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Research article

Bivariate multiquadric quasi-interpolation operators of Lidstone type

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Abstract: In this paper, a kind of bivariate multiquadric quasi-interpolant with the derivatives of a approximated function is studied by combining the known multiquadric quasi-interpolant with the generalized Taylor polynomials that act as the bivariate Lidstone interpolation polynomials. For practical purposes, a kind of improved approximation operator without any derivative of the approximated function is given by using bivariate divided differences to approximate the derivatives. It has the property of high-degree polynomial reproducing. In addition, the improved bivariate quasi-interpolation operators only demand information of the location points rather than the derivatives of the function approximated. Some error bounds in terms of the modulus of continuity of high order and Peano representations for the error are given. Several numerical comparisons with other existing methods are carried out to verify a higher degree of accuracy based on the obtained scheme. Furthermore, the advantage of our method is that the algorithm is very simple and easy to implement.

Keywords: quasi-interpolation; multiquadric functions; Lidstone interpolation polynomials; polynomial reproduction; Peano representations; error estimations
Mathematics Subject Classification: 41A05, 65D05, 65D15

1. Introduction

Although interpolation methods constitute a classical approximation method in numerical mathematics, the interpolation matrix quickly becomes ill-conditioned when the number of interpolation nodes increases. To overcome this problem, the quasi-interpolation method has been confirmed to not solve any linear algebraic equations, and it can achieve the desired convergence order.

Interpolation methods based on a radial basis function constitute an important tool in practical application. In 1971, Hardy [1] studied multiquadrics as a kind of radial basis function. A review by Franke [2] suggested that multiquadric interpolation is one of the best methods among some 29 interpolation methods in terms of accuracy, efficiency, and easy implementation. Micchelli [3] proved the existence of the solution of the associated multiquadric interpolation problem in 1986.

The univariate multiquadric quasi-interpolation of a function $f : [a, b] \to \mathbb{R}$ on some distinct scattered points $a = x_0 < x_1 < \cdots < x_N = b$ has the following form:

$$\mathcal{L}[f : a, b](x, y) = \sum_{i=0}^{N} f(x_i)\psi_i(x), \quad x \in [a, b], \tag{1.1}$$

where $\psi_i(x)$ $(i = 0, 1, \cdots, N)$ denote linear combinations of the multiquadrics introduced by Hardy [1]

$$\varphi_i(x) = [(x - x_i)^2 + c^2]^{1/2}, \quad c > 0. \tag{1.2}$$

Buhmann [4] investigated the accuracy of quasi-interpolation for infinite regular grid data in 1988. Powell [5] studied the scattered points $x_i, i \in \mathbb{Z}$ that extend to both ends of the real line. He defined the normalized second divided difference as follows

$$\psi_i(x) = \frac{\varphi_{i+1}(x) - \varphi_i(x)}{2(x_{i+1} - x_i)} - \frac{\varphi_i(x) - \varphi_{i-1}(x)}{2(x_i - x_{i-1})}, \quad x \in \mathbb{R}, \tag{1.3}$$

and he gave the following multiquadric quasi-interpolation

$$\mathcal{L}[f](x) = \sum_{i=-\infty}^{+\infty} f(x_i)\psi_i(x), \quad x \in \mathbb{R} \tag{1.4}$$

where this method reproduces linear polynomials. Through the above results on finite scattered data, Beatson and Powell [6] first considered a univariate quasi-interpolation operator $\mathcal{L}_B$ that reproduces constants. Wu and Schaback [7] introduced a modified univariate multiquadric quasi-interpolation operator $\mathcal{L}_D$ that reproduces linear polynomials. Wang and Xu [8] studied a kind of Bernoulli-type operator by using a univariate multiquadric quasi-interpolation operator and the generalized Taylor polynomial. Wang et al. [9] introduced a kind of improved quasi-interpolation operator by combining the operator $\mathcal{L}_B$ with univariate Hermite interpolation polynomials $\mathcal{L}_{H_{2m-1}}$, and they got the desired orders of convergence. Based on the superposition idea in [10] and divided difference formula in [11], Feng and Zhou [12] constructed a kind of multiquadric quasi-interpolation operator, and it reproduces any degree polynomials and has the relevant order approximation rates. By combining the operator $\mathcal{L}_B$ with Lidstone interpolating polynomials [13] and applying the divided difference formula in [11], Wu et al. [14] studied Lidstone-type multiquadric quasi-interpolants $\widetilde{\mathcal{L}}_{\Lambda_n}$ and $\mathcal{L}_{\Lambda_n}$ that possess any degree of polynomial reproducibility. Recently, many other works have been studied on the Lidstone interpolation operator; see, for example, [15–17]. At the same time, scholars have studied various interesting areas [18–20].

By using dimension-splitting technology, Ling [21] extended the univariate quasi-interpolant to the bidimensional case. Feng and Zhou [22] and Wu et al. [23] improved the approximation order of the bivariate quasi-interpolant. Feng and Peng [24] constructed the quasi-interpolation scheme for arbitrary dimensional scattered data approximation. To overcome the numerical solution of high-dimensional shockwave equations, in 2019, Zhang et al. [25] proposed the bivariate dimension-splitting multiquadric quasi-interpolant. In 2022, Li et al. [26] introduced the concept of bivariate multiquadric quasi-interpolation on gridded data to solve 2D sine-Gordon equations. Numerical examples verified the effectiveness and high accuracy of the method. However, the convergence orders of their bivariate multiquadric quasi-interpolation operators [25, 26] are low.

To increase the accuracy of the multiquadric quasi-interpolation method for interesting applications, our paper was constructed to present a kind of bivariate multiquadric quasi-interpolation operator with higher accuracy for gridded data. In the paper, by combining the multiquadric quasi-interpolant (see, e.g., [25]) with the generalized Taylor polynomials that act as the bivariate Lidstone interpolation polynomials, we first construct a kind of bivariate Lidstone-type multiquadric quasi-interpolant $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$. For practical purposes, applying the divided difference formula in [11] to the operators $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$, we obtain another kind of bivariate multiquadric quasi-interpolant $\mathcal{L}_{\Lambda_{m,n}}$ which does not require values of the derivatives at nodes. We prove that the constructed operators $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$ and $\mathcal{L}_{\Lambda_{m,n}}$ reproduce all polynomials of degree $(2m-1, 2n-1)$.

The remainder of this paper is organized as follows. In Section 2, we introduce the generalized Taylor polynomial of degree $(2m-1, 2n-1)$ and give new results on the error of approximation that will be used later in the paper. In Section 3, we apply previous results to derive two kinds of Lidstone-type operators with bivariate multiquadrics. In Section 4, we give some error bounds in terms of the modulus of continuity of high order and also apply Peano's theorem. In Section 5, numerical examples are shown to compare the approximation capacity of the new operators with other existing methods. In Section 6, we give the conclusions.

