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Research article

Generalized Newton-Leibniz formula and the embedding of the Sobolev functions with dominating mixed smoothness into Hölder spaces

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Abstract: It is well-known that the embedding of the Sobolev space of weakly differentiable functions into Hölder spaces holds if the integrability exponent is higher than the space dimension. In this paper, the embedding of the Sobolev functions into the Hölder spaces is expressed in terms of the minimal weak differentiability requirement independent of the integrability exponent. The proof is based on the generalization of the Newton-Leibniz formula to the n-dimensional rectangle and the inductive application of the Sobolev trace embedding results. The new method is applied to prove the embedding of the Sobolev spaces with dominating mixed smoothness into Hölder spaces. Counterexamples demonstrate that in terms of minimal weak regularity degree the Sobolev spaces with dominating mixed smoothness present the largest class of weakly differentiable functions with the upgrade of pointwise regularity to continuity. Remarkably, it also presents the largest class of weakly differentiable functions where the generalized Newton-Leibniz formula holds.

Keywords: generalized Newton-Leibniz formula; Sobolev spaces; weakly differentiable functions; Hölder spaces; embedding theorems
Mathematics Subject Classification: 46E35

1. Prelude

Let $W^1_p(\mathbb{R}^n), 1 \leq p \leq \infty$ be a Sobolev space of weakly differentiable functions $u \in L_p(\mathbb{R}^n)$ with first order weak derivatives in $L_p(\mathbb{R}^n), i = 1, ..., n$. Originally discovered in the celebrated paper [1], the concept of Sobolev spaces became a trailblazing idea in many fields of mathematics. The goal of this paper is to analyze embedding of $W^1_p(\mathbb{R}^n)$ into Hölder spaces $C^{0,\mu}(\mathbb{R}^n), 0 \leq \mu \leq 1$ [2]. A standard notation will be employed for the embedding of Banach spaces:

- $B_1 \hookrightarrow B_2$ means bounded embedding of B_1 into B_2, i.e., $B_1 \subset B_2$, and

$$\|u\|_{B_2} \leq C\|u\|_{B_1}, \ \forall u \in B_1, \ \text{for some constant } C.$$

- $B_1 \Subset B_2$ denotes compact embedding of B_1 into B_2, meaning that $B_1 \hookrightarrow B_2$, and every bounded subset of B_1 is precompact in B_2.

If $n = 1$, the equivalency class of elements of $W_p^1(\mathbb{R})$ always contain an absolutely continuous element, which is Hölder continuous with exponent $1 - p^{-1}$, if $p > 1$, i.e., there is a bounded embedding

$$W_p^1(\mathbb{R}) \hookrightarrow C^{0,1-\frac{1}{p}}(\mathbb{R}), \text{ if } p > 1; \;\; W_1^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R}). \tag{1.1}$$

The embedding (1.1) easily follows from the Newton-Leibniz formula

$$u(x') - u(x) = \int_x^{x'} \frac{du(y)}{dx}\, dy \tag{1.2}$$

via the application of the Hölder inequality and compactness argument. The embedding (1.1) fails to be true if $n \geq 2$ and $p \leq n$. However, there is a bounded embedding [3]

$$W_p^1(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{n}{p}}(\mathbb{R}^n) \text{ if } p > n. \tag{1.3}$$

Hence, stretching the integrability exponent p beyond space dimension n implies the Hölder continuity. In particular, elements of the Hilbert space $H^1(\mathbb{R}^n) = W_2^1(\mathbb{R}^n)$, are not continuous in general, if $n \geq 2$. The main goal of this paper is to express the continuity of elements of $W_p^1(\mathbb{R}^n)$ in terms of weak differentiability requirements.

Problem 1.1. *What are the minimal weak differentiability requirements on elements of $W_p^1(\mathbb{R}^n)$ ($1 \leq p \leq n$) to be continuous? In terms of weak differentiability requirements, what is the largest subspace of $W_p^1(\mathbb{R}^n)$ embedded into Hölder spaces for all $p \geq 1$?*

The paper reveals that the anticipated subspace is the Sobolev-Nikol'skii space

$$S_p^1(\mathbb{R}^n) = \Big\{ u \in W_p^1(\mathbb{R}^n) \Big| \; \frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \in L_p(\mathbb{R}^n), i_1 < \cdots < i_k, k = \overline{2,n} \Big\},$$

equipped with the norm

$$\|u\|_{S_p^1(\mathbb{R}^n)} := \begin{cases} \Big(\|u\|^p_{L_p(\mathbb{R}^n)} + \sum\limits_{k=1}^n \sum\limits_{\substack{i_1,\dots,i_k=1 \\ i_1<\dots<i_k}}^n \Big\| \frac{\partial^k u}{\partial x_{i_1}\cdots\partial x_{i_k}} \Big\|^p_{L_p(\mathbb{R}^n)} \Big)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \|u\|_{L_\infty(\mathbb{R}^n)} + \sum\limits_{k=1}^n \sum\limits_{\substack{i_1,\dots,i_k=1 \\ i_1<\dots<i_k}}^n \Big\| \frac{\partial^k u}{\partial x_{i_1}\cdots\partial x_{i_k}} \Big\|_{L_\infty(\mathbb{R}^n)}, & \text{if } p = \infty. \end{cases}$$

The space $S_p^1(\mathbb{R}^n)$ is a special case of Sobolev spaces with dominating mixed smoothness. The class was introduced by Nikol'skii in [4, 5]. There is a vast literature on the analysis of these spaces. We refer to [6–9] and the references therein.

The main result of this paper is twofold. First, we introduce and prove a generalization of the celebrated Newton-Leibniz formula to n-dimensional rectangles (or n-rectangles). Then by using the new formula as a tool, we present a surprisingly simple and elegant proof of the embedding of the Sobolev spaces with dominating mixed smoothness into Hölder spaces. The proof resembles the proof of the embedding (1.1) in the one-dimensional case by using generalized Newton-Leibniz formula, Hölder inequality, and iterative application of the Sobolev trace embedding results. In particular, we prove that the generalized Newton-Leibniz formula is preserved in space $S_p^1(\mathbb{R}^n)$. Counterexamples support the claim that in terms of weak differentiability requirements, S_p^1 is the largest class of Lebesgue's integrable and weakly differentiable functions in $\mathbb{R}^n$ which preserve generalized Newton-Leibniz formula, and upgrades the pointwise regularity to Hölder continuity.

2. Notations

- $C^0(\mathbb{R}^n)$ is a Banach space of continuous and bounded functions with the norm

$$\|u\|_{C^0(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} |u(x)| = \|u\|_{L_\infty(\mathbb{R}^n)}.$$

- For $k \in \mathbb{N}$, $C^k(\mathbb{R}^n)$ is a Banach space of k times continuously differentiable functions, with all derivatives of order k bounded, and with the norm

$$\|u\|_{C^k(\mathbb{R}^n)} := \sum_{j=0}^{k} \sup_{x\in\mathbb{R}^n} |D^j u(x)| = \sum_{j=0}^{k} \|D^j u\|_{L_\infty(\mathbb{R}^n)},$$

where $D^j u$ is a tensor of rank j, dimension n, and

$$|D^j u| = \Big(\sum_{i_1,\dots,i_j=1}^{n} \Big| \frac{\partial^j u(x)}{\partial x_{i_1} \cdots \partial x_{i_j}} \Big|^2 \Big)^{\frac{1}{2}}.$$

- $SC^1(\mathbb{R}^n)$ is a Banach space with the norm

$$\|u\|_{SC^1(\mathbb{R}^n)} := \sum_{\alpha\in\mathbb{Z}_+^n, \alpha_i\le 1} \Big\| \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \Big\|_{C^0(\mathbb{R}^n)}.$$

The following standard notation will be used for Hölder spaces:

- For $0 \le \gamma \le 1$, Hölder space $C^{0,\gamma}(\mathbb{R}^n)$ is the Banach space of elements $u \in C^0(\mathbb{R}^n)$ with finite norm

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} := \|u\|_{C^0(\mathbb{R}^n)} + [u]_{C^{0,\gamma}(\mathbb{R}^n)},$$

where

$$[v]_{C^{0,\gamma}(\mathbb{R}^n)} := \sup_{\substack{x,x'\in\mathbb{R}^n \\ x\neq x'}} \frac{|v(x) - v(x')|}{|x - x'|^\gamma}.$$

The space $C^{0,0}(\mathbb{R}^n)$ is equivalent to $C^0(\mathbb{R}^n)$.

