Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 21138–21160.
DOI: 10.3934/math.20231077
Received: 29 April 2023
Revised: 06 June 2023
Accepted: 15 June 2023
Published: 03 July 2023

Research article

On Mann-type accelerated projection methods for pseudomonotone variational inequalities and common fixed points in Banach spaces

Lu-Chuan Ceng[1], **Yeong-Cheng Liou**[2,*] **and Tzu-Chien Yin**[3,*]

[1] Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

[2] Department of Healthcare Administration and Medical Informatics, and Research Center of Nonlinear Analysis and Optimization, Kaohsiung Medical University, and Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 807, Taiwan

[3] Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: simplex_liou@hotmail.com; yintzuchien@mail.cmuh.org.tw.

Abstract: In this paper, we investigate two Mann-type accelerated projection procedures with line search method for solving the pseudomonotone variational inequality (VIP) and the common fixed-point problem (CFPP) of finitely many Bregman relatively nonexpansive mappings and a Bregman relatively asymptotically nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces. Under mild conditions, we show weak and strong convergence of the proposed algorithms to a common solution of the VIP and CFPP, respectively.

Keywords: pseudomonotone variational inequality; Mann-type accelerated projection method; line-search method; common fixed point; Bregman distance; Bregman projection
Mathematics Subject Classification: 47H09, 47H10, 47J20, 47J25

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\emptyset \neq C \subset H$ be a convex and closed set. Let $\mathrm{Fix}(S)$ be the set of fixed points of a mapping $S : C \to C$, i.e., $\mathrm{Fix}(S) := \{x \in C : x = Sx\}$. S is said to be asymptotically nonexpansive if $\exists\{\theta_n\} \subset [0, +\infty)$ s.t. $\lim_{n\to\infty} \theta_n = 0$ and for all $n \geq 1$,

$$\|S^n u - S^n v\| \leq (1 + \theta_n)\|u - v\|, \ \forall u, v \in C. \tag{1.1}$$

S is nonexpansive when $\theta_n \equiv 0, \ \forall n \geq 1$.

Recall that the variational inequality (VIP) pursues to search $z \in C$ such that

$$\langle Fz, x - z\rangle \geq 0,\ \forall x \in C,$$

where $F : H \to H$ is an operator. Use $\mathrm{VI}(C, F)$ to denote the solution set of VIP.

Korpelevich [11] invented an extragradient method for solving VIP: The sequence $\{w_n\}$ is derived from an initial point $w_0 \in C$ and

$$\begin{cases} z_n = P_C(w_n - \ell F w_n), \\ w_{n+1} = P_C(w_n - \ell F z_n),\ \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\ell \in (0, \frac{1}{L})$ with L being the Lipschitz constant of F. If $\mathrm{VI}(C, F) \neq \emptyset$, then $\{w_n\}$ is convergent weakly to $w^* \in \mathrm{VI}(C, F)$. For solving VIP, many algorithms were introduced and adapted, see [1–3, 5, 7, 8, 10, 12–18, 20, 23, 28, 29, 31, 32]. Within the extragradient method, one needs to compute two projections onto C per iteration. If C is a general convex and closed set, this might result in a prohibitive amount of computation time. To overcome this drawback, Censor et al. [2] presented a subgradient extragradient algorithm in which a half-space is constructed. Reich et al. [13] suggested an iterate for solving the pseudomonotone variational inequality by constructing a hyperplane.

Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E with $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let E^* be the dual space of E. Let J_E^p and $J_{E^*}^q$ be the duality mappings of E and E^*, respectively. Set $f_p(x) = \|x\|^p/p,\ \forall x \in E$. Use D_{f_p} and Π_C to denote the Bregman distance and the Bregman projection from E onto C with respect to (w.r.t) f_p, respectively. Eskandani et al. [18] introduced the hybrid projection method for finding a common solution of the VIP for uniformly continuous pseudomonotone mapping $F : E \to E^*$ and the FPP of Bregman relatively nonexpansive mapping T. Their algorithm is formulated as follows.

Algorithm 1.1 ([18]). *Let $\mu > 0,\ l \in (0, 1),\ \lambda \in (0, \frac{1}{\mu})$ be three constants. Let $x_1 \in C$ be an initial point.*

Step 1. Calculate $y_n = \Pi_C(J_{E^*}^q(J_E^p x_n - \lambda F x_n))$ and $r_\lambda(x_n) := x_n - y_n$. If $Tx_n = x_n$ and $r_\lambda(x_n) = 0$, then stop (in this case $x_n \in \Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C, F)$); otherwise, continue to the next step.

Step 2. Calculate $t_n = x_n - \tau_n r_\lambda(x_n)$ in which $\tau_n := l^{j_n}$ with j_n being the smallest nonnegative integer such that

$$\langle F x_n - F(x_n - l^{j_n} r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \frac{\mu}{2} D_{f_p}(x_n, y_n).$$

Step 3. Calculate $v_n = J_{E^*}^q(\beta_n J_E^p x_n + (1 - \beta_n) J_E^p(T\Pi_{C_n} x_n))$ and $x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1 - \alpha_n) J_E^p v_n))$, where $C_n := \{x \in C : h_n(x_n) \leq 0\}$ and $h_n(x) = \langle F t_n, x - x_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(x_n, y_n)$.

Let $n := n + 1$ and return to Step 1.

Under suitable conditions, they proved the strong convergence of Algorithm 1.1 to $\Pi_\Omega u$. Inspired by the above research works, the main purpose of this paper is to introduce two Mann-type accelerated projection methods for solving the VIP for a uniformly continuous pseudomonotone operator and the CFPP of finitely many Bregman relatively nonexpansive mappings and a Bregman relatively asymptotically nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces. Under mild conditions, we prove weak and strong convergence of the proposed algorithms to a common solution of the VIP and CFPP, respectively. An illustrated example is provided to demonstrate the applicability and implementability of our suggested method. Our algorithms are more advantageous and more flexible than the above Algorithm 1.1 because they involve solving the VIP for uniformly continuous pseudomonotone operator and the CFPP of finitely many Bregman relatively nonexpansive mappings and a Bregman relatively asymptotically nonexpansive mapping. The main theorems presented in this paper are the improvement and extension of the corresponding theorems obtained in [13, 17, 18].

2. Preliminaries

Let $\{x_n\}$ be a sequence of a real Banach space E. Let $\omega_w(x_n)$ be the set of all weak cluster points of $\{x_n\}$, i.e., $\omega_w(x_n) = \{x^\dagger \in E : x_{n_k} \rightharpoonup x^\dagger \text{ for some } \{x_{n_k}\} \subset \{x_n\}\}$.

Let E be a Banach space and $U := \{u \in E : \|u\| = 1\}$. (i) E is strictly convex if $\|u+v\|/2 < 1, \forall u, v \in U$ and $u \neq v$. (ii) E is uniformly convex if $\forall \varepsilon \in (0, 2]$, $\exists \delta > 0$ such that $\|u + v\|/2 \leq 1 - \delta, \forall u, v \in U$ when $\|u - v\| \geq \varepsilon$.

E is uniformly convex $\Leftrightarrow$ for all $\varepsilon \in (0, 2]$, $\delta(\varepsilon) > 0$ where $\delta(\varepsilon) = \inf\{1 - \|u + v\|/2 : u, v \in U \text{ with } \|u - v\| \geq \varepsilon\}$ is the modulus of convexity of E. Moreover, E is p-uniformly convex if $\exists c > 0$ s.t. $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. E is uniformly smooth if $\lim_{\tau\to 0} \rho_E(\tau)/\tau = 0$ where $\rho_E(\tau) = \sup\{(\|u + \tau v\| + \|u - \tau v\|)/2 - 1 : u, v \in U\}$ is the modulus of smoothness of E. E is q-uniformly smooth if $\exists C_q > 0$ s.t. $\rho_E(\tau) \leq C_q\tau^q$, $\forall \tau > 0$. E is p-uniformly convex $\Leftrightarrow$ E^* is q-uniformly smooth. For more details, please refer to [19].

Let $r > 0$ and set $B(0, r) = \{x \in E : \|x\| \leq r\}$. Let $f : E \to \mathbf{R}$ be a function. For $\alpha \in (0, 1)$ and $u, v \in B(0, r)$ with $\|u - v\| = t$. Define

$$\rho_r(t) = \inf\{[\alpha f(u) + (1 - \alpha) f(v) - f(\alpha u + (1 - \alpha)v)]/\alpha(1 - \alpha)\}, \ t \geq 0.$$

E is uniformly convex on bounded set $B(0, r)$ if $\rho_r(t) > 0$ for all $r, t > 0$ (see [9, 18]) .

Set $\frac{1}{p} + \frac{1}{q} = 1$ where $p, q \in (1, \infty)$. The duality mapping $J_E^p : E \to E^*$ is formulated below

$$J_E^p(u) = \{\psi \in E^* : \langle \psi, u\rangle = \|u\|^p \text{ and } \|\psi\| = \|u\|^{p-1}\}, \ \forall u \in E.$$

(i) E is smooth $\Leftrightarrow$ J_E^p is single-valued. (ii) E is reflexive $\Leftrightarrow$ J_E^p is surjective. (ii) E is strictly convex $\Leftrightarrow$ J_E^p is one-to-one.

Let $f : E \to \mathbf{R}$ be a convex function. f is said to be Gâteaux differentiable at x if for each $y \in E$, $\lim_{t\to 0^+} \frac{f(u+tv)-f(u)}{t}$ exists. In this case, define $\langle \nabla f(u), v\rangle = \lim_{t\to 0^+} \frac{f(u+tv)-f(u)}{t}$ for each $v \in E$. Suppose $f : E \to \mathbf{R}$ is Gâteaux differentiable. The Bregman distance ([21]) w.r.t. f is formulated as

$$D_f(u, v) := f(u) - f(v) - \langle \nabla f(v), u - v\rangle, \ \forall u, v \in E.$$

The Bregman distance ensures existence and uniqueness of the Bregman projection and it has also been used to generate generalized proximal point methods for convex optimization and variational inequalities, see [30]. It is easy to check that

$$D_f(u, v) + D_f(v, w) = D_f(u, w) - \langle \nabla f(v) - \nabla f(w), u - v\rangle, \ \forall u, v, w \in E.$$

Note that the Bregman distance w.r.t. f_p is formulated by $\forall u, v \in E$,

$$\begin{aligned} D_{f_p}(u, v) &= \|u\|^p/p - \|v\|^p/p - \langle J_E^p(v), u - v\rangle \\ &= \|u\|^p/p + \|v\|^p/q - \langle J_E^p(v), u\rangle \\ &= (\|v\|^p - \|u\|^p)/q - \langle J_E^p(v) - J_E^p(u), u\rangle. \end{aligned}$$