2. Some remarks about the generalized Taylor polynomial $(2m-1, 2n-1)$

We first recall the univariate Lidstone interpolation problem introduced by Lidstone [13] in 1929

$$\begin{cases} L_m^{(2i)}[f;0,1](0) = f^{(2i)}(0), & i = 0, 1, \cdots, m-1, \\ L_n^{(2i)}[f;0,1](1) = f^{(2i)}(1), & i = 0, 1, \cdots, m-1; \end{cases}$$

it has a unique solution in the space of polynomials of degree not greater than $2m-1$ namely

$$L_m[f;0,1](x) = \sum_{k=0}^{m-1} [\Lambda_k(1-x) f^{(2k)}(0) + \Lambda_k(x) f^{(2k)}(1)]$$

where the polynomials $\Lambda_k(x), k = 0, 1, \cdots$, are Lidstone polynomials [27] defined recursively by

$$\begin{cases} \Lambda_0(x) = x, \\ \Lambda_k''(x) = \Lambda_{k-1}(x), & k \geqslant 1, \\ \Lambda_k(0) = \Lambda_k(1) = 0, & k \geqslant 1. \end{cases} \tag{2.1}$$

Without loss of generality, for the functions $f(x) \in C^{2m-1}[a, b], a, b \in \mathbb{R}, a < b$, the expansion is realized by the equation

$$f(x) = L_m[f;a,b](x) + R_m[f;a,b](x), \quad x \in [a, b], \tag{2.2}$$

where the Lidstone interpolation polynomial $L_m[f;a,b](x)$ is defined as follows

$$L_m[f;a,b](x)=\sum_{k=0}^{n-1}(b-a)^{2k}\left[\Lambda_k\left(\tfrac{b-x}{b-a}\right)f^{(2k)}(a)+\Lambda_k\left(\tfrac{x-a}{b-a}\right)f^{(2k)}(b)\right] \tag{2.3}$$

and $R_m[f;a,b](x)$ denotes the remainder term.

Suppose that $I=[a,b]\times[c,d]$ is a rectangular domain in the plane $\mathbb{R}^2$. Let us denote by $C^{(2m,2n)}(I)$ the space of the function $f:I\to\mathbb{R}^2$ with continuous partial derivatives

$$f^{(i,j)}(x,y)=\frac{\partial^{(i+j)}}{\partial x^i\partial y^j}f(x,y),\quad (x,y)\in I,$$

for all (i,j), $i=0,1,\cdots,2m,\ j=0,1,\cdots,2n$.

For a given function $f\in C^{(2m,2n)}(I)$, the polynomial approximation term $L_{m,n}[f;a,b;c,d](x,y)$ is the bivariate Lidstone interpolation polynomial of degree $(2m-1,2n-1)$ with variables x,y, obtained by the following formula:

$$\begin{aligned}L_{m,n}[f;a,b;c,d](x,y)=\sum_{u=0}^{m-1}\sum_{v=0}^{n-1}h^{2u}k^{2v}\Big[&\Lambda_u\left(\tfrac{b-x}{h}\right)\Lambda_v\left(\tfrac{d-y}{k}\right)f^{(2u,2v)}(a,c)+\Lambda_u\left(\tfrac{b-x}{h}\right)\Lambda_v\left(\tfrac{y-c}{k}\right)f^{(2u,2v)}(a,d)\\&+\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{d-y}{k}\right)f^{(2u,2v)}(b,c)+\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{y-c}{k}\right)f^{(2u,2v)}(b,d)\Big]\end{aligned} \tag{2.4}$$

with $h=b-a$, $k=d-c$.

As in the one dimensional case [14], we also call the bivariate Lidstone interpolation polynomial approximation $L_{m,n}[f;a,b;c,d](x,y)$ the generalized Taylor polynomial of degree $(2m-1,2n-1)$. One can derive the following from a nice property of this operator: its limit when $h\to0$, $k\to0$ is the well-known Taylor polynomial of degree $(2m-1,2n-1)$ of f about point (a,c) [28]:

$$\lim_{h,k\to0}L_{m,n}[f;a,b;c,d](x,y)=T_{2m-1,2n-1}[f;(a,c)](x,y),$$

where

$$T_{2m-1,2n-1}[f;(a,c)](x,y)=\sum_{i=0}^{2m-1}\sum_{j=0}^{2n-1}\frac{(x-a)^i(y-c)^j}{i!j!}f^{(i,j)}(a,c).$$

Moreover, the polynomial $L_{m,n}[f;a,b;c,d](x,y)$ satisfies the following interpolation conditions:

$$\begin{aligned}L_{m,n}^{(2i,2j)}[f;a,b;c,d](a,c)&=f^{(2i,2j)}(a,c),i=0,1,\cdots,m,\ j=0,1,\cdots,n,\\L_{m,n}^{(2i,2j)}[f;a,b;c,d](a,d)&=f^{(2i,2j)}(a,d),i=0,1,\cdots,m,\ j=0,1,\cdots,n,\\L_{m,n}^{(2i,2j)}[f;a,b;c,d](b,c)&=f^{(2i,2j)}(b,c),i=0,1,\cdots,m,\ j=0,1,\cdots,n,\\L_{m,n}^{(2i,2j)}[f;a,b;c,d](b,d)&=f^{(2i,2j)}(b,d),i=0,1,\cdots,m,\ j=0,1,\cdots,n.\end{aligned}$$

We set the k-th power of the argument

$$(\cdot)^k=\begin{cases}(\cdot)^k, & k\ge0\\0, & k<0\end{cases}$$

and give the following theorem.

Theorem 1. *Let $f \in C^{(2m,2n)}(I)$, $m, n \geq 1$. Then at each $(x, y) \in I$ the following identity holds:*

$$f(x,y) = L_{m,n}[f;a,b;c,d](x,y) + R_{m,n}[f;a,b;c,d](x,y) \tag{2.5}$$

where the bivariate Lidstone interpolation polynomial $L_{m,n}[f;a,b;c,d](x,y)$ is defined in (2.4) and the remainder term is

$$\begin{aligned} R_{m,n}[f;a,b;c,d](x,y) = &\sum_{j<2n} \int_a^b f^{(2m,j)}(s,c) H^x_{m,j}(x,y,s)ds \\ &+ \sum_{i<2m} \int_c^d f^{(i,2n)}(a,t) H^y_{i,n}(x,y,t)dt \\ &+ \int_a^b \int_c^d f^{(2m,2n)}(s,t) H^{x,y}_{m,n}(x,y,s,t)dsdt \end{aligned}$$

where $H^x_{m,j}(x,y,s)$, $H^y_{i,n}(x,y,t)$ and $H^{x,y}_{m,n}(x,y,s,t)$ are the Peano's kernels.