- For $k \in \mathbb{N}$, $0 \le \gamma \le 1$, Hölder space $C^{k,\gamma}(\mathbb{R}^n)$ is a subspace of $C^k(\mathbb{R}^n)$ with finite norm

$$\|u\|_{C^{k,\gamma}(\mathbb{R}^n)} := \sum_{j=0}^{k} \|D^j u\|_{C^{0,\gamma}(\mathbb{R}^n)}$$

Throughout the paper we use standard notations for $L_p(Q)$, $1 \le p \le \infty$ spaces; the following standard notations are used for Sobolev spaces [2]:

- For $k \in \mathbb{N}$, $1 \le p \le \infty$, Sobolev space $W_p^k(\mathbb{R}^n)$ is the Banach space of measurable functions on $\mathbb{R}^n$ with finite norm

$$\|u\|_{W_p^k(\mathbb{R}^n)} := \sum_{j=0}^{k} \|D^j u\|_{L_p(\mathbb{R}^n)}.$$

- For $\mathbf{s} = (s_1, ..., s_n) \in \mathbb{Z}_+^n$, $1 \le p \le \infty$, anisotropic Sobolev space $W_p^{\mathbf{s}}(\mathbb{R}^n)$ is the Banach space of measurable functions on $\mathbb{R}^n$ with finite norm

$$\|u\|_{W_p^{\mathbf{s}}(\mathbb{R}^n)} := \|u\|_{L_p(\mathbb{R}^n)} + \sum_{i=1}^{n} \sum_{k=1}^{s_i} \Big\| \frac{\partial^k u}{\partial x_i^k} \Big\|_{L_p(\mathbb{R}^n)}.$$

  Note that the size of the vector $\mathbf{s}$ coincides with the dimension of the space. In particular, for $1 \le k \le n$, and fixed $j \in \{1, ..., k\}$, we consider Sobolev spaces $W_p^{\mathbf{s}}(\mathbb{R}^k)$ of the weakly x_j-differentiable functions on $\mathbb{R}^k$, where $\mathbf{s} = (s_i)_{i=1}^k \in \mathbb{Z}_+^k$ and $s_i = \delta_{ij}$ is a Kronecker symbol.
- For $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}_+^n$, $1 \le p \le \infty$, Sobolev space $S_p^{\mathbf{k}}(\mathbb{R}^n)$ with dominating mixed derivatives is a Banach space of measurable functions on $\mathbb{R}^n$ with the finite norm

$$\|u\|_{S_p^{\mathbf{k}}(\mathbb{R}^n)} := \sum_{\alpha \in \mathbb{Z}_+^n, \alpha_i \le k_i} \Big\| \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \Big\|_{L_p(\mathbb{R}^n)}.$$

  If $k_1 = \cdots = k_n = k \in \mathbb{N}$, we shall write $S_p^k(\mathbb{R}^n) = S_p^{\mathbf{k}}(\mathbb{R}^n)$.
- Let $Q \subset \mathbb{R}^n$ be a bounded domain. For $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}_+^n$, $1 \le p \le \infty$, Sobolev space $S_p^{\mathbf{k}}(Q)$ with dominating mixed derivatives is defined as

$$S_p^{\mathbf{k}}(Q) = \{f \in \mathcal{D}'(Q) : \exists g \in S_p^{\mathbf{k}}(\mathbb{R}^n) \text{ with } g|_Q = f\}$$

  and with

$$\|f\|_{S_p^{\mathbf{k}}(Q)} := \inf \|g\|_{S_p^{\mathbf{k}}(\mathbb{R}^n)},$$

  where the infimum is taken over all $g \in S_p^{\mathbf{k}}(\mathbb{R}^n)$ such that its restriction $g|_Q$ to Q coincides with f in the space of distributions $\mathcal{D}'(Q)$. If $k_1 = \cdots = k_n = k \in \mathbb{N}$, we shall write $S_p^k(Q) = S_p^{\mathbf{k}}(Q)$.

3. Main results

3.1. Generalized Newton-Leibniz formula

Let $x, x' \in \mathbb{R}^n$ with $x_i < x'_i$, $i = \overline{1, n}$ are fixed and P be n-rectangle

$$P = \{\eta \in \mathbb{R}^n : x_i \le \eta_i \le x'_i, i = \overline{1, n}\} \tag{3.1}$$

with vertex x (or x') called a *bottom (or top) corner* of P. For any subset $\{i_1, ..., i_k\} \subset \{1, ..., n\}, k = \overline{1, n}$, let

$$P_{i_1 \dots i_k} = P \cap \{\eta \in \mathbb{R}^n : \eta_l = x_l, l \ne i_j, j = \overline{1, k}\}$$

be a k-rectangle with bottom corner x. Note that $P_{i_1 \dots i_k}$ is invariant with respect to permutation of multi-index $i_1 \cdots i_k$, and it coincides with P if $k = n$.

The following is the generalization of the celebrated Newton-Leibniz formula:

Theorem 3.1. *Any function $u \in SC^1(\mathbb{R}^n)$ satisfies the following generalized Newton-Leibniz formula:*

$$u(x') - u(x) = \sum_{k=1}^{n} \sum_{\substack{i_1, ..., i_k = 1 \\ i_1 < ... < i_k}}^{n} \int_{P_{i_1 \dots i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}} d\eta_{i_1} \cdots d\eta_{i_k}. \tag{3.2}$$

If $n = 1$, (3.2) coincides with the Newton-Leibniz formula (1.2). Note that for $\forall k$ there are $\binom{n}{k}$ integrals in (3.2) along all k-rectangles $P_{i_1\dots i_k}$ with bottom corner x. Therefore, altogether there are

$$\sum_{k=1}^{n}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k} - 1 = (1+1)^n - 1 = 2^n - 1,$$

integrals in (3.2) along all sub-rectangles of P with bottom corner at x.

Having generalized the Newton-Leibniz formula in the class of smooth functions, we can now formulate the major problem of classical analysis generated by the Newton-Leibniz formula:

Problem 3.1. *What is the largest class of Lebesgue integrable and weakly differentiable functions in $\mathbb{R}^n$ which preserve generalized Newton-Leibniz formula?*

In Theorem 3.2 we prove that the formula (3.2) remains valid in spaces $S_p^1(\mathbb{R}^n)$, $1 \le p \le \infty$, where the right-hand side is understood as trace integrals of respective weak derivatives. Counterexamples constructed in Section 3.3 support the claim that in terms of weak differentiability requirements $S_p^1(\mathbb{R}^n)$ is the largest class of Lebesgue's integrable and weakly differentiable functions in $\mathbb{R}^n$ which preserve generalized Newton-Leibniz formula. The formula (3.2) is a key to prove the Hölder continuity of elements of $S_p^1(\mathbb{R}^n)$.