If E is p-uniformly convex and smooth Banach space E, then (see [26])

$$\tau\|u - v\|^p \le D_{f_p}(u, v) \le \langle J_E^p(u) - J_E^p(v), u - v\rangle,\ 2 \le p < \infty, \tau > 0. \tag{2.1}$$

From (2.1) it is readily known that for any bounded sequence $\{x_n\} \subset E$, the following holds:

$$x_n \to u \quad \Leftrightarrow \quad D_{f_p}(u, x_n) \to 0 \quad (n \to \infty).$$

Let E be a reflexive, smooth and strictly convex Banach space and $C \subset E$ a nonempty closed convex set. For every $u \in E$, there exists the unique element denoted by $\Pi_C u \in C$ such that $D_{f_p}(\Pi_C u, u) = \min_{v\in C} D_{f_p}(v, u)$. Π_C is called the Bregman projection w.r.t. f_p. Furthermore, if E is uniformly convex, then ([22, 24])

$$\langle J_E^p(u) - J_E^p(\Pi_C u), v - \Pi_C u\rangle \le 0,\ \forall v \in C, \tag{2.2}$$

which equivalent to

$$D_{f_p}(v, \Pi_C u) + D_{f_p}(\Pi_C u, u) \le D_{f_p}(v, u),\ \forall v \in C. \tag{2.3}$$

Let $V_{f_p} : E \times E^* \to [0, \infty)$ ([18]) be a function defined by

$$V_{f_p}(u, u^*) = \|u\|^p/p - \langle u^*, u\rangle + \|u^*\|^q/q, \quad \forall (u, u^*) \in E \times E^*. \tag{2.4}$$

For all $u \in E$, $u^* \in E^*$ and $v^* \in E^*$, we have $V_{f_p}(u, u^*) = D_{f_p}(u, J_{E^*}^q(u^*))$ and

$$V_{f_p}(u, u^*) + \langle v^*, J_{E^*}^q(u^*) - u\rangle \le V_{f_p}(u, u^* + v^*). \tag{2.5}$$

In addition, $V_{f_p}(x, \cdot)$ is convex. Then, for all $w \in E, \{u_i\}_{i=1}^n \subset E$, $\{t_i\}_{i=1}^n \subset [0, 1]$ and $\sum_{i=1}^n t_i = 1$, we have

$$D_{f_p}(w, J_{E^*}^q(\sum_{i=1}^n t_i J_E^p(u_i))) \le \sum_{i=1}^n t_i D_{f_p}(w, u_i). \tag{2.6}$$

Lemma 2.1 ([24]). *Let E be a uniformly convex Banach space. Let $\{u_n\} \subset E, \{v_n\} \subset E$ be two sequences and $\{u_n\}$ is bounded. Then,* $\lim_{n\to\infty} D_{f_p}(v_n, u_n) = 0 \Rightarrow \lim_{n\to\infty} \|v_n - u_n\| = 0$.

Let $T : C \to C$ be an operator. A point $x \in C$ is an asymptotic fixed point of T ([25]) if $\exists\{x_n\} \subset C$ s.t. $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$. Let $\mathrm{Fix}(T)$ and $\widehat{\mathrm{Fix}}(T)$ be the set of fixed points of T and the set of asymptotic fixed points of T, respectively. T is said to be Bregman relatively asymptotically nonexpansive w.r.t. f_p if $\mathrm{Fix}(T) = \widehat{\mathrm{Fix}}(T) \ne \emptyset$, and $\exists\{\theta_n\} \subset [0, \infty)$ s.t.

$$D_{f_p}(u, T^n v) \le (1 + \theta_n) D_{f_p}(u, v),\ \forall n \ge 1,$$

for all $v \in C$ and $u \in \mathrm{Fix}(T)$.

Recall that an operator $F : C \to E^*$ is said to be

(i) monotone on C if $\langle Fu - Fv, u - v\rangle \geq 0,\ \forall u, v \in C$;

(ii) pseudomonotone if $\langle Fu, v - u\rangle \geq 0 \Rightarrow \langle Fv, v - u\rangle \geq 0,\ \forall u, v \in C$;

(iii) L-Lipschitz continuous if $\exists L > 0$ s.t. $\|Fu - Fv\| \leq L\|u - v\|, \forall u, v \in C$;

(iv) weakly sequentially continuous if for any $\{x_n\} \subset C$, $x_n \rightharpoonup x \Rightarrow Fx_n \rightharpoonup Fx$.

Lemma 2.2 ([18]). *Let E be a Banach space and $f : E \to \mathbf{R}$ be a uniformly convex function on $B(0, r)$. Let $\{x_k\}_{k=1}^n$ be a sequence in $B(0, r)$ and $\{\alpha_k\}_{k=1}^n$ be a real number sequence in $(0, 1)$ such that $\sum_{k=1}^n \alpha_k = 1$. Then,*

$$f(\sum_{k=1}^{n} \alpha_k x_k) \leq \sum_{k=1}^{n} \alpha_k f(x_k) - \alpha_i\alpha_j\rho_r(\|x_i - x_j\|),\ \forall i, j \in \{1, 2, ..., n\}.$$

Lemma 2.3 ([15]). *Let E_1 and E_2 be two Banach spaces. Let $D \subset E_1$ be a bounded set. If $F : E_1 \to E_2$ is uniformly continuous on D, then $F(D)$ is bounded.*

Lemma 2.4 ([6]). *Let C be a nonempty closed convex subset of a real Banach space E. Let $F : C \to E^*$ be a continuous pseudomonotone operator. Then $u \in VI(C, F) \Leftrightarrow \langle Fv, v - u\rangle \geq 0,\ \forall v \in C$.*

Lemma 2.5. *Let $2 \leq p < \infty$ and let E be a smooth and p-uniformly convex Banach space with weakly sequentially continuous duality mapping J_E^p. Let $\{x_n\}$ be a sequence in E and C be a nonempty subset of E. Suppose that $\{D_{f_p}(x, x_n)\}$ converges for every $x \in C$, and $\omega_w(x_n) \subset C$. Then $\{x_n\}$ converges weakly to a point in C.*

Proof. Since the inequality (2.1) leads to $\tau\|x-x_n\|^p \leq D_{f_p}(x, x_n),\ \forall x \in C$, we know that $\{x_n\}$ is bounded. Hence from the reflexivity of E it follows that $\omega_w(x_n) \neq \emptyset$. In what follows, we claim that $\omega_w(x_n)$ is a single-point set. Indeed, let $x, y \in \omega_w(x_n) \subset C$ with $x \neq y$. Then, $\exists\{x_{n_k}\} \subset \{x_n\}$ and $\exists\{x_{m_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup x$ and $x_{m_k} \rightharpoonup y$. By the weakly sequential continuity of J_E^p one has that $J_E^p(x_{n_k}) \rightharpoonup x$ and $J_E^p(x_{m_k}) \rightharpoonup y$. Note that $D_{f_p}(x, y) + D_{f_p}(y, x_n) = D_{f_p}(x, x_n) - \langle J_E^p y - J_E^p x_n, x - y\rangle$. Since $\{D_{f_p}(x, x_n)\}$ and $\{D_{f_p}(y, x_n)\}$ are convergent, we obtain

$$\begin{aligned}
-\langle J_E^p y - J_E^p x, x - y\rangle &= \lim_{k\to\infty}[-\langle J_E^p y - J_E^p x_{n_k}, x - y\rangle] \\
&= \lim_{n\to\infty}[D_{f_p}(x, y) + D_{f_p}(y, x_n) - D_{f_p}(x, x_n)] \\
&= \lim_{k\to\infty}[-\langle J_E^p y - J_E^p x_{m_k}, x - y\rangle] \\
&= -\langle J_E^p y - J_E^p y, x - y\rangle = 0,
\end{aligned}$$

which immediately yields $\langle J_E^p x - J_E^p y, x - y\rangle = 0$. Again from (2.1) we have $0 < \tau\|x - y\|^p \leq D_{f_p}(x, y) \leq \langle J_E^p x - J_E^p y, x - y\rangle = 0$. It is impossible. So, $\omega_w(x_n)$ is a single-point set. □

Lemma 2.6 ([16]). *Let C be a nonempty closed convex subset of a Banach space E. Define $D := \{x \in C : g(x) \leq 0\}$ where g is a real-valued function on E. If $D \neq \emptyset$ and g is Lipschitz continuous on C with modulus $\theta > 0$, then* $\mathrm{dist}(x, D) \geq \theta^{-1} \max\{g(x), 0\},\ \forall x \in C$.

Lemma 2.7 ([4]). *Let $\{a_n\}$ be a sequence of nonnegative numbers such that $a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\mu_n + \nu_n$ $\forall n \geq 1$, where the following hold for sequences $\{\lambda_n\}, \{\mu_n\}, \{\nu_n\} \subset \mathbf{R}$:*

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \mu_n \leq 0$ *and* $\sum_{n=1}^{\infty} |\nu_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 ([27]). *Let $\{\Phi_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists\{\Phi_{n_k}\} \subset \{\Phi_n\}$ s.t. $\Phi_{n_k} < \Phi_{n_k+1}$, $\forall k \geq 1$. Let $n_0 \geq 1$ and $\{\psi(n)\}_{n\geq n_0}$ be integers sequence defined by $\psi(n) = \max\{k \leq n : \Phi_k < \Phi_{k+1}\}$ satisfying $\{k \leq n_0 : \Phi_k < \Phi_{k+1}\} \neq \emptyset$. Then,*

(i) $\psi(n_0) \leq \psi(n_0+1) \leq \cdots$ *and* $\psi(n) \to \infty$;

(ii) for all $n \geq n_0$, $\Phi_{\psi(n)} \leq \Phi_{\psi(n)+1}$ *and* $\Phi_n \leq \Phi_{\psi(n)+1}$.

3. Main results

Let C be a nonempty closed convex subset of of a p-uniformly convex and uniformly smooth Banach space E. Suppose that

(C1) the mapping $T : C \to C$ is Bregman relatively asymptotically nonexpansive with $\{\theta_n\}$ and uniformly continuous.

(C2) the mapping $T_i : C \to C (i = 1, ..., N)$ is Bregman relatively nonexpansive and uniformly continuous and $T_n := T_{n \bmod N}$ for integer $n \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$.

(C3) the mapping $F : E \to E^*$ is uniformly continuous and pseudomonotone such that $\|Fz\| \leq \liminf_{n\to\infty} \|Fx_n\|$ for any $\{x_n\} \subset C$ with $x_n \rightharpoonup z$.

(C4) $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, F) \neq \emptyset$ where $T_0 := T$.

Let $\mu > 0$, $\lambda \in (0, \frac{1}{\mu})$ and $l \in (0, 1)$ be three constants. Let $\{\sigma_n\}, \{\alpha_n\}$ be two sequences in $(0, 1)$ s.t. $\liminf_{n\to\infty} \sigma_n(1-\sigma_n) > 0$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

Algorithm 3.1. *Let $x_1 \in C$ be an initial point.*

Step 1. Calculate $w_n = J^q_{E^*}(\sigma_n J^p_E x_n + (1-\sigma_n)J^p_E(T_n x_n))$, $y_n = \Pi_C(J^q_{E^*}(J^p_E w_n - \lambda F w_n))$ *and* $r_\lambda(w_n) := w_n - y_n$.