Proof. By applying the Peano's theorem for the bidimensional case [29] and using the relation (2.5), we obtain the following equalities:

$$\begin{aligned} H^x_{m,j}(x,y,s) &= R_{m,n}\left[\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!};a,b;c,d\right](x,y), \\ H^y_{i,n}(x,y,t) &= R_{m,n}\left[\frac{(x-a)^i}{i!}\frac{(y-t)_+^{2n-1}}{(2n-1)!};a,b;c,d\right](x,y), \\ H^{x,y}_{m,n}(x,y,t) &= R_{m,n}\left[\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-t)_+^{2n-1}}{(2n-1)!};a,b;c,d\right](x,y) \end{aligned}$$

where

$$z_+ = \begin{cases} z, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

Furthermore,

$$\begin{aligned} H^x_{m,j}(x,y,s) =& \frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!} \\ &- \sum_{u=0}^{m-1}\sum_{v=0}^{n-1} h^{2u}k^{2v}\left[\Lambda_u\left(\tfrac{b-x}{h}\right)\Lambda_v\left(\tfrac{d-y}{k}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=a\\y=c}}\right. \\ &+ \Lambda_u\left(\tfrac{b-x}{h}\right)\Lambda_v\left(\tfrac{y-c}{k}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=a\\y=d}} \\ &+ \Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{d-y}{k}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=b\\y=c}} \\ &\left.+\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{y-c}{k}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=b\\y=d}}\right] \end{aligned}$$

with

$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=a\\y=c}}=0,$$
$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=a\\y=d}}=0,$$
$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=b\\y=c}}=\begin{cases}\frac{(b-s)^{2m-1-2u}}{(2m-1-2u)!}, & j=2v\\ 0, & j\neq 2v\end{cases},$$
$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Big|_{\substack{x=b\\y=d}}=\frac{(b-s)^{2m-1-2u}}{(2m-1-2u)!}\frac{k^{j-2v}}{(j-2v)!},$$

so, we obtain

$$\begin{aligned}H_{m,j}^x(x,y,s)=&\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\\&-\sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\frac{(b-s)^{2m-1-2u}h^{2u}k^{2v}}{(2m-1-2u)!(j-2v)!}\left[\kappa_1\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{d-y}{k}\right)+k^{j-2v}\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{y-c}{k}\right)\right]\end{aligned}$$

where

$$\kappa_1=\begin{cases}1, & j=2v\\ 0, & j\neq 2v\end{cases}.$$

The remaining kernels $H_{i,n}^y(x,y,t)$ and $H_{m,n}^{x,y}(x,y,s,t)$ may be obtained by the analogous arguments as follows:

$$\begin{aligned}H_{i,n}^y(x,y,t)=&\frac{(x-a)^i}{i!}\frac{(y-c)^{2n-1}}{(2n-1)!}\\&-\sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\frac{(d-t)^{2n-1-2v}h^{2u}k^{2v}}{(2n-1-2v)!(i-2u)!}\left[\kappa_2\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{d-y}{k}\right)+h^{i-2u}\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{y-c}{k}\right)\right]\end{aligned}$$

where

$$\kappa_2=\begin{cases}1, & i=2u\\ 0, & i\neq 2u\end{cases},$$

and

$$\begin{aligned}H_{m,n}^{x,y}(x,y,s)=&\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-t)_+^{2n-1}}{(2n-1)!}\\&-\sum_{u=0}^{m-1}\sum_{v=0}^{n-1}h^{2u}k^{2v}\Lambda_u\left(\tfrac{x-a}{h}\right)\Lambda_v\left(\tfrac{y-c}{k}\right)\frac{(b-s)^{2m-1-2u}}{(2m-1-2u)!}\frac{(d-t)^{2n-1-2v}}{(2n-1-2v)!}.\end{aligned}$$

□

Remark 1 (Degree of exactness of the bivariate interpolation polynomial (2.4))**.** *We denote by* $\mathbb{P}^{(2m-1,2n-1)}$*,* $m,n\geq 1$ *the space of polynomials* $P(x,y)$ *of degree* (s,t) *with* $s\leq 2m-1, t\leq 2n-1$*. We can easily prove that the bivariate interpolation polynomial* (2.4) *has the degree of exactness of* $(2m-1,2n-1)$*, i.e.*

$$R_{m,n}[P;a,b;c,d](x,y)=0,\quad \forall P\in\mathbb{P}^{(2m-1,2n-1)}.$$

3. Bivariate multiquadric quasi-interpolation operators of Lidstone type

3.1. A kind of bivariate Lidstone-type quasi-interpolation operator with derivatives

Let us consider a function $f \in C^{(2m+1,2n+1)}(I)$, where $I = [a, b] \times [c, d]$ and there are $N + 1$ distinct points $(x_i, y_j) \in I$, $i = 0, 1, \cdots, N_1, j = 0, 1, \cdots, N_2$; we also set $h_i = x_{i+1} - x_i$ and $k_j = y_{j+1} - y_j$, $i = 0, 1, \cdots, N_1, j = 0, 1, \cdots, N_2$ considering a fictive node $(x_{N_1+1}, y_{N_2+1}) = (x_{N_1-1}, y_{N_2-1})$.

Given data $\{x_i, f_i\}$, $f_i = f(x_i)$, the univariate multiquadric quasi-interpolation operator $\mathcal{L}_B$ was constructed by Beatson and Powell in [6] as follows:

$$(\mathcal{L}_B f)(x) = f(x_0)\psi_0(x) + \sum_{i=1}^{N_1-1} f(x_i)\psi_i(x) + f(x_{N_1})\psi_{N_1}(x), \quad x \in I, \tag{3.1}$$

where

$$\begin{aligned}\psi_0(x) &= \frac{1}{2}c_1^2 \int_{-\infty}^{x_0} \frac{1}{[(x-t)^2 + c_1^2]^{3/2}} dt + \frac{1}{2}c_1^2 \int_{x_0}^{x_1} \frac{(x_1 - t)/(x_1 - x_0)}{[(x-t)^2 + c_1^2]^{3/2}} dt \\ &= \frac{1}{2} + \frac{\phi_1(x) - \phi_0(x)}{2(x_1 - x_0)},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\psi_{N_1}(x) =& \frac{1}{2}c_1^2 \int_{x_{N_1}}^{\infty} \frac{1}{[(x-t)^2 + c_1^2]^{3/2}} dt \\ &+ \frac{1}{2}c_1^2 \int_{x_{N_1-1}}^{x_{N_1}} \frac{(t - x_{N_1-1})/(x_{N_1} - x_{N_1-1})}{[(x-t)^2 + c_1^2]^{3/2}} dt \\ =& \frac{1}{2} - \frac{\phi_{N_1}(x) - \phi_{N_1-1}(x)}{2(x_{N_1} - x_{N_1-1})},\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\psi_i(x) &= \frac{1}{2}c_1^2 \int_{x_{i-1}}^{x_{i+1}} \frac{B_i(t)}{[(x-t)^2 + c_1^2]^{3/2}} dt \\ &= \frac{\phi_{i+1}(x) - \phi_i(x)}{2(x_{i+1} - x_i)} - \frac{\phi_i(x) - \phi_{i-1}(x)}{2(x_i - x_{i-1})}\end{aligned} \tag{3.4}$$

for $i = 1, 2, \cdots, N_1 - 1$, where $\{B_i(t) : t \in \mathbb{R}\}$ is the hat function that has the nodes $\{x_{i-1}, x_i, x_{i+1}\}$, which are identically zero outside of the interval $x_{i-1} \le t \le x_{i+1}$, and that satisfies the normalization condition $B_i(x_i) = 1$. $\phi_i(x) = \sqrt{(x - x_i)^2 + c_1^2}$ is named a multiquadric function, and c_1 is the shape parameter. The operator $\mathcal{L}_B$ reproduces constants.