Remark 3.1. *Some variant of the formula* (3.2) *is proved in [10] in the class of tensor product space, i.e., linear cover of the space of separable (or product form) functions equipped with special norm consisting of some algebraic combination of one-dimensional $W^{1,p}$ norms selected in a way to guarantee the L^p-boundedness of mixed derivatives of product functions.*

3.2. Embedding of the Sobolev spaces with dominating mixed smoothness into Hölder spaces

Theorem 3.2. *The following bounded embedding holds*

$$S_p^1(\mathbb{R}^n) \hookrightarrow C^{0,1-\frac{1}{p}}(\mathbb{R}^n); \quad 1 \le p \le \infty. \tag{3.3}$$

The equivalency class of every element of $S_p^1(\mathbb{R}^n)$ possesses a representative in $C^{0,1-\frac{1}{p}}(\mathbb{R}^n)$, which satisfies the generalized Newton-Leibniz formula (3.2)*, where $P \subset \mathbb{R}^n$ is an n-rectangle with bottom and top corner at x and x' respectively. In particular, $\forall k = 1, ..., n-1$ and $1 \le i_1 < \cdots < i_k \le n$*

$$\frac{\partial^k u}{\partial x_{i_1}\cdots\partial x_{i_k}} \in L_p(P_{i_1\dots i_k}), \tag{3.4}$$

in the sense of traces.

Corollary 3.1. *For $k \in \mathbb{N}$ the following bounded embedding holds*

$$S_p^k(\mathbb{R}^n) \hookrightarrow C^{k-1,1-\frac{1}{p}}(\mathbb{R}^n); \quad 1 \le p \le \infty. \tag{3.5}$$

The following sharp embedding result holds for the anisotropic Sobolev spaces with dominating mixed smoothness:

Corollary 3.2. *Let* $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{N}^n, 1 \le p \le \infty$, *and* $u \in S_p^{\mathbf{k}}(\mathbb{R}^n)$. *Then* $\forall m = 1, ..., n$ *and* $\forall 1 \le i_1 < i_2 < \cdots < i_m \le n$

$$\frac{\partial^{k_{i_1}+\cdots+k_{i_m}-m} u}{\partial x_{i_1}^{k_{i_1}-1} \cdots \partial x_{i_m}^{k_{i_m}-1}} \in C^{0,1-\frac{1}{p}}(\mathbb{R}^n). \tag{3.6}$$

Corollary 3.3. *Let* $Q \subset \mathbb{R}^n$ *be a bounded domain. For* $k \in \mathbb{N}$ *the following bounded and compact embeddings hold*

$$S_p^k(Q) \hookrightarrow C^{k-1,1-\frac{1}{p}}(\overline{Q}), \text{ if } 1 \le p \le \infty; \tag{3.7}$$

$$S_p^k(Q) \Subset C^{k-1,\mu}(\overline{Q}), 0 < \mu < 1 - \frac{1}{p}, \text{ if } 1 < p \le \infty. \tag{3.8}$$

3.3. Counterexamples

The goal of this section is to provide counterexamples to support the claim that $S_p^1(\mathbb{R}^n)$ is a subspace of $W_p^1(\mathbb{R}^n), 1 \le p \le n$ with minimal increase of the weak differentiability requirements in order that every equivalency class has an element which is

- Hölder continuous;
- satisfy generalized Newton-Leibniz formula (3.2);
- satisfy the trace regularity (3.4);

Example 3.1. *Consider a function*

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right) \in W_n^1(B(0,1)) \tag{3.9}$$

where $B(0,1) \subset \mathbb{R}^n$ *is a unit ball with center* 0. *It is discontinuous at* 0. *Direct calculation demonstrates that for arbitrary* $k \in \{1, ..., n-1\}$ *we have*

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \in L_p(B(0,1)), \ 1 \le p \le \frac{n}{k}, \tag{3.10}$$

for all $1 \le i_1 \le \cdots \le i_k \le n$. *However, we have*

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} \notin L_1(B(0,1)). \tag{3.11}$$

Since for all $k \in \{1, ..., n-1\}$ *all k-th order weak derivatives are in* $L_1(B(0,1))$, *by using standard extension theorem [2] function u can be extended to* $\mathbb{R}^n$ *by possessing the regularity*

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \in L_p(\mathbb{R}^n), \ 1 \le p \le \frac{n}{k}, \tag{3.12}$$

for all $1 \le i_1 \le \cdots \le i_k \le n$. *From* (3.11) *it follows that the n-th order mixed derivative of the extended function is not integrable, i.e.,*

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} \notin L_1(\mathbb{R}^n). \tag{3.13}$$

It can also be verified that if P is an n-rectangle with bottom corner at 0*, then* $\forall k = 1, ..., n-1$ *and* $1 \le i_1 < \cdots < i_k \le n$*, we have*

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \notin L_1(P_{i_1 \ldots i_k}). \tag{3.14}$$

Clearly, Newton-Leibniz formula (3.2) *is not satisfied at the origin. Hence, the extended function presents the desired counterexample when the n-th order mixed derivative is removed from the definition of the space* $S^1_p(\mathbb{R}^n)$.

Example 3.2. *Consider a function*

$$u(x) = |x|^{-n} \prod_{k=1}^{n} x_k \in L_\infty(\mathbb{R}^n) \tag{3.15}$$

It is discontinuous at 0*. Along the each hyperplane* $\{x_k = 0\}$ *it is zero, but for* $\forall C > 0$

$$\lim_{x_k = Ct, t\downarrow 0} u = 1. \tag{3.16}$$

Direct calculation demonstrates that for arbitrary $k \in \{1, ..., n-1\}$ *we have*

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \in L_p(B(0,1)),\ 1 \le p < \frac{n}{k}, \tag{3.17}$$

for all $1 \le i_1 \le \cdots \le i_k \le n$*. However, we have*

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} \notin L_1(B(0,1)). \tag{3.18}$$

Since for all $k \in \{1, ..., n-1\}$ *all k-th order weak derivatives are in* $L_1(B(0,1))$*, by using standard extension theorem [2] function u can be extended to* $\mathbb{R}^n$ *by possessing the regularity*

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \in L_p(\mathbb{R}^n),\ 1 \le p < \frac{n}{k}, \tag{3.19}$$

for all $1 \le i_1 \le \cdots \le i_k \le n$*. From* (3.18) *it follows that the n-th order mixed derivative of the extended function is not integrable, i.e.*

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} \notin L_1(\mathbb{R}^n). \tag{3.20}$$

It can also be verified that if P is an n-rectangle with bottom corner at 0*, then* $\forall k = 1, ..., n-1$ *and* $1 \le i_1 < \cdots < i_k \le n$*, we have*

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \equiv 0 \in L_1(P_{i_1 \ldots i_k}). \tag{3.21}$$

In particular, this implies that the Newton-Leibniz formula is satisfied in any k-rectangle $P_{i_1 \ldots i_k}$ *with* $k \le n-1$ *if we assign* $u(0) = 0$*. However, Newton-Leibniz formula is not satisfied in an n-rectangle P due to the fact that*

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} \notin L_1(P). \tag{3.22}$$

Example 3.3. *Consider a function*

$$u(x) = \prod_{s=1}^{k} sign\ x_s \in L_\infty(\mathbb{R}^n), \tag{3.23}$$

with $k \in \{1, ..., n-1\}$. *It is discontinuous along all the coordinates axis* $x_1, ..., x_k$. *For arbitrary* $s \in \{1, ..., k\}$ *the weak derivatives*

$$\frac{\partial^s u}{\partial x_1 \cdots \partial x_s} \tag{3.24}$$

do not exist. All the other mixed derivatives which involve differentiation with respect to any other variables x_l *with* $l \in \{k+1, ..., n\}$ *exists and equal to zero. Clearly, the Newton-Leibniz formula is not satisfied in any n-rectangle whose interior intersects any of the coordinate axis* $x_1, ..., x_k$.

