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Research article

Classical and Bayesian inferences on the stress-strength reliability R = P[Y < X < Z] in the geometric distribution setting

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Abstract: The subject matter described herein includes the analysis of the stress-strength reliability of the system, in which the discrete strength of the system is impacted by two random discrete stresses. The reliability function of such systems is denoted by $R = P[Y < X < Z]$, where X is the strength of the system and Y and Z are the stresses. We look at how X, Y and Z fit into a well-known discrete distribution known as the geometric distribution. The stress-strength reliability of this form is not widely studied in the current literature, and research in this area has only considered the scenario when the strength and stress variables follow a continuous distribution, although it is essentially nil in the case of discrete stress and strength. There are numerous applications wherein a system is exposed to external stress, and its functionality depends on whether its intrinsic physical strength falls within specific stress limits. Furthermore, the continuous measurement of stress and strength variables presents inherent difficulties and inconveniences in such scenarios. For the suggested distribution, we obtain the maximum likelihood estimate of the variable R, as well as its asymptotic distribution and confidence interval. Additionally, in the classical setup, we find the boot-p and boot-t confidence intervals for R. In the Bayesian setup, we utilize the widely recognized Markov Chain Monte Carlo technique and the Lindley approximation method to find the Bayes estimate of R under the squared error loss function. A Monte Carlo simulation study and real data analysis are demonstrated to show the applicability of the suggested model in the real world.

Keywords: Bayes estimation; geometric distribution; Gibbs sampler; Lindley's approximation; maximum likelihood estimation method; stress-strength reliability
Mathematics Subject Classification: 60E05, 62F10, 62N02

1. Introduction

In reliability theory, stress-strength models are one of the most attractive topics. There are several applications, such as engineering, meteorology, quality control, and medicine, where the system is subjected to external stress and its reliability is determined by its strength. Most of the work in the statistical literature has been dedicated to the estimation of the reliability parameter of the form $R = P[Y < X]$, where X represents the strength of the system, Y is the stress imposed upon it, and R is the stress-strength parameter. Many authors consider the estimation of R using various continuous and discrete distributions; for a comprehensive review, see [1]. Other important studies in this context include [2–10].

A different type of stress-strength model may occur in practice, in which the system's strength is impacted by two random stresses. In this context, the reliability measure is defined by $R = P[Y < X < Z]$, where X is the strength and Y and Z are the two stresses. For example, various types of equipment fail to work effectively at high or low temperatures, and a person's blood pressure has systolic and diastolic pressure limits that should not be exceeded (see [11]). The existing literature does not go into great detail about the stress-strength reliability, specifically on $R = P[Y < X < Z]$. Notable works in this field include the following: Singh [12] obtained the minimum variance unbiased, maximum likelihood, and empirical estimates of R, when X, Y and Z are mutually independent random variables from the normal distribution, Dutta et al. [13] considered the estimation of R, when X, Y and Z are from an exponential distribution, and Ivshin [14] obtained the maximum likelihood estimate (MLE) as well as the uniformly minimum variance unbiased estimate (UMVUE) of R under the assumption that X, Y and Z are either uniform or exponential random variables with unknown location meters. Guangming [15] constructed statistical inference for R using the nonparametric normal approximation and the Jackknife empirical likelihood approach. Rasethuntsa [11] considered the parametric estimation of R and its generalizations based on several one- and two-parameter continuous distributions. Choudhary et al. [16] considered the problem of estimating R, when X, Y and Z independently follow the Weibull distribution with different scale parameters and common shape parameters.

The current literature exclusively revolves around the estimation of $R = P[Y < X < Z]$ in the context of continuous probability distributions. However, in real-life situations, there are several instances when it is often difficult or inconvenient to measure the variables on a continuous scale (see [17, 18]). In some real-life situations, stress and strength can have a discrete distribution. The estimation of a stress-strength reliability $R = P[Y < X < Z]$, when Y and Z are two discrete stresses acting on a discrete strength, X, has yet to be addressed in the literature. This type of situation is commonly encountered in practice. Some examples are developed below.

- In a demand-supply system, the minimum supply (Y), maximum supply (Z), and demand (X) can be counted in terms of the number of items. In this case, as long as the demand lies between the minimum supply and maximum supply, the demand-supply system can function by establishing an equilibrium where the quantity demanded matches the quantity supplied.
- Another example of discrete stress and strength is seen in candidate interviews for employment. Here, X represents the number of interviews an employer conducts to find a suitable candidate. The minimum number of candidates required for the interview is denoted by Y, while Z represents the maximum number of candidates attending the interviews. Optimizing the interview process

is crucial, as it requires significant investment in terms of time, effort, and resources. Conducting interviews for too few candidates is inefficient, while interviewing an excessive number leads to resource waste. By establishing an optimal range between Y and Z, employers can strike a balance that ensures efficient resource utilization while maintaining a robust selection process.