Zhang et al. [25] extended the univariate quasi-interpolation $Qf(x) = \sum_i f_i\psi_i(x)$ to bivariate $(Qf)(x, y) = \sum_i \sum_j f_{ij}\psi_i(x)\psi_j(y)$ using the dimension-splitting multiquadric basis function technique. However, it only obtains lower accuracy.

In this paper, in order to increase the approximation capability of the bivariate multiquadric quasi-interpolation operator, we first study a kind of bivariate multiquadric Lidstone-type quasi-interpolation operator $\mathcal{L}_{\Lambda_{m,n}}$ with derivatives of the function f at the endpoints this is achieved by combining the above extended multiquadric operator with the bivariate Lidstone interpolation polynomial

$$\widetilde{\mathcal{L}}_{\Lambda_{m,n}}[f](x, y) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \psi_i(x)\psi_j(y) L_{m,n}[f; x_i, x_{i+1}; y_j, y_{j+1}](x, y), \ (x, y) \in I \tag{3.5}$$

where $\{\psi_i(x)\}_{i=0}^{N_1}$ is given above and $\{\psi_j(y)\}_{j=0}^{N_2}$ is represented as follows:

$$\begin{aligned}
\phi_j(y) &= \sqrt{(y-y_j)^2+c_2^2},\\
\psi_0(y) &= \frac{1}{2}+\frac{\phi_1(y)-\phi_0(y)}{2(y_1-y_0)},\\
\psi_j(y) &= \frac{\phi_{j+1}(y)-\phi_j(y)}{2(y_{j+1}-y_j)}-\frac{\phi_j(y)-\phi_{j-1}(y)}{2(y_j-y_{j-1})}, j=1,2,\cdots,N_2-1,\\
\psi_{N_2}(y) &= \frac{1}{2}-\frac{\phi_{N_2}(x)-\phi_{N_2-1}(x)}{2(x_{N_2}-x_{N_2-1})},
\end{aligned}$$

where c_2 is a small positive constant. $L_{m,n}^{i,j}[f](x,y) := L_{m,n}[f;x_i,x_{i+1};y_j,y_{j+1}](x,y)$ denotes the bivariate Lidstone interpolation polynomial in the rectangle with opposite vertices (x_i,y_j), (x_{i+1},y_{j+1}) and it is given by (2.4).

Theorem 2. *The degree of exactness of the operator* $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}[\cdot]$ *is* $(2m-1,2n-1)$.

Proof. The assertion follows from the following property

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi_i(x)\psi_j(y)=1 \tag{3.6}$$

since the degree of exactness of the Lidstone interpolation operator $L_{m,n}^{i,j}[\cdot]$ is $(2m-1,2n-1)$, $i=0,1,\cdots,N_1$, $j=0,1,\cdots,N_2$. □

3.2. A kind of bivariate improved Lidstone-type quasi-interpolation operator without derivatives

Although the quasi-interpolation operators $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$ has the degree of exactness of $(2m-1,2n-1)$, it requires the derivatives of the function f at the nodes, which are very difficult to measure in practice. By extending the divided difference in one dimension [11] to the bidimensional case, we obtain the following definition. Furthermore, by using the divided difference operator $D_{A_x,A_y}^{(2i,2j)}f$ in Definition 1 to substitute the derivative $f^{(2i,2j)}$ in the operator $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$, we give another kind of bivariate improved Lidstone-type multiquadric quasi-interpolation operator $\mathcal{L}_{\Lambda_{m,n}}$ without derivatives of the function f at the endpoints.

Definition 1. *Let* $\mathcal{F}=\{f|f:\mathbb{R}^2\to\mathbb{R}\}$ *and let* A_x,A_y *be discrete subsets of* $\mathbb{R}$, $i,j\in\mathbb{N}$. *Suppose that* $D^{(2i,2j)}$ *is the order* $(2i,2j)$ *derivative. An operator* $D_{A_x,A_y}^{(2i,2j)}:\mathcal{F}\to\mathcal{F}$ *is said to be a* $\mathbb{P}^{(2m-1,2n-1)}$*-exact* A_x,A_y*-discretization of* $D^{(2i,2j)}$ *if and only if the following is true:*

(i) There exists real vectors $\lambda_x=(\lambda_a)_{a\in A_x}$ *and* $\lambda_y=(\lambda_b)_{b\in A_y}$ *s.t., for any* $f\in\mathcal{F}$,

$$D_{A_x,A_y}^{(2i,2j)}f(\cdot)=\sum_{a\in A_x}\sum_{b\in A_y}\lambda_a\lambda_b f(a+\cdot,b+\cdot),\quad i=1,\cdots,m-1, j=1,\cdots,n-1; \tag{3.7}$$

(ii) For any $p\in\mathbb{P}^{(2m-1,2n-1)}$,

$$D_{A_x,A_y}^{(2i,2j)}p=D^{(2i,2j)}p. \tag{3.8}$$

In such a situation, we also say that $D^{(2i,2j)}_{A_x,A_y}f$ is a $\mathbb{P}^{(2m-1,2n-1)}$-exact A_x, A_y-discretization of $D^{(2i,2j)}f$. Suppose that the points are distinct in the set A_x and A_y. Then $D^{(2i,2j)}_{A_x,A_y}$ is determined uniquely.

Suppose that $|\cdot|$ denotes the number of elements in the set and assume that the points in set $A_x(A_y)$ are distinct and $|A_x| = 2m(|A_y| = 2n)$. Then, by Definition 1 and [11], a $\mathbb{P}^{(2m-1,2n-1)}$-exact A_x, A_y-discretization of the order $(2i, 2j)$ derivative $f^{(2i,2j)}$ is

$$D^{(2i,2j)}_{A_x,A_y}f(x,y) = \sum_{a\in A_x}\sum_{b\in A_y}\lambda_a\lambda_b f(a+x, b+y),\quad i = 1,\cdots,m-1, j = 1,\cdots,n-1, \tag{3.9}$$

where

$$\begin{aligned}\lambda_a &= \frac{(-1)^{2m-2i-1}(2i)!\sum_{\substack{A'_x\subset A_x\backslash\{a\}\\|A'_x|=2m-2i-1}}\prod_{c\in A'_x}(c)}{\prod_{c\in A_x\backslash\{a\}}(a-c)},\\ \lambda_b &= \frac{(-1)^{2n-2j-1}(2j)!\sum_{\substack{A'_y\subset A_y\backslash\{b\}\\|A'_y|=2n-2j-1}}\prod_{c\in A'_y}(c)}{\prod_{c\in A_y\backslash\{b\}}(b-c)}.\end{aligned} \tag{3.10}$$

By virtue of the location of each pair (x_i, y_j), (x_i, y_{i+1}), (x_{i+1}, y_j), (x_{i+1}, y_{j+1}) $(i = 0, 1, \cdots, N_1, j = 0, 1, \cdots, N_2)$, we provide the following theorem.