4. Proof of main results

Proof of Theorem 3.1. Assuming that $u \in SC^1(\mathbb{R}^n)$, we prove (3.2) by induction in terms of the space dimension n. If $n = 1$, it coincides with the Newton-Leibniz formula. Assume that (3.2) is true, and demonstrate that it is true if n is replaced with $n+1$. Let $x, x' \in \mathbb{R}^{n+1}$ with $x_i < x_i'$, $i = \overline{1, n+1}$, are fixed. We have

$$u(x') - u(x) = (u(x') - u(\tilde{x}, x'_{n+1})) + (u(\tilde{x}, x'_{n+1}) - u(x)), \tag{4.1}$$

where $\tilde{x} = (x_1, ..., x_n)$. Applying (3.2) to the first term and the Newton-Leibniz formula to the second term in (4.1), we derive

$$u(x') - u(x) = \sum_{k=1}^{n} \sum_{\substack{i_1,...,i_k=1 \\ i_1<...<i_k}}^{n} \int\limits_{P_{i_1...i_k}} \frac{\partial^k u(\tilde{\eta}, x'_{n+1})}{\partial x_{i_1} \cdots \partial x_{i_k}}\, d\eta_{i_1} \cdots d\eta_{i_k} + \int\limits_{x_{n+1}}^{x'_{n+1}} \frac{\partial u(\tilde{x}, \eta)}{\partial x_{n+1}}\, d\eta. \tag{4.2}$$

Applying the Newton-Leibniz formula to all but the last integrand, we have

$$\begin{aligned} u(x') - u(x) = &\sum_{k=1}^{n} \sum_{\substack{i_1,...,i_k=1 \\ i_1<...<i_k}}^{n} \int\limits_{P_{i_1...i_k}} \int\limits_{x_{n+1}}^{x'_{n+1}} \frac{\partial^{k+1} u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k} \partial x_{n+1}}\, d\eta_{i_1} \cdots d\eta_{i_k}\, d\eta_{n+1} \\ &+ \int\limits_{x_{n+1}}^{x'_{n+1}} \frac{\partial u(\tilde{x}, \eta)}{\partial x_{n+1}}\, d\eta + \sum_{k=1}^{n} \sum_{\substack{i_1,...,i_k=1 \\ i_1<...<i_k}}^{n} \int\limits_{P_{i_1...i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\, d\eta_{i_1} \cdots d\eta_{i_k}, \end{aligned} \tag{4.3}$$

which imply that

$$u(x') - u(x) = \sum_{k=1}^{n+1} \sum_{\substack{i_1,...,i_k=1 \\ i_1<...<i_k}}^{n+1} \int\limits_{P_{i_1...i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\, d\eta_{i_1} \cdots d\eta_{i_k}, \tag{4.4}$$

where we use the same notation for the $(n+1)$-rectangle P, as well as its corresponding sub-rectangles in $\mathbb{R}^{n+1}$. Indeed, divide all $2^{n+1} - 1$ sub-rectangles of P with the bottom corner at x into two groups depending on whether or not the edge p_{n+1} joining vertices x and $(\tilde{x}, x'_{n+1})$ is contained in it. The first two terms on the right-hand side of (4.3) consist of all 2^n terms of (4.4) with integrals along sub-rectangles containing the edge p_{n+1}, and the last term on the right-hand side of (4.3) is identical with the remaining $2^n - 1$ integrals in (4.4) along sub-rectangles which do not contain the edge p_{n+1}. This completes the proof by induction. □

Proof of Theorem 3.2. First, we prove the Theorem assuming that $1 \le p < \infty$. The proof will be pursued in four steps.

Step 1. Prove that for $u \in S^1_p(\mathbb{R}^n)$, each of the $2^n - 1$ integrals on the right-hand side of (3.2) is finite, and in particular, (3.4) is satisfied. Existence of the integral with $k = n$ on the right hand side of (3.2) follows from the definition of $S^1_p(\mathbb{R}^n)$ and Hölder inequality. We prove the existence of the remaining $2^n - 2$ trace integrals in (3.2) by mathematical induction and Sobolev trace embedding result. First, we demonstrate that the claim is true if $k = n - 1$. Then we show that the claim is true for any $k < n - 1$, provided it is true for $k + 1$. Indeed, if $k = n - 1$, for each of the n integrals we select a unique integer j satisfying

$$j \in \{1, ..., n\} \cap \{i_1, ..., i_k\}^c, \tag{4.5}$$

and define a multi-index $\mathbf{s} = (s_1, ..., s_n) \in \mathbb{Z}^n_+$, where $s_i = \delta_{ij}$ is a Kronecker symbol. We have

$$\frac{\partial^{n-1} u}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}} \in W^{\mathbf{s}}_p(P). \tag{4.6}$$

Note that $(n - 1)$-rectangle $P_{i_1 \cdots i_{n-1}}$ is a boundary of P on the hyperplane $\eta_j = x_j$. Existence of the trace

$$\frac{\partial^{n-1} u}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}} \in L_p(P_{i_1 \cdots i_{n-1}}) \tag{4.7}$$

is a consequence of the Sobolev trace embedding result:

$$W^{\mathbf{s}}_p(P) \hookrightarrow L_p(P_{i_1 \cdots i_{n-1}}). \tag{4.8}$$

For completeness, we present a proof of (4.8). Consider a function

$$\zeta(\eta) = 1 - \frac{\eta_j - x_j}{x'_j - x_j}, \tag{4.9}$$

which satisfy

$$0 \le \zeta \le 1, \ \left|\frac{\partial \zeta}{\partial \eta_j}\right| \le \frac{1}{x'_j - x_j}, \ \eta \in P. \tag{4.10}$$

Assuming that $u \in SC^1(P)$, we have

$$\int\limits_{P_{i_1 \dots i_{n-1}}} \left|\frac{\partial^{n-1} u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}}\right|^p d\eta_{i_1} \cdots d\eta_{i_{n-1}}$$
$$= \int\limits_{P_{i_1 \dots i_{n-1}}} \zeta \left|\frac{\partial^{n-1} u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}}\right|^p d\eta_{i_1} \cdots d\eta_{i_{n-1}}$$

$$= -\int\limits_{P_{i_1\cdots i_{n-1}}}\int\limits_{x_j}^{x_j'}\frac{\partial}{\partial x_j}\Big(\zeta\Big|\frac{\partial^{n-1}u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\Big|^p\Big)\,d\eta_j\,d\eta_{i_1}\cdots d\eta_{i_{n-1}}$$

$$= -\int\limits_{P_{i_1\cdots i_{n-1}}}\int\limits_{x_j}^{x_j'}\Big[\frac{\partial\zeta}{\partial x_j}\Big|\frac{\partial^{n-1}u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\Big|^p+\zeta p\Big|\frac{\partial^{n-1}u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\Big|^{p-1}$$

$$\times\, sign\Big(\frac{\partial^{n-1}u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\Big)\frac{\partial^{n}u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_k}\partial x_j}\Big]\,d\eta_j\,d\eta_{i_1}\cdots d\eta_{i_{n-1}} \tag{4.11}$$

