AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 3521–3558.
DOI: 10.3934/math.2024173
Received: 19 November 2023
Revised: 17 December 2023
Accepted: 21 December 2023
Published: 08 January 2024

Research article

Bivariate step-stress accelerated life test for a new three-parameter model under progressive censored schemes with application in medical

Naif Alotaibi[1], A. S. Al-Moisheer[2], Ibrahim Elbatal[1], Salem A. Alyami[1], Ahmed M. Gemeay[3] and Ehab M. Almetwally[1,4,*]

[1] Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia

[2] Department of Mathematics, College of Science, Jouf University, P.O. Box 848, Sakaka 72351, Saudi Arabia

[3] Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

[4] Faculty of Business Administration, Delta University for Science and Technology, Gamasa 11152, Egypt

* **Correspondence:** Email: ehab.metwaly@deltauniv.edu.eg.

Abstract: In this article, a new three-parameter lifetime model called the Gull alpha power exponentiated exponential (GAPEE) distribution is introduced and studied by combining the Gull alpha power family of distributions and the exponentiated exponential distribution. The shapes of the probability density function (PDF) for the GAPEE distribution can be asymmetric shapes, like unimodal, decreasing, and right-skewed. In addition, the shapes of the hazard rate function (hrf) for the GAPEE distribution can be increasing, decreasing, and upside-down shaped. Several statistical features of the GAPEE distribution are computed. Eight estimation methods such as the maximum likelihood, Anderson-Darling, right-tail Anderson-Darling, left-tailed Anderson-Darling, Cramér-von Mises, least-squares, weighted least-squares, and maximum product of spacing are discussed to estimate the parameters of the GAPEE distribution. The flexibility and the importance of the GAPEE distribution were demonstrated utilizing three real-world datasets related to medical sciences. The GAPEE distribution is extremely adaptable and outperforms several well-known statistical models. A bivariate step-stress accelerated life test based on progressive type-I censoring using the model is presented. Minimizing the asymptotic variance of the maximum likelihood estimate of the log of the scale parameter at design stress under progressive type-I censoring yields an expression for the ideal test plan under progressive type-I censoring.

Keywords: exponentiated exponential distribution; Gull alpha power family of distributions; bivariate step-stress; maximum likelihood estimation; progressive type-I censoring; symmetric; asymmetric;

accelerated life test
Mathematics Subject Classification: 62N05, 62H05, 62F15

1. Introduction

Statistical models can be used to describe and forecast real-world occurrences. Several extended distributions have been widely employed in data modeling throughout the last few decades. Recent advances have focused on establishing new families that expand well-known distributions while providing tremendous flexibility in modeling data in practice [62,63]. A large field of statistics aims at developing distributions with innovative characteristics to create flexible models for data interpretation. In reality, a new distribution can provide a new modeling perspective and a deeper description of the underlying mechanisms establishing the data. A more robust family of distributions is produced by these phenomena of parameter addition, which is effectively used to model data sets from the fields of engineering, economics, biological research, and environmental sciences. Consequently, several well-known generating families of distributions in this respect include the generalized odd Burr III-G [31], truncated Cauchy power Weibull-G [8], the generalized transmuted-G [50], generalized inverted Kumaraswamy-G [35], truncated Burr X-G [16], odd generalized N-H-G [3], sine extended odd Fréchet-G [36], generalized odd log-logistic [26], arcsine exponentiated-X family [33], generalized truncated Fréchet [61], tan-G [56], extended cosine-G [46], type II exponentiated half logistic-G [4], logistic-G [59], sine-G [40], cosine-G [57], alpha power transformed family of distributions [41], and for more detail see [4, 5, 17, 43, 51]. In 2020, Ijaz [34] presented a new family of generalized distributions called the GAP family of distributions and they defined its cumulative distribution function (CDF), probability density function (PDF) as

$$F(x) = \tau^{1-G(x)} G(x), \; \tau > 0, x \in R \tag{1.1}$$

and

$$f(x) = \tau^{1-G(x)} g(x) \left[1 - \log(\tau)\, G(x)\right], \quad \tau > 0, x \in R. \tag{1.2}$$

Many authors used the CDF (1.1) and PDF (1.2) to get new generalizations and new sub-models of the Gull alpha power (GAP) family of distributions as the exponentiated generalized Gull alpha power family of distributions [39], Gull alpha power Ampadu family of distributions [12], Kumaraswamy-Gull alpha power Rayleigh distribution [42], and exponentiated Gull alpha power exponential distribution [38].

The exponentiated exponential (EE) distribution has been demonstrated to be useful in a variety of applications such as life testing, survival analysis, and dependability. This distribution, which is a particular case of the exponentiated Weibull distribution [44, 45], was studied in [29]. The CDF and PDF of the EE distribution with scale parameter a and shape parameter b are provided via

$$G(x; a, b) = (1 - e^{-ax})^{b}, \quad x, a, b > 0 \tag{1.3}$$

and

$$g(x; a, b) = abe^{-ax}(1 - e^{-ax})^{b-1}, \quad x, a, b > 0. \tag{1.4}$$

According to the flexibility of the EE distribution, many statisticians utilized it to create new generalizations of the EE distribution, like the beta EE distribution [19], Marshall-Olkin EE distribution [52], half-Cauchy EE distribution [22], odd Lomax EE distribution [54], and modified slashed EE distribution [14].

When investigations, including the lifespan of test units, must end before full observation, censored data emerges in real-world testing trials. For a number of reasons, including time constraints and financial considerations, censoring is a frequent and necessary routine action. The many forms of censorship have been well studied; types I and II censorship are the most common. A generalized censorship technique known as progressive censored schemes has lately garnered significant attention in the literature compared to standard censorship designs because of its effective use of available resources. The (PTIC) is one of several progressive censored Type-II systems. When a specific number of lifetime test units are consistently removed from the test at the end of each post-test interval, this pattern is seen. According to a study by Balakrishnan et al. [15], it can realistically predict the termination time and provides additional design freedom by permitting test units to be terminated during non-terminal time periods.

We now talk about accelerated life tests (ALTs), which are ways to get more data in a shorter amount of time by stressing out items more than they would under normal operating settings. Time and money can be significantly saved with such testing. Hakamipour [30] describes the step-stress accelerated life test (SSALT) as one form of ALT. Typically, the researcher starts with a stress level that is slightly over normal condition and gradually increases it at pre-specified time intervals during the test. The test goes on until the time limit is achieved and censoring takes place, or until the full sample of things fails. For more information about SSALT under PTIC, see [9, 10].

The major goal of this paper is to add to the literature by introducing the Gull alpha power exponentiated exponential distribution (GAPEED) as a novel three-parameter model based on the GAP family of distributions. The subsequent points give adequate cause for examining it:

(1) The GAPEED is a very flexible model whose PDF can be asymmetric (decreasing, unimodal, and right-skewed).
(2) The hazard function (hrf) shape of the GAPEED includes increasing, upside-down and decreasing shapes.
(3) The GAPEED has a closed-form quantile function; it is easy to compute numerous properties and generate random numbers using it.
(4) The parameters of the GAPEED can be estimated utilizing eight different methods of estimation: The maximum likelihood (ML), Anderson-Darling (AD), right-tail Anderson-Darling (RTAD), left-tailed Anderson-Darling (LTAD), Cramér-von Mises (CVM), least-squares (LS), weighted least-squares (WLS), and maximum product of spacing (MPS).
(5) The importance and the flexibility of the GAPEED is discussed using three real datasets, and the GAPEED gives a better fit than well-known distributions such as the Topp-Leone modified Weibull (TLMW), Type II exponentiated half logistic power Lomax (TIIEHLPL), exponential Lomax (EL), Kumaraswamy Weibull (KW), generalized modified Weibull (GMW), Marshall- Olkin alpha power extended Weibull (MOAPEW), exponential Weibull (EW), exponentiated generalized alpha power exponential (EGAPEx), Kavya-Manoharan generalized exponential (KMGEx), exponentiated half logistic inverted Nadarajah- Haghighi (EHLINH), exponentiated exponential (ExEx), and odd Weibull inverse Topp-Leone (OWITL).

(6) We suggest utilizing the GAPEED model to create bivariate SSALTs under PTIC. The optimal test strategy for our suggested bivariate SSALT under PTIC is found by minimizing the asymptotic variance of the MLEs of the scale parameter's.

The remainder of this article is structured as follows: In Section 2, a new three-parameter model utilizing the EE distribution as the parent distribution in the GAP family is presented and discussed. Some important statistical features of the GAPEED are demonstrated in Section 3. Eight different estimation methods, ML, AD, CVM, MPS, LS, RTAD, WLS, and LTAD for the distribution parameters, are proposed in Section 4. In Section 5, we use a Monte Carlo technique to evaluate the quality of different estimators. To illustrate the importance of the GAPEED, we employed three real datasets in Section 6. In Section 7, the bivariate SSALT under the progressive type-I censoring (PTIC) model is discussed. Finally, the paper with concluding remarks.

2. Model formulation

The GAPEED can be formulated by inserting (1.3) and (1.4) into (1.1) and (1.2), and then the CDF of the new suggested model is defined as

$$F(x; a, b, \tau) = (1 - e^{-ax})^b \, \tau^{1-\left(1-e^{-ax}\right)^b}, \quad x > 0, a, b, \tau > 0 \tag{2.1}$$

and its PDF is defined as follows

$$f(x; a, b, \tau) = abe^{-ax}(1 - e^{-ax})^{b-1}\tau^{1-\left(1-e^{-ax}\right)^b}\left[1 - \log(\tau)(1 - e^{-ax})^b\right]. \tag{2.2}$$

The survival function, hazard rate function (hrf), reversed hrf, and cumulative hrf are provided as

$$s(x; a, b, \tau) = 1 - (1 - e^{-ax})^b \, \tau^{1-\left(1-e^{-ax}\right)^b},$$

$$h(x; a, b, \tau) = \frac{abe^{-ax}(1 - e^{-ax})^{b-1}\tau^{1-\left(1-e^{-ax}\right)^b}\left[1 - \log(\tau)(1 - e^{-ax})^b\right]}{1 - (1 - e^{-ax})^b\tau^{1-(1-e^{-ax})^b}},$$

$$\varsigma(x; a, b, \tau) = \frac{ab\left[1 - \log(\tau)(1 - e^{-ax})^b\right]}{e^{ax} - 1}$$

and

$$H(x; a, b, \tau) = -\log\left[1 - (1 - e^{-ax})^b\tau^{1-\left(1-e^{-ax}\right)^b}\right].$$

Figure 1 shows the plots of the PDF and hrf for the GAPEED for different values of parameters. From Figure 1, we can note that the PDF of the GAPEED can be decreasing, unimodal, and right skewed but the hrf can be decreasing, increasing, and up-side-down.

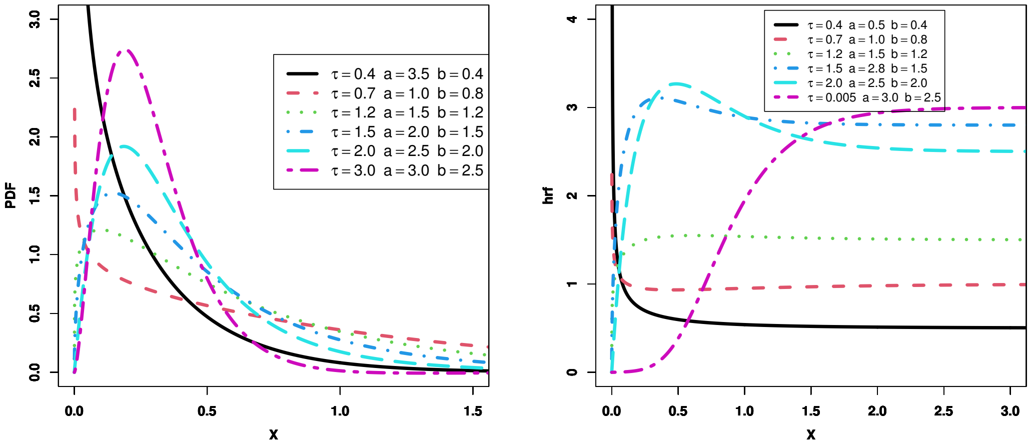

Figure 1. Plots of PDF and hrf for the GAPEED.

3. Statistical properties

3.1. Quantiles

The quantile function, defined as $Q(p; a, b, \tau) = F^{-1}(p; a, b, \tau), p \in (0, 1)$, is computed by inverting Eq (1.1) as

$$p = \frac{\tau G(x)}{\tau^{G(x)}}.$$

Then, we can rewrite the above equation as

$$-\frac{p \log(\tau)}{\tau} = -G(x) \log(\tau)\, e^{-G(x)\log(\tau)}.$$

As a result, through the use of the negative Lambert W function, represented by $W_{-1}(.)$, we obtain the quantile function of the GAPEED as

$$Q(p; a, b, \tau) = \frac{-1}{a} \log\left[1 - \left(\frac{-W_{-1}\left[-\frac{p\log(\tau)}{\tau}\right]}{\log(\tau)}\right)^{\frac{1}{b}}\right].$$

Specifically, by inserting $p = 0.25$, 0.5, and 0.75, we obtain the first, second (median), and third quantiles. Furthermore, relying on the quantiles, Bowley's skewness (α_1) and Moor's kurtosis (α_2) are provided via

$$\alpha_1 = \frac{Q(0.75;\, a,\, b, \tau) - 2Q(0.5;\, a,\, b, \tau) + Q(0.25;\, a,\, b, \tau)}{Q(0.75;\, a,\, b, \tau) - Q(0.25;\, a,\, b, \tau)}$$

and

$$\alpha_2 = \frac{Q(0.875;\, a,\, b, \tau) - Q(0.625;\, a,\, b, \tau) + Q(0.375;\, a,\, b, \tau) - Q(0.125;\, a,\, b, \tau)}{Q(0.75;\, a,\, b, \tau) - Q(0.25;\, a,\, b, \tau)},$$

respectively. These metrics provide helpful details about the GAPEED skewness and kurtosis modeling capabilities and have the benefit of being specified for all parameter values. The plots of α_1 and α_2 for the GAPEED are given in Figure 2.

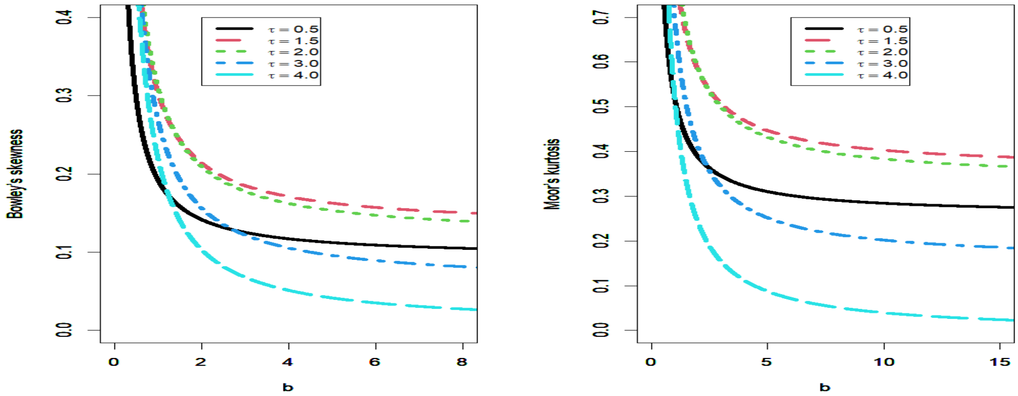

Figure 2. Plots of α_1 and α_2 for the GAPEED at $a = 0.5$.

3.2. *Ordinary moments*

The rth ordinary moments are essential statistics for determining the measures of dispersion for any distribution. Assume that $X \sim$ GAPEED (a, b, τ) for $x > 0$, then the rth ordinary moments of X can computed via

$$\mu_r' = \int_0^\infty x^r f(x)dx = ab \int_0^\infty x^r e^{-ax}(1-e^{-ax})^{b-1}\tau^{1-(1-e^{-ax})^b}\left[1-\log(\tau)(1-e^{-ax})^b\right]dx \tag{3.1}$$

by using the power series

$$\tau^m = \sum_{i=0}^\infty \frac{(\log(\tau))^i}{i!}m^i. \tag{3.2}$$

Inserting (3.2) into (3.1), then

$$\mu_r' = ab\sum_{i=0}^\infty \frac{(\log(\tau))^i}{i!}\int_0^\infty x^r e^{-ax}(1-e^{-ax})^{b-1}\left[1-(1-e^{-ax})^b\right]^i\left[1-\log(\tau)(1-e^{-ax})^b\right]dx. \tag{3.3}$$

Employing the binomial expansion

$$(1-x)^\beta = \sum_{j=0}^\infty (-1)^j \binom{\beta}{j} x^j \tag{3.4}$$

and inserting (3.4) into (3.3), we get

$$\mu_r' = ab\sum_{i,j=0}^\infty (-1)^j \binom{i}{j}\frac{(\log(\tau))^i}{i!}\int_0^\infty x^r e^{-ax}(1-e^{-ax})^{b(j+1)-1}\left[1-\log(\tau)(1-e^{-ax})^b\right]dx. \tag{3.5}$$

We can rewrite the above Eq (3.5) as

$$\mu_r' = \sum_{i,j=0}^{\infty} \pi_{i,j} \int_0^{\infty} x^r \left[e^{-ax}(1-e^{-ax})^{b(j+1)-1} - \log(\tau)\, e^{-ax}(1-e^{-ax})^{b(j+2)-1} \right] dx, \tag{3.6}$$

where $\pi_{i,j} = ab(-1)^j \binom{i}{j} \frac{(\log(\tau))^i}{i!}$. Again, using the binomial expansion (3.4) in (3.6), then the rth ordinary moments of the GAPEED are given by

$$\mu_r' = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \int_0^{\infty} x^r e^{-a(k+1)x} dx = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \frac{\Gamma(r+1)}{[a(k+1)]^{r+1}}, \tag{3.7}$$

where $\pi_{i,j,k} = (-1)^k \pi_{i,j} \left[\binom{b(j+1)-1}{k} - \binom{b(j+2)-1}{k} \log(\tau) \right]$.

Table 1 shows some numerical values of μ_1', μ_2', μ_3', μ_4', variance (σ^2), coefficient of variation (CV), skewness, and kurtosis. Also, some 3D plots of moments are provided in Figure 3.

Table 1. Some numerical values of moments for the GAPEED.

Parameters			Measures							
a	b	τ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	CV	skewness	kurtosis
0.5	0.75	0.25	2.68979	12.4088	79.9101	662.52	5.17387	0.845648	2.52479	6.88807
		0.9	1.74385	6.75901	40.1156	319.514	3.71799	1.10572	4.59144	9.78582
	1.5	1.5	2.0773	7.90377	45.4087	353.528	3.5886	0.911934	4.29041	9.70584
		2.0	1.68603	5.38955	27.4263	198.4	2.54684	0.94653	5.75592	12.5055
0.75	0.75	0.25	1.79319	5.51503	23.6771	130.868	2.2995	0.845648	2.52479	6.88807
		0.9	1.16257	3.004	11.8861	63.1139	1.65244	1.10572	4.59144	9.78582
	1.5	1.5	1.38487	3.51279	13.4544	69.8326	1.59493	0.911934	4.29041	9.70584
		2.0	1.12402	2.39536	8.12631	39.1901	1.13193	0.94653	5.75592	12.5055
1.5	0.75	0.25	0.896596	1.37876	2.95963	8.17926	0.574874	0.845648	2.52479	6.88807
		0.9	0.581283	0.751001	1.48576	3.94462	0.41311	1.10572	4.59144	9.78582
	1.5	1.5	0.692433	0.878196	1.68181	4.36454	0.398733	0.911934	4.29041	9.70584
		2.0	0.562011	0.598839	1.01579	2.44938	0.282982	0.94653	5.75592	12.5055
2.5	0.75	0.25	0.537958	0.496353	0.63928	1.06003	0.206955	0.845648	2.52479	6.88807
		0.9	0.34877	0.27036	0.320925	0.511223	0.14872	1.10572	4.59144	9.78582
	1.5	1.5	0.41546	0.316151	0.36327	0.565644	0.143544	0.911934	4.29041	9.70584
		2.0	0.337207	0.215582	0.21941	0.31744	0.101874	0.94653	5.75592	12.5055

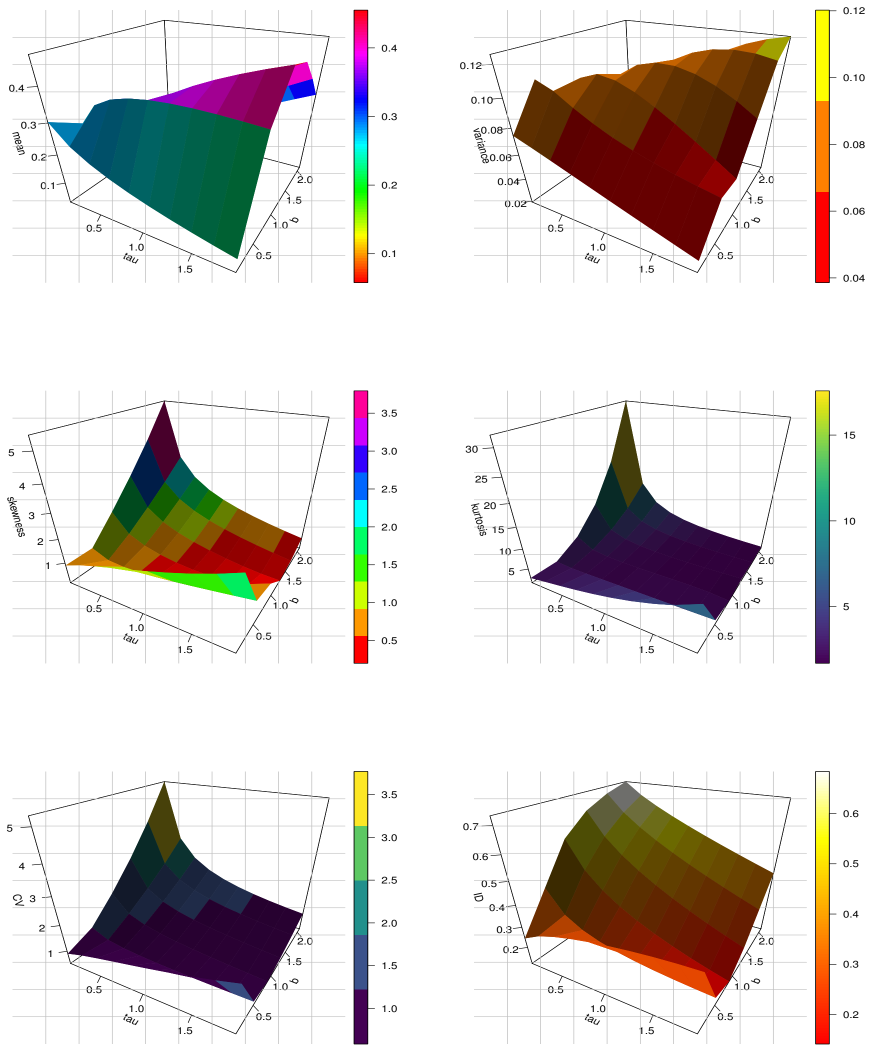

Figure 3. 3D plots of moments for the GAPEED at $a = 0.5$.