Step 2. Calculate $t_n = w_n - \tau_n r_\lambda(w_n)$, *where* $\tau_n := l^{j_n}$ *with* j_n *being the smallest nonnegative integer* j *such that*

$$\langle Fw_n - F(w_n - l^j r_\lambda(w_n)), w_n - y_n\rangle \leq \frac{\mu}{2} D_{f_p}(w_n, y_n). \tag{3.1}$$

Step 3. Calculate $v_n = J^q_{E^*}(\alpha_n J^p_E w_n + (1-\alpha_n)J^p_E(T^n w_n))$ *and* $x_{n+1} = \Pi_{C_n\cap Q_n}(w_n)$, *where* $Q_n := \{x \in C : D_{f_p}(x, v_n) \leq (1+\theta_n)D_{f_p}(x, w_n)\}$, $C_n := \{x \in C : h_n(x) \leq 0\}$ *and*

$$h_n(x) = \langle Ft_n, x - w_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(w_n, y_n). \tag{3.2}$$

Set $n := n + 1$ *and go to Step 1.*

Lemma 3.2. *Suppose that the sequence $\{x_n\}$ is constructed in Algorithm 3.1. Then the inequality holds:* $\langle Fw_n, r_\lambda(w_n)\rangle \geq \frac{1}{\lambda} D_{f_p}(w_n, y_n)$.

Proof. Using the property of Π_C, we obtain

$$\langle J_E^p w_n - \lambda F w_n - J_E^p y_n, w_n - y_n \rangle \leq 0.$$

It follows from (2.1) that

$$D_{f_p}(w_n, y_n) \leq \langle J_E^p w_n - J_E^p y_n, w_n - y_n \rangle \leq \lambda \langle F w_n, w_n - y_n \rangle.$$

□

Lemma 3.3. *The rule* (3.1) *and* $\{x_n\}$ *generated by Algorithm 3.1 are well defined.*

Proof. Note that $\lim_{j\to\infty} \langle F w_n - F(w_n - l^j r_\lambda(w_n)), r_\lambda(w_n) \rangle = 0$. If $r_\lambda(w_n) = 0$, then it is obvious that $j_n = 0$. If $r_\lambda(w_n) \neq 0$, then there is $j_n \geq 0$ fulfilling (3.1).

It is easy to see that for every $n \geq 1$, C_n and Q_n are closed and convex. Next, we show $\Omega \subset C_n \cap Q_n$. Take any $z \in \Omega$. By (2.6) and the Bregman relatively asymptotical nonexpansivity of T, we have

$$\begin{aligned} D_{f_p}(z, v_n) &\leq \alpha_n D_{f_p}(z, w_n) + (1 - \alpha_n) D_{f_p}(z, T^n w_n) \\ &\leq \alpha_n D_{f_p}(z, w_n) + (1 - \alpha_n)(1 + \theta_n) D_{f_p}(z, w_n) \\ &\leq (1 + \theta_n) D_{f_p}(z, w_n), \end{aligned}$$

which immediately yields $z \in Q_n$. Moreover, using Lemma 2.4, we have $\langle F t_n, t_n - z \rangle \geq 0$. Hence

$$\begin{aligned} h_n(z) &= \langle F t_n, z - w_n \rangle + \frac{\tau_n}{2\lambda} D_{f_p}(w_n, y_n) \\ &= -\langle F t_n, w_n - t_n \rangle - \langle F t_n, t_n - z \rangle + \frac{\tau_n}{2\lambda} D_{f_p}(w_n, y_n) \\ &\leq -\tau_n \langle F t_n, r_\lambda(w_n) \rangle + \frac{\tau_n}{2\lambda} D_{f_p}(w_n, y_n). \end{aligned} \tag{3.3}$$

Thanks to (3.1), we have

$$\langle F w_n - F t_n, r_\lambda(w_n) \rangle \leq \frac{\mu}{2} D_{f_p}(w_n, y_n).$$

Using this and Lemma 3.2, we have

$$\begin{aligned} \langle F t_n, r_\lambda(w_n) \rangle &\geq \langle F w_n, r_\lambda(w_n) \rangle - \frac{\mu}{2} D_{f_p}(w_n, y_n) \\ &\geq (\frac{1}{\lambda} - \frac{\mu}{2}) D_{f_p}(w_n, y_n). \end{aligned}$$

Combining this and (3.3) to deduce

$$h_n(z) \leq -\frac{\tau_n}{2}(\frac{1}{\lambda} - \mu) D_{f_p}(w_n, y_n) \leq 0.$$

Consequently, $\Omega \subset C_n \cap Q_n$. Therefore, $\{x_n\}$ is well defined. □

Lemma 3.4. *Let the sequence* $\{w_n\}$ *be defined by Algorithm 3.1. Then* $\lim_{n\to\infty} \|w_n - y_n\| = 0$ *implies that* $\omega_w(w_n) \subset \mathrm{VI}(C, F)$.

Proof. Let $z \in \omega_w(w_n)$. Then, $\exists\{w_{n_k}\} \subset \{w_n\}$, s.t. $w_{n_k} \rightharpoonup z$ and $\lim_{n\to\infty} \|w_{n_k} - y_{n_k}\| = 0$. Hence, it is known that $y_{n_k} \rightharpoonup z$. Since C is convex and closed and $\{y_n\} \subset C$, $z \in C$. Next, we consider two cases. If $Fz = 0$, then $z \in \mathrm{VI}(C, F)$. If $Fz \neq 0$, using the assumption on F, instead of the weakly sequential continuity of F, we get $0 < \|Fz\| \le \liminf_{k\to\infty} \|Fw_{n_k}\|$. So, we might assume that $\|Fw_{n_k}\| \neq 0, \forall k \ge 1$. Using (2.2), we obtain

$$\langle J_E^p w_{n_k} - \lambda F w_{n_k} - J_E^p y_{n_k}, x - y_{n_k}\rangle \le 0,$$

and hence

$$\frac{1}{\lambda}\langle J_E^p w_{n_k} - J_E^p y_{n_k}, x - y_{n_k}\rangle + \langle Fw_{n_k}, y_{n_k} - w_{n_k}\rangle \le \langle Fw_{n_k}, x - w_{n_k}\rangle. \tag{3.4}$$

By Lemma 2.3, $\{Fw_{n_k}\}$ is bounded. Note that $\{y_{n_k}\}$ is also bounded as well. From (3.4) we get

$$\liminf_{k\to\infty}\langle Fw_{n_k}, x - w_{n_k}\rangle \ge 0, \quad \forall x \in C. \tag{3.5}$$

Let $\{\epsilon_k\}$ be a sequence in $(0, 1)$ fulfilling $\epsilon_k \downarrow 0$ as $k \to \infty$. Let l_k be the smallest positive integer satisfying

$$\langle Fw_{n_j}, y - w_{n_j}\rangle + \epsilon_k \ge 0, \quad \forall j \ge l_k. \tag{3.6}$$

Since $\{\epsilon_k\}$ is decreasing, $\{l_k\}$ is increasing. For convenience, we denote $\{Fw_{n_{l_k}}\}$ by $\{Fw_{l_k}\}$. Note that $Fw_{l_k} \neq 0$, $\forall k \ge 1$. Put $\upsilon_{l_k} = \frac{Fw_{l_k}}{\|Fw_{l_k}\|^{\frac{q}{q-1}}}$. We have $\langle Fw_{l_k}, J_{E^*}^q \upsilon_{l_k}\rangle = 1$, $\forall k \ge 1$. Indeed, it is clear that $\langle Fw_{l_k}, J_{E^*}^q \upsilon_{l_k}\rangle = \langle Fw_{l_k}, (\frac{1}{\|Fw_{l_k}\|^{\frac{q}{q-1}}})^{q-1} J_{E^*}^q Fw_{l_k}\rangle = (\frac{1}{\|Fw_{l_k}\|^{\frac{q}{q-1}}})^{q-1}\|Fw_{l_k}\|^q = 1$, $\forall k \ge 1$. So, using (3.6) one has $\langle Fw_{l_k}, y + \epsilon_k J_{E^*}^q \upsilon_{l_k} - w_{l_k}\rangle \ge 0$, $\forall k \ge 1$. Since F is pseudomonotone, we have

$$\langle F(y + \epsilon_k J_{E^*}^q \upsilon_{l_k}), y + \epsilon_k J_{E^*}^q \upsilon_{l_k} - w_{l_k}\rangle \ge 0, \quad \forall y \in C. \tag{3.7}$$

We claim that $\lim_{k\to\infty} \epsilon_k J_{E^*}^q \upsilon_{l_k} = 0$. In fact, since $\{w_{l_k}\} \subset \{w_{n_k}\}$ and $\epsilon_k \downarrow 0$, we have

$$0 \le \limsup_{k\to\infty} \|\epsilon_k J_{E^*}^q \upsilon_{l_k}\| = \limsup_{k\to\infty} \frac{\epsilon_k}{\|Fw_{l_k}\|} \le \frac{\limsup_{k\to\infty} \varsigma_k}{\liminf_{k\to\infty} \|Fw_{n_k}\|} = 0.$$

Hence one gets $\epsilon_k J_{E^*}^q \upsilon_{l_k} \to 0$ as $k \to \infty$. Thus, letting $k \to \infty$ in (3.7) and from (C3), we have $\langle Fy, y - z\rangle \ge 0$, $\forall y \in C$. According to Lemma 2.4 one has $z \in \mathrm{VI}(C, F)$. □

Lemma 3.5. *Let the sequence $\{w_n\}$ be generated by Algorithm 3.1. Then,*

$$\lim_{n\to\infty} \tau_n D_{f_p}(w_n, y_n) = 0 \;\;\Rightarrow\;\; \lim_{n\to\infty} D_{f_p}(w_n, y_n) = 0.$$

Proof. Suppose that $\liminf_{n\to\infty} \tau_n > 0$. In this case, assume that $\exists \tau > 0$ s.t. $\tau_n \ge \tau > 0$, $\forall n \ge 1$. Then,

$$D_{f_p}(w_n, y_n) = \frac{1}{\tau_n}\tau_n D_{f_p}(w_n, y_n) \le \frac{1}{\tau} \cdot \tau_n D_{f_p}(w_n, y_n). \tag{3.8}$$

This together with $\lim_{n\to\infty} \tau_n D_{f_p}(w_n, y_n) = 0$, leads to $\lim_{n\to\infty} D_{f_p}(w_n, y_n) = 0$.