Therefore, the main objective of this article is to address the estimation problem of stress-strength reliability in the form of $R = P[Y < X < Z]$ within the context of discrete data. In our study, we specifically focus on utilizing the geometric distribution. While there are various other discrete distributions available, we have chosen the geometric distribution due to its simplicity, making it a valuable tool for modeling scenarios where the occurrence of an event or failure is governed by a constant probability. Thus, the core focus of this article is to address the estimation problem associated with the stress-strength model denoted by $R = P[Y < X < Z]$, where X, Y, and Z are independent random variables that follow the geometric distribution.

If a random variable X has the geometric distribution with parameter θ, denoted as $Geo(\theta)$, then its probability mass function (PMF) and cumulative distribution function (CDF) are given by, respectively,

$$P[X = x;\ \theta] = \theta(1-\theta)^{x-1},\ \ x = 1, 2, 3, \ldots, \infty \tag{1.1}$$

and

$$F_X(x;\ \theta) = P[X \le x;\ \theta] = 1 - (1-\theta)^x, x = 1, 2, 3, \ldots, \infty, \tag{1.2}$$

where $0 < \theta < 1$. The rest of the paper is organized into the following sections: In Section 2, we derive an expression for the reliability function of the stress-strength model of the form $P[Y < X < Z]$. The MLE of R and its corresponding asymptotic distribution are derived in Sections 3 and 4, respectively. In Section 5, confidence intervals based on bootstrap samples are also obtained. In Section 6, Bayes estimates are constructed under a squared error loss function (SELF), assuming Beta and Jeffreys priors, using the Lindley approximation and Markov Chain Monte Carlo (MCMC) methods. In Section 7, a simulation study is carried out to compare the behavior of the obtained estimates. Section 8 presents applications of the proposed model using real data sets with discussions of the obtained results. Finally, concluding remarks are given in Section 9.

2. Derivation of R

Here, we derive the expression of the stress-strength reliability parameter $R = P[Y < X < Z]$ when the independent random variables X, Y and Z follow the geometric distributions $Geo(\theta_1)$, $Geo(\theta_2)$ and $Geo(\theta_3)$, respectively. By using the standard geometric series formula, we obtain

$$\begin{aligned} R = P[Y < X < Z] &= \sum_{k=1}^{\infty} P[X = k, Y < k, Z > k] = \sum_{k=1}^{\infty} P[X = k].P[Y < k].P[Z > k] \\ &= \sum_{k=1}^{\infty} \theta_1(1-\theta_1)^{k-1}.(1-(1-\theta_2)^{k-1}).(1-\theta_3)^k \\ &= \theta_1(1-\theta_3)\sum_{k=1}^{\infty}(1-\theta_1)^{k-1}(1-\theta_3)^{k-1}(1-(1-\theta_2)^{k-1}) \end{aligned}$$

$$= \theta_1 (1-\theta_3) \sum_{k=1}^{\infty} [\{(1-\theta_1)(1-\theta_3)\}^{k-1} - \{(1-\theta_1)(1-\theta_2)(1-\theta_3)\}^{k-1}]$$
$$= \theta_1(1-\theta_3)\left[\frac{1}{1-(1-\theta_1)(1-\theta_3)} - \frac{1}{1-(1-\theta_1)(1-\theta_2)(1-\theta_3)}\right].$$

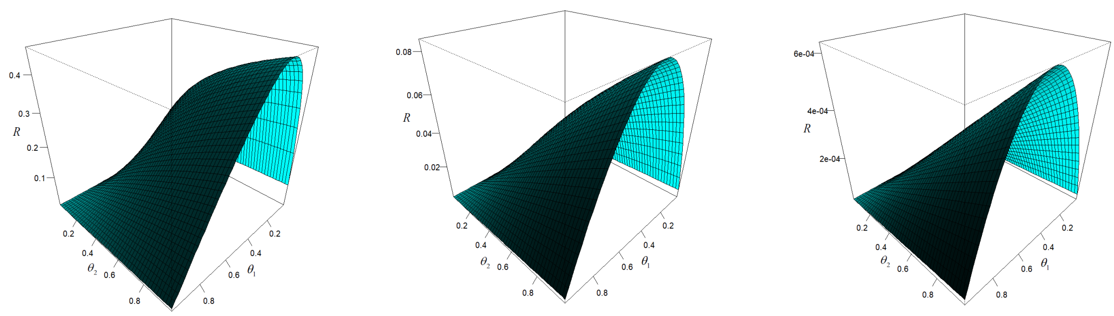
Finally, we get

$$R = \frac{\theta_1\theta_2(1-\theta_1)(1-\theta_3)^2}{(\theta_1+\theta_3-\theta_1\theta_3)(\theta_1+\theta_3-\theta_1\theta_3+\theta_2(1-\theta_1)(1-\theta_3))}. \tag{2.1}$$

Some remarks are formulated below.