3.3. Moment generating function and incomplete moments

The moment-generating function for the GAPEED can be computed from (3.7) as

$$M_x(t) = \int_0^{\infty} e^{tx} f(x)dx = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \int_0^{\infty} x^r e^{-[a(k+1)-t]x} dx = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \frac{\Gamma(r+1)}{[a(k+1)-t]^{r+1}}.$$

The sth lower and upper incomplete moments of the GAPEED are computed as

$$\underset{s}{\omega}(t) = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \int_0^{t} x^s e^{-a(k+1)x} dx = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \frac{\gamma(s+1, a(k+1)t)}{[a(k+1)]^{s+1}},$$

and

$$\underset{s}{\upsilon}(t) = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \int_t^{\infty} x^s e^{-a(k+1)x} dx = \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} \frac{\Gamma(s+1, a(k+1)t)}{[a(k+1)]^{s+1}},$$

where $\gamma(.,.)$ and $\Gamma(.,.)$ are the lower and upper incomplete gamma functions.

4. Estimation of the GAPEED parameters

This section introduces traditional estimation methods specifically designed for estimating the parameters of the GAPEED. These methods are applied in a simulation setting to assess their effectiveness and performance. A total of eight estimation methods are considered for this purpose. Each method involves deriving an estimate by optimizing an objective function to either maximize or minimize a specific value. The estimation setting and the definitions of the functions to be optimized are provided in detail below.

Suppose we have a random sample of values, denoted as $x_1, x_2, \ldots, x_n$, drawn from a random variable that follows the GAPEED. In order to estimate the parameters of the GAPEED, we employ various estimation methods. The first method is maximum likelihood estimation (MLE), where the estimators are obtained by maximizing a specific function which is defined as

$$\begin{aligned}\log L = &\sum_{i=1}^{n} \log\left(\tau^{1-\left(1-e^{-ax_i}\right)^b}\right) + \sum_{i=1}^{n} \log\left(1 - \log(\tau)\left(1-e^{-ax_i}\right)^b\right) + b\sum_{i=1}^{n} \log\left(1-e^{-ax_i}\right) \\ &- \sum_{i=1}^{n} \log\left(e^{ax_i} - 1\right) + n\log(ab).\end{aligned}$$

Next, we utilize Anderson-Darling estimation (ADE) [13] technique for estimating the GAPEED parameters. By considering an ordered sample of values, denoted as $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$, we minimize a certain function to derive the estimators, which is defined as

$$\begin{aligned}A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\Big[&\log(\tau)\left(1-\left(1-e^{-ax_{(i)}}\right)^b\right) + b\log\left(1-e^{-ax_{(i)}}\right) \\ &+ \log\left(1-\left(1-e^{-ax_{(i)}}\right)^b \tau^{1-\left(1-e^{-ax_{(i)}}\right)^b}\right)\Big].\end{aligned}$$

Similarly, we employ the right-tail Anderson-Darling estimation (RADE) [13] by minimizing a specific function, defined as

$$\begin{aligned}R = &\frac{n}{2} - 2\sum_{i=1}^{n}\left(1-e^{-ax_{(i)}}\right)^b \tau^{1-\left(1-e^{-ax_{(i)}}\right)^b} \\ &- \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1-\left(1-e^{-ax_{(i)}}\right)^b \tau^{1-\left(1-e^{-ax_{(i)}}\right)^b}\right).\end{aligned}$$

Additionally, the left-tailed Anderson-Darling estimation (LTADE) [47] is utilized to estimate the GAPEED parameters. This estimation method involves minimizing a particular function to obtain the estimators and is defined as

$$L = -\frac{3}{2}n + 2\sum_{i=1}^{n}\left(1-e^{-ax_{(i)}}\right)^b \tau^{1-\left(1-e^{-ax_{(i)}}\right)^b}$$

$$-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log(\tau)\left(1-(1-e^{-ax_{(i)}})^{b}\right)+b\log\left(1-e^{-ax_{(i)}}\right)\right].$$

Furthermore, we consider Cramér-von Mises estimation (CVME) [23], where the estimators are obtained by minimizing a specific function defined as

$$C=\frac{1}{12n}+\sum_{i=1}^{n}\left[\,(1-e^{-ax_{(i)}})^{b}\,\tau^{1-\left(1-e^{-ax_{(i)}}\right)^{b}}-\frac{2i-1}{2n}\right]^{2}.$$

Another estimation method employed is least-squares estimation (LSE) [58], which involves minimizing a certain function to derive the estimators. This function is defined as follows

$$V=\sum_{i=1}^{n}\left[(1-e^{-ax_{(i)}})^{b}\,\tau^{1-\left(1-e^{-ax_{(i)}}\right)^{b}}-\frac{i}{n+1}\right]^{2}.$$

Additionally, we employ weighted least-squares estimation (WLSE) [58] by minimizing a particular function, and it is defined as

$$W=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left[(1-e^{-ax_{(i)}})^{b}\,\tau^{1-\left(1-e^{-ax_{(i)}}\right)^{b}}-\frac{i}{n+1}\right]^{2}.$$

Lastly, the maximum product of spacing estimation (MPSE) [37] method is utilized, where the estimators are obtained by maximizing a specific function. This function is defined as

$$\Upsilon=\frac{1}{n+1}\sum_{i=1}^{n+1}\log(\vartheta i),$$

where

$$\vartheta i=(1-e^{-ax_{(i)}})^{b}\,\tau^{1-\left(1-e^{-ax_{(i)}}\right)^{b}}-(1-e^{-ax_{(i-1)}})^{b}\,\tau^{1-\left(1-e^{-ax_{(i-1)}}\right)^{b}}.$$

4.1. Confidence intervals

One frequently employed technique for creating confidence intervals (CIs) for parameters relies on the asymptotic normality of MLE and MPS. This method employs the Fisher information matrix, represented as $I(\boldsymbol{\theta})$, where $\boldsymbol{\theta}=(\tau,a,b)$, which is obtained from the second derivatives of the natural logarithm of the likelihood function or product spacing function, calculated at the estimated parameter values $\hat{\boldsymbol{\theta}}=(\hat{\tau},\hat{a},\hat{b})$. The asymptotic variance-covariance matrix of the parameter vector $\boldsymbol{\theta}$ can be articulated as follows:

$$I(\hat{\boldsymbol{\theta}})=-\begin{bmatrix}I_{\hat{a}\hat{a}} & & \\ I_{\hat{b}\hat{a}} & I_{\hat{b}\hat{b}} & \\ I_{\hat{\eta}\hat{a}} & I_{\hat{\eta}\hat{b}} & I_{\hat{\eta}\hat{\eta}},\end{bmatrix}. \tag{4.1}$$

The matrix representing the variances and covariances of the estimated parameters, identified as $V(\hat{\boldsymbol{\theta}})$, is determined by taking the inverse of the Fisher information matrix, denoted as $I^{-1}(\hat{\boldsymbol{\theta}})$. To create CIs for the parameter vector $\boldsymbol{\theta}$ based on the MLE's asymptotic normality, one can calculate a $100(1-\alpha)\%$ confidence interval for each parameter using the following procedure:

To calculate the CI for a use this formula: $\hat{a} \pm Z_{0.025} \sqrt{V_{\hat{a}\hat{a}}}$.
To calculate the CI for b use this formula: $\hat{b} \pm Z_{0.025} \sqrt{V_{\hat{b}\hat{b}}}$.
To calculate the CI for η use this formula: $\hat{\eta} \pm Z_{0.025} \sqrt{V_{\hat{a}\hat{a}}}$.

In this context, $Z_{0.025}$ denotes the critical value from the standard normal distribution's right tail, with a probability of $\frac{\alpha}{2}$. The values $V_{\hat{a}\hat{a}}$, $V_{\hat{b}\hat{b}}$, and $V_{\hat{\eta}\hat{\eta}}$ correspond to the diagonal components of the variance-covariance matrix $V(\hat{\boldsymbol{\theta}})$.

5. Numerical simulation

In our comprehensive simulation study, we investigate the performance of our proposed model using various sample sizes: $n = 35$, 70, 150, 300 and 600. To generate representative datasets, we employ the inversion of the CDF of our proposed model. For each sample size, we randomly generate datasets based on the following parameter values: $\boldsymbol{\theta} = (\tau, a, b) = \{(\tau = 0.5,\ a = 0.25,\ b = 0.75), (\tau = 1.5,\ a = 0.75,\ b = 0.5), (\tau = 2,\ a = 0.5,\ b = 1.5), (\tau = 2,\ a = 1.5,\ b = 2), (\tau = 0.75,\ a = 2,\ b = 3), (\tau = 0.25,\ a = 3,\ b = 0.25)\}$. This process is repeated five thousand of times. By varying the sample sizes and incorporating diverse parameter combinations, our simulation study aims to comprehensively evaluate the performance of the proposed model across different data scenarios.

To thoroughly examine the effectiveness of the considered estimation methods, we employ a range of measures that comprehensively evaluate their performance. These measures serve as valuable benchmarks in assessing the quality of the estimators and provide insights into their accuracy, efficiency, and robustness. The following measures are employed to assess the effectiveness of the estimation methods [20, 53, 60]:

- Average of bias:

$$Bias = \frac{1}{n} \sum_{m=1}^{n} |\widehat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}|,$$

  where L represents the number of iterations and $\widehat{\boldsymbol{\theta}}_i$ is the considered estimate for $\boldsymbol{\theta}$ at the m-th iteration sample.
- Mean squared error:

$$MSE = \frac{1}{n} \sum_{m=1}^{n} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2.$$

- Mean relative error:

$$MRE = \frac{1}{n} \sum_{m=1}^{n} \frac{|\widehat{\boldsymbol{\theta} - \boldsymbol{\theta}}|}{\boldsymbol{\theta}}.$$

- Average absolute difference:

$$D_{abs} = \frac{1}{nk} \sum_{m=1}^{k} \sum_{j=1}^{n} |F(x_{ij}; \boldsymbol{\theta}) - F(x_{ij}; \widehat{\boldsymbol{\theta}})|,$$

  where $F(x; \boldsymbol{\theta}) = F(x)$ and x_{ij} denotes the values obtained at the m-th iteration sample and j-th component of this sample.

- Maximum absolute difference

$$D_{max} = \frac{1}{n}\sum_{m=1}^{n} \max_{j=1,\ldots,n} |F(x_{ij};\boldsymbol{\theta}) - F(x_{ij};\widehat{\boldsymbol{\theta}})|.$$

- Average squared absolute error:

$$ASAE = \frac{1}{n}\sum_{m=1}^{n} \frac{|x_{(i)} - \hat{x}_{(i)}|}{x_{(L)} - x_{(1)}},$$

where $x_{(i)}$ are the ascending ordered observations. The results of simulating the proposed model parameters using different estimation techniques are presented in Tables 2–7. A graphical representation for some numerical values is presented in Figures 4 and 5. A comprehensive analysis of these tables reveals the following key observations:

First, it is important to note that all the parameter estimation methods for the proposed model demonstrate a high level of reliability, with estimated values that are very close to the actual values. This indicates the precision and accuracy of the estimation techniques employed in capturing the underlying characteristics of the proposed model.

Second, as the sample size n increases, each scenario's calculated measures exhibit a decreasing trend. This observation highlights the influence of sample size on the performance of the estimation methods. Larger sample sizes tend to lead to more precise and accurate parameter estimates. In CIs, Asymptotic CI (ACI) approaches are used for MLE and MPS. The length of ACIs can be denoted as LACI. The confidence level is 95%. Also, the coverage probability (CP) are obtained for MLE and MPS methods. See Tables 9 and 10.

Considering the results derived from the simulation analysis and the subsequent evaluations of rankings in Tables 2–8, we can identify several significant conclusions:

- The property of consistency among the estimators was observed in this investigation. This property signifies that, as the sample size, denoted as n, expands, the estimators tend to approach the true parameter values. This convergence not only reaffirms the robustness of these estimators but also underscores their appropriateness for various statistical inference purposes.
- As the sample size "n" increased, a noticeable trend in bias reduction was evident across all the estimating techniques under investigation. This observation highlights larger sample sizes' positive influence on the parameter estimation's precision and impartiality. This phenomenon can be attributed to the decreasing impact of random variations within more extensive datasets.
- With the expansion of the sample size "n", another significant observation pertaining to the consistent reduction is the MSE across all the estimators. The MSE is a comprehensive estimation performance indicator, encompassing bias and variance. The evident decline in the MSE underscores the enhancement in overall estimation accuracy with larger sample sizes.
- For the other measures ($MRE, D_{abs}, D_{max}, ASAE$), we can see that, as the sample size increases, all of these methods' values decrease.
- Based on the overall evaluation of the estimation strategies, the MLE technique emerges as the most effective method for estimating the parameters of the proposed model. As shown in Table 8, which presents the overall ranks for all estimation strategies, the MLE achieves the lowest total score of 67.0. This result further emphasizes the superiority of the MLE technique in accurately estimating the parameters of the proposed model in the context of this study.

Table 2. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max}, and ASAE for ($\tau = 0.5,\ a = 0.25,\ b = 0.75$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE
35	BIAS	$\hat{\tau}$	0.32671[1]	0.58371[5]	0.56143[2]	0.57133[3]	0.60648[6]	0.63277[8]	0.5738[4]	0.61215[7]
		$\hat{a}$	0.04966[1]	0.06013[4]	0.0625[6]	0.05958[3]	0.06715[8]	0.06343[7]	0.05815[2]	0.06028[5]
		$\hat{b}$	0.24844[1]	0.28129[4]	0.28515[5]	0.29203[6]	0.27907[2]	0.29342[7]	0.28051[3]	0.31435[8]
	MSE	$\hat{\tau}$	0.16299[1]	0.59664[5]	0.52583[2]	0.58252[4]	0.61607[6]	0.721[7]	0.55021[3]	0.93333[8]
		$\hat{a}$	0.00411[1]	0.00549[4]	0.00611[6]	0.0054[3]	0.00663[7.5]	0.00663[7.5]	0.00519[2]	0.00574[5]
		$\hat{b}$	0.10323[1]	0.11811[4]	0.12803[6]	0.11815[5]	0.1178[3]	0.14159[8]	0.11156[2]	0.14151[7]
	MRE	$\hat{\tau}$	0.65343[1]	1.16742[5]	1.12285[2]	1.14267[3]	1.21295[6]	1.26554[8]	1.1476[4]	1.22429[7]
		$\hat{a}$	0.19865[1]	0.24052[4]	0.25[6]	0.23833[3]	0.2686[8]	0.2537[7]	0.23258[2]	0.2411[5]
		$\hat{b}$	0.33126[1]	0.37506[4]	0.3802[5]	0.38937[6]	0.37209[2]	0.39123[7]	0.37402[3]	0.41913[8]
	D_{abs}		0.04372[2]	0.04281[1]	0.04653[7]	0.04411[3]	0.04606[5]	0.04624[6]	0.0448[4]	0.04683[8]
	D_{max}		0.07294[3]	0.07168[2]	0.07905[8]	0.07157[1]	0.0766[5]	0.07807[6]	0.07418[4]	0.07819[7]
	ASAE		0.02941[7]	0.02686[2]	0.02879[5]	0.02748[4]	0.02895[6]	0.02682[1]	0.02728[3]	0.03173[8]
	$\sum Ranks$		21[1]	44[3.5]	60[5]	44[3.5]	64.5[6]	79.5[7]	36[2]	83[8]
70	BIAS	$\hat{\tau}$	0.31314[1]	0.47069[3]	0.48998[5]	0.49062[6]	0.50913[7]	0.54111[8]	0.47654[4]	0.46785[2]
		$\hat{a}$	0.03421[1]	0.04143[2]	0.04746[6]	0.04299[3]	0.04804[7]	0.04809[8]	0.04356[4]	0.04532[5]
		$\hat{b}$	0.21631[1]	0.2507[5]	0.23911[2]	0.27064[7]	0.24898[3]	0.24985[4]	0.25229[6]	0.2829[8]
	MSE	$\hat{\tau}$	0.1496[1]	0.41058[4]	0.43118[5]	0.45507[7]	0.44542[6]	0.55159[8]	0.40366[3]	0.39192[2]
		$\hat{a}$	0.00191[1]	0.00286[2]	0.0034[6]	0.00317[4]	0.00368[8]	0.00366[7]	0.00308[3]	0.00328[5]
		$\hat{b}$	0.07529[1]	0.08986[4]	0.0849[2]	0.10267[7]	0.08698[3]	0.09562[6]	0.09[5]	0.11517[8]
	MRE	$\hat{\tau}$	0.62627[1]	0.94139[3]	0.97995[5]	0.98124[6]	1.01826[7]	1.08221[8]	0.95309[4]	0.93571[2]
		$\hat{a}$	0.13684[1]	0.16572[2]	0.18984[6]	0.17197[3]	0.19217[7]	0.19238[8]	0.17425[4]	0.18128[5]
		$\hat{b}$	0.28842[1]	0.33426[5]	0.31881[2]	0.36085[7]	0.33197[3]	0.33314[4]	0.33638[6]	0.3772[8]
	D_{abs}		0.03037[1]	0.03108[3]	0.03275[8]	0.03089[2]	0.03226[5]	0.03245[6]	0.03186[4]	0.03262[7]
	D_{max}		0.05103[2]	0.05227[3]	0.05581[8]	0.05055[1]	0.05432[5]	0.05561[7]	0.0531[4]	0.05469[6]
	ASAE		0.01852[7]	0.01764[3]	0.01828[5]	0.01771[4]	0.0183[6]	0.01677[1]	0.01726[2]	0.02027[8]
	$\sum Ranks$		19[1]	39[2]	60[5]	57[4]	67[7]	75[8]	49[3]	66[6]
150	BIAS	$\hat{\tau}$	0.27897[1]	0.33896[2]	0.4218[7]	0.37504[5]	0.40603[6]	0.43235[8]	0.36118[4]	0.33952[3]
		$\hat{a}$	0.02475[1]	0.02809[2]	0.03377[8]	0.02817[3]	0.03358[7]	0.03171[6]	0.0292[4]	0.03094[5]
		$\hat{b}$	0.17834[1]	0.19969[2]	0.22885[6]	0.23606[8]	0.21943[4]	0.23111[7]	0.20646[3]	0.22692[5]
	MSE	$\hat{\tau}$	0.12003[1]	0.21771[3]	0.32049[7]	0.26977[5]	0.2889[6]	0.35196[8]	0.2381[4]	0.18081[2]
		$\hat{a}$	0.00097[1]	0.00137[2]	0.00186[7]	0.00155[5]	0.00189[8]	0.00175[6]	0.00149[3]	0.00151[4]
		$\hat{b}$	0.05034[1]	0.05811[2]	0.07333[5]	0.08301[8]	0.06651[4]	0.07845[7]	0.06122[3]	0.07669[6]
	MRE	$\hat{\tau}$	0.55795[1]	0.67793[2]	0.84359[7]	0.75008[5]	0.81206[6]	0.86469[8]	0.72235[4]	0.67904[3]
		$\hat{a}$	0.09901[1]	0.11236[2]	0.1351[8]	0.11269[3]	0.13434[7]	0.12685[6]	0.11682[4]	0.12378[5]
		$\hat{b}$	0.23779[1]	0.26626[2]	0.30514[6]	0.31475[8]	0.29257[4]	0.30814[7]	0.27529[3]	0.30257[5]
	D_{abs}		0.02145[2]	0.02295[7]	0.0217[3]	0.02129[1]	0.02288[6]	0.023[8]	0.02213[4]	0.0225[5]
	D_{max}		0.03601[2]	0.03845[6]	0.03771[4]	0.03525[1]	0.03891[7]	0.03973[8]	0.03688[3]	0.03798[5]
	ASAE		0.011[5]	0.01062[3]	0.01139[6]	0.01092[4]	0.01146[7]	0.01039[1]	0.01045[2]	0.01269[8]
	$\sum Ranks$		18[1]	35[2]	74[7]	56[4.5]	72[6]	80[8]	41[3]	56[4.5]
300	BIAS	$\hat{\tau}$	0.20018[1]	0.243[4]	0.29528[6]	0.23781[3]	0.31369[7]	0.33876[8]	0.23778[2]	0.25695[5]
		$\hat{a}$	0.01707[1]	0.01972[4]	0.02215[6]	0.01893[3]	0.02216[7]	0.02177[5]	0.01829[2]	0.02228[8]
		$\hat{b}$	0.13506[1]	0.15636[3]	0.18002[6]	0.17262[5]	0.19427[7]	0.20028[8]	0.15561[2]	0.16985[4]
	MSE	$\hat{\tau}$	0.0643[1]	0.1019[4]	0.14664[6]	0.08922[2]	0.16416[7]	0.21518[8]	0.08995[3]	0.10932[5]
		$\hat{a}$	0.00047[1]	0.00066[4]	0.00082[6]	0.00056[2.5]	0.00088[8]	0.00087[7]	0.00056[2.5]	0.00081[5]
		$\hat{b}$	0.03228[1]	0.03756[3]	0.0464[4]	0.05353[7]	0.05263[6]	0.05765[8]	0.03636[2]	0.04878[5]
	MRE	$\hat{\tau}$	0.40037[1]	0.486[4]	0.59055[6]	0.47561[3]	0.62739[7]	0.67751[8]	0.47557[2]	0.5139[5]
		$\hat{a}$	0.06829[1]	0.07887[4]	0.08859[6]	0.0757[3]	0.08866[7]	0.0871[5]	0.07315[2]	0.08912[8]
		$\hat{b}$	0.18008[1]	0.20848[3]	0.24002[6]	0.23017[5]	0.25903[7]	0.26704[8]	0.20748[2]	0.22646[4]
	D_{abs}		0.01493[1]	0.01579[5]	0.0158[6]	0.0154[3]	0.01595[7]	0.01566[4]	0.01501[2]	0.01623[8]
	D_{max}		0.02495[1]	0.02657[4]	0.0273[6]	0.02576[3]	0.02745[7]	0.02722[5]	0.02546[2]	0.02772[8]
	ASAE		0.00711[5]	0.00685[2]	0.00726[6]	0.007[4]	0.00737[7]	0.0066[1]	0.00688[3]	0.008[8]
	$\sum Ranks$		16[1]	44[4]	70[5]	43.5[3]	84[8]	75[7]	26.5[2]	73[6]
600	BIAS	$\hat{\tau}$	0.14883[1]	0.18347[4]	0.22873[7]	0.16341[2]	0.2235[6]	0.23795[8]	0.17744[3]	0.18749[5]
		$\hat{a}$	0.01222[1]	0.01372[4]	0.01577[7]	0.01259[2]	0.01528[6]	0.01437[5]	0.01333[3]	0.01579[8]
		$\hat{b}$	0.09866[1]	0.12057[3]	0.14941[7]	0.12377[5]	0.14754[6]	0.15886[8]	0.11439[2]	0.12134[4]
	MSE	$\hat{\tau}$	0.03594[1]	0.05294[4]	0.07897[7]	0.04896[2]	0.07454[6]	0.08434[8]	0.04983[3]	0.05618[5]
		$\hat{a}$	0.00024[1]	$3e-04$[4]	0.00039[7]	0.00025[2]	0.00038[6]	0.00033[5]	0.00028[3]	$4e-04$[8]
		$\hat{b}$	0.01685[1]	0.02314[3]	0.03354[7]	0.03316[6]	0.03195[5]	0.03586[8]	0.02149[2]	0.02667[4]
	MRE	$\hat{\tau}$	0.29767[1]	0.36695[4]	0.45746[7]	0.32682[2]	0.447[6]	0.47591[8]	0.35489[3]	0.37498[5]
		$\hat{a}$	0.04889[1]	0.05487[4]	0.06308[7]	0.05037[2]	0.0611[6]	0.05747[5]	0.05332[3]	0.06316[8]
		$\hat{b}$	0.13154[1]	0.16077[3]	0.19922[7]	0.16503[5]	0.19672[6]	0.21182[8]	0.15252[2]	0.16179[4]
	D_{abs}		0.0111[4.5]	0.01086[2]	0.01153[8]	0.01074[1]	0.01151[7]	0.0111[4.5]	0.011[3]	0.01132[6]
	D_{max}		0.01861[3]	0.01858[2]	0.02[8]	0.01805[1]	0.01977[7]	0.01944[5]	0.01862[4]	0.01945[6]
	ASAE		0.00463[5]	0.00449[2]	0.00477[7]	0.00458[4]	0.00468[6]	0.00423[1]	0.00453[3]	0.0053[8]
	$\sum Ranks$		21.5[1]	42[4]	85[8]	34[2.5]	72[6]	72.5[7]	34[2.5]	71[5]