Suppose that $\liminf_{n\to\infty} \tau_n = 0$. In this case, assume that $\limsup_{n\to\infty} D_{f_p}(w_n, y_n) = a > 0$. Then we know that $\exists\{n_k\} \subset \{n\}$ such that

$$\lim_{k\to\infty} \tau_{n_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} D_{f_p}(w_{n_k}, y_{n_k}) = a > 0.$$

We define $\overline{t_{n_k}} = \frac{1}{l}\tau_{n_k} y_{n_k} + (1 - \frac{1}{l}\tau_{n_k})w_{n_k}$ for each $k \geq 1$. Applying (2.1) and noticing that $\lim_{k\to\infty} \tau_{n_k} D_{f_p}(w_{n_k}, y_{n_k}) = 0$, we have $\lim_{k\to\infty} \tau_{n_k} \|w_{n_k} - y_{n_k}\|^p = 0$ and hence

$$\lim_{k\to\infty} \|\overline{t_{n_k}} - w_{n_k}\|^p = \lim_{k\to\infty} \frac{\tau_{n_k}^{p-1}}{l^p} \cdot \tau_{n_k} \|w_{n_k} - y_{n_k}\|^p = 0. \tag{3.9}$$

It follows that

$$\lim_{k\to\infty} \|Fw_{n_k} - F\overline{t_{n_k}}\| = 0. \tag{3.10}$$

So,

$$\langle Fw_{n_k} - F\overline{t_{n_k}}, w_{n_k} - y_{n_k} \rangle > \frac{\mu}{2} D_{f_p}(w_{n_k}, y_{n_k}). \tag{3.11}$$

Now, letting $k \to \infty$ and from (3.10) we have $\lim_{k\to\infty} D_{f_p}(w_{n_k}, y_{n_k}) = 0$. It is a contradiction. Therefore, $\lim_{n\to\infty} D_{f_p}(w_n, y_n) = 0$. □

Theorem 3.6. *Suppose that E is a p-uniformly convex and uniformly smooth Banach space with weakly sequentially continuous duality mapping J_E^p. Let the sequence $\{x_n\}$ be defined by Algorithm 3.1. Then $\{x_n\}$ is convergent weakly to a point in Ω provided $T^n w_n - T^{n+1} w_n \to 0$.*

Proof. Take any $z \in \Omega$. Using Lemma 2.2, we get

$$\begin{aligned}
D_{f_p}(z, w_n) &= V_{f_p}(z, \sigma_n J_E^p x_n + (1-\sigma_n) J_E^p T_n x_n) \\
&\leq \frac{1}{p}\|z\|^p - \sigma_n \langle J_E^p x_n, z\rangle - (1-\sigma_n)\langle J_E^p T_n x_n, z\rangle + \frac{\sigma_n}{q}\|J_E^p x_n\|^q \\
&\quad + \frac{(1-\sigma_n)}{q}\|J_E^p T_n x_n\|^q - \sigma_n(1-\sigma_n)\rho_b^*\|J_E^p x_n - J_E^p T_n x_n\| \\
&= \frac{1}{p}\|z\|^p - \sigma_n \langle J_E^p x_n, z\rangle - (1-\sigma_n)\langle J_E^p T_n x_n, z\rangle + \frac{\sigma_n}{q}\|x_n\|^p \\
&\quad + \frac{(1-\sigma_n)}{q}\|T_n x_n\|^p - \sigma_n(1-\sigma_n)\rho_b^*\|J_E^p x_n - J_E^p T_n x_n\| \\
&= \sigma_n D_{f_p}(z, x_n) + (1-\sigma_n) D_{f_p}(z, T_n x_n) - \sigma_n(1-\sigma_n)\rho_b^*\|J_E^p x_n - J_E^p T_n x_n\| \\
&\leq D_{f_p}(z, x_n) - \sigma_n(1-\sigma_n)\rho_b^*\|J_E^p x_n - J_E^p T_n x_n\|.
\end{aligned}$$

From (2.1) and (2.3), we obtain

$$\begin{aligned}
D_{f_p}(z, x_{n+1}) &\leq D_{f_p}(z, w_n) - D_{f_p}(x_{n+1}, w_n) \\
&= D_{f_p}(z, w_n) - D_{f_p}(\Pi_{C_n\cap Q_n} w_n, w_n) \\
&\leq D_{f_p}(z, w_n) - D_{f_p}(\Pi_{C_n} w_n, w_n) \\
&\leq D_{f_p}(z, w_n) - \tau\|\Pi_{C_n} w_n - w_n\|^p \\
&\leq D_{f_p}(z, w_n) - \tau\|P_{C_n} w_n - w_n\|^p \\
&= D_{f_p}(z, w_n) - \tau[\mathrm{dist}(C_n, w_n)]^p.
\end{aligned}$$

Combining the last two inequalities, we obtain

$$D_{f_p}(z, x_{n+1}) \leq D_{f_p}(z, x_n) - \sigma_n(1 - \sigma_n)\rho_b^*\|J_E^p x_n - J_E^p T_n x_n\| - \tau[\mathrm{dist}(C_n, w_n)]^p. \tag{3.12}$$

This indicates that $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists and the sequence $\{x_n\}$ is bounded. It is easy to check that $\{Fw_n\}, \{y_n\}, \{t_n\}, \{v_n\}, \{T_n x_n\}$ and $\{T^n w_n\}$ are also bounded. Note that $\omega_w(x_n) \neq \emptyset$. Next, we show $\omega_w(x_n) \subset \Omega$. Let $z^* \in \omega_w(x_n)$. Then, $\exists\{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z^*$. From (3.12), we obtain

$$\begin{aligned}
D_{f_p}(x_{n+1}, v_n) &\leq (1 + \theta_n) D_{f_p}(x_{n+1}, w_n) \\
&\leq (1 + \theta_n)[D_{f_p}(z, w_n) - D_{f_p}(z, x_{n+1})] \\
&\leq (1 + \theta_n)[D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1})].
\end{aligned}$$

This implies that $\lim_{n\to\infty} D_{f_p}(x_{n+1}, w_n) = \lim_{n\to\infty} D_{f_p}(x_{n+1}, v_n) = 0$ and hence

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = \lim_{n\to\infty} \|x_{n+1} - v_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.13}$$

Using Lemma 2.2, we get

$$\begin{aligned}
D_{f_p}(z, v_n) &= V_{f_p}(z, \alpha_n J_E^p w_n + (1 - \alpha_n) J_E^p T^n w_n) \\
&\leq \frac{1}{p}\|z\|^p - \alpha_n\langle J_E^p w_n, z\rangle - (1 - \alpha_n)\langle J_E^p T^n w_n, z\rangle + \frac{\alpha_n}{q}\|J_E^p w_n\|^q \\
&\quad + \frac{(1 - \alpha_n)}{q}\|J_E^p T^n w_n\|^q - \alpha_n(1 - \alpha_n)\rho_b^*\|J_E^p w_n - J_E^p T^n w_n\| \\
&= \frac{1}{p}\|z\|^p - \alpha_n\langle J_E^p w_n, z\rangle - (1 - \alpha_n)\langle J_E^p T^n w_n, z\rangle + \frac{\alpha_n}{q}\|w_n\|^p \\
&\quad + \frac{(1 - \alpha_n)}{q}\|T^n w_n\|^p - \alpha_n(1 - \alpha_n)\rho_b^*\|J_E^p w_n - J_E^p T^n w_n\| \\
&= \alpha_n D_{f_p}(z, w_n) + (1 - \alpha_n) D_{f_p}(z, T^n w_n) - \alpha_n(1 - \alpha_n)\rho_b^*\|J_E^p w_n - J_E^p T^n w_n\| \\
&\leq \alpha_n(1 + \theta_n) D_{f_p}(z, w_n) + (1 - \alpha_n)(1 + \theta_n) D_{f_p}(z, w_n) \\
&\quad - \alpha_n(1 - \alpha_n)\rho_b^*\|J_E^p w_n - J_E^p T^n w_n\| \\
&= (1 + \theta_n) D_{f_p}(z, w_n) - \alpha_n(1 - \alpha_n)\rho_b^*\|J_E^p w_n - J_E^p T^n w_n\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha_n(1 - \alpha_n)\rho_b^*\|J_E^p w_n - J_E^p T^n w_n\| &\leq (1 + \theta_n) D_{f_p}(z, w_n) - D_{f_p}(z, v_n) \\
&\leq D_{f_p}(z, w_n) - D_{f_p}(z, v_n) + D_{f_p}(w_n, v_n) + \theta_n D_{f_p}(z, w_n) \\
&= \langle J_E^p v_n - J_E^p w_n, z - w_n\rangle + \theta_n D_{f_p}(z, w_n).
\end{aligned}$$

By (3.13), we get $\lim_{n\to\infty} \rho_b^*\|J_E^p w_n - J_E^p T^n w_n\| = 0$ and hence $\lim_{n\to\infty} \|J_E^p w_n - J_E^p T^n w_n\| = 0$. So,

$$\lim_{n\to\infty} \|w_n - T^n w_n\| = 0. \tag{3.14}$$

In addition, from (3.12) we get $\sigma_n(1-\sigma_n)\rho_b^*\|J_E^p x_n - J_E^p T_n x_n\| \le D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1})$. Noticing the existence of $\lim_{n\to\infty} D_{f_p}(z, x_n)$ and $\liminf_{n\to\infty} \sigma_n(1-\sigma_n) > 0$, we have $\lim_{n\to\infty} \rho_b^*\|J_E^p x_n - J_E^p T_n x_n\| = 0$ and hence $\lim_{n\to\infty} \|J_E^p x_n - J_E^p T_n x_n\| = 0$. Thus,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.15}$$

Since $w_n = J_{E^*}^q(\sigma_n J_E^p x_n + (1-\sigma_n) J_E^p T_n x_n)$, we deduce that

$$\|J_E^p w_n - J_E^p x_n\| = (1-\sigma_n)\|J_E^p T_n x_n - J_E^p x_n\| \le \|J_E^p T_n x_n - J_E^p x_n\| \to 0 \quad (n \to \infty).$$

It follows that

$$\lim_{n\to\infty} \|w_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.16}$$

Now, we prove $z^* \in \mathrm{VI}(C, F)$. Since $\{Ft_n\}$ is bounded, we know that $\exists L > 0$ s.t. $\|Ft_n\| \le L$. This ensures that for any $x, y \in C_n$,

$$|h_n(x) - |h_n(y)| = |\langle Ft_n, x - y\rangle| \le \|Ft_n\|\|x - y\| \le L\|x - y\|.$$

This indicates that $h_n(x)$ is L-Lipschitz in C_n. Applying Lemma 2.6, we have

$$\mathrm{dist}(C_n, w_n) \ge \frac{1}{L} h_n(w_n) = \frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n). \tag{3.17}$$

Using (3.12) and (3.17), we have

$$D_{f_p}(z^*, x_n) - D_{f_p}(z^*, x_{n+1}) \ge \tau[\frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n)]^p. \tag{3.18}$$

Hence $\lim_{n\to\infty} \tau_n D_{f_p}(w_n, y_n) = 0$. By Lemma 3.5, we get $\lim_{n\to\infty} \|w_n - y_n\| = 0$. Besides, combining (3.16) and $x_{n_k} \rightharpoonup z^*$ leads to $w_{n_k} \rightharpoonup z^*$. According to Lemma 3.4, we conclude that $z^* \in \omega_w(w_n) \subset \mathrm{VI}(C, F)$.

