (see [7,20]). The Liouville -Caputo differential operator of fractional order $\alpha > 0$, and $m = \lceil\alpha\rceil$, and $f(t)$ be m −times differentiabl function, $t > a$ is defined as:

$$ {}^{LC}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \left(\frac{d}{ds}\right)^m f(s) ds. \tag{3}$$

Remark (see [21–25]). Some basic properties of fractional calculus are as follows:

1. The fractional (integral and differential) Operator is linear operator.
2. Composition between two Riemann-Liouville integration of orders α and β, is defined as follows:

$$ {}_{a}I_t^{\alpha}\ {}_{a}I_t^{\beta} f(t) = {}_{a}I_t^{\beta}\ {}_{a}I_t^{\alpha} f(t) = I_t^{\alpha+\beta} f(t). \tag{4}$$

3. If $k \geq \alpha$, for $f(t) \in C[a,b]$, and at every point $t \in [a,b]$, then

$$ {}^{RL}_{a}D_t^{k}\left({}_{a}I_t^{\alpha} f(t)\right) = {}^{RL}_{a}D_t^{k-\alpha} f(t). \tag{5}$$

   The relation is done.
4. Composition between fractional (differentiation and integration) of Liouville-Caputo operator of order α, is defined as follows:

$$ {}^{LC}_{a}D_t^{k}\left({}_{a}I_t^{\alpha} f(t)\right) = f(t). \tag{6}$$

5. Composition between fractional (integration and differentiation) of Liouville -Caputo operator of order α, and $m = \lceil\alpha\rceil$ is defined as follows:

$$ {}_{a}I_t^{\alpha}\left({}^{LC}_{a}D_t^{k} f(t)\right) = f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{k!} f^{(k)}(a). \tag{7}$$

   In general, ${}^{LC}_{a}D_t^{k}\left({}_{a}I_t^{\alpha} f(t)\right) \neq {}_{a}I_t^{\alpha}\left({}^{LC}_{a}D_t^{k} f(t)\right)$.

6. Apply fractional integral and differential of Liouville-Caputo operator of function $t^n, n \geq 0$, we get:

$$ {}_{a}I_t^{\alpha} t^n = \frac{\Gamma(1+n)}{\Gamma(1+n+\alpha)} t^{n+\alpha} \ and\ {}^{LC}_{a}D_t^{k} t^n = \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} t^{n-\alpha}. \tag{8}$$

3. Fundamental properties of Rishi transform

In this section, we have defined the Rishi transform and its properties (see [11]). Using several other studies, we will learn an introduction to the fractional formula of the new transform:

Definition 3.1 [Rishi transforms, 11]. The piecewise continuous function of exponential order using the Rishi transform $f(t)$ defined in the interval $[0, \infty)$, is given by:

$$ R\{f(t)\} = \left(\frac{p}{q}\right) \int_0^{\infty} f(t) e^{-\left(\frac{q}{p}\right)t} dt = F(q,p), \qquad q > 0, p > 0.$$

Property 1 (see [24]). Some fundamental functions and apply the Rishi transformation:

$f(t), t>0$	$R\{f(t)\}=F(q,p)$	$f(t), t>0$	$R\{f(t)\}=F(q,p)$
1	$\left(\frac{p}{q}\right)^2$	$sinkt$	$\frac{kp^3}{q(q^2+k^2p^2)}$
e^{kt}	$\frac{p^2}{q(q-kp)}$	$coskt$	$\frac{p^2}{(q^2+p^2k^2)}$
$t^\beta, \beta\in N$	$\beta!\left(\frac{p}{q}\right)^{\beta+2}$	$sinhkt$	$\frac{kp^3}{q(q^2-k^2p^2)}$
$t^\beta, \beta>-1, \beta\in R$	$\Gamma(\beta+1)\left(\frac{p}{q}\right)^{\beta+2}$	$coshkt$	$\frac{p^2}{(q^2-p^2k^2)}$

Property 2 (see [24]). Some fundamental functions, and apply inverse the Rishi transformation:

$F(q,p)$	$R^{-1}\{F(q,p)\}=f(t)$	$F(q,p)$	$R^{-1}\{F(q,p)\}=f(t)$
$\left(\frac{p}{q}\right)^2$	1	$\frac{kp^3}{q(q^2+k^2p^2)}$	$sinkt$
$\frac{p^2}{q(q-kp)}$	e^{kt}	$\frac{p^2}{(q^2+p^2k^2)}$	$coskt$
$\beta!\left(\frac{p}{q}\right)^{\beta+2}$	$t^\beta, \beta\in N$	$\frac{kp^3}{q(q^2-k^2p^2)}$	$sinhkt$
$\Gamma(\beta+1)\left(\frac{p}{q}\right)^{\beta+2}$	$t^\beta, \beta>-1, \beta\in R$	$\frac{p^2}{(q^2-p^2k^2)}$	$coshkt$

Property 3 [Convolution property, 21]. If $R\{f(t)\}=F(q,p)$ and $R\{g(t)\}=G(q,p)$, then

$$R\{f(t)*g(t)\}=\left(\frac{q}{p}\right)F(q,p)\,G(q,p),$$

where * denotes convolution of f and g , then $f(t)*g(t)=\int_0^t f(t-s)\,g(s)ds.$

Property 4 (see [11]). The Rishi transform operator and inverse Rishi transform is linear operator.