Table 3. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max}, and ASAE for ($\tau = 1.5,\ a = 0.75,\ b = 0.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE
35	BIAS	$\hat{\tau}$	0.52028[1]	0.68928[3]	0.70994[5]	0.69622[4]	0.75178[7]	0.67815[2]	0.72398[6]	1.13877[8]
		$\hat{a}$	0.31117[5]	0.30596[4]	0.34262[7]	0.29874[2]	0.32249[6]	0.29648[1]	0.30057[3]	0.40126[8]
		$\hat{b}$	0.10184[1]	0.12191[2]	0.12619[4]	0.12737[5]	0.14134[8]	0.12264[3]	0.13027[7]	0.12801[6]
	MSE	$\hat{\tau}$	0.39328[1]	0.62488[3]	0.63359[4]	0.64832[5]	0.70817[7]	0.59038[2]	0.6678[6]	5.81642[8]
		$\hat{a}$	0.19077[6]	0.16766[4]	0.21757[7]	0.13867[1]	0.1814[5]	0.15924[3]	0.14761[2]	0.28201[8]
		$\hat{b}$	0.01765[1]	0.02526[3]	0.02541[4]	0.02837[7]	0.03057[8]	0.02415[2]	0.02725[5]	0.02764[6]
	MRE	$\hat{\tau}$	0.34685[1]	0.45952[3]	0.47329[5]	0.46415[4]	0.50118[7]	0.4521[2]	0.48266[6]	0.75918[8]
		$\hat{a}$	0.41489[5]	0.40795[4]	0.45682[7]	0.39832[2]	0.42998[6]	0.39531[1]	0.40076[3]	0.53502[8]
		$\hat{b}$	0.20368[1]	0.24381[2]	0.25237[4]	0.25474[5]	0.28267[8]	0.24528[3]	0.26055[7]	0.25601[6]
	D_{abs}		0.04223[1]	0.04403[2]	0.04672[8]	0.04455[3]	0.04648[7]	0.04513[4]	0.04515[5]	0.04614[6]
	D_{max}		0.07079[1]	0.07367[3]	0.07922[8]	0.07196[2]	0.07766[6]	0.07539[5]	0.07491[4]	0.07795[7]
	ASAE		0.02942[7]	0.02673[4]	0.02904[5]	0.02425[1]	0.02924[6]	0.02505[2]	0.02572[3]	0.03359[8]
	$\sum Ranks$		31[2]	37[3]	68[6]	41[4]	81[7]	30[1]	57[5]	87[8]
70	BIAS	$\hat{\tau}$	0.44843[1]	0.55647[2]	0.60703[6]	0.59899[5]	0.61789[7]	0.59399[4]	0.5885[3]	0.81136[8]
		$\hat{a}$	0.21823[1]	0.23547[2]	0.28101[7]	0.24595[4]	0.27126[6]	0.24022[3]	0.25611[5]	0.34138[8]
		$\hat{b}$	0.07119[1]	0.07899[2]	0.09493[7]	0.09127[4]	0.09464[6]	0.09441[5]	0.08918[3]	0.09618[8]
	MSE	$\hat{\tau}$	0.29594[1]	0.41655[2]	0.47968[5]	0.50918[7]	0.49378[6]	0.47083[4]	0.46093[3]	1.11606[8]
		$\hat{a}$	0.08284[1]	0.08655[2]	0.13227[7]	0.08818[3]	0.12046[6]	0.09121[4]	0.10373[5]	0.1922[8]
		$\hat{b}$	0.00891[1]	0.01107[2]	0.01553[5.5]	0.01731[8]	0.01553[5.5]	0.01424[4]	0.01391[3]	0.01668[7]
	MRE	$\hat{\tau}$	0.29895[1]	0.37098[2]	0.40469[6]	0.39933[5]	0.41193[7]	0.39599[4]	0.39233[3]	0.5409[8]
		$\hat{a}$	0.29097[1]	0.31396[2]	0.37468[7]	0.32794[4]	0.36168[6]	0.32029[3]	0.34148[5]	0.45517[8]
		$\hat{b}$	0.14238[1]	0.15797[2]	0.18985[7]	0.18254[4]	0.18928[6]	0.18881[5]	0.17837[3]	0.19237[8]
	D_{abs}		0.03152[2.5]	0.03098[1]	0.03327[6]	0.03152[2.5]	0.03365[7]	0.03254[5]	0.0324[4]	0.03395[8]
	D_{max}		0.05225[3]	0.05176[1]	0.0565[6]	0.05216[2]	0.05677[7]	0.05497[5]	0.05425[4]	0.05832[8]
	ASAE		0.01684[5]	0.0164[4]	0.01827[7]	0.01516[2]	0.01819[6]	0.01497[1]	0.01597[3]	0.02063[8]
	$\sum Ranks$		19.5[1]	24[2]	76.5[7]	50.5[5]	75.5[6]	47[4]	44[3]	95[8]
150	BIAS	$\hat{\tau}$	0.35036[1]	0.41902[2]	0.48817[6]	0.48135[5]	0.50235[7]	0.47052[4]	0.45634[3]	0.61217[8]
		$\hat{a}$	0.15767[1]	0.18619[2]	0.21414[6]	0.20254[5]	0.22223[7]	0.18666[3]	0.18946[4]	0.26043[8]
		$\hat{b}$	0.04827[1]	0.05232[3]	0.06485[7]	0.05135[2]	0.06782[8]	0.06386[6]	0.05683[4]	0.06102[5]
	MSE	$\hat{\tau}$	0.18777[1]	0.25158[2]	0.32527[5]	0.352[7]	0.34253[6]	0.31557[4]	0.28732[3]	0.5778[8]
		$\hat{a}$	0.04089[1]	0.05247[3]	0.07072[6]	0.06092[5]	0.07443[7]	0.05186[2]	0.05349[4]	0.10477[8]
		$\hat{b}$	0.00404[1]	0.00479[2]	0.00774[7]	0.00548[3]	0.00827[8]	0.00696[6]	0.00551[4]	0.00665[5]
	MRE	$\hat{\tau}$	0.23357[1]	0.27934[2]	0.32544[6]	0.3209[5]	0.3349[7]	0.31368[4]	0.30422[3]	0.40811[8]
		$\hat{a}$	0.21023[1]	0.24825[2]	0.28552[6]	0.27005[5]	0.29631[7]	0.24888[3]	0.25261[4]	0.34723[8]
		$\hat{b}$	0.09655[1]	0.10464[3]	0.12969[7]	0.1027[2]	0.13565[8]	0.12772[6]	0.11365[4]	0.12204[5]
	D_{abs}		0.02102[1]	0.02184[3]	0.02231[5]	0.02208[4]	0.02368[8]	0.02251[6]	0.02177[2]	0.02282[7]
	D_{max}		0.03532[1]	0.03657[2]	0.03853[6]	0.0366[3]	0.04015[8]	0.03831[5]	0.03668[4]	0.03957[7]
	ASAE		0.00991[5]	0.0094[3]	0.0108[7]	0.00918[2]	0.01075[6]	0.00871[1]	0.00976[4]	0.01271[8]
	$\sum Ranks$		16[1]	29[2]	74[6]	48[4]	87[8]	50[5]	43[3]	85[7]
300	BIAS	$\hat{\tau}$	0.26655[1]	0.33434[3]	0.37325[6]	0.34711[4]	0.39467[7]	0.36499[5]	0.33138[2]	0.48508[8]
		$\hat{a}$	0.12193[1]	0.14744[4]	0.16883[6]	0.15015[5]	0.17776[7]	0.14515[2]	0.14696[3]	0.21805[8]
		$\hat{b}$	0.03505[1]	0.03905[4]	0.04169[5]	0.03599[2]	0.04485[7]	0.04629[8]	0.03769[3]	0.04173[6]
	MSE	$\hat{\tau}$	0.11494[1]	0.16891[3]	0.19839[4]	0.22068[7]	0.21592[6]	0.20154[5]	0.16587[2]	0.37566[8]
		$\hat{a}$	0.0244[1]	0.03273[4]	0.0429[6]	0.03661[5]	0.04653[7]	0.03141[2]	0.03249[3]	0.07079[8]
		$\hat{b}$	0.00192[1]	0.00236[4]	0.00295[6]	0.00211[2]	0.00333[7]	0.00349[8]	0.00226[3]	0.00285[5]
	MRE	$\hat{\tau}$	0.1777[1]	0.22289[3]	0.24883[6]	0.23141[4]	0.26311[7]	0.24333[5]	0.22092[2]	0.32339[8]
		$\hat{a}$	0.16257[1]	0.19658[4]	0.2251[6]	0.2002[5]	0.23701[7]	0.19353[2]	0.19595[3]	0.29074[8]
		$\hat{b}$	0.0701[1]	0.07811[4]	0.08338[5]	0.07198[2]	0.08971[7]	0.09259[8]	0.07539[3]	0.08345[6]
	D_{abs}		0.0149[1]	0.01559[4]	0.0158[5]	0.01556[3]	0.01608[7]	0.01621[8]	0.01505[2]	0.01595[6]
	D_{max}		0.02507[1]	0.02657[4]	0.02735[5]	0.02614[3]	0.02778[7]	0.02768[6]	0.02561[2]	0.02806[8]
	ASAE		0.00607[5]	0.00598[4]	0.0069[7]	0.00571[2]	0.00682[6]	0.00559[1]	0.00593[3]	0.00804[8]
	$\sum Ranks$		16[1]	45[4]	67[6]	44[3]	82[7]	60[5]	31[2]	87[8]
600	BIAS	$\hat{\tau}$	0.19544[1]	0.23541[3]	0.30543[6]	0.22954[2]	0.30719[7]	0.2498[5]	0.24212[4]	0.36224[8]
		$\hat{a}$	0.08322[1]	0.10415[4]	0.13813[6]	0.10194[2]	0.13953[7]	0.1021[3]	0.10419[5]	0.1712[8]
		$\hat{b}$	0.02563[2]	0.02682[3]	0.03267[8]	0.02544[1]	0.03225[7]	0.03151[6]	0.0275[4]	0.02908[5]
	MSE	$\hat{\tau}$	0.06188[1]	0.09047[2]	0.14058[7]	0.12175[5]	0.13924[6]	0.10305[4]	0.09293[3]	0.21849[8]
		$\hat{a}$	0.01115[1]	0.01707[3]	0.02836[6]	0.01967[5]	0.02862[7]	0.01637[2]	0.01746[4]	0.04409[8]
		$\hat{b}$	0.00105[2]	0.00116[3]	0.00174[8]	0.00103[1]	0.00162[7]	0.00159[6]	0.00118[4]	0.00135[5]
	MRE	$\hat{\tau}$	0.13029[1]	0.15694[3]	0.20362[6]	0.15302[2]	0.20479[7]	0.16653[5]	0.16141[4]	0.24149[8]
		$\hat{a}$	0.11096[1]	0.13886[4]	0.18417[6]	0.13592[2]	0.18604[7]	0.13613[3]	0.13892[5]	0.22827[8]
		$\hat{b}$	0.05126[2]	0.05365[3]	0.06534[8]	0.05088[1]	0.06449[7]	0.06302[6]	0.055[4]	0.05817[5]
	D_{abs}		0.01057[1]	0.01082[3]	0.01131[7]	0.0107[2]	0.01154[8]	0.01116[5]	0.01097[4]	0.01124[6]
	D_{max}		0.01792[1]	0.01849[3]	0.01965[6]	0.01801[2]	0.02001[8]	0.01931[5]	0.01869[4]	0.01979[7]
	ASAE		0.00367[3]	0.00378[5]	0.00448[7]	0.00364[2]	0.00445[6]	0.0035[1]	0.00376[4]	0.0053[8]
	$\sum Ranks$		17[1]	39[3]	81[6]	27[2]	84[7.5]	51[5]	49[4]	84[7.5]

Table 4. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max}, and ASAE for ($\tau = 2,\ a = 0.5,\ b = 1.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE
35	BIAS	$\hat{\tau}$	0.53211[2]	0.62026[4]	0.75084[7]	0.47271[1]	0.63274[6]	0.62076[5]	0.55091[3]	0.88774[8]
		$\hat{a}$	0.17954[7]	0.14419[3]	0.15786[5]	0.13798[1]	0.16003[6]	0.14116[2]	0.14743[4]	0.20721[8]
		$\hat{b}$	0.29362[3]	0.2843[2]	0.33553[5]	0.27643[1]	0.34292[7]	0.34139[6]	0.29688[4]	0.35737[8]
	MSE	$\hat{\tau}$	0.48119[1]	3.73487[7]	5.47953[8]	0.75293[2]	1.38498[5]	1.30859[4]	0.81897[3]	1.74615[6]
		$\hat{a}$	0.05597[7]	0.03354[3]	0.04117[5]	0.02854[1]	0.04153[6]	0.03226[2]	0.03661[4]	0.06468[8]
		$\hat{b}$	0.15312[3]	0.13705[2]	0.20378[7]	0.11466[1]	0.19627[6]	0.19503[5]	0.16026[4]	0.25206[8]
	MRE	$\hat{\tau}$	0.26605[2]	0.31013[4]	0.37542[7]	0.23636[1]	0.31637[6]	0.31038[5]	0.27546[3]	0.44387[8]
		$\hat{a}$	0.35909[7]	0.28838[3]	0.31573[5]	0.27597[1]	0.32006[6]	0.28232[2]	0.29486[4]	0.41442[8]
		$\hat{b}$	0.19574[3]	0.18953[2]	0.22369[5]	0.18428[1]	0.22862[7]	0.2276[6]	0.19792[4]	0.23825[8]
	D_{abs}		0.04307[1]	0.0449[4]	0.04669[7]	0.04388[2]	0.04491[5]	0.04571[6]	0.04461[3]	0.04673[8]
	D_{max}		0.07024[1]	0.07378[3]	0.07974[7]	0.07148[2]	0.07603[5]	0.07639[6]	0.07422[4]	0.08325[8]
	ASAE		0.03049[7]	0.02701[3]	0.02932[6]	0.02714[4]	0.02773[5]	0.0261[1]	0.02631[2]	0.03283[8]
	$\sum$ *Ranks*		44[4]	40[2]	74[7]	18[1]	70[6]	50[5]	42[3]	94[8]
70	BIAS	$\hat{\tau}$	0.51021[6]	0.46181[3]	0.55823[7]	0.32583[1]	0.50722[5]	0.4729[4]	0.44219[2]	0.72147[8]
		$\hat{a}$	0.14444[7]	0.11709[3]	0.12932[6]	0.10479[1]	0.12807[5]	0.10663[2]	0.11767[4]	0.16304[8]
		$\hat{b}$	0.21298[4]	0.19057[1]	0.22769[7]	0.19871[2]	0.22471[5]	0.22894[8]	0.20491[3]	0.22533[6]
	MSE	$\hat{\tau}$	0.42994[5]	0.35419[2]	0.59732[7]	0.22679[1]	0.51238[6]	0.42894[4]	0.35475[3]	0.92497[8]
		$\hat{a}$	0.03539[7]	0.02113[4]	0.02531[5]	0.01663[1]	0.02539[6]	0.01784[2]	0.02057[3]	0.0383[8]
		$\hat{b}$	0.07882[5]	0.06102[2]	0.08373[6]	0.05998[1]	0.07835[4]	0.08602[7]	0.07017[3]	0.08803[8]
	MRE	$\hat{\tau}$	0.25511[6]	0.2309[3]	0.27912[7]	0.16292[1]	0.25361[5]	0.23645[4]	0.2211[2]	0.36073[8]
		$\hat{a}$	0.28889[7]	0.23418[3]	0.25864[6]	0.20958[1]	0.25613[5]	0.21326[2]	0.23534[4]	0.32609[8]
		$\hat{b}$	0.14198[4]	0.12705[1]	0.15179[7]	0.13248[2]	0.14981[5]	0.15263[8]	0.13661[3]	0.15022[6]
	D_{abs}		0.03036[2]	0.03145[4]	0.03273[6]	0.02999[1]	0.03228[5]	0.03339[8]	0.03132[3]	0.03284[7]
	D_{max}		0.05009[2]	0.05184[3]	0.05627[7]	0.04922[1]	0.05469[5]	0.05545[6]	0.05225[4]	0.0576[8]
	ASAE		0.01841[6]	0.01732[3]	0.0189[7]	0.01759[4]	0.01822[5]	0.01671[1]	0.01717[2]	0.0212[8]
	$\sum$ *Ranks*		61[5.5]	32[2]	78[7]	17[1]	61[5.5]	56[4]	36[3]	91[8]
150	BIAS	$\hat{\tau}$	0.43313[5]	0.38216[3]	0.4568[6]	0.2421[1]	0.46034[7]	0.38646[4]	0.36914[2]	0.55156[8]
		$\hat{a}$	0.11342[7]	0.09493[4]	0.10673[5]	0.07841[1]	0.10869[6]	0.09064[2]	0.09329[3]	0.13012[8]
		$\hat{b}$	0.13544[3]	0.13508[2]	0.15218[6]	0.12937[1]	0.15998[8]	0.14971[5]	0.13655[4]	0.15561[7]
	MSE	$\hat{\tau}$	0.3042[5]	0.23781[3]	0.35945[6]	0.11844[1]	0.37455[7]	0.26349[4]	0.23662[2]	0.47541[8]
		$\hat{a}$	0.02075[7]	0.01375[4]	0.01716[5]	0.00947[1]	0.01836[6]	0.013[2]	0.01327[3]	0.02369[8]
		$\hat{b}$	0.03131[4]	0.02853[2]	0.03592[5]	0.02519[1]	0.03888[8]	0.03726[7]	0.03049[3]	0.03632[6]
	MRE	$\hat{\tau}$	0.21657[5]	0.19108[3]	0.2284[6]	0.12105[1]	0.23017[7]	0.19323[4]	0.18457[2]	0.27578[8]
		$\hat{a}$	0.22685[7]	0.18986[4]	0.21346[5]	0.15681[1]	0.21737[6]	0.18128[2]	0.18657[3]	0.26023[8]
		$\hat{b}$	0.0903[3]	0.09005[2]	0.10145[6]	0.08625[1]	0.10665[8]	0.09981[5]	0.09104[4]	0.10374[7]
	D_{abs}		0.02029[1]	0.02158[4]	0.02244[8]	0.02061[2]	0.02192[6]	0.02201[7]	0.0213[3]	0.02178[5]
	D_{max}		0.0336[1]	0.0357[4]	0.0381[8]	0.03368[2]	0.03725[6]	0.03676[5]	0.03557[3]	0.03761[7]
	ASAE		0.01084[5]	0.01034[2]	0.01173[7]	0.01071[4]	0.01128[6]	0.00996[1]	0.01055[3]	0.01309[8]
	$\sum$ *Ranks*		53[5]	37[3]	73[6]	17[1]	81[7]	48[4]	35[2]	88[8]
300	BIAS	$\hat{\tau}$	0.38726[6]	0.33359[2]	0.38992[7]	0.18992[1]	0.37128[5]	0.35983[4]	0.34388[3]	0.46204[8]
		$\hat{a}$	0.09738[7]	0.0794[2]	0.09391[6]	0.06178[1]	0.08945[5]	0.08286[3]	0.08352[4]	0.11042[8]
		$\hat{b}$	0.09272[1]	0.10093[4]	0.11688[7]	0.09282[2]	0.11978[8]	0.10638[5]	0.10064[3]	0.11367[6]
	MSE	$\hat{\tau}$	0.23046[6]	0.1782[2]	0.24446[7]	0.08426[1]	0.2303[5]	0.21335[4]	0.19028[3]	0.31359[8]
		$\hat{a}$	0.01471[7]	0.00985[2]	0.01326[6]	0.0063[1]	0.01233[5]	0.01072[4]	0.01051[3]	0.01694[8]
		$\hat{b}$	0.01347[2]	0.0154[3]	0.02142[7]	0.0127[1]	0.02166[8]	0.01781[5]	0.01588[4]	0.02005[6]
	MRE	$\hat{\tau}$	0.19363[6]	0.16679[2]	0.19496[7]	0.09496[1]	0.18564[5]	0.17992[4]	0.17194[3]	0.23102[8]
		$\hat{a}$	0.19476[7]	0.1588[2]	0.18781[6]	0.12356[1]	0.17891[5]	0.16573[3]	0.16703[4]	0.22084[8]
		$\hat{b}$	0.06181[1]	0.06728[4]	0.07792[7]	0.06188[2]	0.07985[8]	0.07092[5]	0.0671[3]	0.07578[6]
	D_{abs}		0.01458[2]	0.01466[3]	0.01637[8]	0.01442[1]	0.01585[5]	0.01586[6]	0.01526[4]	0.01617[7]
	D_{max}		0.02429[2]	0.02455[3]	0.02772[8]	0.0237[1]	0.02703[6]	0.02665[5]	0.0255[4]	0.02769[7]
	ASAE		0.00677[5]	0.0067[3]	0.00736[7]	0.00671[4]	0.00725[6]	0.00629[1]	0.00664[2]	0.00836[8]
	$\sum$ *Ranks*		52[5]	32[2]	83[7]	17[1]	71[6]	49[4]	40[3]	88[8]
600	BIAS	$\hat{\tau}$	0.33439[6]	0.29901[2]	0.33425[5]	0.12434[1]	0.34928[7]	0.307[3]	0.31014[4]	0.38531[8]
		$\hat{a}$	0.08166[6]	0.0727[3]	0.07959[5]	0.04277[1]	0.08482[7]	0.07253[2]	0.0737[4]	0.09209[8]
		$\hat{b}$	0.0686[2]	0.07273[4]	0.0851[7]	0.06697[1]	0.08853[8]	0.07975[5]	0.07184[3]	0.08318[6]
	MSE	$\hat{\tau}$	0.16634[5]	0.13955[2]	0.17655[6]	0.05129[1]	0.18951[7]	0.15596[4]	0.1493[3]	0.20575[8]
		$\hat{a}$	0.01011[6]	0.00794[2]	0.00965[5]	0.0035[1]	0.01076[7]	0.00839[4]	0.00828[3]	0.01148[8]
		$\hat{b}$	0.00734[2]	0.00804[3]	0.01118[7]	0.00685[1]	0.01207[8]	0.00953[5]	0.00809[4]	0.01058[6]
	MRE	$\hat{\tau}$	0.1672[6]	0.14951[2]	0.16712[5]	0.06217[1]	0.17464[7]	0.1535[3]	0.15507[4]	0.19266[8]
		$\hat{a}$	0.16331[6]	0.1454[3]	0.15918[5]	0.08554[1]	0.16963[7]	0.14506[2]	0.14741[4]	0.18418[8]
		$\hat{b}$	0.04573[2]	0.04849[4]	0.05674[7]	0.04465[1]	0.05902[8]	0.05317[5]	0.0479[3]	0.05545[6]
	D_{abs}		0.01052[1]	0.01059[3]	0.01098[5]	0.01057[2]	0.01117[7]	0.0113[8]	0.01068[4]	0.01106[6]
	D_{max}		0.01751[2]	0.01784[3]	0.01888[5]	0.01736[1]	0.01914[7]	0.01915[8]	0.01798[4]	0.01898[6]
	ASAE		0.00424[4]	0.0042[2]	0.00469[7]	0.00438[5]	0.00466[6]	0.00396[1]	0.00422[3]	0.00525[8]
	$\sum$ *Ranks*		48[4]	33[2]	69[6]	17[1]	86[7.5]	50[5]	43[3]	86[7.5]