Next, we show $z^* \in \bigcap_{i=0}^N \mathrm{Fix}(T_i)$ with $T_0 := T$. Indeed, we first show that $\lim_{n\to\infty} \|x_n - T_r x_n\| = 0$ for $r = 1, ..., N$. Actually, according to the definition of T_n, we obtain that $T_n \in \{T_1, ..., T_N\}$ $\forall n \ge 1$, which hence leads to $T_{n+i} \in \{T_1, ..., T_N\}$ $\forall n \ge 1, i = 1, ..., N$. Observe that

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \sum_{j=1}^N \|T_j x_{n+i} - T_j x_n\|.\end{aligned}$$

Thanks to (3.15) and (3.16), we have $x_{n+i} - T_{n+i}x_{n+i} \to 0$ and $T_j x_{n+i} - T_j x_n \to 0$ for $i, j = 1, ..., N$. Thus, we get $\lim_{n\to\infty} \|x_n - T_{n+i}x_n\| = 0$ for $i = 1, ..., N$. This immediately implies that

$$\lim_{n\to\infty} \|x_n - T_r x_n\| = 0, \quad \text{for } r = 1, ..., N. \tag{3.19}$$

So it follows from (3.19) and $x_{n_k} \rightharpoonup z^*$ that $z \in \widehat{\mathrm{Fix}}(T_r) = \mathrm{Fix}(T_r)$ for $r = 1, ..., N$. Therefore, $z \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. In addition, observe also that

$$\|w_n - Tw_n\| \le \|w_n - T^n w_n\| + \|T^n w_n - T^{n+1} w_n\| + \|T^{n+1} w_n - Tw_n\|. \tag{3.20}$$

Noticing the uniform continuity of T on C, we conclude from (3.14) that $Tw_n - T^{n+1}w_n \to 0$. Thus, using the assumption $T^n w_n - T^{n+1}w_n \to 0$, from (3.20) we get $\lim_{n\to\infty} \|w_n - Tw_n\| = 0$. Again from (3.16) and $x_{n_k} \rightharpoonup z^*$, one has that $w_{n_k} \rightharpoonup z^*$. Hence, we obtain $z^* \in \widehat{\text{Fix}}(T) = \text{Fix}(T)$. Consequently, $z^* \in \bigcap_{i=0}^N \text{Fix}(T_i)$, and hence $z^* \in \Omega = \bigcap_{i=0}^N \text{Fix}(T_i) \cap \text{VI}(C, F)$. This means that $\omega_w(x_n) \subset \Omega$. Accordingly, applying Lemma 2.5 we conclude that $x_n \rightharpoonup z^*$. □

Next, we show a strong convergence result.

Algorithm 3.7. *Let $x_1 \in C$, $\mu > 0$, $l \in (0,1)$ and $\lambda \in (0, \frac{1}{\mu})$. Choose $\{\sigma_n\}, \{\alpha_n\}, \{\beta_n\} \subset (0,1)$ s.t. (i) $\liminf_{n\to\infty} \sigma_n(1-\sigma_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, and (ii) $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \theta_n/\alpha_n = 0$ and $\sum_{n=1}^\infty \theta_n < \infty$.*

Step 1. Set $w_n = J_{E^*}^q(\sigma_n J_E^p x_n + (1-\sigma_n)J_E^p(T_n x_n))$, and calculate $y_n = \Pi_C(J_{E^*}^q(J_E^p w_n - \lambda F w_n))$ and $r_\lambda(w_n) := w_n - y_n$.

Step 2. Calculate $t_n = w_n - \tau_n r_\lambda(w_n)$ in which $\tau_n := l^{j_n}$ with j_n being the smallest nonnegative integer j fulfilling

$$\langle Fw_n - F(w_n - l^j r_\lambda(w_n)), w_n - y_n\rangle \le \frac{\mu}{2} D_{f_p}(w_n, y_n).$$

Step 3. Set $z_n = \Pi_{C_n}(w_n)$, and compute $v_n = J_{E^*}^q(\beta_n J_E^p w_n + (1-\beta_n)J_E^p(T^n z_n))$ and $x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)$, where $C_n := \{x \in C : h_n(x) \le 0\}$ and

$$h_n(x) = \langle Ft_n, x - w_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(w_n, y_n).$$

Let $n := n+1$ and go to Step 1.

Theorem 3.8. *Suppose that the conditions (C1)–(C4) are satisfied. Then, the sequence $\{x_n\}$ constructed in Algorithm 3.7 converges strongly to $\Pi_\Omega u$ provided $T^n z_n - T^{n+1} z_n \to 0$.*

Proof. We divide our proof into four claims.

Claim 1. The sequence $\{x_n\}$ is bounded. Indeed, set $\hat{u} = \Pi_\Omega u$. According to Theorem 3.6 and Lemma 2.2 , we have

$$D_{f_p}(\hat{u}, w_n) \le D_{f_p}(\hat{u}, x_n) - \sigma_n(1-\sigma_n)\rho_b^* \|J_E^p x_n - J_E^p T_n x_n\|. \tag{3.21}$$

Using (2.3), (2.6) and (3.21), we deduce

$$\begin{aligned} D_{f_p}(\hat{u}, x_{n+1}) &\le D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n) D_{f_p}(\hat{u}, v_n) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n) D_{f_p}(\hat{u}, T^n z_n)] \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)(1+\theta_n) D_{f_p}(\hat{u}, z_n)] \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)(1+\theta_n) D_{f_p}(\hat{u}, w_n)] \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)(1+\theta_n) D_{f_p}(\hat{u}, x_n) \\ &\le \max\{D_{f_p}(\hat{u}, u), (1+\theta_n) D_{f_p}(\hat{u}, x_n)\}. \end{aligned} \tag{3.22}$$

From (3.22) that

$$\begin{aligned}
D_{f_p}(\hat{u}, x_{n+2}) &\le \max\{D_{f_p}(\hat{u}, u), (1+\theta_{n+1})D_{f_p}(\hat{u}, x_{n+1})\}\\
&\le \max\{D_{f_p}(\hat{u}, u), (1+\theta_{n+1})\max\{\prod_{i=2}^{n}(1+\theta_i)D_{f_p}(\hat{u}, u), \prod_{i=1}^{n}(1+\theta_i)D_{f_p}(\hat{u}, x_1)\}\}\\
&\le \max\{\prod_{i=2}^{n+1}(1+\theta_i)D_{f_p}(\hat{u}, u), \prod_{i=1}^{n+1}(1+\theta_i)D_{f_p}(\hat{u}, x_1)\}.
\end{aligned}$$

Noticing $\sum_{n=1}^{\infty}\theta_n < \infty$, we obtain that $\{D_{f_p}(\hat{u}, x_n)\}$ is bounded. This together with (2.1), implies that $\{x_n\}$ is bounded. Hence, $\{T_n x_n\}, \{Fw_n\}, \{w_n\}, \{y_n\}, \{t_n\}, \{z_n\}, \{T^n z_n\}$ and $\{v_n\}$ are also bounded.

Claim 2. We show $(1-\beta_n)(1+\theta_n)D_{f_p}(z_n, w_n) \le (1+\theta_n)D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p\hat{u}, s_n - \hat{u}\rangle$. Set $b = \sup_{n\ge1}\{\|w_n\|^{p-1}, \|T^n z_n\|^{p-1}\}$. By Lemma 2.2, we obtain

$$\begin{aligned}
D_{f_p}(\hat{u}, v_n) &= V_{f_p}(\hat{u}, \beta_n J_E^p w_n + (1-\beta_n)J_E^p T^n z_n)\\
&\le \frac{1}{p}\|\hat{u}\|^p - \beta_n\langle J_E^p w_n, \hat{u}\rangle - (1-\beta_n)\langle J_E^p T^n z_n, \hat{u}\rangle + \frac{\beta_n}{q}\|J_E^p w_n\|^q\\
&\quad + \frac{(1-\beta_n)}{q}\|J_E^p T^n z_n\|^q - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\|\\
&= \frac{1}{p}\|\hat{u}\|^p - \beta_n\langle J_E^p w_n, \hat{u}\rangle - (1-\beta_n)\langle J_E^p T^n z_n, \hat{u}\rangle + \frac{\beta_n}{q}\|w_n\|^p\\
&\quad + \frac{(1-\beta_n)}{q}\|T^n z_n\|^p - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\|\\
&= \beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)D_{f_p}(\hat{u}, T^n z_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\|\\
&\le \beta_n(1+\theta_n)D_{f_p}(\hat{u}, w_n) + (1-\beta_n)(1+\theta_n)D_{f_p}(\hat{u}, z_n)\\
&\quad - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\|\\
&\le (1+\theta_n)D_{f_p}(\hat{u}, w_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\|.
\end{aligned} \tag{3.23}$$

Set $s_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n$. Using (2.5), we have

$$\begin{aligned}
D_{f_p}(\hat{u}, x_{n+1}) &\le D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)\\
&= V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)\\
&\le V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n - \alpha_n(J_E^p u - J_E^p\hat{u})) + \alpha_n\langle J_E^p u - J_E^p\hat{u}, s_n - \hat{u}\rangle\\
&\le \alpha_n D_{f_p}(\hat{u}, \hat{u}) + (1-\alpha_n)D_{f_p}(\hat{u}, v_n) + \alpha_n\langle J_E^p u - J_E^p\hat{u}, s_n - \hat{u}\rangle\\
&= (1-\alpha_n)D_{f_p}(\hat{u}, v_n) + \alpha_n\langle J_E^p u - J_E^p\hat{u}, s_n - \hat{u}\rangle\\
&\le (1-\alpha_n)[(1+\theta_n)D_{f_p}(\hat{u}, w_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\|]\\
&\quad + \alpha_n\langle J_E^p u - J_E^p\hat{u}, s_n - \hat{u}\rangle\\
&= (1-\alpha_n)(1+\theta_n)D_{f_p}(\hat{u}, w_n) - (1-\alpha_n)\beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\|\\
&\quad + \alpha_n\langle J_E^p u - J_E^p\hat{u}, s_n - \hat{u}\rangle\\
&\le (1-\alpha_n)(1+\theta_n)D_{f_p}(\hat{u}, w_n) + \alpha_n\langle J_E^p u - J_E^p\hat{u}, s_n - \hat{u}\rangle.
\end{aligned} \tag{3.24}$$