1. Rishi Transform is linear operator

$R\{\sum_{i=0}^{n} k_i f_i(t)\}=\sum_{i=0}^{n} k_i R\{f_i(t)\}$. Where k_i are arbitrary constant.

2. Inverse Rishi Transform is linear operator

If $f_i(t)=R^{-1}\{F_i(q,p)\}$, then $R^{-1}\{\sum_{i=0}^{n} k_i F_i(q,p)\}=\sum_{i=0}^{n} k_i R^{-1}\{F_i(q,p)\}$, where k_i are arbitrary constant.

4. Rishi transform of fractional integrals and derivatives

In this section, we derived fractional formula of Rishi transform for fractional integral and fractional derivative using those properties.

Property 5: (see [11]). The Rishi transformation for integer order derivative of $f(t)$ is:

$$R\{f^{(m)}(t)\}=\left(\frac{q}{p}\right)^m F(q,p)-\sum_{k=0}^{m-1}\left(\frac{q}{p}\right)^{k-1} f^{(m-1-k)}(0),$$

It is easy prove by Mathematical Induction.

Theorem 4.1: If fractional order $\alpha \in [m-1, m)$, and apply the Rishi transform of the fractional integral is $R\{I_t^\alpha f(t)\} = \left(\frac{p}{q}\right)^\alpha F(q,p).$

Proof: From definition of Riemann-Liouville integral of Eq (1), and apply Rishi transform

$$R\{I_t^\alpha f(t)\} = R\{{}_0^{RL}D_t^{-\alpha} f(t)\} = R\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds\right\} = \frac{1}{\Gamma(\alpha)} R\left\{\int_0^t (t-s)^{\alpha-1} f(s)ds\right\} .$$

By using properties (3), and (1), we get:

$$R\{I_t^\alpha f(t)\} = \frac{1}{\Gamma(\alpha)}\frac{q}{p} R\{t^{\alpha-1}\} F(q,p) = \frac{1}{\Gamma(\alpha)}\frac{q}{p}\left(\frac{p}{q}\right)^{\alpha+1}\Gamma(\alpha) F(q,p).$$

Hence $\quad R\{I_t^\alpha f(t)\} = \left(\frac{p}{q}\right)^\alpha F(q,p).$

Theorem 4.2: If $f(t)$ is a function and $F(q,p)$ is a Rishi transform for Reimann-Liouville fractional derivative of order $\alpha > 0$ is:

$$R\{{}_0^{RL}D_t^\alpha f(t)\} = \left(\frac{q}{p}\right)^\alpha F(q,p) - \sum_{k=0}^{m-1}\left(\frac{q}{p}\right)^{k-1}[{}_0^{RL}D_t^{\alpha-k-1} f(t)]_{t=0}.$$

Proof: From definition of Riemann-Liouville fractional derivative, we have:

$$R\{{}_0^{RL}D_t^\alpha f(t)\} = R\{D_t^m I_t^{m-\alpha} f(t)\},$$

Now, from property (5), and apply Rishi transform of above relation, we get:

$$R\{D_t^m I_t^{m-\alpha} f(t)\} = \left(\frac{q}{p}\right)^m R\{I_t^{m-\alpha} f(t)\} - \sum_{k=0}^{m-1}\left(\frac{q}{p}\right)^{k-1} D_t^{m-k-1}[I_t^{m-\alpha} f(t)]_{t=0},$$

Using Theorem (4.1), and Eq (5), we get:

$$= \left(\frac{q}{p}\right)^m \left(\frac{p}{q}\right)^{m-\alpha} F(q,p) - \sum_{k=0}^{m-1}\left(\frac{q}{p}\right)^{k-1}[{}_0^{RL}D_t^{\alpha-k-1} f(t)]_{t=0}.$$

Hence,

$$R\{{}_0^{RL}D_t^\alpha f(t)\} = \left(\frac{q}{p}\right)^\alpha F(q,p) - \sum_{k=0}^{m-1}\left(\frac{q}{p}\right)^{k-1}[{}_0^{RL}D_t^{\alpha-k-1} f(t)]_{t=0}.$$

Theorem 4.3: If $f(t)$ is a function and $F(q,p)$ is a Rishi transform then the Rishi transform for Liouville-Caputo fractional derivative of order $\alpha > 0$ is:

$$R\{{}_0^{LC}D_t^\alpha f(t)\} = \left(\frac{q}{p}\right)^\alpha F(q,p) - \sum_{k=0}^{m-1}\left(\frac{p}{q}\right)^{m-k-\alpha+1} f^{(m-k-1)}(0).$$