Table 5. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max}, and ASAE for ($\tau = 2,\ a = 1.5,\ b = 2$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE
35	BIAS	$\hat{\tau}$	0.51619[2]	0.63113[6]	0.67423[7]	0.51172[1]	0.61334[4]	0.62651[5]	0.55707[3]	0.88577[8]
		$\hat{a}$	0.4211[6]	0.39524[5]	0.43673[7]	0.34427[1]	0.38438[3]	0.36625[2]	0.38901[4]	0.57667[8]
		$\hat{b}$	0.42122[2]	0.42702[3]	0.52173[8]	0.40397[1]	0.48285[6]	0.45722[5]	0.44181[4]	0.4914[7]
	MSE	$\hat{\tau}$	0.44269[1]	1.40915[6]	0.97253[5]	0.79973[4]	0.78856[3]	4.65109[8]	0.70986[2]	1.63292[7]
		$\hat{a}$	0.28044[6]	0.24296[5]	0.31317[7]	0.18465[1]	0.23352[4]	0.21626[2]	0.23222[3]	0.50968[8]
		$\hat{b}$	0.32216[2]	0.33661[4]	0.5635[8]	0.25378[1]	0.4679[6]	0.37052[5]	0.3353[3]	0.47121[7]
	MRE	$\hat{\tau}$	0.2581[2]	0.31556[6]	0.33711[7]	0.25586[1]	0.30667[4]	0.31326[5]	0.27853[3]	0.44288[8]
		$\hat{a}$	0.28073[6]	0.26349[5]	0.29115[7]	0.22951[1]	0.25625[3]	0.24417[2]	0.25934[4]	0.38445[8]
		$\hat{b}$	0.21061[2]	0.21351[3]	0.26087[8]	0.20198[1]	0.24143[6]	0.22861[5]	0.2209[4]	0.2457[7]
	D_{abs}		0.04173[1]	0.04563[4]	0.04698[8]	0.04308[2]	0.0467[7]	0.04585[5]	0.04476[3]	0.04629[6]
	D_{max}		0.06876[1]	0.07601[4]	0.08059[7]	0.07069[2]	0.07795[6]	0.07656[5]	0.07406[3]	0.08173[8]
	ASAE		0.03095[7]	0.02759[4]	0.02937[6]	0.02711[3]	0.02798[5]	0.02651[1]	0.02653[2]	0.03252[8]
	$\sum Ranks$		38[2.5]	55[5]	85[7]	19[1]	57[6]	50[4]	38[2.5]	90[8]
70	BIAS	$\hat{\tau}$	0.47926[5]	0.46754[4]	0.5632[7]	0.3673[1]	0.55214[6]	0.46172[2]	0.46383[3]	0.67025[8]
		$\hat{a}$	0.36435[7]	0.30602[3]	0.3521[6]	0.27605[1]	0.33303[5]	0.29119[2]	0.31509[4]	0.43801[8]
		$\hat{b}$	0.29223[3]	0.29115[2]	0.3284[7]	0.28673[1]	0.33862[8]	0.31389[5]	0.30284[4]	0.31591[6]
	MSE	$\hat{\tau}$	0.37454[3]	0.36448[2]	0.56211[7]	0.2657[1]	0.53431[6]	0.39635[5]	0.37914[4]	0.72289[8]
		$\hat{a}$	0.20482[7]	0.14296[3]	0.18606[6]	0.11389[1]	0.17027[5]	0.1266[2]	0.15246[4]	0.27299[8]
		$\hat{b}$	0.14415[3]	0.13708[2]	0.17917[7]	0.13026[1]	0.19342[8]	0.16635[5]	0.15758[4]	0.1689[6]
	MRE	$\hat{\tau}$	0.23963[5]	0.23377[4]	0.2816[7]	0.18365[1]	0.27607[6]	0.23086[2]	0.23191[3]	0.33512[8]
		$\hat{a}$	0.2429[7]	0.20401[3]	0.23473[6]	0.18403[1]	0.22202[5]	0.19413[2]	0.21006[4]	0.29201[8]
		$\hat{b}$	0.14612[3]	0.14557[2]	0.1642[7]	0.14337[1]	0.16931[8]	0.15694[5]	0.15142[4]	0.15796[6]
	D_{abs}		0.03043[1]	0.03146[3]	0.03257[5]	0.03117[2]	0.03291[7]	0.03307[8]	0.03149[4]	0.03263[6]
	D_{max}		0.04996[1]	0.05229[4]	0.0556[7]	0.05107[2]	0.05559[6]	0.05467[5]	0.05214[3]	0.05719[8]
	ASAE		0.01855[6]	0.01739[3]	0.01871[7]	0.01758[4]	0.01813[5]	0.01678[1]	0.01729[2]	0.02109[8]
	$\sum Ranks$		51[5]	35[2]	79[7]	17[1]	75[6]	44[4]	43[3]	88[8]
150	BIAS	$\hat{\tau}$	0.4392[5]	0.39379[3]	0.45566[7]	0.27549[1]	0.44834[6]	0.38128[2]	0.41046[4]	0.54089[8]
		$\hat{a}$	0.31014[7]	0.26139[3]	0.28645[5]	0.21856[1]	0.28762[6]	0.24802[2]	0.26741[4]	0.3628[8]
		$\hat{b}$	0.19985[3]	0.19591[1]	0.21751[6]	0.19621[2]	0.22876[8]	0.21446[5]	0.20679[4]	0.2256[7]
	MSE	$\hat{\tau}$	0.31057[5]	0.2424[2]	0.33171[7]	0.13533[1]	0.32597[6]	0.25702[3]	0.26429[4]	0.43436[8]
		$\hat{a}$	0.15063[7]	0.10087[3]	0.12317[6]	0.07577[1]	0.12274[5]	0.09392[2]	0.10589[4]	0.18107[8]
		$\hat{b}$	0.06714[4]	0.05805[1]	0.07483[6]	0.05816[2]	0.08113[7]	0.07187[5]	0.06713[3]	0.08227[8]
	MRE	$\hat{\tau}$	0.2196[5]	0.19689[3]	0.22783[7]	0.13775[1]	0.22417[6]	0.19064[2]	0.20523[4]	0.27044[8]
		$\hat{a}$	0.20676[7]	0.17426[3]	0.19097[5]	0.14571[1]	0.19175[6]	0.16535[2]	0.17827[4]	0.24186[8]
		$\hat{b}$	0.09993[3]	0.09795[1]	0.10875[6]	0.09811[2]	0.11438[8]	0.10723[5]	0.10339[4]	0.1128[7]
	D_{abs}		0.02083[1]	0.02159[4]	0.02178[5]	0.02125[2.5]	0.02218[6]	0.02294[8]	0.02125[2.5]	0.02263[7]
	D_{max}		0.03432[1]	0.03586[4]	0.03724[5]	0.03487[2]	0.03795[6]	0.03803[7]	0.03566[3]	0.03867[8]
	ASAE		0.0109[5]	0.01075[3.5]	0.01143[7]	0.01075[3.5]	0.0111[6]	0.01011[1]	0.01051[2]	0.01296[8]
	$\sum Ranks$		53[5]	31.5[2]	72[6]	20[1]	76[7]	44[4]	42.5[3]	93[8]
300	BIAS	$\hat{\tau}$	0.38358[5]	0.35025[3]	0.40996[7]	0.20918[1]	0.38933[6]	0.35514[4]	0.34544[2]	0.47301[8]
		$\hat{a}$	0.25949[7]	0.22513[3]	0.25895[6]	0.15794[1]	0.25286[5]	0.2244[2]	0.23092[4]	0.31424[8]
		$\hat{b}$	0.14384[3]	0.14219[2]	0.16653[7]	0.13428[1]	0.16744[8]	0.15275[5]	0.14455[4]	0.16515[6]
	MSE	$\hat{\tau}$	0.22395[5]	0.18899[3]	0.25726[7]	0.08634[1]	0.23811[6]	0.21275[4]	0.18524[2]	0.31153[8]
		$\hat{a}$	0.10146[7]	0.07569[2]	0.0982[6]	0.04298[1]	0.09473[5]	0.07846[3]	0.07943[4]	0.13087[8]
		$\hat{b}$	0.03255[4]	0.0313[2]	0.04249[7]	0.02688[1]	0.04388[8]	0.03491[5]	0.03196[3]	0.04248[6]
	MRE	$\hat{\tau}$	0.19179[5]	0.17513[3]	0.20498[7]	0.10459[1]	0.19466[6]	0.17757[4]	0.17272[2]	0.2365[8]
		$\hat{a}$	0.17299[7]	0.15009[3]	0.17264[6]	0.10529[1]	0.16858[5]	0.1496[2]	0.15394[4]	0.20949[8]
		$\hat{b}$	0.07192[3]	0.07109[2]	0.08327[7]	0.06714[1]	0.08372[8]	0.07638[5]	0.07227[4]	0.08258[6]
	D_{abs}		0.01502[2]	0.01519[4]	0.01582[6]	0.01471[1]	0.01536[5]	0.01593[7]	0.01513[3]	0.01595[8]
	D_{max}		0.02487[2]	0.02543[4]	0.02719[7]	0.02418[1]	0.02652[5]	0.02678[6]	0.02539[3]	0.02738[8]
	ASAE		0.00687[5]	0.00666[2]	0.00739[7]	0.00686[4]	0.00722[6]	0.00639[1]	0.0067[3]	0.00837[8]
	$\sum Ranks$		55[5]	33[2]	80[7]	15[1]	73[6]	48[4]	38[3]	90[8]
600	BIAS	$\hat{\tau}$	0.34337[5]	0.29599[2]	0.35608[7]	0.12837[1]	0.34858[6]	0.30958[4]	0.30911[3]	0.4042[8]
		$\hat{a}$	0.22922[7]	0.19345[3]	0.22519[5]	0.09817[1]	0.22778[6]	0.19235[2]	0.20261[4]	0.27243[8]
		$\hat{b}$	0.10161[2]	0.10663[3]	0.12649[8]	0.09413[1]	0.12277[6]	0.11287[5]	0.10742[4]	0.12442[7]
	MSE	$\hat{\tau}$	0.17598[6]	0.13529[2]	0.18281[7]	0.04448[1]	0.17429[5]	0.15427[4]	0.14265[3]	0.22018[8]
		$\hat{a}$	0.07869[7]	0.05647[2]	0.07356[6]	0.02176[1]	0.07329[5]	0.05829[3]	0.06038[4]	0.09527[8]
		$\hat{b}$	0.01627[2]	0.01766[3]	0.02538[8]	0.01352[1]	0.02319[6]	0.01936[5]	0.01824[4]	0.02403[7]
	MRE	$\hat{\tau}$	0.17169[5]	0.14799[2]	0.17804[7]	0.06418[1]	0.17429[6]	0.15479[4]	0.15455[3]	0.2021[8]
		$\hat{a}$	0.15282[7]	0.12897[3]	0.15012[5]	0.06545[1]	0.15185[6]	0.12823[2]	0.13507[4]	0.18162[8]
		$\hat{b}$	0.0508[2]	0.05331[3]	0.06325[8]	0.04706[1]	0.06139[6]	0.05644[5]	0.05371[4]	0.06221[7]
	D_{abs}		0.01056[2]	0.0106[3]	0.01125[7]	0.0103[1]	0.01164[8]	0.01124[6]	0.0107[4]	0.01089[5]
	D_{max}		0.01782[2]	0.01801[3]	0.01931[7]	0.017[1]	0.01973[8]	0.01893[6]	0.01812[4]	0.01874[5]
	ASAE		0.00429[2]	0.0043[3]	0.00478[7]	0.0045[5]	0.00457[6]	0.00406[1]	0.00431[4]	0.00534[8]
	$\sum Ranks$		49[5]	32[2]	82[7]	16[1]	74[6]	47[4]	45[3]	87[8]

Table 6. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max}, and ASAE for ($\tau = 0.75,\ a = 2,\ b = 3$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE
35	BIAS	$\hat{\tau}$	0.41263[1]	0.63117[3]	0.61646[2]	0.68971[6]	0.66123[4]	0.69605[7]	0.6614[5]	0.73592[8]
		$\hat{a}$	0.3458[1]	0.40339[3]	0.40077[2]	0.43258[6]	0.43443[7]	0.46564[8]	0.41928[5]	0.40721[4]
		$\hat{b}$	0.90474[2]	0.95946[6]	0.95309[4]	0.95489[5]	0.92977[3]	0.88[1]	0.96175[7]	1.06824[8]
	MSE	$\hat{\tau}$	0.23196[1]	0.53768[3]	0.47197[2]	0.65139[7]	0.55116[4]	0.64812[6]	0.57172[5]	0.87411[8]
		$\hat{a}$	0.19495[1]	0.25654[3]	0.25032[2]	0.28251[5]	0.28292[6]	0.33262[8]	0.27497[4]	0.28509[7]
		$\hat{b}$	1.40385[7]	1.37779[5]	1.33026[3]	1.37841[6]	1.24057[1]	1.28937[2]	1.33079[4]	1.7756[8]
	MRE	$\hat{\tau}$	0.55017[1]	0.84156[3]	0.82195[2]	0.91961[6]	0.88164[4]	0.92807[7]	0.88187[5]	0.98123[8]
		$\hat{a}$	0.1729[1]	0.20169[3]	0.20039[2]	0.21629[6]	0.21721[7]	0.23282[8]	0.20964[5]	0.20361[4]
		$\hat{b}$	0.30158[2]	0.31982[6]	0.3177[4]	0.3183[5]	0.30992[3]	0.29333[1]	0.32058[7]	0.35608[8]
	D_{abs}		0.04253[1]	0.04336[2]	0.04718[8]	0.04338[3]	0.04551[5]	0.04671[7]	0.04479[4]	0.04664[6]
	D_{max}		0.07112[2]	0.07194[3]	0.0792[8]	0.07007[1]	0.07453[5]	0.07777[6]	0.07355[4]	0.07787[7]
	ASAE		0.02959[7]	0.02756[3]	0.02928[6]	0.02713[2]	0.02829[5]	0.02772[4]	0.02691[1]	0.03129[8]
	$\sum Ranks$		27[1]	43[2]	45[3]	58[6]	54[4]	65[7]	56[5]	84[8]
70	BIAS	$\hat{\tau}$	0.37728[1]	0.57428[3]	0.57379[2]	0.61526[7]	0.60356[5]	0.60726[6]	0.58819[4]	0.61897[8]
		$\hat{a}$	0.25016[1]	0.31949[2]	0.33757[4]	0.35113[7]	0.34537[6]	0.35421[8]	0.33813[5]	0.3285[3]
		$\hat{b}$	0.68242[1]	0.79998[5]	0.77034[3]	0.82325[6]	0.83417[8]	0.69139[2]	0.78972[4]	0.82557[7]
	MSE	$\hat{\tau}$	0.20498[1]	0.47839[4]	0.45179[2]	0.57434[7]	0.47675[3]	0.53826[6]	0.48684[5]	0.5989[8]
		$\hat{a}$	0.09985[1]	0.15258[2]	0.17907[4]	0.20305[8]	0.18005[5]	0.18882[7]	0.17375[3]	0.18104[6]
		$\hat{b}$	0.77461[2]	1.01864[6]	0.8902[3]	1.13061[8]	1.01306[5]	0.74376[1]	0.95657[4]	1.08045[7]
	MRE	$\hat{\tau}$	0.50303[1]	0.76571[3]	0.76505[2]	0.82035[7]	0.80474[5]	0.80968[6]	0.78426[4]	0.8253[8]
		$\hat{a}$	0.12508[1]	0.15974[2]	0.16879[4]	0.17557[7]	0.17268[6]	0.1771[8]	0.16906[5]	0.16425[3]
		$\hat{b}$	0.22747[1]	0.26666[5]	0.25678[3]	0.27442[6]	0.27806[8]	0.23046[2]	0.26324[4]	0.27519[7]
	D_{abs}		0.03062[2]	0.03064[3]	0.03404[8]	0.03001[1]	0.03304[7]	0.03289[6]	0.03251[4]	0.03252[5]
	D_{max}		0.05131[2]	0.05151[3]	0.05754[8]	0.04967[1]	0.05555[7]	0.05553[6]	0.05366[4]	0.0546[5]
	ASAE		0.01854[7]	0.01731[4]	0.01828[6]	0.01722[2]	0.01814[5]	0.01725[3]	0.01692[1]	0.01936[8]
	$\sum Ranks$		21[1]	42[2]	49[4]	67[6]	70[7]	61[5]	47[3]	75[8]
150	BIAS	$\hat{\tau}$	0.31212[1]	0.45159[2]	0.49391[4]	0.50158[6]	0.49767[5]	0.51173[8]	0.47926[3]	0.50467[7]
		$\hat{a}$	0.18389[1]	0.24619[2]	0.26366[5]	0.26623[6]	0.27263[8]	0.26155[3]	0.26205[4]	0.26637[7]
		$\hat{b}$	0.51055[1]	0.58914[2]	0.64746[7]	0.59531[4]	0.65501[8]	0.62579[5]	0.59382[3]	0.63847[6]
	MSE	$\hat{\tau}$	0.14708[1]	0.33233[2]	0.35351[3]	0.44757[8]	0.36746[5]	0.41305[7]	0.36255[4]	0.4025[6]
		$\hat{a}$	0.05157[1]	0.09975[2]	0.10789[3]	0.12791[8]	0.11535[6]	0.1123[5]	0.11096[4]	0.1174[7]
		$\hat{b}$	0.47277[1]	0.61572[4]	0.6529[5]	0.7086[8]	0.67921[6]	0.59584[2]	0.60046[3]	0.70453[7]
	MRE	$\hat{\tau}$	0.41616[1]	0.60212[2]	0.65854[4]	0.66877[6]	0.66356[5]	0.68231[8]	0.63901[3]	0.67289[7]
		$\hat{a}$	0.09195[1]	0.1231[2]	0.13183[5]	0.13311[6]	0.13632[8]	0.13078[3]	0.13102[4]	0.13318[7]
		$\hat{b}$	0.17018[1]	0.19638[2]	0.21582[7]	0.19844[4]	0.21834[8]	0.2086[5]	0.19794[3]	0.21282[6]
	D_{abs}		0.02081[1]	0.02156[3]	0.02279[7.5]	0.02171[4]	0.02269[6]	0.02221[5]	0.02123[2]	0.02279[7.5]
	D_{max}		0.03496[1]	0.0362[4]	0.0389[8]	0.03607[3]	0.03834[5]	0.03836[6]	0.03583[2]	0.03862[7]
	ASAE		0.01105[5]	0.0105[3]	0.0111[7]	0.01079[4]	0.01108[6]	0.01049[2]	0.01039[1]	0.01196[8]
	$\sum Ranks$		16[1]	30[2]	65.5[5]	67[6]	76[7]	59[4]	36[3]	82.5[8]
300	BIAS	$\hat{\tau}$	0.26159[1]	0.33734[2]	0.40449[6]	0.3744[4]	0.41347[7]	0.42097[8]	0.34325[3]	0.37827[5]
		$\hat{a}$	0.14993[1]	0.18532[3]	0.20625[7]	0.19513[4]	0.20695[8]	0.19719[5]	0.18169[2]	0.20501[6]
		$\hat{b}$	0.37223[1]	0.41779[2]	0.508[7]	0.44601[4]	0.50594[6]	0.52145[8]	0.42076[3]	0.45303[5]
	MSE	$\hat{\tau}$	0.10953[1]	0.20094[2]	0.26331[5]	0.29301[7]	0.28568[6]	0.30513[8]	0.20537[3]	0.24655[4]
		$\hat{a}$	0.03556[1]	0.05977[3]	0.06975[4]	0.07768[8]	0.07525[7]	0.07119[5]	0.05699[2]	0.07212[6]
		$\hat{b}$	0.29126[1]	0.31942[2]	0.43464[6]	0.48268[8]	0.44893[7]	0.43375[5]	0.32104[3]	0.39031[4]
	MRE	$\hat{\tau}$	0.34879[1]	0.44978[2]	0.53932[6]	0.49921[4]	0.5513[7]	0.56129[8]	0.45767[3]	0.50436[5]
		$\hat{a}$	0.07496[1]	0.09266[3]	0.10313[7]	0.09757[4]	0.10347[8]	0.09859[5]	0.09085[2]	0.1025[6]
		$\hat{b}$	0.12408[1]	0.13926[2]	0.16933[7]	0.14867[4]	0.16865[6]	0.17382[8]	0.14025[3]	0.15101[5]
	D_{abs}		0.01478[1]	0.01563[4]	0.01625[8]	0.01541[3]	0.01591[6]	0.01623[7]	0.01493[2]	0.01584[5]
	D_{max}		0.02519[1]	0.02694[4]	0.02821[8]	0.02592[3]	0.02751[6]	0.02814[7]	0.02552[2]	0.0274[5]
	ASAE		0.00695[4]	0.00682[3]	0.00721[7]	0.00697[5]	0.00704[6]	0.00671[1]	0.0068[2]	0.00775[8]
	$\sum Ranks$		15[1]	32[3]	78[7]	58[4]	80[8]	75[6]	30[2]	64[5]
600	BIAS	$\hat{\tau}$	0.19336[1]	0.23349[2]	0.28978[6]	0.23372[3]	0.30777[7]	0.30949[8]	0.23507[4]	0.2662[5]
		$\hat{a}$	0.10937[1]	0.12304[2]	0.14902[6]	0.12842[4]	0.15724[8]	0.13825[5]	0.12621[3]	0.15464[7]
		$\hat{b}$	0.26801[1]	0.30194[4]	0.37263[6]	0.27088[2]	0.38556[7]	0.42662[8]	0.29441[3]	0.30703[5]
	MSE	$\hat{\tau}$	0.06126[1]	0.10127[2]	0.15044[6]	0.13095[4]	0.16716[8]	0.16478[7]	0.10547[3]	0.13164[5]
		$\hat{a}$	0.0189[1]	0.02821[2]	0.04017[6]	0.03537[4]	0.04408[8]	0.03582[5]	0.02981[3]	0.04188[7]
		$\hat{b}$	0.12688[1]	0.15549[2]	0.25123[6]	0.18801[4]	0.28486[7]	0.29794[8]	0.15848[3]	0.18934[5]
	MRE	$\hat{\tau}$	0.25781[1]	0.31131[2]	0.38638[6]	0.31162[3]	0.41036[7]	0.41265[8]	0.31343[4]	0.35494[5]
		$\hat{a}$	0.05468[1]	0.06152[2]	0.07451[6]	0.06421[4]	0.07862[8]	0.06912[5]	0.06311[3]	0.07732[7]
		$\hat{b}$	0.08934[1]	0.10065[4]	0.12421[6]	0.09029[2]	0.12852[7]	0.14221[8]	0.09814[3]	0.10234[5]
	D_{abs}		0.01062[2]	0.01055[1]	0.01157[7]	0.01098[3]	0.01158[8]	0.01113[5]	0.01112[4]	0.0113[6]
	D_{max}		0.0181[1]	0.01823[2]	0.02019[8]	0.01873[3]	0.0201[7]	0.01976[6]	0.01902[4]	0.01972[5]
	ASAE		0.00457[5]	0.00443[3]	0.00471[6]	0.00456[4]	0.00473[7]	0.0044[2]	0.00435[1]	0.00516[8]
	$\sum Ranks$		17[1]	28[2]	75[6.5]	40[4]	89[8]	75[6.5]	38[3]	70[5]