On the other hand, we have

$$\begin{aligned} D_{f_p}(\hat{u}, v_n) &\le \beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)(1+\theta_n) D_{f_p}(\hat{u}, z_n) \\ &\le \beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)(1+\theta_n)[D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n)] \\ &\le (1+\theta_n) D_{f_p}(\hat{u}, w_n) - (1-\beta_n)(1+\theta_n) D_{f_p}(z_n, w_n). \end{aligned}$$

This together with (3.24) implies that

$$\begin{aligned} D_{f_p}(\hat{u}, x_{n+1}) &\le (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle \\ &\le (1+\theta_n) D_{f_p}(\hat{u}, w_n) - (1-\beta_n)(1+\theta_n) D_{f_p}(z_n, w_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle. \end{aligned}$$

This immediately arrives at

$$\begin{aligned} (1-\beta_n)(1+\theta_n) D_{f_p}(z_n, w_n) \le\ & (1+\theta_n) D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}) \\ & + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle. \end{aligned} \tag{3.25}$$

Claim 3. We show that

$$(1-\alpha_n)(1-\beta_n)(1+\theta_n)\tau[\frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n)]^p \le \alpha_n D_{f_p}(\hat{u}, u) + (1+\theta_n) D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}).$$

Using the similar inferences to these of (3.18) in Theorem 3.6, we get

$$D_{f_p}(\hat{u}, z_n) \le D_{f_p}(\hat{u}, w_n) - \tau[\frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n)]^p. \tag{3.26}$$

Applying (3.26), we get

$$\begin{aligned} D_{f_p}(\hat{u}, x_{n+1}) &\le D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n) D_{f_p}(\hat{u}, v_n) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)(1+\theta_n) D_{f_p}(\hat{u}, z_n)] \\ &= \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)\beta_n D_{f_p}(\hat{u}, w_n) + (1-\alpha_n)(1-\beta_n)(1+\theta_n) D_{f_p}(\hat{u}, z_n) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)\beta_n D_{f_p}(\hat{u}, w_n) - \tau[\frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n)]^p] \\ &\quad + (1-\alpha_n)(1-\beta_n)(1+\theta_n)[D_{f_p}(\hat{u}, w_n) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) - (1-\alpha_n)(1-\beta_n)(1+\theta_n)\tau[\frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n)]^p \\ &\quad + (1+\theta_n) D_{f_p}(\hat{u}, w_n). \end{aligned} \tag{3.27}$$

Claim 4. Finally, we prove $x_n \to \hat{u}$ as $n \to \infty$. Note that $\omega_w(x_n) \neq \emptyset$. Let $z^\dagger \in \omega_w(x_n)$. Then, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z^\dagger$. For each $n \ge 1$, we write $\Phi_n = D_{f_p}(\hat{u}, x_n)$. Put $\Phi_n = \|x_n - z^*\|^2$.

Case 1. Suppose that $\{\Phi_n\}_{n=n_0}^\infty$ is nonincreasing for some $n_0 \ge 1$. Then $\lim_{n\to\infty} \Phi_n = d < +\infty$ and $\lim_{n\to\infty}(\Phi_n - \Phi_{n+1}) = 0$. By (3.21) and (3.25) we get

$$\begin{aligned} (1-\beta_n)(1+\theta_n) D_{f_p}(z_n, w_n) &\le (1+\theta_n) D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle \\ &\le (1+\theta_n)[D_{f_p}(\hat{u}, x_n) - \sigma_n(1-\sigma_n)\rho_b^* \|J_E^p x_n - J_E^p T_n x_n\|] \\ &\quad - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle \\ &\le (1+\theta_n) D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle \\ &\quad - (1+\theta_n)\sigma_n(1-\sigma_n)\rho_b^* \|J_E^p x_n - J_E^p T_n x_n\|, \end{aligned}$$

which immediately yields

$$
\begin{aligned}
(1-\beta_n)(1+\theta_n)D_{f_p}(z_n, w_n) &+ (1+\theta_n)\sigma_n(1-\sigma_n)\rho_b^*\|J_E^p x_n - J_E^p T_n x_n\| \\
&\le (1+\theta_n)D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle \\
&= (1+\theta_n)\Phi_n - \Phi_{n+1} + \alpha_n\langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle.
\end{aligned}
$$

Since $\lim_{n\to\infty}\theta_n = 0$, $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\liminf_{n\to\infty}\sigma_n(1-\sigma_n) > 0$, $\lim_{n\to\infty}\Phi_n = d$ and the sequences $\{s_n\}$ is bounded, we obtain that $\lim_{n\to\infty} D_{f_p}(z_n, w_n) = 0$ and $\lim_{n\to\infty}\|J_E^p x_n - J_E^p T_n x_n\| = 0$. Noticing $w_n = J_{E^*}^q(\sigma_n J_E^p x_n + (1-\sigma_n)J_E^p T_n x_n)$, we also have $\lim_{n\to\infty}\|J_E^p w_n - J_E^p x_n\| = 0$. So it follows from (2.1) that

$$
\lim_{n\to\infty}\|z_n - w_n\| = \lim_{n\to\infty}\|x_n - T_n x_n\| = \lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.28}
$$

Furthermore, from (3.24) we have

$$
\begin{aligned}
&(1-\alpha_n)\beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p T^n z_n\| \\
&\le (1-\alpha_n)(1+\theta_n)D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u}\rangle.
\end{aligned}
$$

By the similar inferences, we infer that $\lim_{n\to\infty}\|J_E^p w_n - J_E^p T^n z_n\| = 0$, which hence leads to $\lim_{n\to\infty}\|J_E^p v_n - J_E^p w_n\| = 0$. Then,

$$
\lim_{n\to\infty}\|w_n - T^n z_n\| = \lim_{n\to\infty}\|v_n - w_n\| = 0. \tag{3.29}
$$

Combining with (3.28), we have

$$
\lim_{n\to\infty}\|z_n - T^n z_n\| = \lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.30}
$$

Since $s_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)$, it can be readily seen from (3.30) that

$$
\lim_{n\to\infty}\|s_n - x_n\| = 0. \tag{3.31}
$$

In addition, using (2.3) and (3.23), we achieve

$$
\begin{aligned}
D_{f_p}(\hat{u}, x_{n+1}) &\le D_{f_p}(\hat{u}, s_n) - D_{f_p}(x_{n+1}, s_n) \\
&\le D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n) - D_{f_p}(x_{n+1}, s_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)D_{f_p}(\hat{u}, v_n) - D_{f_p}(x_{n+1}, s_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)(1+\theta_n)D_{f_p}(\hat{u}, w_n) - D_{f_p}(x_{n+1}, s_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)(1+\theta_n)D_{f_p}(\hat{u}, x_n) - D_{f_p}(x_{n+1}, s_n),
\end{aligned}
$$

which hence arrives at

$$
\begin{aligned}
D_{f_p}(x_{n+1}, s_n) &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)(1+\theta_n)D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)(1+\theta_n)\Phi_n - \Phi_{n+1}.
\end{aligned}
$$

Then, $\lim_{n\to\infty} D_{f_p}(x_{n+1}, s_n) = 0$ and hence $\lim_{n\to\infty}\|x_{n+1} - s_n\| = 0$. This together with (3.31), leads to

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.32}
$$

Note that

$$\|z_n - Tz_n\| \le \|z_n - T^n z_n\| + \|T^n z_n - T^{n+1} z_n\| + \|T^{n+1} z_n - Tz_n\|. \tag{3.33}$$

By (3.30), we have $Tz_n - T^{n+1}z_n \to 0$. This together with (3.33) implies that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. Again from (3.28) and $x_{n_k} \rightharpoonup z^\dagger$, one has that $z_{n_k} \rightharpoonup z^\dagger$. Hence, we obtain $z^\dagger \in \widehat{\text{Fix}}(T) = \text{Fix}(T)$. In the meantime, let us show that $z \in \bigcap_{i=1}^N \text{Fix}(T_i)$. Indeed, by the definition of T_n, we know that $T_n \in \{T_1, ..., T_N\}$, $\forall n \ge 1$, which hence leads to $T_{n+i} \in \{T_1, ..., T_N\}$, $\forall n \ge 1, i = 1, ..., N$. Observe that

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \sum_{j=1}^N \|T_j x_{n+i} - T_j x_n\|.\end{aligned}$$

Since each T_j is uniformly continuous on C, we deduce from (3.28) and (3.32) that $x_{n+i} - T_{n+i}x_{n+i} \to 0$ and $T_j x_{n+i} - T_j x_n \to 0$ for $i, j = 1, ..., N$. Thus, we get $\lim_{n\to\infty} \|x_n - T_{n+i}x_n\| = 0$ for $i = 1, ..., N$. Hence, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 (i = 1, ..., N)$ and so $z^\dagger \in \widehat{\text{Fix}}(T_i) = \text{Fix}(T_i)$ for $i = 1, ..., N$. Consequently, $z^\dagger \in \bigcap_{i=0}^N \text{Fix}(T_i)$ with $T_0 := T$.

Next, we prove $z^\dagger \in \text{VI}(C, F)$. Using (3.21) and (3.27), we have

$$(1 - \alpha_n)(1 - \beta_n)(1 + \theta_n)\tau[\frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n)]^p \le \alpha_n D_{f_p}(\hat{u}, u) + (1 + \theta_n) D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}).$$

Then, $\lim_{n\to\infty} \frac{\tau_n}{2\lambda L} D_{f_p}(w_n, y_n) = 0$ and hence $\lim_{n\to\infty} \tau_n D_{f_p}(w_n, y_n) = 0$. Using Lemma 3.5, we infer that

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{3.34}$$

By Lemma (3.34) and 3.4, we obtain that $z^\dagger \in \text{VI}(C, F)$, and hence $z^\dagger \in \Omega = \bigcap_{i=0}^N \text{Fix}(T_i) \cap \text{VI}(C, F)$ with $T_0 := T$. This means that $\omega_w(x_n) \subset \Omega$. Lastly, we show that $\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u} \rangle \le 0$. We can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u} \rangle = \lim_{j\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u} \rangle.$$

Without loss of generality, assume that $x_{n_j} \rightharpoonup \tilde{z}$. So it follows from (2.2) and $\tilde{z} \in \Omega$ that

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u} \rangle = \lim_{j\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u} \rangle = \langle J_E^p u - J_E^p \hat{u}, \tilde{z} - \hat{u} \rangle \le 0. \tag{3.35}$$

This together with (3.31) ensures that

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u} \rangle \le 0. \tag{3.36}$$

Using (3.21) and (3.24), we get

$$\begin{aligned}D_{f_p}(x_{n+1}, \hat{u}) &\le (1 - \alpha_n)(1 + \theta_n) D_{f_p}(\hat{u}, x_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u} \rangle \\ &\le (1 - \alpha_n) D_{f_p}(\hat{u}, x_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, s_n - \hat{u} \rangle + \theta_n D_{f_p}(\hat{u}, x_n).\end{aligned}$$

Since $\{D_{f_p}(\hat{u}, x_n)\}$ is bounded and $\sum_{n=1}^{\infty} \theta_n < \infty$, one has that $\sum_{n=1}^{\infty} \theta_n D_{f_p}(\hat{u}, x_n) < \infty$. Noticing $\{\alpha_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, by Lemma 2.7 and (3.36), we conclude that $\lim_{n\to\infty} D_{f_p}(\hat{u}, x_n) = 0$ and $\lim_{n\to\infty} \|\hat{u} - x_n\| = 0$.