Proof: From definition of Liouville -Caputo fractional derivative, we have:

$$R\{{}_0^{LC}D_t^\alpha f(t)\} = R\{I_t^{m-\alpha} D_t^m f(t)\}.$$

Now, by using theorem (4.1), we get:

$$R\{I_t^{m-\alpha} D_t^m f(t)\} = \left(\frac{p}{q}\right)^{m-\alpha} R\{D_t^m f(t)\}.$$

Also, using property (5), we get:

$$\left(\frac{p}{q}\right)^{m-\alpha} R\{D_t^m f(t)\} = \left(\frac{p}{q}\right)^{m-\alpha}\left[(\frac{q}{p})^m F(q,p) - \sum_{k=0}^{m-1}\left(\frac{q}{p}\right)^{k-1} f^{(m-k-1)}(0)\right]$$

$$= \left(\frac{q}{p}\right)^{\alpha} F(q,p) - \sum_{k=0}^{m-1}\left(\frac{p}{q}\right)^{m-k-\alpha+1} f^{(m-k-1)}(0).$$

Hence,

$$R\{{}_0^{LC}D_t^{\alpha} f(t)\} = \left(\frac{q}{p}\right)^{\alpha} F(q,p) - \sum_{k=0}^{m-1}\left(\frac{p}{q}\right)^{m-k-\alpha+1} f^{(m-k-1)}(0).$$

5. Illustrative examples

In this section, contains four analytical problems for explaining the utility of Rishi transform for determining the exact (analytic) solution of multi-high order linear fractional differential equations of Riemann-Liouville and Liouville -Caputo Sense.

Example 5.1: Consider the linear fractional differential equation for Riemann-Liouville's fractional derivative

$${}_0^{RL}D_t^{1.3} f(t) + 2\frac{d}{dt} f(t) = \frac{6}{\Gamma(2.7)} t^{1.7} - \frac{2}{\Gamma(1.7)} t^{0.7} + 6t^2 - 4t,$$

with the initial conditions: $f(0) = 0$, $[{}_0^{RL}D_t^{0.3}(f(t))]_{t=0} = 0$ *and* $[{}_0^{RL}D_t^{-0.7}(f(t))]_{t=0} = 0$.

Solution: Apply Rishi transform for both sides, by using property (4), we get:

$$R\{{}_0^{RL}D_t^{1.3} y(t)\} + R\left\{2\frac{d}{dt} y(t)\right\} = R\left\{\frac{6}{\Gamma(2.7)} t^{1.7} - \frac{2}{\Gamma(1.7)} t^{0.7} + 6t^2 - 4t\right\}.$$

By using theorem (4.2) and property (1), we get:

$$\left(\left(\frac{q}{p}\right)^{1.3} F(q,p) - \sum_{k=0}^{1}\left(\frac{q}{p}\right)^{k-1}[{}_0^{RL}D_t^{1.3-k-1} f(t)]_{t=0}\right) + 2\left(\frac{q}{p}F(q,p) - \sum_{k=0}^{0}\left(\frac{q}{p}\right)^{k-1} f^{(1-1-k)}(0)\right) =$$
$$\frac{6}{\Gamma(2.7)}\left(\frac{p}{q}\right)^{3.7}\Gamma(2.7) - \frac{2}{\Gamma(1.7)}\left(\frac{p}{q}\right)^{2.7}\Gamma(1.7) + 12\left(\frac{p}{q}\right)^{4} - 4\left(\frac{p}{q}\right)^{3},$$

$$\left(\frac{q}{p}\right)^{1.3} F(q,p) - \left(\frac{q}{p}\right)^{-1}[{}_0^{RL}D_t^{0.3} f(t)]_{t=0} - [{}_0^{RL}D_t^{-0.7} f(t)]_{t=0} + 2\frac{q}{p}F(q,p) - 2\left(\frac{q}{p}\right)^{-1} f(0) =$$
$$6\left(\frac{p}{q}\right)^{3.7} - 2\left(\frac{p}{q}\right)^{2.7} + 12\left(\frac{p}{q}\right)^{4} - 4\left(\frac{p}{q}\right)^{3},$$

$$F(q,p)\left(\left(\frac{q}{p}\right)^{0.3} + 2\right)\left(\frac{p}{q}\right)^{-1} = 6\left(\frac{p}{q}\right)^{3.7} + 12\left(\frac{p}{q}\right)^{4} - 2\left(\frac{p}{q}\right)^{2.7} - 4\left(\frac{p}{q}\right)^{3},$$

$$F(q,p)\left(\left(\frac{q}{p}\right)^{0.3} + 2\right) = 6\left(\frac{p}{q}\right)^{4.7} + 12\left(\frac{p}{q}\right)^{5} - 2\left(\frac{p}{q}\right)^{3.7} - 4\left(\frac{p}{q}\right)^{4},$$

$$F(q,p) = \frac{1}{\left(\frac{q}{p}\right)^{0.3}+2}\left(6\left(\frac{p}{q}\right)^{5}\left(\left(\frac{p}{q}\right)^{-0.3} + 2\right) - 2\left(\frac{p}{q}\right)^{4}\left(\left(\frac{p}{q}\right)^{-0.3} + 2\right)\right),$$