Theorem 3. *For any $A_x, A_y \subset \mathbb{R}$ and $x, y \in \mathbb{R}$, let A_x, A_y be a $\mathbb{P}^{(2m-1,2n-1)}$-unisolvent sets and $A^x_x = A_x - x(A^y_y = A_y - y)$ denote the set of points $e \in \mathbb{R}$ of the form $e = a - x(e = b - y)$ where $a \in A_x(b \in A_y)$. If $A_{x_i} = \{x_{i-m+1}, x_{i-m+2}, \cdots, x_i, x_{i+1}, \cdots, x_{i+m}\}, i = m-1, m, \cdots, N_1 - m$ $(A_{y_j} = \{y_{j-n+1}, y_{j-n+2}, \cdots, y_j, y_{j+1}, \cdots, y_{j+n}\},\ j = n-1, n, \cdots N_2 - n)$, then we have, for each $i = 1, 2, \cdots, m-1, j = 1, 2, \cdots, n-1$,*

$$D^{(2i,2j)}_{A^{\alpha}_{x_i},A^{\beta}_{y_j}}f(\alpha,\beta) = \sum_{a\in A_{x_i}}\sum_{b\in A_{y_j}}\lambda_{a-\alpha}\lambda_{b-\beta}f(a,b),$$

where

$$\begin{aligned}\lambda_{a-\alpha} &= \frac{(-1)^{2m-2i-1}(2i)!\sum_{\substack{A'_{x_i}\subset A_{x_i}\backslash\{a\}\\|A'_{x_i}|=2m-2i-1}}\prod_{c\in A'_{x_i}}(c-\alpha)}{\prod_{c\in A_{x_i}\backslash\{a\}}(a-c)} \text{ and } \alpha = x_i, x_{i+1},\\ \lambda_{b-\beta} &= \frac{(-1)^{2n-2j-1}(2j)!\sum_{\substack{A'_{y_j}\subset A_{y_j}\backslash\{b\}\\|A'_{y_j}|=2n-2j-1}}\prod_{c\in A'_{y_j}}(c-\beta)}{\prod_{c\in A_{y_j}\backslash\{b\}}(b-c)} \text{ and } \beta = y_j, y_{j+1}.\end{aligned} \tag{3.11}$$

Let $A_{x_i} = \{x_{i-m+1}, x_{i-m+2}, \cdots, x_i, x_{i+1}, \cdots, x_{i+m}\}$ $(i = 0, 1, \cdots, m-2$ or $i = N_1-m+1, N_1-m+2, \cdots, N_1)$ such that $x_l = x_{l+2m}$, $l = -m+1, -m+2, \cdots, -1(x_l = x_l - 2m, l = N_1+1, N_1+2, \cdots, N_1+m)$ and $A_{y_j} = \{y_{j-n+1}, y_{j-n+2}, \cdots, y_j, y_{j+1}, \cdots, y_{j+n}\}$ $(j = 0, 1, \cdots, n-2$ or $j = N_2-n+1, N_2-n+2, \cdots, N_2)$ such that $y_l = y_{l+2n}$, $l = -n+1, -n+2, \cdots, -1(y_l = y_l - 2n, l = N_2+1, N_2+2, \cdots, N_2+n)$; then, we have, for $i = 1, 2, \cdots, m-1, j = 1, 2, \cdots, n-1$,

$$D^{(2i,2j)}_{A^{\alpha}_{x_i},A^{\beta}_{y_j}}f(\alpha,\beta) = \sum_{a\in A_{x_i}}\sum_{b\in A_{y_j}}\lambda_{a-\alpha}\lambda_{b-\beta}f(a,b)$$

hold for $\lambda_{a-\alpha}$ and α ($\lambda_{b-\beta}$ and β) are defined in (3.11).

Proof. For each pair $\alpha(\beta) \in \mathbb{R}$, we set $A_{x_i}^{\alpha} = \{x_{i-m+1}-\alpha, x_{i-m+2}-\alpha, \cdots, x_i-\alpha, x_{i+1}-\alpha, \cdots, x_{i+m}-\alpha\}, i = m-1, m, \cdots, N_1-m$ ($A_{y_i}^{\beta} = \{y_{j-n+1}-\beta, y_{j-n+2}-\beta, \cdots, y_j-\beta, y_{j+1}-\beta, \cdots, y_{j+n}-\beta\}, j = n-1, n, \cdots, N_2-n$). According to (3.7) and (3.10), we get

$$D^{(2i,2j)}_{A_{x_i}^{\alpha}, A_{y_j}^{\beta}} f(0,0) = \sum_{a\in A_{x_i}} \sum_{b\in A_{y_j}} \lambda_{a-\alpha}\lambda_{b-\beta} f(a-\alpha, b-\beta),$$

where

$$\lambda_{a-\alpha} = \frac{(-1)^{2m-2i-1}(2i)! \sum_{\substack{A'_{x_i}\subset A_{x_i}\backslash\{a\} \\ |A'_{x_i}|=2m-2i-1}} \prod_{c\in A'_{x_i}}(c-\alpha)}{\prod_{c\in A_{x_i}\backslash\{a\}}(a-c)} \text{ and } \alpha = x_i, x_{i+1},$$

$$\lambda_{b-\beta} = \frac{(-1)^{2n-2j-1}(2j)! \sum_{\substack{A'_{y_j}\subset A_{y_j}\backslash\{b\} \\ |A'_{y_j}|=2n-2j-1}} \prod_{c\in A'_{y_j}}(c-\beta)}{\prod_{c\in A_{y_j}\backslash\{b\}}(b-c)} \text{ and } \beta = y_j, y_{j+1}.$$

Therefore, we have

$$D^{(2i,2j)}_{A_{x_i}^{\alpha}, A_{y_j}^{\beta}} f(\alpha,\beta) = \sum_{a\in A_{x_i}} \sum_{b\in A_{y_j}} \lambda_{a-\alpha}\lambda_{b-\beta} f(a, b),$$

Similar to the discussion, we get the proof of Theorem 3. □

Next, by applying Theorem 3 to the operator $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$, we introduce improved multiquadric quasi-interpolation operator $\mathcal{L}_{\Lambda_{m,n}}$ without derivatives as follows:

$$\mathcal{L}_{\Lambda_{m,n}}[f](x,y) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{i_1=0}^{2m-1}\sum_{j_1=0}^{2n-1}\prod_{\substack{k_1=0\\k_1\neq i_1}}^{2m-1}\prod_{\substack{k_2=0\\k_2\neq j_1}}^{2n-1} \frac{(x-x_{i-m+1+k_1})(y-y_{j-n+1+k_2})}{(x_{i-m+1+i_1}-x_{i-m+1+k_1})(y_{j-n+1+j_1}-y_{j-n+1+k_2})}\psi_i(x)\psi_j(y) \quad (3.12)$$

where

$$\begin{cases} x_l = x_{l+2m}, & l = -m+1, -m+2, \cdots, -1 \\ x_l = x_{l-2m}, & l = N_1+1, N_1+2, \cdots, N_1+m \end{cases}, \begin{cases} y_l = y_{l+2n}, & l = -n+1, -n+2, \cdots, -1 \\ y_l = y_{l-2n}, & l = N_2+1, N_2+2, \cdots, N_2+n \end{cases}.$$

Theorem 4. *The degree of exactness of the operator* $\mathcal{L}_{\Lambda_{m,n}}[\cdot]$ *is* $(2m-1, 2n-1)$.