Case 2. Suppose that $\exists\{\Phi_{n_k}\} \subset \{\Phi_n\}$ s.t. $\Phi_{n_k} < \Phi_{n_k+1}, \ \forall k \in \mathbb{N}$. Define an operator $\psi : \mathbb{N} \to \mathbb{N}$ by

$$\psi(n) := \max\{k \le n : \Phi_k < \Phi_{k+1}\}.$$

Based on Lemma 2.8, we have

$$\Phi_{\psi(n)} \le \Phi_{\psi(n)+1} \quad \text{and} \quad \Phi_n \le \Phi_{\psi(n)+1}. \tag{3.37}$$

From (3.21) and (3.25), we have

$$\begin{aligned}
&(1 - \beta_{\psi(n)})(1 + \theta_{\psi(n)})D_{f_p}(z_{\psi(n)}, w_{\psi(n)}) + (1 + \theta_{\psi(n)})\sigma_{\psi(n)}(1 - \sigma_{\psi(n)})\rho_b^*\|J_E^p x_{\psi(n)} - J_E^p T_{\psi(n)} x_{\psi(n)}\| \\
&\le (1 + \theta_{\psi(n)})D_{f_p}(\hat{u}, x_{\psi(n)}) - D_{f_p}(\hat{u}, x_{\psi(n)+1}) + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, s_{\psi(n)} - \hat{u}\rangle \\
&= (1 + \theta_{\psi(n)})\Phi_{\psi(n)} - \Phi_{\psi(n)+1} + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, s_{\psi(n)} - \hat{u}\rangle.
\end{aligned}$$

Noticing $w_{\psi(n)} = J_{E^*}^q(\sigma_{\psi(n)} J_E^p x_{\psi(n)} + (1 - \sigma_{\psi(n)}) J_E^p T_{\psi(n)} x_{\psi(n)})$, we deduce that

$$\lim_{n\to\infty} \|z_{\psi(n)} - w_{\psi(n)}\| = \lim_{n\to\infty} \|x_{\psi(n)} - T_{\psi(n)} x_{\psi(n)}\| = \lim_{n\to\infty} \|w_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.38}$$

Furthermore, by (3.21) and (3.24), we obtain

$$\begin{aligned}
&(1 - \alpha_{\psi(n)})\beta_{\psi(n)}(1 - \beta_{\psi(n)})\rho_b^*\|J_E^p w_{\psi(n)} - J_E^p T^{\psi(n)} z_{\psi(n)}\| \\
&\le (1 - \alpha_{\psi(n)})(1 + \theta_{\psi(n)})D_{f_p}(\hat{u}, x_{\psi(n)}) - D_{f_p}(\hat{u}, x_{\psi(n)+1}) + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, s_{\psi(n)} - \hat{u}\rangle.
\end{aligned}$$

Since $v_{\psi(n)} = J_{E^*}^q(\beta_{\psi(n)} J_E^p w_{\psi(n)} + (1 - \beta_{\psi(n)}) J_E^p T^{\psi(n)} z_{\psi(n)})$, by (3.38) we get

$$\lim_{n\to\infty} \|z_{\psi(n)} - T^{\psi(n)} z_{\psi(n)}\| = \lim_{n\to\infty} \|v_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.39}$$

Noticing $s_{\psi(n)} = J_{E^*}^q(\alpha_{\psi(n)} J_E^p u + (1 - \alpha_{\psi(n)}) J_E^p v_{\psi(n)})$, from (3.39) we get

$$\lim_{n\to\infty} \|s_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.40}$$

Using (3.37) and exploiting the similar inferences to those in the proof of Case 1, we conclude that $\lim_{n\to\infty} \|x_{\psi(n)+1} - x_{\psi(n)}\| = 0$, $\lim_{n\to\infty} \|x_{\psi(n)} - T_r x_{\psi(n)}\| = 0$ for $r = 1, ..., N$,

$$\lim_{n\to\infty} \|z_{\psi(n)} - T z_{\psi(n)}\| = \lim_{n\to\infty} \|w_{\psi(n)} - y_{\psi(n)}\| = 0, \tag{3.41}$$

and

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, s_{\psi(n)} - \hat{u}\rangle \le 0. \tag{3.42}$$

Using (3.21) and (3.24) we have

$$\Phi_{\psi(n)+1} \le (1 - \alpha_{\psi(n)})\Phi_{\psi(n)} + \theta_{\psi(n)}\Phi_{\psi(n)} + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, s_{\psi(n)} - \hat{u}\rangle. \tag{3.43}$$

Combining with (3.37), we have

$$\begin{aligned}\alpha_{\psi(n)}\Phi_{\psi(n)} &\leq \Phi_{\psi(n)} - \Phi_{\psi(n)+1} + \theta_{\psi(n)}\Phi_{\psi(n)} + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, s_{\psi(n)} - \hat{u}\rangle \\ &\leq \theta_{\psi(n)}\Phi_{\psi(n)} + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, s_{\psi(n)} - \hat{u}\rangle.\end{aligned}$$

Since $\frac{\theta_{\psi(n)}}{\alpha_{\psi(n)}} \to 0$, from (3.42) we deduce that

$$\lim_{n\to\infty} \Phi_{\psi(n)} = 0. \tag{3.44}$$

From (3.42), (3.43) and (3.44), we get that

$$\lim_{n\to\infty} \Phi_{\psi(n)+1} = 0. \tag{3.45}$$

Now from (3.37), we deduce $\lim_{n\to\infty} D_{f_p}(\hat{u}, x_n) = \lim_{n\to\infty} \Phi_n = 0$. Hence $\lim_{n\to\infty} \|x_n - \hat{u}\| = 0$. □

Putting $F = 0$ in Algorithm 3.1, we have the following corollary.

Corollary 3.9. *Let E be a p-uniformly convex and uniformly smooth Banach space with weakly sequentially continuous duality mapping J_E^p. Let $T : C \to C$ be a uniformly continuous and Bregman relatively asymptotically nonexpansive mapping with and $T_i : C \to C (i = 1, ..., N)$ be a uniformly continuous and Bregman relatively nonexpansive mapping. Assume that $\Omega := \bigcap_{i=0}^N \mathrm{Fix}(T_i) \neq \emptyset$ with $T_0 := T$. For an initial $x_1 \in C$, let $\{x_n\}$ be the sequence constructed by*

$$\begin{cases} w_n = J_{E^*}^q(\sigma_n J_E^p x_n + (1-\sigma_n)J_E^p(T_n x_n)), \\ v_n = J_{E^*}^q(\alpha_n J_E^p w_n + (1-\alpha_n)J_E^p(T^n w_n)), \\ Q_n = \{x \in C : D_{f_p}(x, v_n) \leq (1+\theta_n)D_{f_p}(x, w_n)\}, \\ x_{n+1} = \Pi_{Q_n}(w_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\sigma_n\}, \{\alpha_n\} \subset (0,1)$ s.t. $\liminf_{n\to\infty} \sigma_n(1-\sigma_n) > 0$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point in Ω provided $T^n w_n - T^{n+1} w_n \to 0$.

Setting $T = I$ in Algorithm 3.7, we obtain the following algorithm and corollary.

Algorithm 3.10. *Let $x_1 \in C$, $\mu > 0$, $l \in (0,1)$ and $\lambda \in (0, \frac{1}{\mu})$. Choose $\{\sigma_n\}, \{\alpha_n\}, \{\beta_n\} \subset (0,1)$ s.t. (i) $\liminf_{n\to\infty} \sigma_n(1-\sigma_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, and (ii) $\sum_{n=1}^\infty \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$.*

Step 1. Set $w_n = J_{E^*}^q(\sigma_n J_E^p x_n + (1-\sigma_n)J_E^p(T_n x_n))$, and compute $y_n = \Pi_C(J_{E^*}^q(J_E^p w_n - \lambda F w_n))$ and $r_\lambda(w_n) := w_n - y_n$.

Step 2. Calculate $t_n = w_n - \tau_n r_\lambda(w_n)$ in which $\tau_n := l^{j_n}$ with j_n being the smallest nonnegative integer j fulfilling

$$\langle F w_n - F(w_n - l^j r_\lambda(w_n)), w_n - y_n\rangle \leq \frac{\mu}{2} D_{f_p}(w_n, y_n).$$

Step 3. Compute $v_n = J_{E^*}^q(\beta_n J_E^p w_n + (1-\beta_n)J_E^p(\Pi_{C_n} w_n))$ and $x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)$, where $C_n := \{x \in C : h_n(x) \leq 0\}$ and

$$h_n(x) = \langle F t_n, x - w_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(w_n, y_n).$$

Set $n := n + 1$ and go to Step 1.

Corollary 3.11. *Suppose that (C1)–(C4) are satisfied. Then, $\{x_n\}$ constructed in Algorithm 3.7 converges strongly to $\Pi_\Omega u$.*

4. Examples

In this section, we provide an illustrated example to demonstrate the applicability and implementability of our proposed method. We first provide an example of Lipschitz continuous and pseudomonotone monotone mapping F, Bregman relatively asymptotically nonexpansive mapping T and Bregman relatively nonexpansive mapping T_1 with $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, F) \neq \emptyset$.

Let $C = [-3, 3]$ and $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The initial point x_1 is randomly chosen in C. Put $\mu = 1,\ l = \lambda = \frac{1}{3}$ and $\sigma_n = \frac{1}{2}$.