If $p > 1$, by using Young's inequality and (4.10), from (4.11) it follows

$$\Big\|\frac{\partial^{n-1}u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\Big\|_{L_p(P_{i_1\cdots i_{n-1}})} \le C\Big\|\frac{\partial^{n-1}u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\Big\|_{W_p^{\mathbf{s}}(P)}, \tag{4.12}$$

where $C = \max(p-1+|x_j'-x_j|^{-1};1)$. If $p = 1$, (4.12) follows directly from (4.10) and (4.11). In general, we can approximate $u \in S_p^1(\mathbb{R}^n)$ with the sequence $u^\epsilon = u * \phi^\epsilon \in C_{loc}^\infty(\mathbb{R}^n)$, where ϕ^ϵ is a standard rescaled mollifier, and derive (4.12) for u^ϵ. Since u^ϵ converges to u in the norm given on the right-hand side of (4.12), it is so in the norm of the left-hand side as well, and passing to the limit as $\epsilon \to 0$, (4.12), (4.8) and (4.7) follow. Hence, n relations of (3.4) with $k = n-1$ are established. Next, we prove that the claim is true for k if it is so for $k+1$. For any of the $\binom{n}{k}$ integrals in (3.2) along the k-dimensional prism $P_{i_1\cdots i_k}$ we select any integer j satisfying (4.5), and define a multiindex $\mathbf{s} = (s_1, ..., s_{k+1}) \in \mathbb{Z}_+^{k+1}$, where $s_i = \delta_{ij}$ is a Kronecker symbol. Noting that $P_{i_1\cdots i_{k+1}}$ is invariant with respect to permutations of the multi-index $i_1\cdots i_{k+1}$, and due to the induction assumption we have

$$\frac{\partial^k u}{\partial x_{i_1}\cdots\partial x_{i_k}} \in W_p^{\mathbf{s}}(P_{i_1\cdots i_k j}). \tag{4.13}$$

k-rectangle $P_{i_1\cdots i_k}$ is a boundary of $(k+1)$-*rectangle* $P_{i_1\cdots i_k j}$ on the hyperplane $x_j = const$. Sobolev trace embedding result implies:

$$W_p^{\mathbf{s}}(P_{i_1\cdots i_k j}) \hookrightarrow L_p(P_{i_1\cdots i_k}), \tag{4.14}$$

The proof of (4.14) is identical to the proof of (4.8). Hence, (3.4) is proved for all k-dimensional integrals.

Step 2. In this step we prove that any $u \in S_p^1(\mathbb{R}^n)\cap SC_{loc}^1(\mathbb{R}^n)$, $p > 1$ satisfies the estimate

$$|u(x)-u(x')| \le \Big[\Big((1+p)^{\frac{1}{p}}+|x-x'|^{\frac{p-1}{p}}\Big)^n-(1+p)^{\frac{n}{p}}\Big]\|u\|_{S_p^1(\mathbb{R}^n)}, \tag{4.15}$$

for all $x, x' \in \mathbb{R}^n$. Similarly, any $u \in S_1^1(\mathbb{R}^n)\cap SC_{loc}^1(\mathbb{R}^n)$ satisfy the estimate

$$|u(x)-u(x')| \le (3^n-2^n)\|u\|_{S_1^1(\mathbb{R}^n)}, \tag{4.16}$$

for all $x, x' \in \mathbb{R}^n$. Note that the estimate (4.16) is a formal limit of the estimate (4.15) as $p \to 1$.

To prove (4.15) (or (4.16)) without loss of generality we can assume that $x_i < x_i'$, $i = \overline{1,n}$. Indeed, if $x_i \ne x_i'$, $i = \overline{1,n}$, then we can transform the space via finitely many translations

$$\tilde{y} : \mathbb{R}^n \to \mathbb{R}^n,\ \tilde{y}_i = \begin{cases} y_i, & \text{if } x_i < x_i', \\ -y_i, & \text{if } x_i > x_i', \end{cases} \tag{4.17}$$

and note that the space $S_p^1(\mathbb{R}^n)$ is invariant under this transformation. Then we can apply (4.15) (or (4.16)) to the ϵ-mollification of the transformed function $\tilde{u}(\tilde{x}) = u(\tilde{x})$, and passing to limit as $\epsilon \to 0$ deduce (4.15) (or (4.16)) for $\tilde{u}$. Applying inverse transformation of (4.17) implies (4.15) (or (4.16)) for u. If, on the other side $x_i = x_i'$ for some i, we can replace x_i' with $x_i' + \delta$, prove (4.15) (or (4.16)) and pass to limit as $\delta \to 0$.

The proof of (4.15) and (4.16) under the assumption that $x_i < x_i'$, $i = \overline{1,n}$ is based on the generalized Newton-Leibniz formula (3.2). The following is the proof of the estimate (4.15). Let P be a n-rectangle (3.1), and

$$P^1 := \{\eta \in \mathbb{R}^n : x_i \le \eta_i \le x_i' + 1, \ i = \overline{1,n}\}.$$

By using Hölder inequality the integral on the right-hand side of (3.2) with $k = n$ is estimated as follows

$$\Big|\int_P \frac{\partial^n u(\eta)}{\partial x_1 \cdots \partial x_n} d\eta\Big| \le |P|^{\frac{p-1}{p}} \Big\|\frac{\partial^n u}{\partial x_1 \cdots \partial x_n}\Big\|_{L_p(P)}, \tag{4.18}$$

where $|P|$ denotes volume of the n-rectangle P. For $k = 1, ..., n-1$, estimation of any of the k-dimensional integrals on the right-hand side of (3.2) will be pursued in $n-k$ steps. Consider typical k-dimensional integral in (3.2) along the k-rectangle $P_{i_1 \cdots i_k}$. The idea is based on successive application of the trace embedding result (4.14) $n-k$ times. First, we select any integer j from (4.5) and assign it to the multi-index component i_{k+1}. By using Hölder inequality we have

$$\Big|\int_{P_{i_1 \cdots i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}} d\eta_{i_1} \cdots d\eta_{i_k}\Big| \le |P_{i_1 \cdots i_k}|^{\frac{p-1}{p}} \Big\|\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big\|_{L_p(P_{i_1 \cdots i_k})}. \tag{4.19}$$

Consider a function

$$\zeta(\eta) = 1 - \frac{\eta_{i_{k+1}} - x_{i_{k+1}}}{x'_{i_{k+1}} - x_{i_{k+1}} + 1}, \tag{4.20}$$

which satisfy

$$0 \le \zeta \le 1, \ \Big|\frac{\partial \zeta}{\partial \eta_{i_{k+1}}}\Big| \le 1. \tag{4.21}$$