$F(q,p) = \frac{1}{\left(\frac{q}{p}\right)^{0.3}+2}\left(6\left(\frac{p}{q}\right)^{5} - 2\left(\frac{p}{q}\right)^{4}\right)\left(\left(\frac{q}{p}\right)^{0.3} + 2\right).$

Hence

$$F(q,p) = 6\left(\frac{p}{q}\right)^{5} - 2\left(\frac{p}{q}\right)^{4},$$

Take Inverse Rishi transform for both sides $R^{-1}\{F(q,p)\} = R^{-1}\left\{6\left(\frac{p}{q}\right)^{5} - 2\left(\frac{p}{q}\right)^{4}\right\}$.

The exact solution is: $f(t) = t^2(t-1)$. Furthermore, the graph of the exact solution has been demonstrated in Figure 1.

Example 5.2 (see [15]). Consider the linear fractional differential equation for Riemann-Liouville's fractional derivative, given by:

$${}_{0}^{RL}D_{t}^{\frac{1}{2}} f(t) + f(t) = \frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}},$$

with the initial conditions: $\left[{}_{0}^{RL}D_{t}^{-\frac{1}{2}} f(t)\right]_{t=0} = 0.$

Solution: Apply Rishi transform for both sides of problem above, using property (4), we get:

$$R\left\{{}_{0}^{RL}D_{t}^{\frac{1}{2}} f(t)\right\} + R\{f(t)\} = R\left\{\frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}}\right\} ,$$

By using theorem (4.2), property (1) and definition (3.1), we can say:

$$R\left\{{}_{0}^{RL}D_{t}^{\frac{1}{2}} f(t)\right\} = \left(\frac{q}{p}\right)^{\frac{1}{2}} F(q,p) - \sum_{k=0}^{0}\left(\frac{q}{p}\right)^{k-1}\left[{}_{0}^{RL}D_{t}^{\frac{1}{2}-k-1} f(t)\right]_{t=0} : R\{f(t)\} = F(q,p),$$

$$R\left\{\frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}}\right\} = \frac{1}{2}\left(\frac{p}{q}\right)^{3} + \frac{1}{\sqrt{\pi}}\left(\frac{p}{q}\right)^{\frac{5}{2}}\Gamma\left(\frac{3}{2}\right),$$

So, the above equation as follows:

$$\left(\frac{q}{p}\right)^{\frac{1}{2}} F(q,p) - \sum_{k=0}^{0}\left(\frac{q}{p}\right)^{k-1}\left[{}_{0}^{RL}D_{t}^{\frac{1}{2}-k-1} f(t)\right]_{t=0} + F(q,p) = \frac{1}{2}\left(\frac{p}{q}\right)^{3} + \frac{1}{\sqrt{\pi}}\left(\frac{p}{q}\right)^{\frac{5}{2}}\Gamma\left(\frac{3}{2}\right) ,$$

$$F(q,p)\left(\left(\frac{q}{p}\right)^{\frac{1}{2}} + 1\right) = \frac{1}{2}\left(\frac{p}{q}\right)^{3} + \frac{1}{2}\left(\frac{p}{q}\right)^{\frac{5}{2}} = \frac{1}{2}\left(\frac{p}{q}\right)^{3}\left(1 + \left(\frac{q}{p}\right)^{\frac{1}{2}}\right),$$

$$F(q,p) = \frac{1}{2}\left(\frac{p}{q}\right)^{3}.$$

Now, take inverse Rishi transform for both sides, we get:

$R^{-1}\{F(q,p)\} = R^{-1}\left\{\frac{1}{2}\left(\frac{p}{q}\right)^{3}\right\}$. Hence, the exact solution is $f(t) = \frac{1}{2}t$. Furthermore, the graph of this exact solution has been demonstrated in Figure 2.

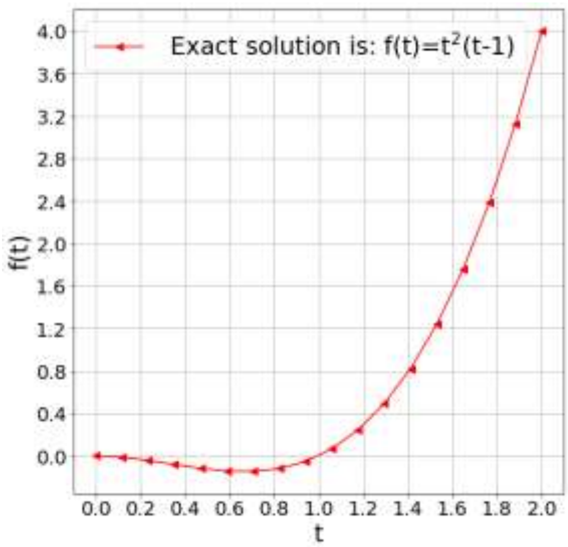

Figure 1. Exact solution of Example 5.1.