Table 7. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max}, and ASAE for ($\tau = 0.25,\ a = 3,\ b = 0.25$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE
35	BIAS	$\hat{\tau}$	0.28261[2]	0.4647[5]	0.48379[7]	0.41025[3]	0.42315[4]	0.51017[8]	0.47272[6]	0.26393[1]
		$\hat{a}$	0.70395[1]	0.78929[3]	0.91784[8]	0.77174[2]	0.90659[7]	0.82398[4]	0.83444[5]	0.84481[6]
		$\hat{b}$	0.11892[2]	0.13452[6]	0.13845[7]	0.12898[4]	0.11407[1]	0.14639[8]	0.13051[5]	0.12794[3]
	MSE	$\hat{\tau}$	0.14259[1]	0.54086[5]	0.58623[6]	0.45928[3]	0.52049[4]	0.67661[8]	0.60141[7]	0.18215[2]
		$\hat{a}$	0.92929[2]	0.99608[3]	1.35102[8]	0.89166[1]	1.27505[7]	1.08694[4]	1.10922[5]	1.17679[6]
		$\hat{b}$	0.02632[3]	0.03369[6]	0.03609[7]	0.02757[4]	0.02515[2]	0.03881[8]	0.03305[5]	0.02449[1]
	MRE	$\hat{\tau}$	1.13045[2]	1.85879[5]	1.93515[7]	1.64101[3]	1.6926[4]	2.04067[8]	1.8909[6]	1.05573[1]
		$\hat{a}$	0.23465[1]	0.2631[3]	0.30595[8]	0.25725[2]	0.3022[7]	0.27466[4]	0.27815[5]	0.2816[6]
		$\hat{b}$	0.47569[2]	0.53808[6]	0.5538[7]	0.5159[4]	0.45626[1]	0.58556[8]	0.52205[5]	0.51177[3]
	D_{abs}		0.04268[1]	0.04508[3]	0.04693[8]	0.04333[2]	0.04525[4]	0.04586[6]	0.0455[5]	0.04675[7]
	D_{max}		0.0706[1]	0.07457[3]	0.07976[8]	0.0712[2]	0.07566[5]	0.07738[7]	0.07522[4]	0.07734[6]
	ASAE		0.02998[6]	0.02782[4]	0.02947[5]	0.02765[3]	0.03091[7]	0.02581[1]	0.02751[2]	0.03566[8]
	$\sum Ranks$		24[1]	52[4]	86[8]	33[2]	53[5]	74[7]	60[6]	50[3]
70	BIAS	$\hat{\tau}$	0.26535[2]	0.35171[5]	0.40366[7]	0.29134[3]	0.35289[6]	0.42263[8]	0.34207[4]	0.24784[1]
		$\hat{a}$	0.47336[1]	0.55386[3]	0.64736[7]	0.55627[4]	0.64889[8]	0.61187[6]	0.54143[2]	0.59785[5]
		$\hat{b}$	0.09971[1]	0.11155[5]	0.11542[6]	0.10741[3]	0.10532[2]	0.12794[8]	0.10844[4]	0.11607[7]
	MSE	$\hat{\tau}$	0.12711[1]	0.28471[4]	0.4177[7]	0.22138[3]	0.33996[6]	0.48266[8]	0.30172[5]	0.13702[2]
		$\hat{a}$	0.36529[1]	0.48714[4]	0.68965[8]	0.48295[2]	0.66234[7]	0.6146[6]	0.48359[3]	0.58995[5]
		$\hat{b}$	0.01575[1]	0.02146[5]	0.02598[7]	0.01737[2]	0.02098[4]	0.03058[8]	0.02191[6]	0.01909[3]
	MRE	$\hat{\tau}$	1.06141[2]	1.40685[5]	1.61465[7]	1.16535[3]	1.41156[6]	1.6905[8]	1.3683[4]	0.99135[1]
		$\hat{a}$	0.15779[1]	0.18462[3]	0.21579[7]	0.18542[4]	0.2163[8]	0.20396[6]	0.18048[2]	0.19928[5]
		$\hat{b}$	0.39883[1]	0.44619[5]	0.46169[6]	0.42965[3]	0.42127[2]	0.51178[8]	0.43376[4]	0.46429[7]
	D_{abs}		0.02997[1]	0.03175[4]	0.03324[8]	0.03081[2]	0.03247[5]	0.0327[7]	0.03127[3]	0.03251[6]
	D_{max}		0.0499[1]	0.05326[4]	0.05658[8]	0.05081[2]	0.05486[6]	0.05572[7]	0.05218[3]	0.05438[5]
	ASAE		0.01808[5]	0.0179[4]	0.01884[6]	0.01751[3]	0.0192[7]	0.01618[1]	0.01733[2]	0.02197[8]
	$\sum Ranks$		18[1]	51[4]	84[8]	34[2]	67[6]	81[7]	42[3]	55[5]
150	BIAS	$\hat{\tau}$	0.20572[2]	0.23878[4]	0.31697[8]	0.216[3]	0.29435[7]	0.28901[6]	0.25867[5]	0.20305[1]
		$\hat{a}$	0.30956[1]	0.34668[2]	0.42934[7]	0.34894[3]	0.4327[8]	0.39302[5]	0.36418[4]	0.41109[6]
		$\hat{b}$	0.07839[1]	0.08716[2]	0.09845[8]	0.08897[3]	0.09366[5]	0.09844[7]	0.09049[4]	0.09505[6]
	MSE	$\hat{\tau}$	0.0763[2]	0.11584[4]	0.24604[8]	0.08934[3]	0.21433[6]	0.22492[7]	0.14363[5]	0.07388[1]
		$\hat{a}$	0.15388[1]	0.18414[2]	0.29541[8]	0.19105[3]	0.29359[7]	0.25171[5]	0.20937[4]	0.26876[6]
		$\hat{b}$	0.00994[1]	0.01226[3]	0.01875[7]	0.01132[2]	0.01714[6]	0.01897[8]	0.014[5]	0.01332[4]
	MRE	$\hat{\tau}$	0.82287[2]	0.95511[4]	1.26786[8]	0.86398[3]	1.17741[7]	1.15604[6]	1.03466[5]	0.81219[1]
		$\hat{a}$	0.10319[1]	0.11556[2]	0.14311[7]	0.11631[3]	0.14423[8]	0.13101[5]	0.12139[4]	0.13703[6]
		$\hat{b}$	0.31354[1]	0.34864[2]	0.39378[8]	0.35589[3]	0.37463[5]	0.39376[7]	0.36195[4]	0.38019[6]
	D_{abs}		0.02072[1]	0.02107[2]	0.0225[7]	0.02189[4]	0.02263[8]	0.02228[6]	0.02197[5]	0.02181[3]
	D_{max}		0.03401[1]	0.03512[2]	0.03847[8]	0.03585[3]	0.03844[7]	0.038[6]	0.0367[4]	0.03682[5]
	ASAE		0.01108[5]	0.0106[3]	0.01135[6]	0.01106[4]	0.01179[7]	0.00992[1]	0.01047[2]	0.01254[8]
	$\sum Ranks$		19[1]	32[2]	90[8]	37[3]	81[7]	69[6]	51[4]	53[5]
300	BIAS	$\hat{\tau}$	0.16066[1]	0.18134[3]	0.23938[8]	0.17022[2]	0.22051[6]	0.23877[7]	0.1881[4]	0.18849[5]
		$\hat{a}$	0.22654[2]	0.24264[3]	0.2944[7]	0.22178[1]	0.28817[6]	0.26568[5]	0.25777[4]	0.30304[8]
		$\hat{b}$	0.06214[1]	0.07012[2]	0.08156[6]	0.07737[4]	0.07758[5]	0.08871[8]	0.07058[3]	0.08471[7]
	MSE	$\hat{\tau}$	0.04415[2]	0.05978[4]	0.11788[7]	0.04234[1]	0.10883[6]	0.13383[8]	0.06789[5]	0.05456[3]
		$\hat{a}$	0.08313[2]	0.09201[3]	0.14205[8]	0.07858[1]	0.13512[6]	0.11565[5]	0.10225[4]	0.13876[7]
		$\hat{b}$	0.00617[1]	0.00773[2]	0.01181[7]	0.00837[4]	0.01107[6]	0.015[8]	0.00814[3]	0.01064[5]
	MRE	$\hat{\tau}$	0.64263[1]	0.72534[3]	0.95752[8]	0.68088[2]	0.88205[6]	0.95509[7]	0.75242[4]	0.75394[5]
		$\hat{a}$	0.07551[2]	0.08088[3]	0.09813[7]	0.07393[1]	0.09606[6]	0.08856[5]	0.08592[4]	0.10101[8]
		$\hat{b}$	0.24856[1]	0.28049[2]	0.32624[6]	0.30949[4]	0.31033[5]	0.35482[8]	0.2823[3]	0.33885[7]
	D_{abs}		0.01473[2]	0.01494[3]	0.01581[6]	0.01432[1]	0.01598[7]	0.01578[5]	0.01551[4]	0.01624[8]
	D_{max}		0.02441[2]	0.02498[3]	0.02726[8]	0.02345[1]	0.027[5]	0.02708[6]	0.02606[4]	0.02724[7]
	ASAE		0.00706[5]	0.00686[3]	0.00722[6]	0.00694[4]	0.00749[7]	0.00632[1]	0.00684[2]	0.0084[8]
	$\sum Ranks$		22[1]	34[3]	84[8]	26[2]	71[5]	73[6]	44[4]	78[7]
600	BIAS	$\hat{\tau}$	0.13045[1]	0.14467[4]	0.1922[7]	0.13076[2]	0.18277[6]	0.19589[8]	0.15452[5]	0.14197[3]
		$\hat{a}$	0.1464[1]	0.17091[3]	0.19656[6]	0.15933[2]	0.20011[7]	0.18356[5]	0.17299[4]	0.21255[8]
		$\hat{b}$	0.05408[1]	0.05771[2]	0.0699[7]	0.06108[4]	0.06848[6]	0.07427[8]	0.06095[3]	0.06422[5]
	MSE	$\hat{\tau}$	0.02716[2]	0.03419[4]	0.06868[7]	0.02593[1]	0.0615[6]	0.08038[8]	0.03947[5]	0.03024[3]
		$\hat{a}$	0.03481[1]	0.04524[3]	0.06127[6]	0.04288[2]	0.06229[7]	0.05226[5]	0.04678[4]	0.06879[8]
		$\hat{b}$	0.00463[1]	0.00511[2]	0.00825[7]	0.00581[4]	0.00774[6]	0.01049[8]	0.0057[3]	0.00661[5]
	MRE	$\hat{\tau}$	0.52182[1]	0.57868[4]	0.76881[7]	0.52302[2]	0.73109[6]	0.78357[8]	0.61806[5]	0.56786[3]
		$\hat{a}$	0.0488[1]	0.05697[3]	0.06552[6]	0.05311[2]	0.0667[7]	0.06119[5]	0.05766[4]	0.07085[8]
		$\hat{b}$	0.21631[1]	0.23082[2]	0.27958[7]	0.24431[4]	0.27392[6]	0.29709[8]	0.24381[3]	0.25689[5]
	D_{abs}		0.00998[1]	0.01059[3]	0.01162[8]	0.01045[2]	0.01138[7]	0.01125[6]	0.01098[4]	0.01118[5]
	D_{max}		0.01645[1]	0.01762[3]	0.01993[8]	0.01726[2]	0.01949[7]	0.01935[6]	0.01833[4]	0.01893[5]
	ASAE		0.00442[3]	0.00443[4]	0.00475[7]	0.00444[5]	0.00472[6]	0.00408[1]	0.00436[2]	0.00545[8]
	$\sum Ranks$		15[1]	37[3]	83[8]	32[2]	77[7]	76[6]	46[4]	66[5]

Table 8. Partial and overall ranks of all the methods of estimation of proposed distribution by various values of τ, a, and b.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE
$\tau = 0.5, a = 0.25, b = 0.75$	35	4	2	7	1	6	5	3	8
	70	5.5	2	7	1	5.5	4	3	8
	150	5	3	6	1	7	4	2	8
	300	5	2	7	1	6	4	3	8
	600	4	2	6	1	7.5	5	3	7.5
$\tau = 1.5, a = 0.75, b = 0.5$	35	2.5	5	7	1	6	4	2.5	8
	70	5	2	7	1	6	4	3	8
	150	5	2	6	1	7	4	3	8
	300	5	2	7	1	6	4	3	8
	600	5	2	7	1	6	4	3	8
$\tau = 2, a = 0.5, b = 1.5$	35	1	3.5	5	3.5	6	7	2	8
	70	1	2	5	4	7	8	3	6
	150	1	2	7	4.5	6	8	3	4.5
	300	1	4	5	3	8	7	2	6
	600	1	4	8	2.5	6	7	2.5	5
$\tau = 2, a = 1.5, b = 2$	35	2	3	6	4	7	1	5	8
	70	1	2	7	5	6	4	3	8
	150	1	2	6	4	8	5	3	7
	300	1	4	6	3	7	5	2	8
	600	1	3	6	2	7.5	5	4	7.5
$\tau = 0.75, a = 2, b = 3$	35	1	2	3	6	4	7	5	8
	70	1	2	4	6	7	5	3	8
	150	1	2	5	6	7	4	3	8
	300	1	3	7	4	8	6	2	5
	600	1	2	6.5	4	8	6.5	3	5
$\tau = 0.25, a = 3, b = 0.25$	35	1	4	8	2	5	7	6	3
	70	1	4	8	2	6	7	3	5
	150	1	2	8	3	7	6	4	5
	300	1	3	8	2	5	6	4	7
	600	1	3	8	2	7	6	4	5
$\sum$ Ranks		67.0	80.5	193.5	82.5	195.5	159.5	95.0	206.5
Overall Rank		1	2	6	3	7	5	4	8

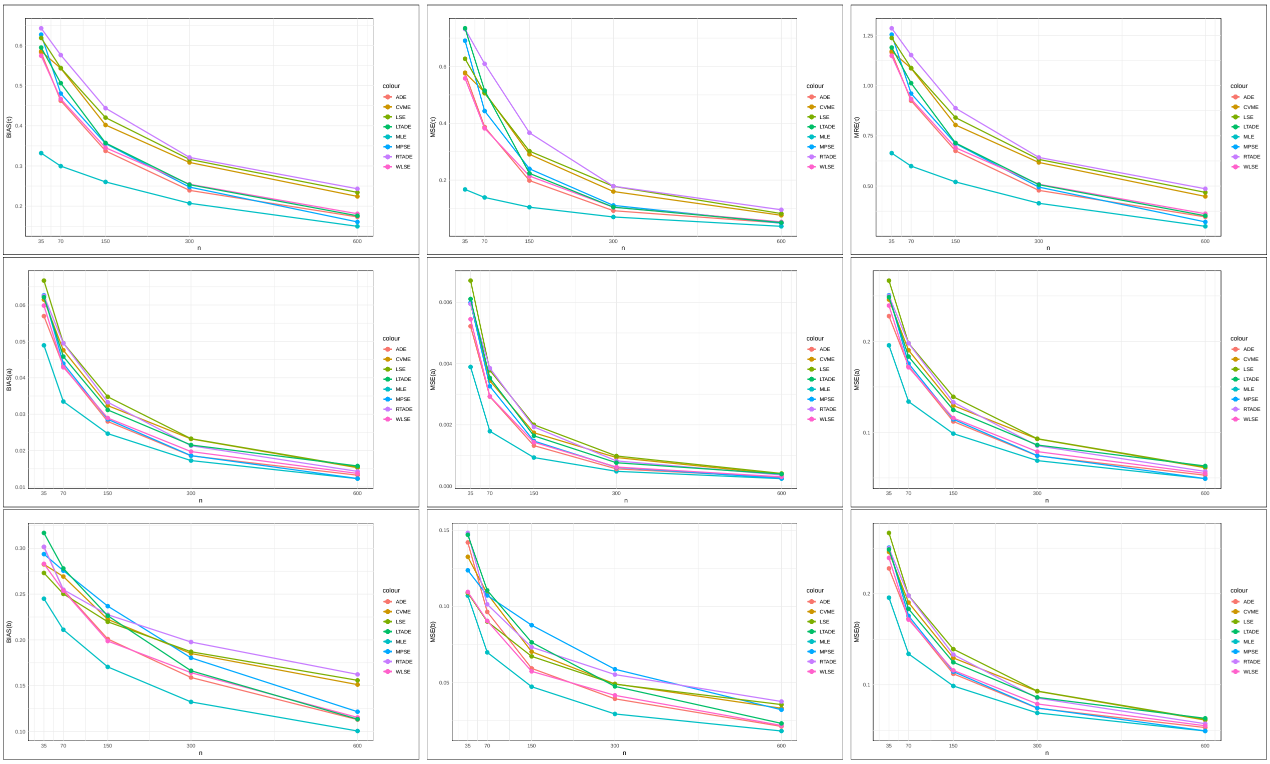

Figure 4. Graphical representation of BIAS, MSE, and MRE values in Table 2.