We have

$$\begin{aligned}
\int_{P_{i_1 \ldots i_k}} \Big|\frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big|^p d\eta_{i_1} \cdots d\eta_{i_k} &= \int_{P_{i_1 \ldots i_k}} \zeta \Big|\frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big|^p d\eta_{i_1} \cdots d\eta_{i_k} \\
&= -\int_{P_{i_1 \ldots i_k}} \int_{x_{i_{k+1}}}^{x'_{i_{k+1}}+1} \frac{\partial}{\partial x_{i_{k+1}}}\Big(\zeta \Big|\frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big|^p\Big) d\eta_{i_{k+1}} d\eta_{i_1} \cdots d\eta_{i_k} \\
&= -\int_{P_{i_1 \ldots i_k}} \int_{x_{i_{k+1}}}^{x'_{i_{k+1}}+1} \Big[\frac{\partial \zeta}{\partial x_{i_{k+1}}} \Big|\frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big|^p + \zeta p \Big|\frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big|^{p-1} \\
&\quad \times sign\Big(\frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big) \frac{\partial^{k+1} u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k} \partial x_{i_{k+1}}}\Big] d\eta_{i_{k+1}} d\eta_{i_1} \cdots d\eta_{i_k}
\end{aligned} \tag{4.22}$$

By using Young's inequality and (4.21), from (4.22) it follows

$$\int\limits_{P_{i_1\cdots i_k}} \Big|\frac{\partial^k u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_k}}\Big|^p d\eta_{i_1}\cdots d\eta_{i_k} \le \int\limits_{P_{i_1\cdots i_k}} \int\limits_{x_{i_{k+1}}}^{x'_{i_{k+1}}+1} \Big[\Big|\frac{\partial^{k+1} u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_k}\partial x_{i_{k+1}}}\Big|^p + p\,\Big|\frac{\partial^k u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_k}}\Big|^p\Big] d\eta_{i_{k+1}}\, d\eta_{i_1}\cdots d\eta_{i_k}. \tag{4.23}$$

From (4.19), (4.23) it follows that

$$\Big|\int\limits_{P_{i_1\cdots i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_k}}\, d\eta_{i_1}\cdots d\eta_{i_k}\Big| \le |P_{i_1\cdots i_k}|^{\frac{p-1}{p}} \times \Big(\Big\|\frac{\partial^{k+1} u}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}\Big\|^p_{L_p(P^1_{i_1\cdots i_k})} + p\Big\|\frac{\partial^k u}{\partial x_{i_1}\cdots\partial x_{i_k}}\Big\|^p_{L_p(P^1_{i_1\cdots i_k})}\Big)^{\frac{1}{p}}, \tag{4.24}$$

where $P^1_{i_1\cdots i_k} = P_{i_1\cdots i_k} \times (x_{i_{k+1}}, x'_{i_{k+1}} + 1)$ is a $(k+1)$-rectangle. This completes one out of $n-k$ steps for the estimation of the k-dimensional integral in (3.2) along the k-*rectangle* $P_{i_1\cdots i_k}$. In the next step, we select any integer j from (4.5) with k replaced with $k+1$ and assign it to multi-index component i_{k+2}. Then for each of the $k+1$-dimensional integrals on the right-hand side of (4.24) we derive the estimation similar to (4.23), where $P_{i_1\cdots i_k}$ is replaced with $(k+1)$-*rectangle* $P^1_{i_1\cdots i_k}$, and integration interval $(x_{i_{k+1}}, x'_{i_{k+1}} + 1)$ is replaced accordingly with $(x_{i_{k+2}}, x'_{i_{k+2}} + 1)$. Application of these estimations to the right-hand side of (4.24) would complete the second out of $n-k$ steps. By repeating the procedure after $m = 1, ..., n-k$ steps we derive the following estimate:

$$\Big|\int\limits_{P_{i_1\cdots i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_k}}\, d\eta_{i_1}\cdots d\eta_{i_k}\Big| \le |P_{i_1\cdots i_k}|^{\frac{p-1}{p}} \times \Big[\sum_{j=0}^{m}\binom{m}{j}p^j\Big\|\frac{\partial^{k+m-j} u}{\partial x_{i_1}\cdots\partial x_{i_{k+m-j}}}\Big\|^p_{L_p(P^m_{i_1\cdots i_k})}\Big]^{\frac{1}{p}}, \tag{4.25}$$

where

$$P^m_{i_1\cdots i_k} = P_{i_1\cdots i_k} \times (x_{i_{k+1}}, x'_{i_{k+1}} + 1) \times \cdots \times (x_{i_{k+m}}, x'_{i_{k+m}} + 1),$$

be a $(k+m)$-rectangle. Let us prove the estimation (4.25) by induction. If $m = 1$, the estimation (4.25) coincides with (4.24). Prove that (4.25) is true for $m+1$ if it is so for any $m < n-k$. Each of the $k+m$-dimensional integrals on the right-hand side of (4.25) satisfy the following estimate

$$\int\limits_{P^m_{i_1\cdots i_k}} \Big|\frac{\partial^{k+m-j} u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{k+m-j}}}\Big|^p d\eta_{i_1}\cdots d\eta_{i_{k+m}} \le \int\limits_{P^m_{i_1\cdots i_k}} \int\limits_{x_{i_{k+1+m}}}^{x'_{i_{k+1+m}}+1} \Big[\Big|\frac{\partial^{k+1+m-j} u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{k+m-j}}\partial x_{i_{k+1+m-j}}}\Big|^p + p\,\Big|\frac{\partial^{k+m-j} u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_{k+m-j}}}\Big|^p\Big] d\eta_{i_{k+1+m}}\, d\eta_{i_1}\cdots d\eta_{i_{k+m}}. \tag{4.26}$$

Using (4.26), we have

$$\sum_{j=0}^{m}\binom{m}{j}p^j\Big\|\frac{\partial^{k+m-j}u}{\partial x_{i_1}\cdots\partial x_{i_{k+m-j}}}\Big\|^p_{L_p(P^m_{i_1\cdots i_k})}$$
$$\le\sum_{j=0}^{m}\binom{m}{j}p^j\Big[\Big\|\frac{\partial^{k+1+m-j}u}{\partial x_{i_1}\cdots\partial x_{i_{k+1+m-j}}}\Big\|^p_{L_p(P^{m+1}_{i_1\cdots i_k})}+p\Big\|\frac{\partial^{k+m-j}u}{\partial x_{i_1}\cdots\partial x_{i_{k+m-j}}}\Big\|^p_{L_p(P^{m+1}_{i_1\cdots i_k})}\Big]$$
$$=\sum_{j=0}^{m}\binom{m}{j}p^j\Big\|\frac{\partial^{k+1+m-j}u}{\partial x_{i_1}\cdots\partial x_{i_{k+1+m-j}}}\Big\|^p_{L_p(P^{m+1}_{i_1\cdots i_k})}$$
$$+\sum_{j=1}^{m+1}\binom{m}{j-1}p^j\Big\|\frac{\partial^{k+1+m-j}u}{\partial x_{i_1}\cdots\partial x_{i_{k+1+m-j}}}\Big\|^p_{L_p(P^{m+1}_{i_1\cdots i_k})}. \tag{4.27}$$

Since

$$\binom{m}{j}+\binom{m}{j-1}=\binom{m+1}{j},\ \ j=1,...,m,$$

from (4.27) it follows

$$\sum_{j=0}^{m}\binom{m}{j}p^j\Big\|\frac{\partial^{k+m-j}u}{\partial x_{i_1}\cdots\partial x_{i_{k+m-j}}}\Big\|^p_{L_p(P^m_{i_1\cdots i_k})}\le\sum_{j=0}^{m+1}\binom{m+1}{j}p^j\Big\|\frac{\partial^{k+m+1-j}u}{\partial x_{i_1}\cdots\partial x_{i_{k+1+m-j}}}\Big\|^p_{L_p(P^{m+1}_{i_1\cdots i_k})}, \tag{4.28}$$

which completes the proof of (4.25) by mathematical induction. By choosing $m = n - k$ in (4.25), we derive an upper bound of the right-hand side by replacing the integration domain with $\mathbb{R}^n$:

$$\Big|\int\limits_{P_{i_1\cdots i_k}}\frac{\partial^k u(\eta)}{\partial x_{i_1}\cdots\partial x_{i_k}}\,d\eta_{i_1}\cdots d\eta_{i_k}\Big|\le|P_{i_1\cdots i_k}|^{\frac{p-1}{p}}\Big[\sum_{j=0}^{n-k}\binom{n-k}{j}p^j\Big]^{\frac{1}{p}}\|u\|_{S^1_p(\mathbb{R}^n)}$$
$$\le|x-x'|^{\frac{k(p-1)}{p}}(1+p)^{\frac{n-k}{p}}\|u\|_{S^1_p(\mathbb{R}^n)}. \tag{4.29}$$