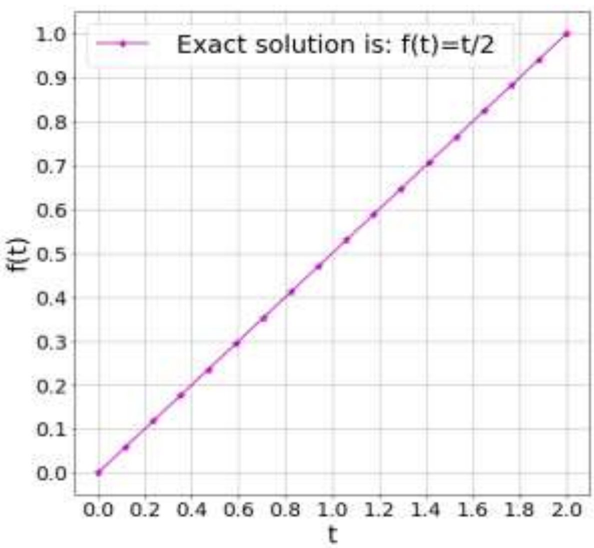

Figure 2. Exact solution of Example 5.2.

Remark: The above example was proved by Kamal transform in [12]. Exact solutions were obtained from both methods.

Example 5.3 Consider the linear fractional differential equation for Liouville -Caputo fractional derivative is given by:

$$ {}^{LC}_{0}D_t^{1.2} f(t) + {}^{LC}_{0}D_t^{0.2} f(t) + f(t) = \frac{\Gamma(2.5)}{\Gamma(1.3)} t^{0.3} + \frac{\Gamma(2.5)}{\Gamma(2.3)} t^{1.3} + t^{1.5} + 1, $$

with the initial conditions: $f(0) = 1$, $f'(0) = 0$.

Solution: Take Rishi transform for both sides, by using property (4), we obtain:

$$ R\{{}^{LC}_{0}D_t^{1.2} f(t) + {}^{LC}_{0}D_t^{0.2} f(t)\} + R\{f(t)\} = R\left\{\frac{\Gamma(2.5)}{\Gamma(1.3)} t^{0.3} + \frac{\Gamma(2.5)}{\Gamma(2.3)} t^{1.3} + t^{1.5} + 1\right\}. $$

By using theorem (4.3), property (1) and definition (3.1), we get:

$$ \left(\frac{q}{p}\right)^{1.2} F(q,p) - \sum_{k=0}^{1} \left(\frac{p}{q}\right)^{1.8-k} f^{(1-k)}(0) + \left(\frac{q}{p}\right)^{0.2} F(q,p) - \sum_{k=0}^{0} \left(\frac{p}{q}\right)^{0.8-k} f^{(-k)}(0) + F(q,p) $$
$$ = \frac{\Gamma(2.5)}{\Gamma(1.3)} \left(\frac{p}{q}\right)^{2.3} \Gamma(1.3) + \frac{\Gamma(2.5)}{\Gamma(2.3)} \left(\frac{p}{q}\right)^{3.3} \Gamma(2.3) + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^{2}. $$

Simplify the statement

$$ \left(\frac{q}{p}\right)^{1.2} F(q,p) - \left(\frac{p}{q}\right)^{1.8} f'(0) - \left(\frac{p}{q}\right)^{0.8} f(0) + \left(\frac{q}{p}\right)^{0.2} F(q,p) - \left(\frac{p}{q}\right)^{1.8} f(0) + F(q,p) = $$
$$ \Gamma(2.5) \left(\frac{p}{q}\right)^{2.3} + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.3} + \Gamma(2.5) \left(\frac{p}{q}\right)^{3.5} + \left(\frac{p}{q}\right)^{2}. $$

After input initial conditions, we get:

$F(q,p)\left(\left(\frac{q}{p}\right)^{1.2}+\left(\frac{q}{p}\right)^{0.2}+1\right)=\Gamma(2.5)\left(\frac{p}{q}\right)^{2.3}+\Gamma(2.5)\left(\frac{p}{q}\right)^{3.3}+\Gamma(2.5)\left(\frac{p}{q}\right)^{3.5}+\left(\frac{p}{q}\right)^{2}+\left(\frac{p}{q}\right)^{0.8}+\left(\frac{p}{q}\right)^{1.8},$