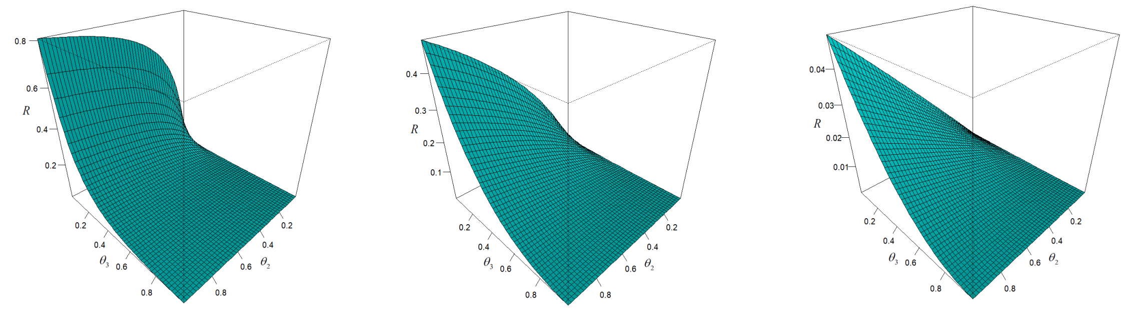
- The stress-strength reliability parameter R is in a closed form, and it is a function of θ_1, θ_2 and θ_3.
- To make inferences on R, provided the estimates hold the invariance property, we simply need to estimate the parameters θ_1, θ_2 and θ_3, and plug in these estimated values in Eq (2.1).
- As the strength parameter θ_1 increases, R first increases and then decreases for the fixed values of the stress parameters θ_2 and θ_3.
- As the stress parameter θ_2 increases, R increases for the fixed values of strength parameter θ_1 and stress parameter θ_3. On the other hand, when the stress parameter θ_3 increases, R decreases for the fixed values of strength parameter θ_1 and stress parameter θ_2.

A graphical depiction of the stress-strength reliability parameter R for different values of θ_1, θ_2, and θ_3 is given in Figures 1–3. It is evident that, with varying values of the parameters, the stress-strength reliability parameter R can take a variety of values, exhibiting different concave and convex shapes in terms of multidimensional function.



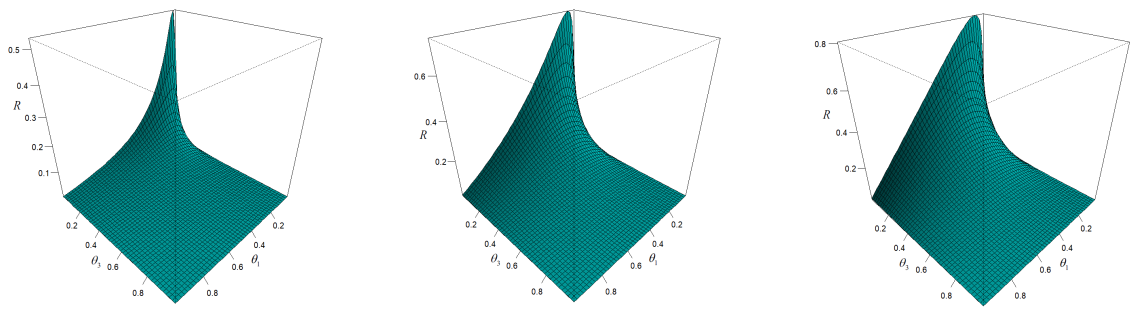
(a) a (b) b (c) c

Figure 1. 3D plots of R for a fix value of (a) $\theta_3 = 0.1$, (b) $\theta_3 = 0.5$, (c) $\theta_3 = 0.95$.



(a) a (b) b (c) c

Figure 2. 3D plots of R for a fix value of (a) $\theta_1 = 0.1$, (b) $\theta_1 = 0.5$, (c) $\theta_1 = 0.95$.



(a) a (b) b (c) c

Figure 3. 3D plots of R for a fix value of (a) $\theta_2 = 0.1$, (b) $\theta_2 = 0.5$, (c) $\theta_2 = 0.95$.