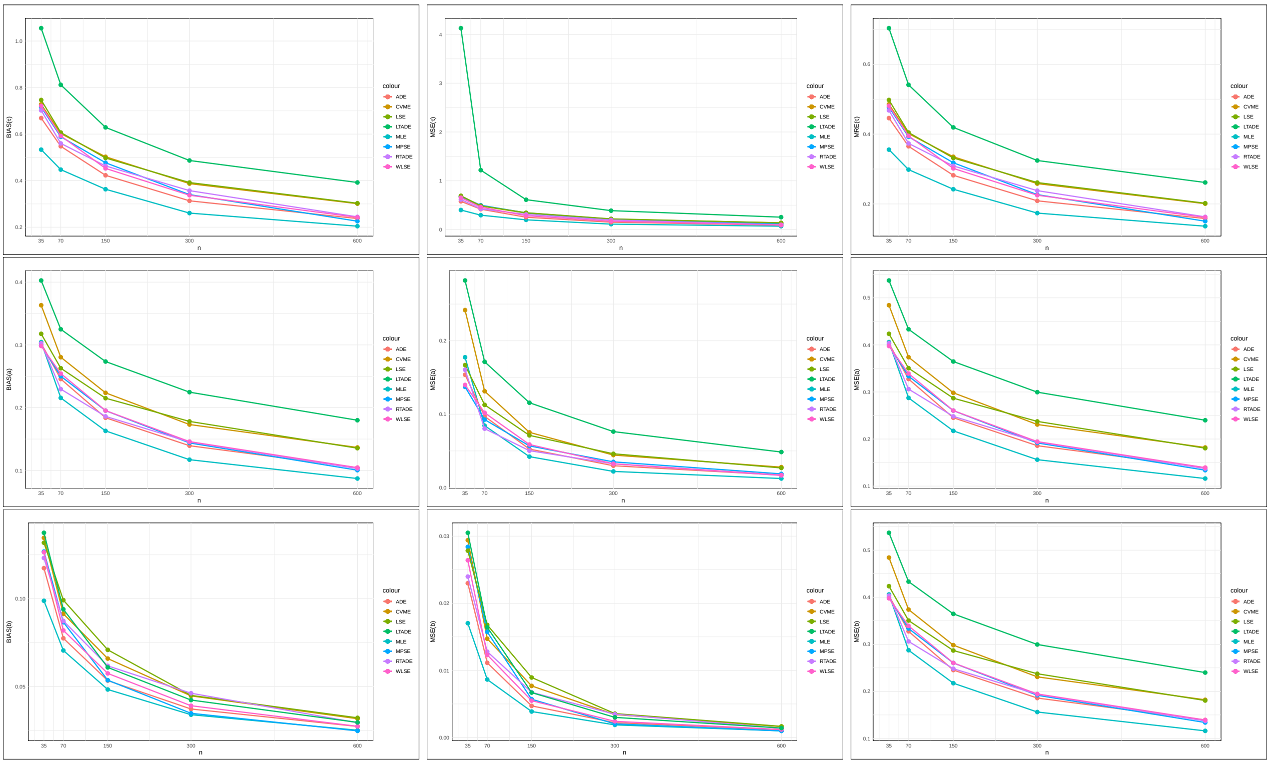

Figure 5. Graphical representation of BIAS, MSE, and MRE values in Table 3.

Table 9. Lower, upper, length of CP, and CP for GAPEED parameters by MLE and MPS.

	n		MLE Lower	MLE Upper	MLE LACI	MLE CP	MPS Lower	MPS Upper	MPS LACI	MPS CP
$a = 0.25$	35	a	0.1424	0.3840	0.2416	95.2%	0.1240	0.3716	0.2475	97.4%
		b	0.3074	1.3956	1.0881	96.2%	0.1761	1.3511	1.1751	98.6%
		τ	-0.1474	1.3940	1.5415	94.6%	-0.2409	1.3937	1.6346	95.6%
	70	a	0.1680	0.3336	0.1655	95.8%	0.1571	0.3300	0.1730	96.0%
		b	0.3442	1.2438	0.8996	94.6%	0.2876	1.1783	0.8907	96.8%
		τ	-0.0981	1.3002	1.3983	94.2%	-0.0982	1.1923	1.2906	96.8%
$b = 0.75$	150	a	0.1940	0.3084	0.1144	94.2%	0.1890	0.3050	0.1160	96.4%
		b	0.4466	1.0598	0.6132	93.2%	0.4220	1.0220	0.6001	96.4%
		τ	0.0900	0.9525	0.8625	94.0%	0.1039	0.8881	0.7841	95.0%
$\tau = 0.5$	300	a	0.2082	0.2896	0.0814	95.2%	0.2092	0.2852	0.0761	96.4%
		b	0.4659	1.0316	0.5657	94.8%	0.5239	0.9391	0.4153	96.2%
		τ	0.0907	0.9503	0.8596	93.6%	0.2218	0.7761	0.5543	95.8%
	600	a	0.2172	0.2825	0.0654	94.2%	0.2162	0.2805	0.0643	95.4%
		b	0.5764	0.9262	0.3498	93.6%	0.5853	0.8925	0.3073	95.2%
		τ	0.2346	0.7904	0.5559	94.6%	0.2858	0.7160	0.4302	96.0%
$a = 0.75$	35	a	0.3749	1.7629	1.3880	96.8%	0.2215	1.6978	1.4762	98.2%
		b	0.1733	0.7843	0.6110	91.0%	0.1273	0.7737	0.6464	92.8%
		τ	0.0167	2.3436	2.3269	99.8%	-0.1434	2.5520	2.6954	100.0%
	70	a	0.5547	1.4283	0.8736	95.2%	0.4283	1.4450	1.0167	97.8%
		b	0.1933	0.7132	0.5198	92.0%	0.1226	0.7452	0.6226	93.8%
		τ	0.1180	2.1860	2.0680	94.2%	-0.1264	2.4289	2.5553	100.0%
$b = 0.5$	150	a	0.6664	1.3496	0.6831	94.8%	0.5957	1.3420	0.7462	97.8%
		b	0.2169	0.6506	0.4337	93.0%	0.2726	0.6031	0.3306	93.2%
		τ	0.1519	1.9431	1.7911	94.4%	0.3050	1.8915	1.5865	93.0%
$\tau = 1.5$	300	a	0.7354	1.2244	0.4890	95.2%	0.6828	1.2239	0.5411	96.6%
		b	0.3296	0.5786	0.2489	95.4%	0.3910	0.5441	0.1531	89.8%
		τ	0.5329	1.7045	1.1716	95.6%	0.7327	1.6352	0.9025	90.8%
	600	a	0.8075	1.1805	0.3730	95.2%	0.7656	1.1558	0.3902	98.6%
		b	0.3372	0.5607	0.2236	97.4%	0.4507	0.4944	0.0438	49.6%
		τ	0.5921	1.5764	0.9843	96.6%	0.9952	1.3684	0.3732	65.0%
$a = 0.5$	35	a	0.0841	3.0983	3.0142	99.4%	0.5964	2.4213	1.8249	97.0%
		b	0.7792	3.3199	2.5407	96.4%	0.9277	2.9819	2.0541	96.2%
		τ	-4.1397	11.2084	15.3481	96.8%	0.5888	5.0887	4.5000	96.8%
	70	a	0.3513	2.9435	2.5922	98.6%	0.7959	2.3551	1.5592	97.6%
		b	1.1282	2.8535	1.7253	95.8%	1.2526	2.6453	1.3927	95.4%
		τ	-2.3903	7.7478	10.1381	94.6%	0.7896	3.9179	3.1283	96.6%
$b = 1.5$	150	a	0.4155	3.0630	2.6474	95.6%	0.8704	2.4470	1.5766	98.4%
		b	1.2344	2.6437	1.4093	95.8%	1.4967	2.3945	0.8979	95.0%
		τ	-1.8044	6.2974	8.1017	93.8%	0.7386	3.4610	2.7224	99.2%
$\tau = 2$	300	a	0.7300	2.9444	2.2144	89.4%	0.9837	2.4903	1.5066	96.4%
		b	1.4469	2.5415	1.0947	93.2%	1.6720	2.3314	0.6594	95.2%
		τ	-0.7896	4.4351	5.2247	92.8%	0.8241	2.8969	2.0728	96.8%
	600	a	0.8443	3.0427	2.1984	88.0%	1.1938	2.4879	1.2942	91.4%
		b	1.4851	2.4993	1.0142	91.4%	1.8123	2.2369	0.4246	89.2%
		τ	-0.9598	4.2972	5.2570	91.6%	0.8832	2.5574	1.6742	90.4%

Table 10. Lower, upper, length of CP, and CP for GAPEED parameters by MLE and MPS.

	n		MLE				MPS			
			Lower	Upper	LACI	CP	Lower	Upper	LACI	CP
$a = 1.5$	35	a	0.0841	3.0983	3.0142	99.4%	0.5964	2.4213	1.8249	97.0%
		b	0.7792	3.3199	2.5407	96.4%	0.9277	2.9819	2.0541	96.2%
		τ	-4.1397	11.2084	15.3481	96.8%	0.5888	5.0887	4.5000	96.8%
	70	a	0.3513	2.9435	2.5922	98.6%	0.7959	2.3551	1.5592	97.6%
		b	1.1282	2.8535	1.7253	95.8%	1.2526	2.6453	1.3927	95.4%
		τ	-2.3903	7.7478	10.1381	94.6%	0.7896	3.9179	3.1283	96.6%
$b = 2$	150	a	0.4155	3.0630	2.6474	95.6%	0.8704	2.4470	1.5766	98.4%
		b	1.2344	2.6437	1.4093	95.8%	1.4967	2.3945	0.8979	95.0%
		τ	-1.8044	6.2974	8.1017	93.8%	0.7386	3.4610	2.7224	99.2%
$\tau = 2$	300	a	0.7300	2.9444	2.2144	89.4%	0.9837	2.4903	1.5066	96.4%
		b	1.4469	2.5415	1.0947	93.2%	1.6720	2.3314	0.6594	95.2%
		τ	-0.7896	4.4351	5.2247	92.8%	0.8241	2.8969	2.0728	96.8%
	600	a	0.8443	3.0427	2.1984	88.0%	1.1938	2.4879	1.2942	91.4%
		b	1.4851	2.4993	1.0142	91.4%	1.8123	2.2369	0.4246	89.2%
		τ	-0.9598	4.2972	5.2570	91.6%	0.8832	2.5574	1.6742	90.4%
$a = 2$	35	a	1.4516	3.0180	1.5664	95.2%	1.2272	3.0073	1.7800	98.6%
		b	1.2482	5.2132	3.9651	95.8%	0.8513	4.9181	4.0668	99.2%
		τ	-0.2282	1.6153	1.8435	95.6%	-0.4111	1.7960	2.2071	94.0%
	70	a	1.6144	2.7305	1.1160	95.0%	1.4843	2.7390	1.2547	97.6%
		b	1.4910	4.6090	3.1180	96.2%	1.1561	4.5050	3.3489	99.0%
		τ	-0.0943	1.4316	1.5259	95.2%	-0.2493	1.5585	1.8078	94.8%
$b = 3$	150	a	1.7896	2.5391	0.7496	95.4%	1.7365	2.5418	0.8053	96.2%
		b	1.7926	4.1551	2.3625	96.2%	1.6447	4.0422	2.3974	96.8%
		τ	0.0645	1.1806	1.1161	96.6%	0.0097	1.1917	1.1821	97.0%
$\tau = 0.75$	300	a	1.9039	2.4287	0.5249	95.8%	1.8799	2.4292	0.5493	97.4%
		b	2.1524	3.7909	1.6385	94.2%	1.9889	3.7889	1.8000	97.2%
		τ	0.1813	1.0293	0.8480	95.4%	0.1349	1.0399	0.9050	97.0%
	600	a	1.9724	2.3730	0.4006	95.6%	1.9612	2.3715	0.4103	96.0%
		b	2.3438	3.5541	1.2102	94.2%	2.3352	3.4916	1.1564	95.4%
		τ	0.2739	0.9082	0.6344	95.0%	0.2774	0.8905	0.6131	96.2%
$a = 3$	35	a	1.9729	4.6161	2.6432	95.8%	1.8782	4.3354	2.4572	97.0%
		b	-0.0293	0.7353	0.7646	97.0%	-0.0398	0.6755	0.7154	98.0%
		τ	-0.4116	1.4397	1.8512	94.0%	-0.4786	1.4140	1.8926	95.4%
	70	a	2.4063	3.9743	1.5680	94.2%	2.2750	3.9640	1.6890	94.2%
		b	-0.0284	0.5486	0.5770	94.8%	-0.0534	0.5696	0.6230	97.4%
		τ	-0.3834	1.0028	1.3862	93.4%	-0.4462	1.0793	1.5255	93.6%
$b = 0.25$	150	a	2.6660	3.7299	1.0639	93.4%	2.6852	3.6862	1.0010	94.4%
		b	0.0336	0.4138	0.3803	94.2%	-0.0209	0.5050	0.5260	91.6%
		τ	-0.2258	0.6358	0.8616	93.8%	-0.3280	0.8359	1.1640	88.4%
$\tau = 0.25$	300	a	2.8201	3.5448	0.7248	94.0%	2.9279	3.5496	0.6216	93.8%
		b	0.0642	0.3265	0.2624	92.6%	0.2438	0.3026	0.0587	98.2%
		τ	-0.1524	0.4342	0.5866	92.8%	0.3140	0.3670	0.0530	97.3%
	600	a	2.9387	3.4003	0.4616	94.8%	2.9697	3.1695	0.1998	98.1%
		b	0.1070	0.2593	0.1523	95.4%	0.3550	0.1831	-0.1719	96.9%
		τ	-0.0465	0.2697	0.3162	95.4%	0.4718	0.1116	-0.3601	97.5%

6. Application

To prove that the proposed model is better than previous distributions, a comparison must be made using some data from previous studies. From the previous studies, we note that the basic distribution under study focused on the data of medical field. The three datasets under study have a basic relationship with the medical field in various ways. The GAPEED was applied to the various three datasets, and the outcomes of this comparison were produced with the TLMW [11], TIIEHLPL [32], EL [27], KW [25], GMW [21], MOAPEW [6], EW [24], EGAPEx [28], KMGEx [1], EHLINH [1], ExEx [48], and OWITL [7].

Furthermore, in order to assess the EGAPE model's validity in comparison to other competing models, we utilized various goodness-of-fit metrics, including the Kolmogorov-Smirnov (K-S) statistic with its p-value, Cramer-von Mises (CVM), and Anderson-Darling (AD), as well as other criteria measures like Bayesian information (BI), Akaike information (AI), corrected AI (CAI), and Hannan-Quinn information (HQI). All goodness-of-fit metrics are all taken into account when comparing the fits of all models. We utilize R software and the Maximum Likelihood Estimation (MLE) method to estimate the parameters of the specified distributions and to assess the goodness-of-fit metrics.

Data I: This data set was utilized as "1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0" by Barco et al. [18], and the data shows how quickly 20 people felt better after taking an analgesic.

Data II: The most recent data cited by [2, 11], showing the number of daily confirmed death cases linked to COVID-19. The data consists of 89 observed values with an average daily death rate of 18.72. The data set is given as follows: "1, 1, 2, 4, 5, 1, 1, 3, 6, 6, 4, 1, 5, 6, 6, 8, 5, 7, 7, 9, 9, 15, 17, 11, 13, 5, 14, 5, 13, 9, 19, 15, 11, 14, 12, 11, 7, 13, 10, 20, 22, 21, 12, 14, 9, 14, 7, 16, 17, 13, 21, 11, 11, 8, 11, 12, 15, 21, 20, 18, 15, 14, 21, 16, 11, 28, 29, 19, 14, 19, 29, 34, 34, 46, 46, 47, 36, 38, 40, 32, 39, 34, 35, 36, 35, 45, 62." Recently, papers [2, 11] used this data, for which the Topp-Leone modified Weibull (TLMW) [11] has the best results where the KSPV reached 0.7280, while in this paper the KSPV reached 0.7453, and this is better than the another comparative models.

Data III: Survival rates for Guinea pigs infected with virulent tubercle bacilli are shown in the set of data [55]. Guinea pigs were chosen for this experiment for a number of reasons, one of which is that it is believed that they are extremely vulnerable to human tuberculosis. The information set is as follows: 0.10, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24 1.30, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Tables 11–16 present the MLE of the parameters for the GAPEED as well as other distributions, together with standard error (SE) values of the parameters and goodness of fit metrics for each distribution. In order to obtain the likelihood with SE, we first used the "maxLik" package, which implements the Newton Raphson (NR) method of maximization, and the variance covariance matrix. Second, we compare the fits with other distributions to determine if the relevant datasets genuinely fit the GAPEED or not using the goodness-of-fit test. Among all the models that have been fitted to these datasets, the GAPEED provides the lowest values for the KSD, AI, BI, CAI, HAI, CVM, and AD statistics. It also provides the highest value for the P-value when compared to other distributions. Figures 6–14 have been discussed for GAPEED.

Profile likelihood and uniqueness proof of GAPEED parameters have been discussed in Figures 7, 8, 10, 11, 13, and 14. The results in Tables 12, 14, and 16 show that the GAPEED is the most effective model to fit these datasets when compared to the other distributions indicated in Tables 12, 14, and 16. Graphical representations in Figures 6, 9, and 12 reflect these findings.

Table 11. Estimates and SE for parameters of each model: Taking an analgesic, data I.

		α	β	τ	θ	λ
EGAPE	Estimates	1.8897	29.0863	1.7697		
	SE	0.5609	21.0642	0.8618		
EL	Estimates	77.2175	12.0930	3.6927		
	SE	116.8405	17.6372	7.7470		
KW	Estimates	30.4293	0.3994	1.7768	1.4045	
	SE	35.9424	0.4654	0.8620	0.6895	
EW	Estimates	2.757653	13.05099	11.26919		
	SE	0.425237	16.18943	25.32466		
MOAPEW	Estimates	0.0048	0.4068	0.1943	0.4860	0.0038
	SE	0.0070	0.1936	0.0756	0.2005	0.0011
KMGE	Estimates	32.4295	2.0003			
	SE	20.6526	0.4056			
EHLINH	Estimates	6.7046	28.4439	0.0674		
	SE	2.0967	65.6860	0.1601		
ExEx	Estimates	133.3134	0.0028			
	SE	78.3222	0.0015			
OWITL	Estimates	2.9015	79.0976	0.3261		
	SE	0.4311	115.5561	0.1408		

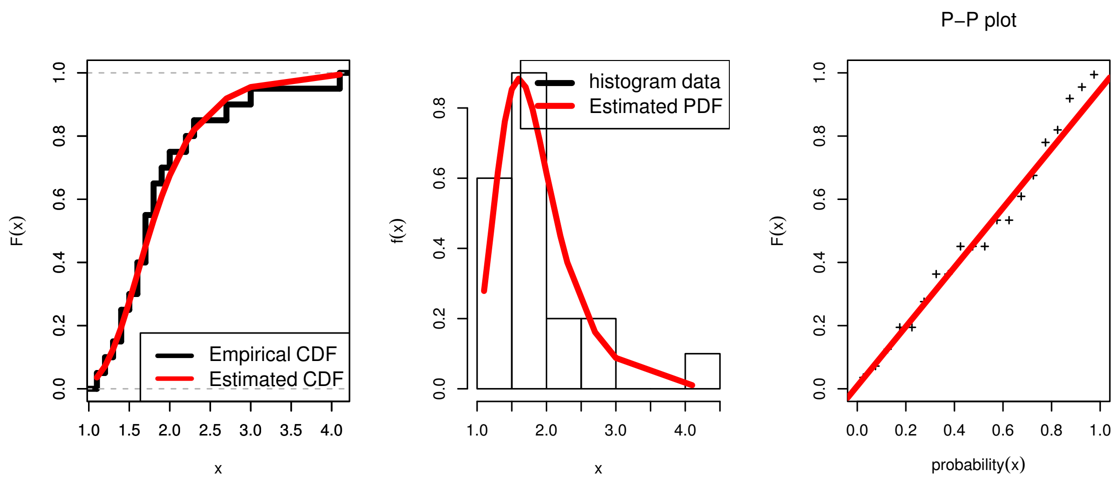

Figure 6. Estimation of GAPEED model of taking an analgesic, data I.

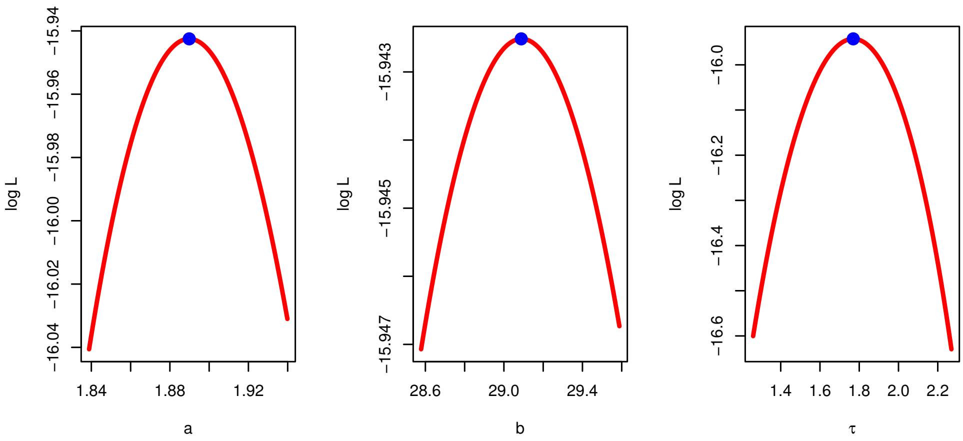

Figure 7. Profile likelihood of GAPEED parameter for taking an analgesic, data I.