$F(q,p)\left(\left(\frac{q}{p}\right)^{1.2}+\left(\frac{q}{p}\right)^{0.2}+1\right)=\Gamma(2.5)\left(\frac{p}{q}\right)^{3.5}\left(\left(\frac{q}{p}\right)^{1.2}+\left(\frac{q}{p}\right)^{0.2}+1\right)+\left(\frac{p}{q}\right)^{2}\left(\left(\frac{q}{p}\right)^{1.2}+\left(\frac{q}{p}\right)^{0.2}+1\right),$

$$F(q,p)\left(\left(\frac{q}{p}\right)^{1.2}+\left(\frac{q}{p}\right)^{0.2}+1\right)=\left(\left(\frac{q}{p}\right)^{1.2}+\left(\frac{q}{p}\right)^{0.2}+1\right)\left[\Gamma(2.5)\left(\frac{p}{q}\right)^{3.5}+\left(\frac{p}{q}\right)^{2}\right].$$

Implies that $F(q,p)=\Gamma(2.5)\left(\frac{p}{q}\right)^{3.5}+\left(\frac{p}{q}\right)^{2}$.

Now, take inverse Rishi transform for both sides, we get:

$$R^{-1}\{F(q,p)\}=R^{-1}\left\{\Gamma(2.5)\left(\frac{p}{q}\right)^{3.5}+\left(\frac{p}{q}\right)^{2}\right\}.$$

Hence, the exact solution is: $f(t)=1+t\sqrt{t}$. Figure 3 demonstrates the graph of the exact solution for Example 5.3.

Example 5.4 Consider the linear fractional differential equation given by

$${}_{0}^{LC}D_{t}^{1.9}f(t)+f(t)=\frac{2}{\Gamma(1.1)}t^{0.1}+t^{2}+1,$$

with the initial conditions: $f(0)=1, f'(0)=0$

Solution: Take Rishi transform for both sides, and by property (4), we obtain:

$$R\{{}_{0}^{LC}D_{t}^{1.9}f(t)\}+R\{f(t)\}=R\left\{\frac{2}{\Gamma(1.1)}t^{0.1}+t^{2}+1\right\}.$$

By using theorem (4.3) and property (1), we get:

$$\left(\frac{q}{p}\right)^{1.9}F(q,p)-\sum_{k=0}^{1}\left(\frac{p}{q}\right)^{1.1-k}f^{(1-k)}(0)+F(q,p)=\frac{2}{\Gamma(1.1)}\left(\frac{p}{q}\right)^{2.1}\Gamma(1.1)+\left(\frac{p}{q}\right)^{4}\Gamma(3)+\left(\frac{p}{q}\right)^{2},$$

$$\left(\frac{q}{p}\right)^{1.9}F(q,p)-\left(\frac{p}{q}\right)^{1.1}f'(0)-\left(\frac{p}{q}\right)^{0.1}f(0)+F(q,p)=2\left(\frac{p}{q}\right)^{2.1}+2\left(\frac{p}{q}\right)^{4}+\left(\frac{p}{q}\right)^{2}.$$

After input initial conditions, we get:

$$F(q,p)\left(\left(\frac{q}{p}\right)^{1.9}+1\right)=2\left(\frac{p}{q}\right)^{2.1}+2\left(\frac{p}{q}\right)^{4}+\left(\frac{p}{q}\right)^{2}+\left(\frac{p}{q}\right)^{0.1},$$

$$F(q,p)=\frac{1}{\left(\frac{q}{p}\right)^{1.9}+1}\left(2\left(\frac{p}{q}\right)^{4}\left(\left(\frac{q}{p}\right)^{1.9}+1\right)+\left(\frac{p}{q}\right)^{2}\left(\left(\frac{q}{p}\right)^{1.9}+1\right)\right),$$

$$F(q,p)=\left(\frac{1}{\left(\frac{q}{p}\right)^{1.9}+1}\right)\left(\left(\frac{q}{p}\right)^{1.9}+1\right)\left(2\left(\frac{p}{q}\right)^{4}+\left(\frac{p}{q}\right)^{2}\right),$$

$$F(q,p)=\left(\frac{p}{q}\right)^{2}+2\left(\frac{p}{q}\right)^{4}.$$

Finally, take inverse Rishi transform for both sides, we get:

$$R^{-1}\{F(q,p)\} = R^{-1}\left\{\left(\frac{p}{q}\right)^2 + 2\left(\frac{p}{q}\right)^4\right\}$$

Hence, the exact solution is: $f(t) = 1 + t^2$. In addition, the graph of the exact solution has been demonstrated in Figure 4 for Example 5.4.