Proof. Based on the proof of Theorem 3 and Definition 1, we get the proof of Theorem 4 immediately. □

4. Error estimation of the operators

First, based on the mesh length, we obtain an estimation of the approximation error by using the modulus of smoothness of order k. Let us recall the following theorem. Some detailed definition is introduced in [30].

Theorem 5 (see, for example [30], Th. 7.3, p. 225)**.** *Given a quasi-interpolation operator Q of order r, for each $f \in C[a,b]$, we derive the following estimation:*

$$\|f - Qf\|_\infty \le C_r\omega_r(f;\delta)_\infty$$

where C_r is a constant, $\omega_r(f;\delta)_\infty$ denotes the r-th modulus of smoothness of a function f (see [31]) and δ is defined by

$$\delta = \max_{0\leq i\leq N} |x_{i+1} - x_i|.$$

Consider the functions $f \in C^{2m}[a,b]$ and $g \in C^{2n}[c,d]$ and the corresponding univariate Lidstone interpolation polynomials $L_m[f;a,b](x)$ and $L_n[g;c,d](y)$, for $m,n \geq 1$, introduced by (2.3). Since the operators L_m and L_n are quasi-interpolation operators of order $2m$ and $2n$ respectively (see the definitions on p. 144–146 of [30]), from Theorem 5, we obtain the following estimates

$$\begin{aligned} &\|f - L_m[f]\|_\infty \leq C_{2m}\omega_{2m}(f;\delta_1)_\infty, \\ &\|f - L_n[g]\|_\infty \leq C_{2n}\omega_{2n}(f;\delta_2)_\infty \end{aligned} \tag{4.1}$$

where $\omega_k(f;t)_\infty$ denotes the k-th modulus of the smoothness of a function f (see [31]) having

$$\begin{aligned} \delta_1 &= \max_{0\leq i\leq N_1} |x_{i+1} - x_i|, \\ \delta_2 &= \max_{0\leq j\leq N_2} |y_{j+1} - y_j|, \end{aligned} \tag{4.2}$$

and C_{2m}, C_{2n} are some constants.

Based on this information, we give an estimation of the error associated with approximating by using the bivariate quasi-interpolants of Lidstone type, in terms of the modulus of smoothness of high order.

Theorem 6. *Let $f \in C^{(2m,2n)}(I)$, $I = [a,b]\times[c,d]$; then,*

$$\begin{aligned} \|f - \widetilde{\mathcal{L}}_{\Lambda_{m,n}}[f]\|_\infty \leq & C_{2m} \max_{y\in[c,d]} \omega_{2m}(f(\cdot,y);\delta_1)_\infty \\ &+ C_{2n} \max_{x\in[a,b]} \omega_{2n} f(x,\cdot);\delta_{2\infty} \\ &+ C_{2m} \max_{y\in[c,d]} \omega_{2m}((f - L_n[f])(\cdot,y);\delta_1)_\infty \end{aligned}$$

where δ_1, δ_2 are expressed in (4.2) and C_{2m}, C_{2n} are constants.

Proof. By the relation (3.6), we have

$$\begin{aligned} f(x,y)-&\widetilde{\mathcal{L}}_{\Lambda_{m,n}}[f](x,y) \\ &= f(x,y) - \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y) L_{m,n}^{i,j}[f](x,y) \\ &= \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y)\left[f(x,y) - L_{m,n}^{i,j}[f](x,y)\right]. \end{aligned}$$

We know from the relation (2.4) that

$$L_{m,n}[f](x,y) = L_n\left[L_m[f]\right](x,y) = L_m\left[L_n[f]\right](x,y),$$

so

$$
\begin{aligned}
f(x,y)-&L_{m,n}[f](x,y)\\
&= f(x,y) - L_m[f](x,y) + f(x,y) - L_n[f](x,y)\\
&\quad + L_m[f - L_n[f]](x,y) - (f - L_n[f])(x,y).
\end{aligned}
$$

We have

$$
\begin{aligned}
&f(x,y) - \widetilde{\mathcal{L}}_{\Lambda_{m,n}}[f](x,y)\\
&= \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y)(f(x,y) - L_m^i[f](x,y))\\
&+ \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y)(f(x,y) - L_n^j[f](x,y))\\
&+ \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y)\left(L_m^i[f - L_n^j[f]](x,y) - (f - L_n^j[f])(x,y)\right)
\end{aligned}
\tag{4.3}
$$

with $L_m^i[f] := L_m[f; x_i, x_{i+1}]$ and $L_n^j[f] = L_n[f; y_j, y_{j+1}]$. It follows from the relation (4.3) that

$$
\begin{aligned}
&|f(x,y) - \widetilde{\mathcal{L}}_{\Lambda_{m,n}}[f](x,y)|\\
&\leq \max_{y\in[c,d]} \|f(\cdot,y) - L_m[f](\cdot,y)\|_\infty \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y)\\
&\quad + \max_{x\in[a,b]} \|f(x,\cdot) - L_n[f](x,\cdot)\|_\infty \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y)\\
&\quad + \max_{y\in[c,d]} \|(f - L_n[f])(\cdot,y) - L_m[f - L_n[f]](\cdot,y)\|_\infty\\
&\quad \times \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \psi_i(x)\psi_j(y).
\end{aligned}
$$

By using the relations (3.6) and (4.2), we can finish the proof. □

Next, by applying the Peano's theorem we give the following integral representations of the error.

Theorem 7. *If $f \in C^{(2m,2n)}(I)$ and $I = [a,b] \times [c,d]$, then, for the remainder term*

$$
R_{L_{m,n}}[f](x,y) = f(x,y) - \widetilde{\mathcal{L}}_{\Lambda_{m,n}}[f](x,y) \tag{4.4}
$$

we have

$$
\begin{aligned}
R_{L_{m,n}}[f](x,y) &= \sum_{q<2n} \int_a^b f^{(2m,q)}(s,c) K_{m,q}^x(x,y,s)ds\\
&\quad + \sum_{p<2m} \int_c^d f^{(p,2n)}(a,t) K_{p,n}^y(x,y,t)dt\\
&\quad + \int_a^b \int_c^d f^{(2m,2n)}(s,t) K_{m,n}^{x,y}(x,y,s,t)dsdt
\end{aligned}
$$

where $K_{m,q}^x(x,y,s)$, $K_{p,n}^y(x,y,t)$, and $K_{m,n}^{x,y}(x,y,s,t)$ are the Peano's kernels.