Let $F : H \to H$ and $T, T_1 : C \to C$ be defined as $Fx := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$, $Tx := \frac{3}{4}\sin x$ and $T_1 x := \sin x$ for all $x \in C$. Now, we first show that F is pseudomonotone and Lipschitz continuous. Indeed, for all $x, y \in H$ we have

$$\begin{aligned}
\|Fx - Fy\| &= |\frac{1}{1 + \|\sin x\|} - \frac{1}{1 + \|x\|} - \frac{1}{1 + \|\sin y\|} + \frac{1}{1 + \|y\|}| \\
&\leq |\frac{\|y\| - \|x\|}{(1 + \|x\|)(1 + \|y\|)}| + |\frac{\|\sin y\| - \|\sin x\|}{(1 + \|\sin x\|)(1 + \|\sin y\|)}| \\
&\leq \|x - y\| + \|\sin x - \sin y\| \\
&\leq 2\|x - y\|.
\end{aligned}$$

This implies that F is Lipschitz continuous. Next, we show that F is pseudomonotone. For each $x, y \in H$, it is easy to see that

$$\langle Fx, y - x\rangle = (\frac{1}{1 + |\sin x|} - \frac{1}{1 + |x|})(y - x) \geq 0$$

which implies that

$$\langle Fy, y - x\rangle = (\frac{1}{1 + |\sin y|} - \frac{1}{1 + |y|})(y - x) \geq 0.$$

Furthermore, it is easy to check that T is asymptotically nonexpansive with $\theta_n = (\frac{3}{4})^n\ \forall n \geq 1$, and for each $\{p_n\} \subset C$ one has $\|T^{n+1}p_n - T^n p_n\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\|T^n x - T^n y\| \leq \frac{3}{4}\|T^{n-1}x - T^{n-1}y\| \leq \cdots \leq (\frac{3}{4})^n\|x - y\| \leq (1 + \theta_n)\|x - y\|,$$

and

$$\begin{aligned}
\|T^{n+1}p_n - T^n p_n\| &\leq (\frac{3}{4})^{n-1}\|T^2 p_n - T p_n\| \\
&= (\frac{3}{4})^{n-1}\|\frac{3}{4}\sin(Tp_n) - \frac{3}{4}\sin p_n\| \\
&\leq 2(\frac{3}{4})^n \to 0\ \ (n \to \infty).
\end{aligned}$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and T is also Bregman relatively asymptotically nonexpansive with $\theta_n = (\frac{3}{4})^n\ \forall n \geq 1$. In addition, it is clear that $\mathrm{Fix}(T_1) = \{0\}$ and T_1 is Bregman relatively nonexpansive. Therefore, $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$.

Example 4.1. Putting $\alpha_n = \frac{n}{2(n+1)}, \forall n \geq 1$, we can rewrite Algorithm 3.1 as follows:

$$\begin{cases} w_n = \frac{1}{2}x_n + \frac{1}{2}T_1 x_n, \\ y_n = P_C(w_n - \frac{1}{3}Fw_n), \\ t_n = (1-\tau_n)w_n + \tau_n y_n, \\ v_n = \frac{n}{2(n+1)}w_n + \frac{n+2}{2(n+1)}T^n w_n, \\ Q_n = \{x \in C : |x - v_n|^2 \leq (1 + (\frac{3}{4})^n)|x - w_n|^2\}, \\ x_{n+1} = P_{C_n \cap Q_n} w_n, \forall n \geq 1, \end{cases}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.1. Then, by Theorem 3.6, we deduce that $\{x_n\}$ converges to $0 \in \Omega = \text{Fix}(T_1) \cap \text{Fix}(T) \cap \text{VI}(C, A)$.

Example 4.2. Putting $\alpha_n = \frac{1}{2(n+1)}$ and $\beta_n = \frac{n}{2(n+1)}, \forall n \geq 1$, we can rewrite Algorithm 3.7 as follows:

$$\begin{cases} w_n = \frac{1}{2}x_n + \frac{1}{2}T_1 x_n, \\ y_n = P_C(w_n - \frac{1}{3}Fw_n), \\ t_n = (1-\tau_n)w_n + \tau_n y_n, \\ v_n = \frac{n}{2(n+1)}w_n + \frac{n+2}{2(n+1)}T^n P_{C_n} w_n, \\ x_{n+1} = P_C(\frac{1}{2(n+1)}u + \frac{2n+1}{2(n+1)}v_n), \forall n \geq 1, \end{cases}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.7. Then, by Theorem 3.8, we obtain that $\{x_n\}$ converges to $0 \in \Omega = \text{Fix}(T_1) \cap \text{Fix}(T) \cap \text{VI}(C, A)$.

5. Conclusions

In this paper, we investigate iterative algorithms for solving the variational inequality and the common fixed-point problem in p-uniformly convex and uniformly smooth Banach spaces. With the help of accelerated projection methods and line search technique, we construct two algorithms for finding a common solution of the pseudomonotone variational inequality and the common fixed-point problem of finitely many Bregman relatively nonexpansive mappings and a Bregman relatively asymptotically nonexpansive mapping. We provide convergence analysis of the proposed algorithms by using standard conditions and new techniques.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Yeong-Cheng Liou was supported in part by the MOST Project in Taiwan (110-2410-H-037-001) and the grant from NKUST and KMU joint R&D Project (110KK002).

Conflict of interest

The authors declare no conflict of interest.

References

1. L. C. Ceng, A. Petrusel, X. Qin, J. C. Yao, Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints, *Optimization*, **70** (2021), 1337–1358. https://doi.org/10.1080/02331934.2020.1858832
2. Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, *J. Optim. Theory Appl.,* **148** (2011), 318–335. https://doi.org/10.1007/s10957-010-9757-3
3. Y. Yao, N. Shahzad, J. C. Yao, Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems, *Carpathian J. Math.*, **37** (2021), 541–550. https://doi.org/10.37193/CJM.2021.03.15
4. H. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.*, **66** (2002), 240–256. https://doi.org/10.1112/S0024610702003332
5. L. He, Y. L. Cui, L. C. Ceng, T. Y. Zhao, D. Q. Wang, H. Y. Hu, Strong convergence for monotone bilevel equilibria with constraints of variational inequalities and fixed points using subgradient extragradient implicit rule, *J. Inequal. Appl.*, **2021** (2021), 146. https://doi.org/10.1186/s13660-021-02683-y
6. R. W. Cottle, J. C. Yao, Pseudo-monotone complementarity problems in Hilbert space, *J. Optim. Theory Appl.*, **75** (1992), 281–295. https://doi.org/10.1007/BF00941468
7. Y. Yao, M. Postolache, J. C. Yao, Iterative algorithms for generalized variational inequalities, *U.P.B. Sci. Bull., Series A*, **81** (2019), 3–16.
8. D. V. Thong, D. V. Hieu, Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems, *Numer. Algor.*, **80** (2019), 1283–1307. https://doi.org/10.1007/s11075-018-0527-x
9. K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, New York: Marcel Dekker, 1984.
10. L. C. Ceng, C. F. Wen, Systems of variational inequalities with hierarchical variational inequality constraints for asymptotically nonexpansive and pseudocontractive mappings, *RACSAM,* **113** (2019), 2431–2447. https://doi.org/10.1007/s13398-019-00631-6
11. G. M. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon*, **12** (1976), 747–756.
12. L. C. Ceng, A. Petrusel, X. Qin, J. C. Yao, Pseudomonotone variational inequalities and fixed points, *Fixed Point Theory*, **22** (2021), 543–558.
13. S. Reich, D. V. Thong, Q. L. Dong, X. H. Li, V. T. Dung, New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings, *Numer. Algor.*, **87** (2021), 527–549. https://doi.org/10.1007/s11075-020-00977-8

14. Y. Yao, O. S. Iyiola, Y. Shehu, Subgradient extragradient method with double inertial steps for variational inequalities, *J. Sci. Comput.*, **90** (2022), 71. https://doi.org/10.1007/s10915-021-01751-1

15. A. N. Iusem, M. Nasri, Korpelevich's method for variational inequality problems in Banach spaces, *J. Glob. Optim.*, **50** (2011), 59–76.

16. Y. He, A new double projection algorithm for variational inequalities, *J. Comput. Appl. Math.*, **185** (2006), 166–173. https://doi.org/10.1016/j.cam.2005.01.031

17. L. C. Ceng, J. C. Yao, Y. Shehu, On Mann-type subgradient-like extragradient method with linear-search process for hierarchical variational inequalities for asymptotically nonexpansive mappings, *Mathematics*, **9** (2021), 3322. https://doi.org/10.3390/math9243322

18. G. Z. Eskandani, R. Lotfikar, M. Raeisi, Hybrid projection methods for solving pseudomonotone variational inequalities in Banach spaces, *Fixed Point Theory*, In press.

19. Y. Takahashi, K. Hashimoto, M. Kato, On sharp uniform convexity, smoothness, and strong type, cotype inequalities, *J. Nonlinear Convex Anal.*, **3** (2002), 267–281.

20. L. J. Zhu, Y. Yao, Algorithms for approximating solutions of split variational inclusion and fixed point problems, *Mathematics,* **11** (2023), 641. https://doi.org/10.3390/math11030641

21. D. Reem, S. Reich, A. De Pierro, Re-examination of Bregman functions and new properties of their divergences, *Optimization*, **68** (2019), 279–348. https://doi.org/10.1080/02331934.2018.1543295

22. D. Butnariu, E. Resmerita, Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.*, **2006** (2006), 084919. https://doi.org/10.1155/AAA/2006/84919

23. X. Zhao, Y. Yao, Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems, *Optimization*, **69** (2020), 1987–2002. https://doi.org/10.1080/02331934.2019.1711087

24. D. Butnariu, A. N. Iusem, E. Resmerita, Total convexity for powers of the norm in uniformly convex Banach spaces, *J. Convex Anal.*, **7** (2000), 319–334.

25. S. Reich, A weak convergence theorem for the alternating method with Bregman distances, In: *Theory and applications of nonlinear operators*, New York: Marcel Dekker, 1996, 313–318.

26. F. Schöpfer, T. Schuster, A. K. Louis, An iterative regularization method for the solution of the split feasibility problem in Banach spaces, *Inverse Probl.*, **24** (2008), 055008, 20. https://doi.org/10.1088/0266-5611/24/5/055008

27. P. E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.,* **16** (2008), 899–912.

28. S. Jabeen, M. A. Noor, K. I. Noor, Inertial methods for solving system of quasi variational inequalities, *J. Adv. Math. Stud.*, **15** (2022), 1–10.

29. J. Zheng, J. Chen, X. Ju, Fixed-time stability of projection neurodynamic network for solving pseudomonotone variational inequalities, *Neurocomputing*, **505** (2022), 402–412. https://doi.org/10.1016/j.neucom.2022.07.034

30. J. Chen, Z. Wan, L. Yuan, Y. Zheng, Approximation of fixed points of weak Bregman relatively nonexpansive mappings in Banach spaces, *Int. J. Math. Math. Sci.*, **2011** (2011), 420192. https://doi.org/10.1155/2011/420192

31. B. Tan, X. Qin, J. C. Yao, Strong convergence of inertial projection and contraction methods for pseudomonotone variational inequalities with applications to optimal control problems, *J. Glob. Optim.*, **82** (2022), 523–557. https://doi.org/10.1007/s10898-021-01095-y

32. B. Tan, S. Y. Cho, J. C. Yao, Accelerated inertial subgradient extragradient algorithms with non-monotonic step sizes for equilibrium problems and fixed point problems, *J. Nonlinear Var. Anal.*, **6** (2022), 89–122. https://doi.org/10.23952/jnva.6.2022.1.06