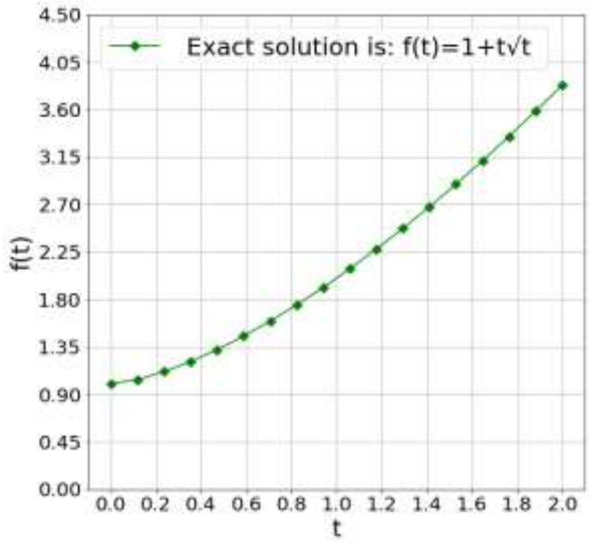

Figure 3. Exact solution of Example 5.3.

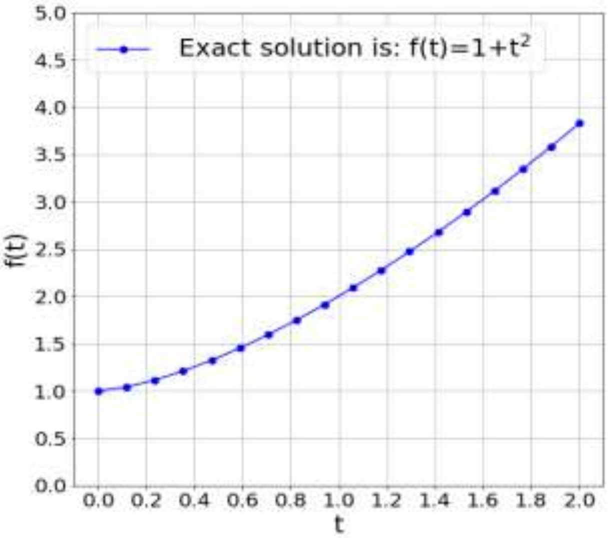

Figure 4. Exact solution of Example 5.4.

In this work, we proposed a transform for solving the fractional differential equations, but more studies will be required for solving the real-world problems modeled as fractional differential equations using our proposed method.

6. Conclusions

After analyzing the literature regarding fractional differential equations (FDEs), it can be concluded that the Rishi transformation method is a feasible approach for solving such equations. This method involves transforming the FDEs into a simpler form. The Rishi transformation method offers several advantages, such as the reduction of the order of the equation, simplification of the form of the equation and the ability to solve the FDEs with constant coefficients. What we have been able to achieve is that we have derived Rishi transform formulas for fractional derivation that can be used for later research. It is an applicable method for the analysis of many types of fractional differential equations, and several studies have shown that the Rishi transformation method is an effective tool for solving different types of FDEs.

In conclusion, the Rishi transformation method is a promising method for solving FDEs, and with further research and development, it can become an even more efficient and effective tool for solving increasingly complex FDEs.

Concerning future work, we plan to extend the numerical schemes presented here to apply them to other types of fractional differential equations, including exponential and Mittag-Leffler in kernels, to establish similar results (visit [21,26] to see these operators).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Authors' contributions

Conceptualization, H.H.; Data curation, P.O.M.; Formal analysis, N.C.; Funding acquisition, J.L.G.G.; Investigation, H.H., S.J., P.O.M., J.L.G.G. and N.C.; Methodology, A.T.; Project administration, A.T. and H.H.; Resources, S.J.; Software, S.J.; Supervision, P.O.M. and N.C.; Validation, J.L.G.G.; Visualization, S.J. and N.C.; Writing – original draft, A.T. and J.L.G.G.; Writing-review & editing, P.O.M. and H.H. All authors have read and agreed to the final version of the manuscript.

Acknowledgments

Researchers Supporting Project number (RSP2024R153), King Saud University, Riyadh, Saudi Arabia.

Conflicts of interest

The authors declare no conflicts of interest.