Proof. By applying the Peano's theorem for the bidimensional case [29] and using the relation (3.6), we give the following equalities:

$$K^x_{m,q}(x,y,s) = R_{L_{m,n}}\left[\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!}\right](x,y),$$
$$K^y_{p,n}(x,y,t) = R_{L_{m,n}}\left[\frac{(x-a)^p}{p!}\frac{(y-t)_+^{2n-1}}{(2n-1)!}\right](x,y),$$
$$K^{x,y}_{m,n}(x,y,t) = R_{L_{m,n}}\left[\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-t)_+^{2n-1}}{(2n-1)!}\right](x,y),$$

such that

$$K^x_{m,q}(x,y,s) = \frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!} - \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\psi_i(x)\psi_j(y) \times L_{m,n}\left[\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!}; x_i, x_{i+1}; y_j, y_{j+1}\right](x,y)$$

with

$$\begin{aligned}
&L_{m,n}\left[\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!}; x_i, x_{i+1}; y_j, y_{j+1}\right](x,y)\\
&= \sum_{u=0}^{m-1}\sum_{v=0}^{n-1} h_i^{2u}k^{2v}\Bigg[\Lambda_u\left(\tfrac{x_{i+1}-x}{h_i}\right)\Lambda_v\left(\tfrac{y_{j+1}-y}{k_j}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_i\\ y=y_j}}\\
&+ \Lambda_u\left(\tfrac{x_{i+1}-x}{h_i}\right)\Lambda_v\left(\tfrac{y-y_j}{k_j}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_i\\ y=y_{j+1}}}\\
&+ \Lambda_u\left(\tfrac{x-x_i}{h_i}\right)\Lambda_v\left(\tfrac{y_{j+1}-y}{k_j}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_{i+1}\\ y=y_j}}\\
&+\Lambda_u\left(\tfrac{x-x_i}{h_i}\right)\Lambda_v\left(\tfrac{y-y_j}{k_j}\right)\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^j}{j!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_{i+1}\\ y=y_{j+1}}}\Bigg]
\end{aligned}$$

with

$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_i\\ y=y_j}} = \frac{(x_i-s)_+^{2m-1-2u}}{(2m-1-2u)!}\frac{(y_j-c)^{q-2v}}{(q-2v)!},$$
$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_i\\ y=y_{j+1}}} = \frac{(x_i-s)_+^{2m-1-2u}}{(2m-1-2u)!}\frac{(y_{j+1}-c)^{q-2v}}{(q-2v)!},$$
$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_{i+1}\\ y=y_j}} = \frac{(x_{i+1}-s)_+^{2m-1-2u}}{(2m-1-2u)!}\frac{(y_j-c)^{q-2v}}{(q-2v)!},$$
$$\left(\frac{(x-s)_+^{2m-1}}{(2m-1)!}\frac{(y-c)^q}{q!}\right)^{(2u,2v)}\Bigg|_{\substack{x=x_{i+1}\\ y=y_{j+1}}} = \frac{(x_{i+1}-s)_+^{2m-1-2u}}{(2m-1-2u)!}\frac{(y_{j+1}-c)^{q-2v}}{(q-2v)!}.$$

The rest of Peano's kernels, $K^y_{i,n}(x,y,t)$ and $K^{x,y}_{m,n}(x,y,s,t)$ are obtained in the same manner. □

By considering the discussion for Theorems 6 and 7, we can obtain analogous theorems for the error estimation by the improved multiquadric quasi-interpolation operator $\mathcal{L}_{\Lambda_{m,n}}$.

5. Numerical experiments

To test the bivariate Lidstone-type multiquadric quasi-interpolation operators $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$ and $\mathcal{L}_{\Lambda_{m,n}}$, we consider the following test functions (see, e.g., [32]) on the computational domain $[0,1]\times[0,1]$:

$$\text{Gentle}\ \ f_1(x,y) = \frac{1}{3}e^{-\frac{81}{16}\left((x-0.5)^2+(y-0.5)^2\right)}, \tag{5.1}$$

$$\text{Sphere}\ \ f_2(x,y) = \frac{\sqrt{64-81\left((x-0.5)^2+(y-0.5)^2\right)}}{9} - 0.5, \tag{5.2}$$

$$\text{Saddle}\ \ f_3(x,y) = \frac{1.25+\cos(5.4y)}{6+6(3x-1)^2}, \tag{5.3}$$

$$\text{Steep}\ \ f_4(x,y) = \frac{1}{3}e^{-\frac{81}{4}\left((x-0.5)^2+(y-0.5)^2\right)}, \tag{5.4}$$

$$\text{Cliff}\ \ f_5(x,y) = \frac{\tanh(9\text{y}-9\text{x})+1}{9}, \tag{5.5}$$

$$\begin{aligned}\text{Exponential}\ \ f_6(x,y) =& 0.75e^{-\frac{1}{4}\left((9x-2)^2-(9y-2)^2\right)}\\ &+0.75e^{-\left(\frac{(9x+1)^2}{49}+\frac{(9y+1)^2}{10}\right)}\\ &+0.5e^{-\frac{1}{4}\left((9x-7)^2-(9y-3)^2\right)}\\ &-0.2e^{-\left((9x-4)^2+(9y-7)^2\right)}.\end{aligned} \tag{5.6}$$

For each function $f_i, i = 1,\cdots,6$, we will compare the numerical results of our operators $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$ and $\mathcal{L}_{\Lambda_{m,n}}$ with those of another bivariate multiquadric quasi-interpolation operator Q (see for [25, 26]) on the application of partial differential equations.

We use uniform grids of 6×6 nodes for the operators $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$ $\mathcal{L}_{\Lambda_{m,n}}$ and Q (see for [25, 26]) with $c_1 = c_2 = (0.1)^l, l = 2,3,4$ respectively. In order to estimate the errors as accurately as possible, we compute the approximating functions at the points $(\frac{i}{21},\frac{j}{21}), (i = 1,2,\cdots,20, j = 1,2,\cdots,20)$. Tables 1–6 display the mean and max errors for the different approximation operators above. The numerical results show that the bivariate Lidstone type multiquadric quasi-interpolation operators $\widetilde{\mathcal{L}}_{\Lambda_{m,n}}$ and $\mathcal{L}_{\Lambda_{m,n}}$ have good approximating behavior.

Table 1. Gentle.

	$\widetilde{\mathcal{L}}_{\Lambda_{m,n}}f_1$		$\mathcal{L}_{\Lambda_{m,n}}f_1$		Qf_1 [25]		Qf_1 [26]	
(l, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.03014	0.00796	0.03014	0.00796	0.03063	0.00653	0.03074	0.00653
(2,1,2)	0.01798	0.00471	0.01840	0.00471	0.03063	0.00653	0.03074	0.00653
(2,2,2)	0.00314	0.00071	0.00393	0.00059	0.03063	0.00653	0.03074	0.00653
(3,1,1)	0.02989	0.00771	0.02989	0.00771	0.03019	0.00631	0.03020	0.00631
(3,1,2)	0.01781	0.00459	0.01826	0.00460	0.03019	0.00631	0.03020	0.00631
(3,2,2)	0.00319	0.00073	0.00398	0.00060	0.03019	0.00631	0.03020	0.00631
(4,1,1)	0.02989	0.00771	0.02989	0.00771	0.03019	0.00630	0.03019	0.00630
(4,1,2)	0.01781	0.00459	0.01825	0.00460	0.03019	0.00630	0.03019	0.00630
(4,2,2)	0.00319	0.00073	0.00398	0.00060	0.03019	0.00630	0.03019	0.00630

Table 2. Sphere.