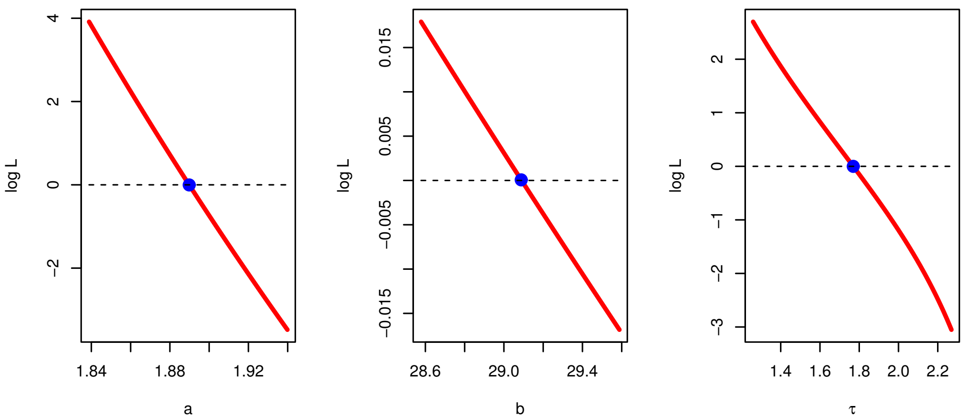

Figure 8. Uniqueness proof of GAPEED parameter for taking an analgesic, data I.

Table 12. Statistical measures of each comparative models: Taking an analgesic data I.

	KSD	KSPV	AI	BI	CAI	HQI	CVM	AD
GAPEED	0.1163	0.9495	37.8850	40.8722	39.3850	38.4682	0.0427	0.2510
EL	0.1211	0.9308	37.5124	40.4996	39.0124	38.0955	0.0391	0.2260
KW	0.1392	0.8329	39.9867	43.9696	42.6534	40.7642	0.0498	0.2913
MOAPEW	0.1853	0.4984	47.2771	50.2643	48.7771	47.8603	0.1866	1.0986
EW	0.1853	0.4984	47.2771	50.2643	48.7771	47.8603	0.1866	1.0986
KMGE	0.1206	0.9330	35.9024	37.8938	36.6082	36.2911	0.0438	0.2576
EHLINH	0.1294	0.8912	37.9113	40.8985	39.4113	38.4944	0.0457	0.2641
ExEx	0.4041	0.0029	59.5574	61.5489	60.2633	59.9461	0.1761	1.0400
OWITL	0.1783	0.5481	44.5537	47.5409	46.0537	45.1369	0.1441	0.8519

Table 13. Estimates and SE for parameters of each model: COVID-19, data II.

		α	β	τ	θ	λ
EGAPE	Estimates	0.0886	1.4401	0.6050		
	SE	0.0157	0.5555	0.6115		
TLMW	Estimates	0.0106	0.0101	1.2689	1.2680	
	SE	0.0740	0.0276	0.2493	1.0647	
TIIEHLPL	Estimates	1.7143	0.1844	28.8074	166.7427	
	SE	2.9734	0.2303	71.7154	27.4533	
EL	Estimates	1.8125	11.2464	123.1732		
	SE	0.3163	8.1168	102.2685		
KW	Estimates	1.2083	2.3127	0.0326	1.1786	
	SE	0.9050	6.4453	0.0641	0.6493	
GMW	Estimates	0.0370	1.2290	0.0015	1.1750	
	SE	0.0939	0.9993	0.0140	0.7523	
MOAPEW	Estimates	0.3553	0.2575	0.1384	0.0058	0.0087
	SE	0.5066	0.0104	0.1008	0.0018	0.0078
EW	Estimates	0.2312	0.0085	0.2914		
	SE	0.0152	0.0058	0.1531		
KMGE	Estimates	1.8212	0.0675			
	SE	0.2588	0.0091			
EHLINH	Estimates	19.5686	0.2837	1589.2263		
	SE	15.3431	0.0490	237.2804		
ExEx	Estimates	3.4494	0.0117			
	SE	1.9636	0.0079			
OWITL	Estimates	1.1721	0.0508	1.1382		
	SE	0.4598	0.0350	0.5937		

Table 14. Statistical measures of each comparative models for COVID-19, data II.

	KSD	KSPV	AI	BI	CAI	HQI	CVM	AD
GAPEED	0.0728	0.7453	662.0716	669.4693	662.3608	665.0504	0.0869	0.5958
TLMW	0.0740	0.7280	663.9288	673.7924	664.4166	667.9005	0.0901	0.6041
TIIEHLPL	0.0816	0.6084	665.4572	675.3208	665.9450	669.4290	0.0814	0.6494
EL	0.0845	0.5635	663.7241	671.1218	664.0132	666.7029	0.0761	0.6190
KW	0.0752	0.7090	663.9278	673.7914	664.4156	667.8996	0.0920	0.6121
GMW	0.0768	0.6834	663.8639	673.7276	664.3517	667.8357	0.0943	0.6201
MOAPEW	0.0762	0.6929	665.7249	678.0545	666.4657	670.6897	0.0879	0.5993
EW	0.1110	0.2336	667.4458	674.8435	667.7349	670.4246	0.2186	1.2041
KMGE	0.0861	0.5389	662.3907	669.8323	662.5335	665.3766	0.0763	0.6177
EHLINH	0.0864	0.5345	664.3826	671.7803	664.6718	667.3615	0.0853	0.7083
ExEx	0.0919	0.4547	662.8435	669.7753	662.9863	665.8294	0.1603	0.9021
OWITL	0.0771	0.6787	662.6932	669.5910	662.9824	665.6721	0.0996	0.6511

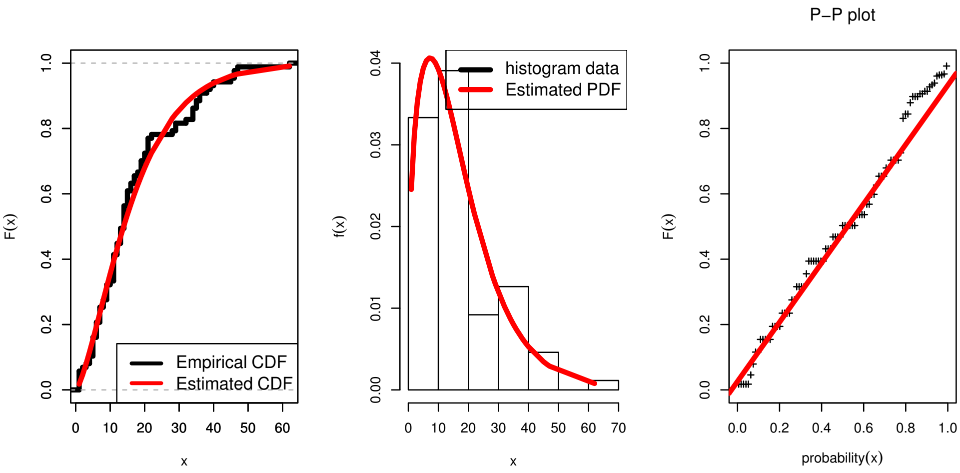

Figure 9. Estimation of GAPEED model of COVID-19, data II.

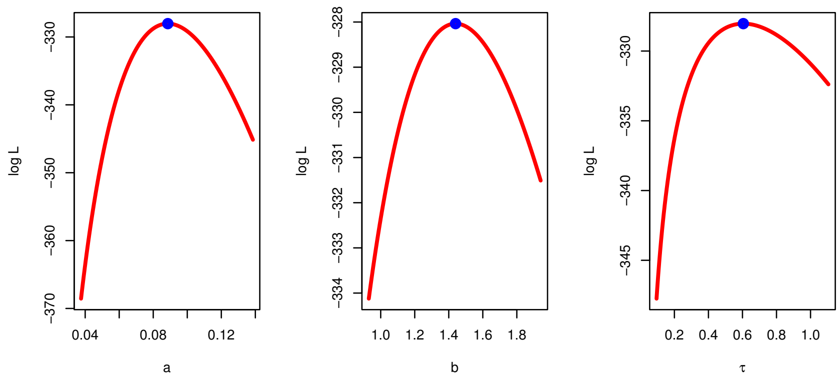

Figure 10. Profile likelihood of GAPEED parameter for COVID-19, data II.

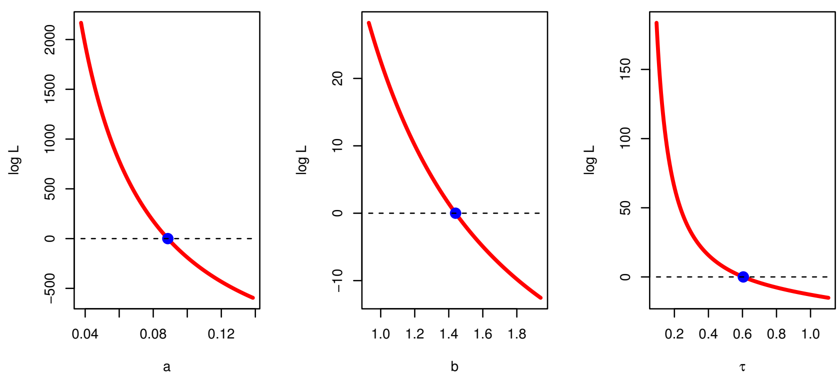

Figure 11. Uniqueness of GAPEED parameter for COVID-19, data II.

Table 15. Estimates and SE for parameters of each model: Guinea pigs, data III.

		α	β	τ	θ
EGAPE	Estimates	1.2948	0.9091	0.0079	
	SE	0.1631	0.6367	0.0242	
TLMW	Estimates	0.2497	0.2004	1.2916	2.7723
	SE	0.9087	0.7554	0.7622	1.5924
TIIEHLPL	Estimates	0.0927	1.3381	2.4967	138.0944
	SE	0.2557	0.8698	1.5475	532.9272
EL	Estimates	3.8657	36.6762	30.4730	
	SE	0.8248	61.0255	53.2398	
KW	Estimates	3.9049	3.8098	0.6329	0.7832
	SE	9.9178	25.2951	0.8289	1.7951
GMW	Estimates	1.4999	7.0403	0.1177	0.5813
	SE	0.7081	2.0431	0.0250	0.1381
EW	Estimates	1.8162	36.6594	5.3695	
	SE	0.1607	70.2187	9.0321	
EGAPEx	Estimates	2.2303	3.0157	3.0038	0.4497
	SE	4.3322	1.7338	3.8605	0.5913
KMGE	Estimates	3.7890	0.9720		
	SE	0.7019	0.1221		
EHLINH	Estimates	34.1057	0.3627	94.1204	
	SE	38.2904	0.0934	165.6271	
ExEx	Estimates	70.0000	0.0051		
	SE	81.8420	0.0059		
OWITL	Estimates	1.8011	19.0880	0.3149	
	SE	0.1713	23.2625	0.1740	

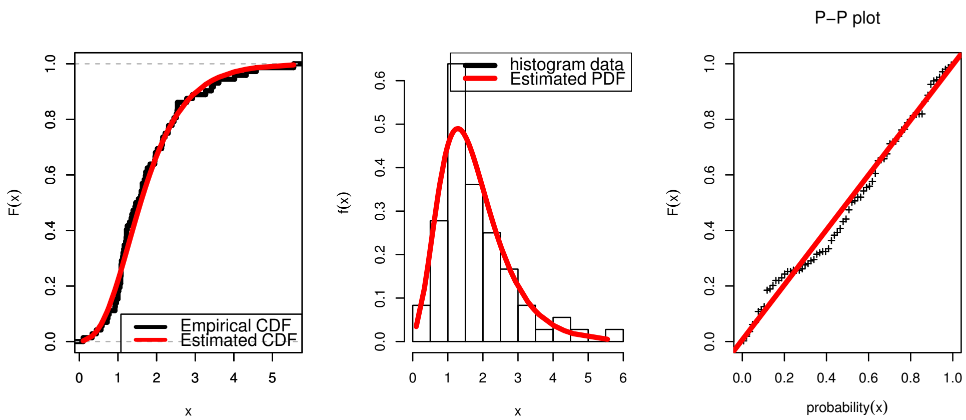

Figure 12. Estimation of GAPEED model of Guinea pigs, dataset III.

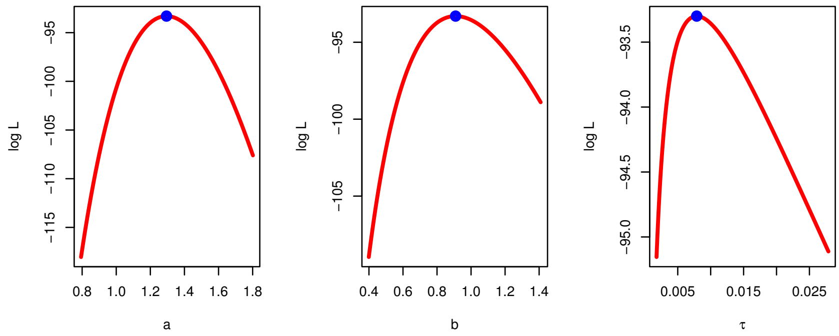

Figure 13. Profile likelihood of GAPEED parameter for Guinea pigs, dataset III.

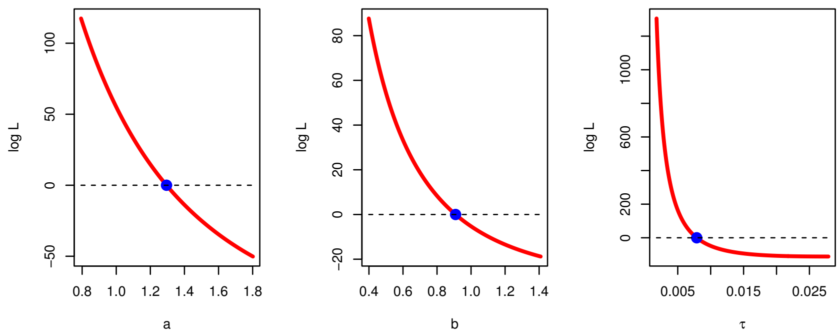

Figure 14. Uniqueness of GAPEED parameter for Guinea pigs, dataset III.

Table 16. Statistical measures of each comparative models for Guinea pigs, dataset III.

	KSD	KSPV	AI	BI	CAI	HQI	CVM	AD
GAPEED	0.0826	0.7094	192.5995	199.4295	192.9524	195.3185	0.0881	0.5118
TLMW	0.0885	0.6253	196.1265	205.2332	196.7235	199.7519	0.0915	0.5657
TIIEHLPL	0.0874	0.6408	196.0386	205.1453	196.6356	199.6640	0.0747	0.4823
EL	0.0944	0.5429	194.7195	201.5495	195.0725	197.4386	0.0770	0.5188
KW	0.0896	0.6103	196.1880	205.2947	196.7850	199.8134	0.0933	0.5735
GMW	0.0905	0.5967	197.2302	206.3369	197.8272	200.8556	0.1064	0.6601
EW	0.1056	0.3984	197.6848	204.5148	198.0377	200.4038	0.1662	0.9792
EGAPEx	0.0874	0.6411	196.1340	205.2406	196.7310	199.7594	0.0917	0.5652
KMGE	0.0906	0.5961	193.4319	200.9853	193.6058	196.2446	0.0970	0.5771
EHLINH	0.1011	0.4537	195.7417	202.5717	196.0946	198.4607	0.0976	0.6161
ExEx	0.2118	0.0031	210.6588	215.2121	210.8327	212.4715	0.2429	1.4240
OWITL	0.0929	0.5634	194.6419	201.4719	194.9949	197.3610	0.0921	0.5773

7. SSALT based on PTIC

The majority of research employs just one accelerating stress variable. There are situations when increasing one stress variable does not produce enough failure data. For further acceleration, two stress factors might be required. Two stress variables are examined in this paper. It will be possible to better comprehend the impact of two stress variables functioning simultaneously if two stress variables are included in a test design. Furthermore, the test units' failure time is assumed by the author to follow a GAPEED model. The bivariate SSALT under PTIC is discussed in this section. The MLE of the model parameters is also examined.

7.1. Model assumptions

The bivariate SSALT under PTIC is as follows: Each stress variable (SV) has two levels when using the bivariate SSALT. Let H_s represent the variable l's stress level (SL) s, where $l = 1, 2$ and $s = 0, 1, 2$. H_{10} and H_{20} illustrate typical operational scenarios. Allow the experiment to run for T_1, during which n_1 failures will be logged, with all n units starting at the 1^{st} step with SLs (K_{11}, K_{21}).

The 1^{st} SV is raised from H_{11} to H_{12} at time T_1, c_1 units are removed at random from the remaining $N - n_1$ units, and the 1^{st} SV is raised from H_{11} to H_{12}. Until the predetermined time tau_2 is calculated at time tau_2 from the remaining $N - n_1 - c_1 - n_2$ units, the *second* phase is repeated.

The other SV is increased from H_{21} to H_{22} at the conclusion of the 2^{nd} step. Up until when T is reached, at which point n_2 units fail this stage, the test is repeated. All of the surviving units $c_3 = N - n_1 - c_1 - n_2 - c_2 - n_3$ are taken out of the test at time T.

In the first phase, GAPEED with cdf in Eq (2.1) is used to calculate the life of test units. A log-linear function of SLs exists for the scale parameter α_i at test step i for i =1, 2, and 3.

Step 1. $\ln(\alpha_1) = B_0 + B_1 H_{11} + B_2 H_{21}$;
Step 2. $\ln(\alpha_2) = B_0 + B_1 H_{12} + B_2 H_{21}$;
Step 3. $\ln(\alpha_3) = B_0 + B_1 H_{12} + B_2 H_{22}$,

where B_0, B_1, and B_2 are unidentified parameters that vary based on the test technique and the product. The two pressures are thought to be unrelated to one another.

The model of cumulative exposure is also considered. Regardless of how the chance is calculated, the remaining life in this model is completely based on the current cumulative failure probability and the current SL [49].

The shape parameter β is constant for all SLs. The cumulative distribution function (cdf) of the test unit lifespan for the bivariate SSALT and cumulative exposure models is then:

$$F_i(x) = \begin{cases} \left(1 - e^{-a_1 x}\right)^b \tau^{1-\left(1-e^{-a_1 x}\right)^b} & 0 \le x \le T_1, \\ \left(1 - e^{-[a_1 T_1 + a_2(x - T_1)]}\right)^b \tau^{1-\left(1-e^{-[a_1 T_1 + a_2(x-T_1)]}\right)^b} & T_1 \le x \le T_2, \\ \left(1 - e^{-[a_1 T_1 + a_2(T_2 - T_1) + a_3(x - T_2)]}\right)^b \tau^{1-\left(1-e^{-[a_1 T_1 + a_2(T_2-T_1) + a_3(x-T_2)]}\right)^b} & T_2 \le x \le T, \end{cases} \tag{7.1}$$

where $i = 1, 2, 3$. The pdf of bivariate SSALT for this can be written as

$$f_1(x) = \frac{1}{e^{a_1 x} - 1} \left\{ a_1 b \left(1 - e^{-a_1 x}\right)^b \tau^{1-\left(1-e^{-a_1 x}\right)^b} \left[1 - \log(\tau)\left(1 - e^{-a_1 x}\right)^b\right] \right\}, \quad 0 \le x \le T_1, \tag{7.2}$$

$$f_2(x) = \frac{1}{e^{(a_1T_1+a_2(x-T_1))}-1}\left\{a_2b\left(1-e^{-(a_1T_1+a_2(x-T_1))}\right)^b \tau^{1-\left(1-e^{-(a_1T_1+a_2(x-T_1))}\right)^b}\right.$$
$$\left.\left[1-\log(\tau)\left(1-e^{-(a_1T_1+a_2(x-T_1))}\right)^b\right]\right\}, \quad T_1 \le x \le T_2, \tag{7.3}$$

and

$$f_3(x) = \frac{1}{e^{(a_1T_1+a_2(T_2-T_1)+a_3(x-T_2))}-1}\left\{a_3b\left(1-e^{-(a_1T_1+a_2(T_2-T_1)+a_3(x-T_2))}\right)^b \tau^{1-\left(1-e^{-(a_1T_1+a_2(T_2-T_1)+a_3(x-T_2))}\right)^b}\right.$$
$$\left.\left[1-\log(\tau)\left(1-e^{-(a_1T_1+a_2(T_2-T_1)+a_3(x-T_2))}\right)^b\right]\right\} \quad T_2 \le x \le T_3. \tag{7.4}$$

7.2. Likelihood function of bivariate SSALT model

Assume that in a bivariate SSALT, x_{ij} represents the observations produced from a PTIC sample with random deletions, where $i = 1, 2, 3$, $j = 1, 2, ..., n_i$. Each unit is excluded from the test with the same probability p, and the number of items removed from the test at any one time is distributed binomially. In other words,

$$C_i = \begin{cases} c_1 \approx binomial(N-n_1, p), \\ c_2|c_1 = c_2 \approx binomial(N-n_1-n_2-c_1, p), \\ c_3 = N-n_1-n_2-n_3-c_1-c_2. \end{cases} \tag{7.5}$$

The joint log-LLF of the bivariate SSALT model under the PTIC sample is as follows if C_i is independent of x_{ij} for all i.

$$L(\Theta, p|C) = L_1(\Theta)\, P(c_1, p) P(c_2|c_1, p), \tag{7.6}$$

where

$$L_1(\Theta) = \prod_{i=1}^{3}\prod_{j=1}^{n_i} f_i(x_{ij})\left[1-F_i(x_{ij})\right]^{c_i}, \tag{7.7}$$

where in (7.1)–(7.4), $F_i(x_{ij})$ and $f_i(x_{ij})$ will be replaced for $F_i(x_{ij})$ and $f_i(x_{ij})$, respectively.

The ML estimators of the model under bivariate SSALT based on the PTIC sample are shown in Tables 17 and 18, respectively, when p=0 and p=0.2. The results in Tables 17 and 18 indicate that the model's effectiveness rises as the probability of binomial elimination rises and the AINC and BINC values fall.

Table 17. MLE under bivariate SSALT based on PTIC sample when p=0.