References

1. J. T. Machado, V. Kiryakova, F. Mainardi, Recent History of Fractional Calculus, *Sci. Numer. Simul.,* **16** (2011), 1140–1153. https://doi.org/10.1016/j.cnsns.2010.05.027
2. K. B. Oldham, J. Spanier, The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic press, INC: New York, (1974), ISBN: 9780080956206.
3. M. Y. Almusawa, P. O. Mohammed, Approximation of sequential fractional systems of Liouville-Caputo type by discrete delta difference operators, *Chaos Soliton. Fract.,* **176** (2023), 114098. https://doi.org/10.1016/j.chaos.2023.114098
4. P. O. Mohammed, D. Baleanu, E. Al-Sarairah, T. Abdeljawad, N. Chorfi, Theoretical and numerical computations of convexity analysis for fractional differences using lower boundedness, *Fractals,* **31** (2023), 2340183. https://doi.org/10.1142/S0218348X23401837
5. R. Hilfer, Applications of Fractional Calculus in Physics, *Appl. Fract. Calc. Phys.,* **90** (2000), https://doi.org/10.1142/3779
6. Y. Povstenko, Essentials of fractional calculus, in *Fractional Thermoelasticity,* Springer, **219** (2015), 1–11. https://doi.org/10.1007/978-3-319-15335-3_1
7. K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, *John-Wily Sons,* (1993), ISBN-100471588849.
8. H. HamaRashid, H. M. Srivastava, M. Hama, P. O. Mohammed, E. Al-Sarairah, M. Y. Almusawa, New numerical results on existence of Volterra-Fredholm Integral equation of nonlinear boundary integro-differential type, *Symmetry,* **15** (2023), 1144. https://doi.org/10.3390/sym15061144
9. A. Jan, H. M. Srivastava, A. Khan, P. O. Mohammed, R. Jan, Y. S. Hamed, In Vivo HIV Dynamics, Modeling the Interaction of HIV and Immune System via Non-Integer Derivatives, *Fractal Fract.,* **7** (2023), 361. https://doi.org/10.3390/fractalfract7050361

10. P. O. Mohammed, R. Dahal, C. S. Goodrich, Y. S. Hamed, D. Baleanu, Analytical and numerical negative boundedness of fractional differences with Mittag--Leffler kernel, *AIMS Math.,* **8** (2023), 5540–5550. https://doi.org/10.3934/math.2023279
11. R. kumar, J. Chandel, S. Aggarwal, A new integral transform “Rishi Transform” with application, *J. Sci. Res.,* **14** (2022), 521–532. https://doi.org/doi:10.3329/jsr.v2i3.4899
12. S. A. Ahmad, S. K. Rafiq, H. D. Hilmi, H. U. Ahmed, Mathematical modeling techniques to predict the compressive strength of pervious concrete modified with waste glass powders, *Asian J. Civ. Eng.,* **8** (2023). https://doi.org/10.1007/s42107-023-00811-1
13. V. R. Hosseini, A. A. Mehrizi, H. Karimi-Maleh, M. Naddafi, A numerical solution of fractional reaction-convection-diffusion for modeling PEM fuel cells based on a meshless approach, *Eng. Anal. Bound. Elem*., **155** (2023), 707–716. https://doi.org/ 10.1016/j.enganabound.2023.06.016
14. V. R. Hosseini, W. Zou, The peridynamic differential operator for solving time-fractional partial differential equations, *Nonlinear Dyn.,* **109** (2022), 1823–1850. https// doi.org/10.1007/s11071-022-07424-4
15. R. Khandelwal, P. Choudhary, Y. Khandelwal, Solution of fractional ordinary differential equation by Kamal transform, *Int. J. Stat. Appl. Math.*, **3** (2018), 279–284.
16. K. S. Aboodh, The new integral transform “Aboodh Transform,” *Glob. J. Pure Appl. Math.,* **9** (2013), 35–43.
17. V. Daftardar-Gejji, Fractional Calculus and Fractional Differential Equations, Eds., Birkhauser, (2019). https://doi.org/10.1007/978-981-13-9227-6
18. M. F. Kazem, A. Al-Fayadh, Solving Fredholm Integro-Differential Equation of Fractional Order by Using Sawi Homotopy Perturbation Method, *J. Phys. Conf. Ser.,* (2022). https// doi.org/10.1088/1742-6596/2322/1/012056
19. S. Aggarwal, R. kumar, J. Chandel, Solution of linear Volterra integral equation of second kind via Rishi transform, *J. Sci. Res.,* **15** (2023), 111–119. https://doi.org/10.3329/jsr.v2i3.4899
20. K. Diethelm, The analysis of fractional differential equations, Springer, (2004), https://doi.org/10.1007/978-3-642-14574-2
21. T. Abdeljawad, D. Baleanu, On fractional derivatives with generalized Mittag-Leffler kernels, *Adv. Differ. Equ.,* **468** (2018). https://doi.org/10.1186/s13662-018-1914-2
22. G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer, Berlin, (2012). https://doi.org/10.1007/978-3-642-65690-3
23. S. Ahmed, Solving Fractional Differential Equations by Using Conformable Fractional Derivatives Definition, (2016).
24. K. jwamer, H. Hilmi, Asymptotic Behavior of Eigenvalues and Eigenfunctions of T, Regge Fractional Problem, *J. Al-Qadisiyah Comput. Sci. Math.,* **14** (2022), 89–100. https://doi.org/10.29304/jqcm.2022.14.3.1031
25. H. Hilmi, K. jwamer, Existence and Uniqueness Solution of Fractional Order Regge Problem, *J. Babylon. Res.,* **30** (2022), 80–96. https://doi.org/10.29196/jubpas.v30i2.4186
26. T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Differ. Equ.*, **313** (2017). https://doi.org/10.1186/s13662-017-1285-0

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)