	$\widetilde{\mathcal{L}}_{\Lambda_{m,n}} f_2$		$\mathcal{L}_{\Lambda_{m,n}} f_2$		Qf_2 [25]		Qf_2 [26]	
(l, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.02032	0.00757	0.02032	0.00757	0.02034	0.01060	0.02070	0.01080
(2,1,2)	0.01024	0.00516	0.01170	0.00505	0.02034	0.01060	0.02070	0.01080
(2,2,2)	0.00151	0.00027	0.00214	0.00041	0.02034	0.01060	0.02070	0.01080
(3,1,1)	0.02044	0.00738	0.02044	0.00738	0.02044	0.01029	0.02044	0.01029
(3,1,2)	0.01019	0.00503	0.01177	0.00491	0.02044	0.01029	0.02044	0.01029
(3,2,2)	0.00148	0.00026	0.00215	0.00041	0.02044	0.01029	0.02044	0.01029
(4,1,1)	0.02044	0.00738	0.02044	0.00738	0.02044	0.01028	0.02044	0.01028
(4,1,2)	0.01019	0.00502	0.01177	0.00491	0.02044	0.01028	0.02044	0.01028
(4,2,2)	0.00148	0.00026	0.00215	0.00041	0.02044	0.01028	0.02044	0.01028

Table 3. Saddle.

	$\widetilde{\mathcal{L}}_{\Lambda_{m,n}} f_3$		$\mathcal{L}_{\Lambda_{m,n}} f_3$		Qf_3 [25]		Qf_3 [26]	
(l, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.02782	0.00695	0.02782	0.00695	0.04541	0.00765	0.04534	0.00763
(2,1,2)	0.02679	0.00427	0.02949	0.00434	0.04541	0.00765	0.04534	0.00763
(2,2,2)	0.01128	0.00155	0.01270	0.00216	0.04541	0.00765	0.04534	0.00763
(3,1,1)	0.02768	0.00687	0.02768	0.00687	0.04507	0.00743	0.04507	0.00743
(3,1,2)	0.02649	0.00420	0.02923	0.00429	0.04507	0.00743	0.04507	0.00743
(3,2,2)	0.01155	0.00156	0.01264	0.00217	0.04507	0.00743	0.04507	0.00743
(4,1,1)	0.02767	0.00687	0.02767	0.00687	0.04507	0.00743	0.04507	0.00743
(4,1,2)	0.02649	0.00420	0.02923	0.00429	0.04507	0.00743	0.04507	0.00743
(4,2,2)	0.01155	0.00156	0.01264	0.00217	0.04507	0.00743	0.04507	0.00743

Table 4. Steep.

	$\widetilde{\mathcal{L}}_{\Lambda_{m,n}} f_4$		$\mathcal{L}_{\Lambda_{m,n}} f_4$		Qf_4 [25]		Qf_4 [26]	
(l, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.07654	0.01451	0.07654	0.01451	0.10411	0.01037	0.10416	0.01035
(2,1,2)	0.05826	0.00990	0.05828	0.01329	0.10411	0.01037	0.10416	0.01035
(2,2,2)	0.04660	0.00502	0.04238	0.01200	0.10411	0.01037	0.10416	0.01035
(3,1,1)	0.07625	0.01451	0.07625	0.01451	0.10345	0.01023	0.10345	0.01023
(3,1,2)	0.05785	0.00990	0.05817	0.01327	0.10345	0.01023	0.10345	0.01023
(3,2,2)	0.04716	0.00502	0.04245	0.01197	0.10345	0.01023	0.10345	0.01023
(4,1,1)	0.07625	0.01451	0.07625	0.01451	0.10344	0.01023	0.10344	0.01023
(4,1,2)	0.05785	0.00990	0.05817	0.01327	0.10344	0.01023	0.10344	0.01023
(4,2,2)	0.04717	0.00502	0.04245	0.01197	0.10344	0.01023	0.10344	0.01023

Table 5. Cliff.

	$\widetilde{\mathcal{L}}_{\Lambda_{m,n}} f_5$		$\mathcal{L}_{\Lambda_{m,n}} f_5$		Qf_5 [25]		Qf_5 [26]	
(l, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.02814	0.00810	0.02814	0.00810	0.03299	0.01012	0.03300	0.01012
(2,1,2)	0.02778	0.00755	0.03053	0.00851	0.03299	0.01012	0.03300	0.01012
(2,2,2)	0.02741	0.00788	0.02187	0.00774	0.03299	0.01012	0.03300	0.01012
(3,1,1)	0.02813	0.00804	0.02813	0.00804	0.03282	0.00986	0.03282	0.00986
(3,1,2)	0.02776	0.00751	0.03056	0.00848	0.03282	0.00986	0.03282	0.00986
(3,2,2)	0.02732	0.00783	0.02188	0.00774	0.03282	0.00986	0.03282	0.00986
(4,1,1)	0.02813	0.00804	0.02813	0.00804	0.03282	0.00986	0.03282	0.00986
(4,1,2)	0.02776	0.00751	0.03056	0.00848	0.03282	0.00986	0.03282	0.00986
(4,2,2)	0.02732	0.00783	0.02188	0.00774	0.03282	0.00986	0.03282	0.00986

Table 6. Exponential.

	$\widetilde{\mathcal{L}}_{\Lambda_{m,n}} f_6$		$\mathcal{L}_{\Lambda_{m,n}} f_6$		Qf_6 [25]		Qf_6 [26]	
(l, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.23296	0.03791	0.23296	0.03791	0.14396	0.03171	0.14420	0.03175
(2,1,2)	0.16047	0.03346	0.20249	0.03577	0.14396	0.03171	0.14420	0.03175
(2,2,2)	0.10739	0.01895	0.16252	0.03170	0.14396	0.03171	0.14420	0.03175
(3,1,1)	0.23339	0.03783	0.23339	0.03783	0.14287	0.03112	0.14287	0.03113
(3,1,2)	0.15867	0.03341	0.20222	0.03570	0.14287	0.03112	0.14287	0.03113
(3,2,2)	0.10669	0.01881	0.16174	0.03163	0.14287	0.03112	0.14287	0.03113
(4,1,1)	0.23339	0.03783	0.23339	0.03783	0.14286	0.03112	0.14286	0.03112
(4,1,2)	0.15864	0.03342	0.20221	0.03570	0.14286	0.03112	0.14286	0.03112
(4,2,2)	0.10668	0.01881	0.16174	0.03163	0.14286	0.03112	0.14286	0.03112

6. Conclusions

In this paper, a kind of bivariate multiquadric quasi-interpolant is proposed by combining a bivariate multiquadric quasi-interpolant with the generalized Taylor polynomials that act as the Lidstone interpolation polynomials. However, it requires the derivatives of the approximated function at the endpoints, which is not very convenient for practical purposes. Using linear combinations of the shifts of the approximated function to approximate the derivatives of the approximated function, we have studied another kind of bivariate multiquadric quasi-interpolant which does not require values of the derivatives at nodes. Some results on the some error bounds of the new operators are given. Numerical tests show that the operators give higher accuracy. Furthermore, the associated algorithm is easily implemented.

In our future work, we plan to apply it to solve partial differential equations, and good results may be obtained. Moreover, we could construct stochastic quasi-interpolation operators with Lidstone interpolation polynomials.

Use of AI tools declaration

The author declares that he has not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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