Note that the estimation (4.29) holds for k=n as well in view of (4.18). By using (4.29) from the generalized Newton-Leibniz formula (3.2) it follows the estimate

$$|u(x')-u(x)|\le\sum_{k=1}^{n}\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n}|x-x'|^{\frac{k(p-1)}{p}}(1+p)^{\frac{n-k}{p}}\|u\|_{S^1_p(\mathbb{R}^n)}$$
$$=\sum_{k=1}^{n}\binom{n}{k}|x-x'|^{\frac{k(p-1)}{p}}(1+p)^{\frac{n-k}{p}}\|u\|_{S^1_p(\mathbb{R}^n)}$$
$$=\Big[\sum_{k=0}^{n}\binom{n}{k}|x-x'|^{\frac{k(p-1)}{p}}(1+p)^{\frac{n-k}{p}}-(1+p)^{\frac{n}{p}}\Big]\|u\|_{S^1_p(\mathbb{R}^n)}$$
$$=\Big[\Big((1+p)^{\frac{1}{p}}+|x-x'|^{\frac{p-1}{p}}\Big)^n-(1+p)^{\frac{n}{p}}\Big]\|u\|_{S^1_p(\mathbb{R}^n)}, \tag{4.30}$$

which proves the desired estimate (4.15). The proof of the estimate (4.16) is almost identical to the proof of (4.15).

Step 3. In this step we prove

- the uniform $C^{0,1-\frac{1}{p}}(\mathbb{R}^n)$-estimate for any $u \in S^1_p(\mathbb{R}^n) \cap SC^1_{loc}(\mathbb{R}^n)$, $1 < p < \infty$;
- the uniform $C^0(\mathbb{R}^n)$-estimate for any $u \in S^1_1(\mathbb{R}^n) \cap SC^1_{loc}(\mathbb{R}^n)$.

Assume $p > 1$ and fix $x, x' \in \mathbb{R}^n$ such that $|x - x'| \leq 1$. From (4.15) it follows that

$$\begin{aligned}
|u(x) - u(x')| &\leq \Big[\Big((1+p)^{\frac{1}{p}} + |x-x'|^{\frac{p-1}{p}}\Big)^n - (1+p)^{\frac{n}{p}}\Big]\|u\|_{S^1_p(\mathbb{R}^n)} \\
&= \sum_{k=1}^{n} \binom{n}{k} |x-x'|^{\frac{k(p-1)}{p}} (1+p)^{\frac{n-k}{p}} \|u\|_{S^1_p(\mathbb{R}^n)} \\
&= \sum_{k=1}^{n} \binom{n}{k} |x-x'|^{\frac{(k-1)(p-1)}{p}} (1+p)^{\frac{n-k}{p}} \|u\|_{S^1_p(\mathbb{R}^n)} |x-x'|^{\frac{p}{p-1}} \\
&\leq \sum_{k=1}^{n} \binom{n}{k} (1+p)^{\frac{n-k}{p}} \|u\|_{S^1_p(\mathbb{R}^n)} |x-x'|^{\frac{p}{p-1}} \\
&= \Big[\sum_{k=0}^{n} \binom{n}{k} (1+p)^{\frac{n-k}{p}} - (1+p)^{\frac{n}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)} |x-x'|^{\frac{p}{p-1}} \\
&= \Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)} |x-x'|^{\frac{p}{p-1}}. \qquad (4.31)
\end{aligned}$$

Hence, we have

$$\sup_{\substack{|x-x'|\leq 1 \\ x\neq x'}} \frac{|u(x)-u(x')|}{|x-x'|^{\frac{p}{p-1}}} \leq \Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)}. \qquad (4.32)$$

Now fix $x \in \mathbb{R}^n$. By using (4.32) and Hölder inequality we deduce

$$\begin{aligned}
|u(x)| &\leq \fint_{|y-x|\leq 1} |u(x) - u(y)|\, dy + \fint_{|y-x|\leq 1} |u(y)|\, dy \\
&\leq \Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)} + \Gamma_n^{-\frac{1}{p}} \|u\|_{L_p(\mathbb{R}^n)} \\
&\leq \Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)}. \qquad (4.33)
\end{aligned}$$

where Γ_n is a volume of the unit ball in $\mathbb{R}^n$. Hence, we have

$$\|u\|_{C^0(\mathbb{R}^n)} \leq \Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)}. \qquad (4.34)$$

From (4.34) it follows that

$$\sup_{|x-x'|\geq 1} \frac{|u(x)-u(x')|}{|x-x'|^{\frac{p}{p-1}}} \leq 2\|u\|_{C^0(\mathbb{R}^n)} \leq 2\Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)}. \qquad (4.35)$$

From (4.32) and (4.35) we deduce the following Hölder seminorm estimate for u:

$$[u]_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq 2\Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)}. \qquad (4.36)$$

Finally, (4.34), (4.36) imply the following Hölder norm estimate for $u \in S^1_p(\mathbb{R}^n) \cap SC^1_{loc}(\mathbb{R}^n)$, $1 < p < \infty$:

$$\|u\|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq 3\Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big] \|u\|_{S^1_p(\mathbb{R}^n)}. \qquad (4.37)$$

If $p = 1$ from the estimate (4.16) with the similar argument as in (4.33) we derive the following $C^0(\mathbb{R}^n)$-estimate for any $u \in S_1^1(\mathbb{R}^n) \cap SC^1_{loc}(\mathbb{R}^n)$:

$$\|u\|_{C^0(\mathbb{R}^n)} \leq \left[3^n - 2^n + \Gamma_n^{-1}\right]\|u\|_{S_1^1(\mathbb{R}^n)}. \tag{4.38}$$

Step 4. We complete the proof of the embedding (3.3) by using estimates (4.37), (4.38) and smooth approximation of elements of $S_p^1(\mathbb{R}^n)$. Given $u \in S_p^1(\mathbb{R}^n)$, $1 \leq p < \infty$, we select a sequence $v_m \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|v_m - u\|_{S_p^1(\mathbb{R}^n)} \to 0, \text{ as } m \to \infty. \tag{4.39}$$

For example, the sequence v_m can be given explicitly as in [11] (Lemma 23):

$$v_m(x) = u^{\frac{1}{m}}(x)\eta\Big(\frac{x}{m}\Big),$$

where $u^{\frac{1}{m}} = u * \phi^{\frac{1}{m}} \in C^\infty_{loc}(\mathbb{R}^n) \cap S_p^1(\mathbb{R}^n)$ is the $\frac{1}{m}$-mollification of u, $\phi^{\frac{1}{m}}$ is a standard rescaled mollifier, $\eta \in C_0^\infty(\mathbb{R}^n)$ be a compactly supported function which equals 1 near the origin. If $p > 1$, then by applying the estimate (4.37) to v_m, we have

$$\|v_m\|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq 3\Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big]\|v_m\|_{S_p^1(\mathbb{R}^n)}. \tag{4.40}$$