Data	T_1	T_2	T_3	n_1	n_2	n_3	α_1	α_2	α_3	β	τ	Llog	AI	BI
I	1.6	1.9	3	6	7	5	2.5353	4.2746	3.3228	1.8354	0.0020	-7.2028	24.4055	29.3842
			3.5			6	2.4710	3.8668	2.3909	2.0295	0.0031	-10.4644	30.9287	35.9074
		2.2	3		9	3	2.5416	3.5990	5.0954	1.9838	0.0026	-7.1745	24.3490	29.3277
			3.5			4	2.5352	3.0753	2.9460	1.9840	0.0008	-10.8706	31.7412	36.7198
	1.8	1.9	3	11	2	5	2.8502	4.7026	3.3463	1.4596	0.0034	-7.6071	25.2142	30.1929
			3.5			6	2.7474	4.0312	2.4071	1.7368	0.0024	-10.7940	31.5879	36.5666
		2.2	3		4	3	2.9864	3.0214	5.1299	1.3835	0.0007	-7.4430	24.8859	29.8646
			3.5			4	2.8427	2.4470	2.9661	1.9249	0.0025	-10.9378	31.8756	36.8543
II	8	14	22	22	21	24	0.0842	0.1251	0.3435	1.4147	0.8461	-207.0863	424.1727	436.5022
			38			36	0.0463	0.0689	0.1152	1.3591	1.4858	-278.0608	566.1216	578.4511
		18	30		35	14	0.1071	0.1562	0.2666	1.2760	0.3879	-231.5655	473.1311	485.4606
			38			22	0.0929	0.1258	0.1248	1.2874	0.4486	-278.8941	567.7881	580.1177
	10	14	30	28	15	28	0.0546	0.0970	0.2060	1.4255	1.5355	-229.6798	469.3596	481.6891
			38			36	0.0473	0.0790	0.1146	1.3702	1.5078	-277.6793	565.3585	577.6881
		18	30		29	14	0.0908	0.1687	0.2639	1.3715	0.6488	-230.6346	471.2691	483.5987
			38			22	0.0642	0.1176	0.1199	1.4065	1.0223	-278.2243	566.4487	578.7782
III	1.1	1.6	2.4	21	17	18	1.9285	1.9888	3.3652	2.1804	0.0428	-42.3850	94.7701	106.1534
			3			26	1.8135	1.6050	2.1656	2.1324	0.0408	-61.6277	133.2554	144.6387
		1.9	2.4		25	10	1.9262	2.1187	4.8965	2.1808	0.0427	-41.3863	92.7727	104.1560
			3			18	1.8140	1.6102	2.5898	2.1043	0.0396	-60.9120	131.8240	143.2073
	1.3	1.6	2.4	30	8	18	2.0586	1.6644	3.3873	2.4176	0.0402	-42.2304	94.4608	105.8442
			3			26	1.8815	1.2926	2.1763	2.2743	0.0399	-61.2000	132.4000	143.7833
		1.9	2.4		16	10	2.0571	1.9902	4.9177	2.4155	0.0401	-41.4372	92.8744	104.2577
			3			18	1.8835	1.4376	2.6025	2.2621	0.0392	-60.5813	131.1625	142.5459

Table 18. MLE under bivariate SSALT based on PTIC sample when p=0.2.

Data	T_1	T_2	T_3	n_1	n_2	n_3	α_1	α_2	α_3	β	τ	Llog	AI	BI
I	1.6	1.9	3	6	7	2	2.7800	6.6237	5.1808	1.5332	0.0019	-1.3345	12.6689	17.6476
			3.5			3	2.6606	5.5631	2.0821	1.5061	0.0026	-5.5431	21.0862	26.0648
		2.2	3.1		9	2	2.5638	4.3445	2.5229	1.7539	0.0021	-6.6903	23.3806	28.3592
			3.5			2	2.5638	4.3445	2.5229	1.7539	0.0001	-6.6903	23.3806	28.3592
	1.8	1.9	3	11	1	2	3.3879	5.5803	2.5550	1.4507	0.0012	-4.0139	18.0277	23.0064
			3.5			3	3.5248	3.7242	1.6542	2.0645	0.0018	-7.1073	24.2145	29.1932
		2.2	3		3	1	3.2535	4.6798	10.2574	1.5629	0.0019	-3.0837	16.1675	21.1461
			3.5			1	3.2535	4.6798	10.2574	1.5629	0.0029	-3.0837	16.1675	21.1461
II	8	14	22	22	18	16	0.1221	0.1552	0.3637	1.3846	0.5564	-171.7641	353.5283	365.8578
			38			25	0.0621	0.0835	0.1128	1.4401	1.4256	-226.8644	463.7287	476.0583
		18	22		29	5	0.1362	0.2049	0.5140	1.3076	0.3588	-171.8571	353.7143	366.0438
			38			15	0.1127	0.1369	0.1093	1.2970	0.4134	-230.3447	470.6895	483.0190
	10	14	22	28	12	16	0.0711	0.1187	0.3461	1.5427	1.6947	-169.7180	349.4361	361.7656
			38			24	0.0584	0.0858	0.1120	1.4557	1.6740	-220.9976	451.9953	464.3248
		18	22		24	5	0.1297	0.2358	0.5680	1.3375	0.4332	-173.7040	357.4080	369.7376
			38			13	0.1069	0.1581	0.1051	1.3487	0.5302	-224.0241	458.0481	470.3777
III	1.1	1.6	2.4	21	13	9	2.2016	2.4738	4.1560	2.3470	0.0482	-27.9595	65.9191	77.3024
			3			13	2.1023	2.0038	2.3040	2.2828	0.0463	-39.9871	89.9742	101.3575
		1.9	2.4		21	5	2.0900	2.4406	5.7452	2.3197	0.0480	-32.3110	74.6221	86.0054
			3			9	2.0104	1.9690	2.9062	2.2560	0.0457	-43.2342	96.4684	107.8517
	1.3	1.6	2.4	30	6	10	2.3723	1.9721	3.2473	2.7184	0.0406	-32.2770	74.5541	85.9374
			3			14	2.2271	1.5491	2.2233	2.5817	0.0408	-42.5040	95.0080	106.3914
		1.9	2.4		12	8	2.2277	1.8942	4.5920	2.5778	0.0406	-36.5661	83.1323	94.5156
			3			10	2.1666	1.6701	3.5295	2.5178	0.0404	-41.6866	93.3732	104.7565

8. Conclusions

In this article, we derived and studied a new three-parameter lifetime distribution called the GAPEED. Some important statistical and mathematical features (quantile function, ordinary moments, incomplete moments, and moment generating function) were computed. Eight different estimation methods for the distribution parameters, ML, AD, CVM, MPS, LS, RTAD, WLS, and LTAD, were proposed. The Monte Carlo technique was employed to evaluate the quality of different estimators. The importance and flexibility of the GAPEED were demonstrated by utilizing three real datasets. For the GAPEED model, a bivariate SSALT based on PTIC was presented. An optimal test plan under PTIC is expressed by minimizing the asymptotic variance of the MLE of the log of the scale parameter at design stress. Tables 17 and 18 compare the approaches based on various binomial removal values. We conclude from these findings that the effectiveness of this model increases as the value of the binomial removals rises.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RG23142).

Funding statement

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RG23142).

Conflicts of interest

The authors declare no conflict of interest.

References

1. M. M. Abdelwahab, A. B. Ghorbal, A. S. Hassan, M. Elgarhy, E. M. Almetwally, A. F. Hashem, Classical and bayesian inference for the Kavya-Manoharan generalized exponential distribution under generalized progressively hybrid censored data, *Symmetry*, **15** (2023). https://doi.org/10.3390/sym15061193
2. A. A. Alahmadi, M. Alqawba, W. Almutiry, A. W. Shawki, S. Alrajhi, S. Al-Marzouki, et al., A new version of weighted Weibull distribution: Modelling to COVID-19 data, *Discrete Dyn. Nat. Soc.*, **2022** (2022). https://doi.org/10.1155/2022/3994361
3. Z. Ahmad, M. Elgarhy, G. Hamedani, N. S. Butt, Odd generalized NH generated family of distributions with application to exponential model, *Pak. J. Stat. Oper. Res.*, **16** (2020), 53–71. https://doi.org/10.18187/pjsor.v16i1.2295
4. H. Al-Mofleh, M. Elgarhy, A. Afify, M. Zannon, Type II exponentiated half logistic generated family of distributions with applications, *Electron. J. Appl. Stat.*, **13** (2020), 536–561.
5. A. M. Basheer, E. M. Almetwally, H. M. Okasha, Marshall-olkin alpha power inverse Weibull distribution: Non Bayesian and Bayesian estimations, *J. Stat. Appl. Probab.*, **10** (2021), 327–345. https://doi.org/10.18576/jsap/100205
6. E. M. Almetwally, Marshall olkin alpha power extended Weibull distribution: Different methods of estimation based on type I and type II censoring, *Gazi U. J. Sci.*, **35** (2022), 293–312. https://doi.org/10.35378/gujs.741755
7. E. M. Almetwally, The odd Weibull inverse Topp-Leone distribution with applications to COVID-19 data, *Ann. Data Sci.*, **9** (2022), 121–140. https://doi.org/10.1007/s40745-021-00329-w
8. N. Alotaibi, I. Elbatal, E. M. Almetwally, S. A. Alyami, A. S. Al-Moisheer, M. Elgarhy, Truncated Cauchy power Weibull-G class of distributions: Bayesian and non-Bayesian inference modelling for COVID-19 and carbon fiber data, *Mathematics*, **10** (2022). https://doi.org/10.3390/math10091565
9. N. Alotaibi, I. Elbatal, E. M. Almetwally, S. A. Alyami, A. Al-Moisheer, M. Elgarhy, Bivariate step-stress accelerated life tests for the Kavya-Manoharan exponentiated Weibull model under progressive censoring with applications, *Symmetry*, **14** (2022). https://doi.org/10.3390/sym14091791
10. R. Alotaibi, A. Al Mutairi, E. M. Almetwally, C. Park, H. Rezk, Optimal design for a bivariate step-stress accelerated life test with alpha power exponential distribution based on type-I progressive censored samples, *Symmetry*, **14** (2022). https://doi.org/10.3390/sym14040830

11. S. A. Alyami, I. Elbatal, N. Alotaibi, E. M. Almetwally, H. M. Okasha, M. Elgarhy, Topp-Leone modified Weibull model: Theory and applications to medical and engineering data, *Appl. Sci.*, **12** (2022). https://doi.org/10.3390/app122010431

12. C. B. Ampadu, Gull Alpha power of the Ampadu-type: Properties and applications, *Earthline J. Math. Sci.*, **6** (2021), 187–207. https://doi.org/10.34198/ejms.6121.187207

13. T. W. Anderson, D. A. Darling, Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes, *Ann. Math. Stat.*, **23** (1952), 193–212. https://doi.org/10.1214/aoms/1177729437

14. J. M. Astorga, Y. A. Iriarte, H. W. Gómez, H. Bolfarine, Modified slashed generalized exponential distribution, *Commun. Stat.-Theory M.*, **49** (2020), 4603–4617. https://doi.org/10.1080/03610926.2019.1604959

15. N. Balakrishnan, D. Han, G. Iliopoulos, Exact inference for progressively type-I censored exponential failure data, *Metrika*, **73** (2011), 335–358. https://doi.org/10.1007/s00184-009-0281-0

16. R. A. R. Bantan, C. Chesneau, F. Jamal, I. Elbatal, M. Elgarhy, The truncated burr XG family of distributions: Properties and applications to actuarial and financial data, *Entropy*, **23** (2021), 1088. https://doi.org/10.3390/e23081088

17. R. A. Bantan, F. Jamal, C. Chesneau, M. Elgarhy, On a new result on the ratio exponentiated general family of distributions with applications, *Mathematics*, **8** (2020), 598. https://doi.org/10.3390/math8040598

18. K. V. P. Barco, J. Mazucheli, V. Janeiro, The inverse power Lindley distribution, *Commun. Stat.-Simul. C.*, **46** (2017), 6308–6323. https://doi.org/10.1080/03610918.2016.1202274

19. W. B. Souza, A. H. Santos, G. M. Cordeiro, The beta generalized exponential distribution, *J. Stat. Comput. Sim.*, **80** (2010), 159–172. https://doi.org/10.1080/00949650802552402

20. M. Capanu, M. Giurcanu, C. B. Begg, M. Gönen, Subsampling based variable selection for generalized linear models, *Comput. Stat. Data An.*, **184** (2023), 107740. https://doi.org/10.1016/j.csda.2023.107740

21. J. M. Carrasco, E. M. Ortega, G. M. Cordeiro, A generalized modified Weibull distribution for lifetime modeling, *Computat. Stat. Data An.*, **53** (2008), 450–462. https://doi.org/10.1016/j.csda.2008.08.023

22. A. K. Chaudhary, L. P. Sapkota, V. Kumar, Half-Cauchy generalized exponential distribution: Theory and application, *J. Nepal Math. Soc.*, **5** (2022), 1–10. https://doi.org/10.3126/jnms.v5i2.50018

23. K. Choi, W. G. Bulgren, An estimation procedure for mixtures of distributions, *J. Roy. Stat. Soc. B*, **30** (1968), 444–460. https://doi.org/10.1111/j.2517-6161.1968.tb00743.x

24. G. M. Cordeiro, E. M. Ortega, A. J. Lemonte, The exponential-Weibull lifetime distribution, *J. Stat. Comput. Sim.*, **84** (2014), 2592–2606. https://doi.org/10.1080/00949655.2013.797982

25. G. M. Cordeiro, E. M. Ortega, S. Nadarajah, The Kumaraswamy Weibull distribution with application to failure data, *J. Franklin I.*, **347** (2010), 1399–1429. https://doi.org/10.1016/j.jfranklin.2010.06.010

26. G. M. Cordeiro, M. Alizadeh, G. Ozel, B. Hosseini, E. M. M. Ortega, E. Altun, The generalized odd log-logistic class of distributions: Properties, regression models and applications, *J. Stat. Comput. Sim.*, **87** (2017), 908–932. https://doi.org/10.1080/00949655.2016.1238088

27. A. H. El-Bassiouny, N. F. Abdo, H. S. Shahen, Exponential Lomax distribution, *Int. J. Comput. Appl.*, **121** (2015). https://doi.org/10.5120/21602-4713

28. E. A. ElSherpieny, E. M. Almetwally, The exponentiated generalized Alpha power family of distribution: Properties and applications, *Pak. J. Stat. Oper. Res.*, **18** (2022), 349–367. https://doi.org/10.18187/pjsor.v18i2.3515

29. R. C. Gupta, P. L. Gupta, R. D. Gupta, Modeling failure time data by Lehman alternatives, *Commun. Stat.-Theor. M.*, **27** (1998), 887–904. https://doi.org/10.1080/03610929808832134

30. N. Hakamipour, Approximated optimal design for a bivariate step-stress accelerated life test with generalized exponential distribution under type-I progressive censoring, *Int. J. Qual. Reliab. Ma.*, **38** (2021), 1090–1115. https://doi.org/10.1108/IJQRM-05-2020-0150

31. M. A. Haq, M. Elgarhy, S. Hashmi, The generalized odd Burr III family of distributions: Properties, applications and characterizations, *J. Taibah Univ. Sci.*, **13** (2019), 961–971. https://doi.org/10.1080/16583655.2019.1666785

32. E. A. Hassan, M. Elgarhy, E. A. Eldessouky, O. H. M. Hassan, E. A. Amin, E. M. Almetwally, Different estimation methods for new probability distribution approach based on environmental and medical data, *Axioms*, **12** (2023), 220. https://doi.org/10.3390/axioms12020220

33. W. He, Z. Ahmad, A. Z. Afify, H. Goual, The arcsine exponentiated-X family: Validation and insurance application, *Complexity*, **20** (2020), 8394815. https://doi.org/10.1155/2020/8394815

34. M. Ijaz, S. M. A. Alamgir, M. Farooq, S. A. Khan, S. Manzoor, A Gull Alpha Power Weibull distribution with applications to real and simulated data, *Plos One*, **15** (2020). https://doi.org/10.1371/journal.pone.0233080

35. F. Jamal, M. A. Nasir, G. Ozel, M. Elgarhy, N. M. Khan, Generalized inverted Kumaraswamy generated family of distributions: Theory and applications, *J. Appl. Stat.*, **46** (2019), 2927–2944. https://doi.org/10.1080/02664763.2019.1623867

36. F. Jamal, C. Chesneau, K. Aidi, The sine extended odd Fréchet-G family of distribution with applications to complete and censored data, *Math. Slovaca*, **71** (2021), 961–982. https://doi.org/10.1515/ms-2021-0033

37. J. H. K. Kao, Computer methods for estimating Weibull parameters in reliability studies, *IRE T. Reliab. Qual. Contr.*, 1958, 15–22. https://doi.org/10.1109/IRE-PGRQC.1958.5007164

38. H. A. Khogeer, A. Alrumayh, M. M. Abd El-Raouf, M. Kilai, R. Aldallal, Exponentiated gull alpha exponential distribution with application to COVID-19 data, *J. Math.*, **2022** (2022). https://doi.org/10.1155/2022/4255079

39. M. Kilai, G. A. Waititu, W. A. Kibira, H. M. Alshanbari, M. El-Morshedy, A new generalization of Gull Alpha Power family of distributions with application to modeling COVID-19 mortality rates, *Results Phys.*, **36** (2022), 105339. https://doi.org/10.1016/j.rinp.2022.105339

40. D. Kumar, U. Singh, S. K. Singh, A new distribution using sine function its application to bladder cancer patients data, *J. Stat. Appl. Pro.*, **4** (2015), 417–427.

41. A. Mahdavi, D. Kundu, A new method for generating distributions with an application to exponential distribution, *Commun. Stat.-Theor. M.*, **46** (2017), 6543–6557. https://doi.org/10.1080/03610926.2015.1130839

42. M. Kpangay, L. O. Odongo, G. O. Orwa, The Kumaraswamy-Gull Alpha Power Rayleigh distribution: Properties and application to HIV/AIDS data, *Int. J. Sci. Res. Eng.*, **37** (2023), 431–442.

43. E. M. Almetwally, R. Alotaibi, H. Rezk, Estimation and prediction for Alpha-Power Weibull distribution based on hybrid censoring, *Symmetry*, **15** (2023), 1687. https://doi.org/10.3390/sym15091687

44. G. S. Mudholkar, D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE T. Reliab.*, **42** (1993), 299–302. https://doi.org/10.1109/24.229504

45. G. S. Mudholkar, D. K. Srivastava, M. Freimer, The exponentiated Weibull family: A reanalysis of the bus-motor-failure data, *Technometrics*, **37** (1995), 436–445. https://doi.org/10.1080/00401706.1995.10484376

46. M. Muhammad, R. A. R. Bantan, L. Liu, C. Chesneau, M. H. Tahir, F. Jamal, et al., The truncated Burr XG family of distributions: Properties and applications to actuarial and financial data, *Mathematics*, **9** (2021), 2758. https://doi.org/10.3390/math9212758

47. M. S. Mukhtar, M. El-Morshedy, M. S. Eliwa, H. M. Yousof, Expanded Fréchet model: Mathematical properties, copula, different estimation methods, applications and validation testing, *Mathematics*, **8** (2020), 1949. https://doi.org/10.3390/math8111949

48. S. Nadarajah, The exponentiated exponential distribution: A survey, *ASTA-Adv. Stat. Anal.*, **95** (2011), 219–251. https://doi.org/10.1007/s10182-011-0154-5

49. W. B. Nelson, *Accelerated testing: Statistical models, test plans, and data analysis*, John Wiley & Sons, 2009.

50. Z. M. Nofal, A. Z. Afify, H. M. Yousof, G. M. Cordeiro, The generalized transmuted-G family of distributions, *Commun. Stat.-Theor. M.*, **46** (2017), 4119–4136. https://doi.org/10.1080/03610926.2015.1078478

51. A. H. Tolba, A. H. Muse, A. Fayomi, H. M. Baaqeel, E. M. Almetwally, The Gull Alpha Power Lomax distributions: Properties, simulation, and applications to modeling COVID-19 mortality rates, *Plos One*, **18** (2023). https://doi.org/10.1371/journal.pone.0283308

52. M. M. Ristic, D. Kundu, Marshall-Olkin generalized exponential distribution, *J. Stat. Comput. Sim.*, **73** (2015), 317–333. https://doi.org/10.1007/s40300-014-0056-x

53. T. Ruzgas, M. Lukauskas, G. Čepkauskas, Nonparametric multivariate density estimation: Case study of Cauchy mixture model, *Mathematics*, **9** (2021), 2717. https://doi.org/10.3390/math9212717

54. L. P. Sapkota, V. Kumar, Odd lomax generalized exponential distribution: Application to engineering and COVID-19 data, *Pak. J. Stat. Oper. Res.*, **18** (2022), 883–900. https://doi.org/10.18187/pjsor.v18i4.4149

55. D. C. U. Sivakumar, R. Kanaparthi, G. S. Rao, K. Kalyani, The odd generalized exponential log-logistic distribution group acceptance sampling plan, *Stat. Transition New Series*, **20** (2019), 103–116. https://doi.org/10.21307/stattrans-2019-006

56. L. Souza, W. Junior, C. de Brito, C. Chesneau, R. Fernandes, T. Ferreira, Tan-G class of trigonometric distributions and its applications, *Cubo*, **23** (2021), 1–20. https://doi.org/10.4067/S0719-06462021000100001

57. L. Souza, W. R. O. Junior, C. C. R. de Brito, C. Chesneau, T. A. E. Ferreira, L. Soares, General properties for the Cos-G class of distributions with applications, *Eurasian Bull. Math.*, **2** (2019), 63–79.

58. J. J. Swain, S. Venkatraman, J. R. Wilson, Least-squares estimation of distribution functions in Johnson's translation system, *J. Stat. Comput. Sim.*, **29** (1988), 271–297. https://doi.org/10.1080/00949658808811068

59. H. Torabi, N. H. Montazeri, The logistic-uniform distribution and its application, *Commun. Stat.-Simul. C.*, **43** (2014), 2551–2569. https://doi.org/10.1080/03610918.2012.737491

60. X. Romao, R. Delgado, A. Costa, An empirical power comparison of univariate goodness-of-fit tests for normality, *J. Stat. Computat. Sim.*, **80** (2010), 545–591. https://doi.org/10.1080/00949650902740824

61. R. A. ZeinEldin, C. Chesneau, F. Jamal, M. Elgarhy, A. M. Almarashi, S. Al-Marzouki, Generalized truncated Fréchet generated family distributions and their applications, *CMES-Comp. Model. Eng.*, **126** (2021), 791–819. https://doi.org/10.32604/cmes.2021.012169

62. C. Zhao, R. Zhuang, Statistical solutions and Liouville theorem for the second order lattice systems with varying coefficients, *J. Differ. Equations*, **372** (2023), 194–234. https://doi.org/10.1016/j.jde.2023.06.040

63. S. Zhou, A. Xu, Y. Tang, L. Shen, Fast Bayesian inference of reparameterized Gamma process with random effects, *IEEE T. Reliab.*, **2023** (2023) 1–14. https://doi.org/10.1109/TR.2023.3263940

AIMS Press