Equivalently, we have

$$\|v_m - v_l\|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq 3\Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big]\|v_m - v_l\|_{S_p^1(\mathbb{R}^n)}, \tag{4.41}$$

for all $m, l \geq 1$, whence there exists a function $u_* \in C^{0,1-\frac{1}{p}}(\mathbb{R}^n)$ such that

$$\|v_m - u_*\|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \to 0, \text{ as } m \to \infty. \tag{4.42}$$

From (4.39) it follows that $u_* = u$, a.e. on $\mathbb{R}^n$, so that u_* is in the equivalency class of u. Passing to limit as $m \to \infty$, from (4.40) it also follows that

$$\|u_*\|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq 3\Big[\Big(1 + (1+p)^{\frac{1}{p}}\Big)^n - (1+p)^{\frac{n}{p}} + \Gamma_n^{-\frac{1}{p}}\Big]\|u\|_{S_p^1(\mathbb{R}^n)}. \tag{4.43}$$

which proves the bounded embedding (3.3). Step 1 of the proof implies that the traces of u_* satisfy (3.4), and each of them is an L_p-limit of the corresponding sequence of traces of v_m. Therefore, writing (3.2) for v_m, and passing to limit as $m \to \infty$, it follows that u_* satisfies the generalized Newton-Leibniz formula (3.2).

Proof in the case $p = 1$ is identical by using an estimate (4.38). This completes the proof of the theorem in the case $1 \leq p < \infty$.

Assume that $p = \infty$. In this case, the embedding (3.3) is not new, and it is contained in the well-known fact that [2]

$$W^1_\infty(\mathbb{R}^n) \hookrightarrow C^{0,1}(\mathbb{R}^n).$$

Since $S^1_\infty(\mathbb{R}^n)$ is a subspace of $W^1_\infty(\mathbb{R}^n)$, its elements are bounded and Lipschitz continuous functions and the embedding (3.3) holds. The assertion that u satisfies (3.2) follows from the proof given for the case $p < \infty$. It only remains to show that (3.4) holds with $p = \infty$. Note that from the given proof it follows that (3.4) holds for any $p < \infty$. In particular for the smoothing sequence $u^\epsilon = u * \phi^\epsilon \in S^1_\infty(\mathbb{R}^n) \cap C^\infty_{loc}(\mathbb{R}^n)$ all the traces indicated on the left hand side of (3.4) are uniformly bounded in $L_\infty(P_{i_1 \cdots i_k})$, and converge to corresponding traces of u in $L_p(P_{i_1 \cdots i_k})$ with any $1 < p < \infty$. Such limits are also limits in the sense of distributions. Since $L_\infty(P_{i_1 \cdots i_k})$ is a dual space of $L_1(P_{i_1 \cdots i_k})$, distributional limit of the sequence bonded in $L_\infty(P_{i_1 \cdots i_k})$ remains in $L_\infty(P_{i_1 \cdots i_k})$. Therefore, (3.4) holds with $p = \infty$. Theorem is proved. □

Corollaries 3.1, 3.2 and 3.3 are the direct consequence of the Theorem 3.2 due to the fact that if $u \in S^k_p(\mathbb{R}^n)$, then all the weak partial derivatives of order $k-1$ are elements of $S^1_p(\mathbb{R}^n)$, and if $u \in S^{\mathbf{k}}_p(\mathbb{R}^n)$ the indicated partial derivative on the left hand side of (3.6) is an element of $S^1_p(\mathbb{R}^n)$. The bounded embedding (3.7) is a direct consequence of (3.5) and the definition of the space $S^1_p(Q)$. The compact embedding (3.8) follows from (3.7) and Arzela-Ascoli's theorem.

The following remarks are added following on the recommendation of the reviewer:

Remark 4.1. *Reviewer of the paper writes: The techniques used to prove the Theorem 3.2 are nice alternative to Fourier-based methods. The results of Theorem 3.2 and Corollary 3.1 are new if p* $= 1$. *If* $1 < p < \infty$, *the results can be derived from the known embedding results of Besov and Triebel-Lizorkin spaces as follows: The theorem on page 104 in [7] tells us that* $S^m_p W(\mathbb{R}^n) = S^m_{p,2} F(\mathbb{R}^n)$ *if* $m \in \mathbb{N}$ *and* $1 < p < \infty$. *By Proposition 1.15 in [9] we have* $S^m_{p,2}(\mathbb{R}^n) \hookrightarrow S^m_{p,\max(p,2)} B(\mathbb{R}^n) \hookrightarrow S^m_{p,\infty} B(\mathbb{R}^n)$. *Next, by the theorem on p.131 in [7] we have* $S^m_{p,\infty} B(\mathbb{R}^n) \hookrightarrow S^{m-1/p}_{\infty,\infty} B(\mathbb{R}^n)$. *The latter space* $S^{m-1/p}_{\infty,\infty} B(\mathbb{R}^n)$ *is a dominating mixed smoothness type Hölder-Zygmund space and sometimes denoted by* $S^{m-1,1-1/p} C(\mathbb{R}^n)$. *Either way, relying on Theorem 3.1 from [12] we get the embedding* $S^{m-1,1-1/p} C(\mathbb{R}^n) \hookrightarrow B^{m-1/p}_{\infty,\infty}(\mathbb{R}^n) = C^{m-1,1-1/p}(\mathbb{R}^n)$, *where the last equality is a classical result (see e.g., [13]). In summary, we have the embedding* $S^m_p W(\mathbb{R}^n) \hookrightarrow C^{m-1,1-1/p}(\mathbb{R}^n)$. *Regarding the assertion on traces, the cases with p* $= 1, \infty$ *are novel in the literature. But the cases* $1 < p < \infty$ *are contained in Lemma 4.1 of [14].*

Remark 4.2. *The task of embedding large subspaces of the Sobolev spaces* W^m_p *into Lebesgue spaces* L_q *is touched on e.g., in [15] or [6].*

5. Conclusions

The concept of Sobolev spaces became a trailblazing idea in many fields of mathematics. The goal of this paper is to gain insight into the embedding of the Sobolev spaces into Hölder spaces-a very powerful concept that reveals the connection between weak differentiability and integrability (or weak regularity) of the function with its pointwise regularity. It is well-known that the embedding of the Sobolev space of weakly differentiable functions into Hölder spaces holds if the integrability exponent is higher than the space dimension. Otherwise speaking, one can trade one degree of weak regularity with an integrability exponent higher than the space dimension to upgrade the pointwise regularity to Hölder continuity. In this paper, the embedding of the Sobolev functions into the Hölder spaces is expressed in terms of the minimal weak differentiability requirement independent of the integrability exponent. Precisely, the question asked is what is the minimal weak regularity degree of Sobolev functions which upgrades the pointwise regularity to Hölder continuity independent of the integrability exponent. The paper presents proof of the embedding of the Sobolev spaces with dominating mixed smoothness into Hölder spaces. The new method of proof is based on the generalization of the

Newton-Leibniz formula to the n-dimensional rectangle and inductive application of the Sobolev trace embedding results. Counterexamples demonstrate that in terms of minimal weak regularity degree the Sobolev spaces with dominating mixed smoothness present the largest class of weakly differentiable functions which preserve generalized Newton-Leibniz formula, and upgrades the pointwise regularity to Hölder continuity.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author states that there is no conflict of interest.

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