3. Maximum likelihood estimation

In this section, we find the MLE of R. To acheive this aim, we must first derive the MLEs of θ_1, θ_2 and θ_3. Let $\underline{x} = (x_1, x_2, \ldots, x_{n_1})$, $\underline{y} = (y_1, y_2, \ldots, y_{n_2})$ and $\underline{z} = (z_1, z_2, \ldots, z_{n_3})$ be random samples drawn from the following geometric distributions $Geo(\theta_1)$, $Geo(\theta_2)$ and $Geo(\theta_3)$, respectively. Then the likelihood function of the observed sample is

$$L(\underline{x}, \underline{y}, \underline{z}, \theta_1, \theta_2, \theta_3) = {\theta_1}^{n_1}(1-\theta_1)^{\sum_{i=1}^{n_1}(x_i - n_1)}{\theta_2}^{n_2}(1-\theta_2)^{\sum_{j=1}^{n_2}(y_j - n_2)}{\theta_3}^{n_3}(1-\theta_3)^{\sum_{k=1}^{n3}(z_k - n_3)}. \tag{3.1}$$

The log-likelihood function is given by

$$\begin{aligned} l &= \log L(\underline{x}, \underline{y}, \underline{z}, \theta_1, \theta_2, \theta_3) \\ &= n_1 \log\theta_1 + \sum_{i=1}^{n_1}(x_i - n_1)\log(1-\theta_1) + n_2\log\theta_2 + \sum_{j=1}^{n_2}(y_j - n_2)\log(1-\theta_2) + n_3\log\theta_3 \\ &+ \sum_{k=1}^{n_3}(z_k - n_3)\log(1-\theta_3), \end{aligned} \tag{3.2}$$

where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the means of the random samples $\underline{x}$, $\underline{y}$ and $\underline{z}$.
The MLEs of $\theta_1,\ \theta_2$ and θ_3 are the values $\hat{\theta}_1,\ \hat{\theta}_2$ and $\hat{\theta}_3$ that maximize the likelihood function. Formally, these MLEs are the components of the following vector:

$$\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3 = \underset{\theta_1,\theta_2,\theta_3}{\arg\max}\ L(\underline{x}, \underline{y}, \underline{z}, \theta_1, \theta_2, \theta_3).$$

Now, we know that the argmax of a function is the same as the argmax of the log of the function. Therefore, we use the following normal equations in order to calculate the MLEs of θ_1, θ_2 and θ_3:

$$\frac{\partial l}{\partial \theta_1} = 0,\ \ \frac{\partial l}{\partial \theta_2} = 0,\ \ \frac{\partial l}{\partial \theta_3} = 0.$$

Upon solving the above normal equation, we get

$$\hat{\theta}_1 = \frac{1}{\bar{x}},\ \hat{\theta}_2 = \frac{1}{\bar{y}},\ \hat{\theta}_3 = \frac{1}{\bar{z}}\,.$$

After obtaining the MLEs of $\theta_1,\ \theta_2$ and θ_3 using the invariance property, the MLE of R can be derived as

$$\hat{R} = \frac{(\bar{x}-1)\,(\bar{z}-1)^2}{(\bar{x}+\bar{z}-1)\,((\bar{y}-1)\,(\bar{x}+\bar{z}-1)+\bar{x}\,\bar{z})}.$$

4. Asymptotic distribution and confidence interval for R

In this section, we first derive the asymptotic distribution of $\hat{R}$. Then, based on it, we will derive the confidence interval for R. This can be easily followed by the following two theorems:

Theorem 1. *Suppose that the ratios $\frac{n_1}{n_2}$ and $\frac{n_1}{n_3}$ converge to the numbers p and q, respectively, as $n_1, n_2,\ n_3 \to \infty$. Then we have*

$$\left(\sqrt{n_1}\left(\hat{\theta}_1-\theta\right),\ \sqrt{n_2}\left(\hat{\theta}_2-\theta\right),\ \sqrt{n_3}\left(\hat{\theta}_3-\theta\right)\right) \xrightarrow[n\to\infty]{D} N_3\left[0, J\left(\theta_1,\ \theta_2,\ \theta_3\right)\right],$$

where

$$J\left(\theta_1,\ \theta_2,\ \theta_3\right) = \begin{bmatrix} {\theta_1}^2\left(1-\theta_1\right) & 0 & 0 \\ 0 & p{\theta_2}^2\left(1-\theta_2\right) & 0 \\ 0 & 0 & q{\theta_3}^2\left(1-\theta_3\right) \end{bmatrix}.$$

Proof. First of all, it is clear that

$$E\left[-\frac{\partial^2 l}{\partial \theta_1}\right] = \frac{n_1}{{\theta_1}^2\left(1-\theta_1\right)};\ E\left[-\frac{\partial^2 l}{\partial \theta_2}\right] = \frac{n_2}{{\theta_2}^2\left(1-\theta_2\right)};\ E\left[-\frac{\partial^2 l}{\partial \theta_3}\right] = \frac{n_3}{{\theta_3}^2\left(1-\theta_3\right)}.$$

From the asymptotic normality of the MLE (see [19]), we have

$$\sqrt{n_1}\left(\hat{\theta}_1-\theta\right) \xrightarrow[n_1\to\infty]{D} N\left(0, {\theta_1}^2\left(1-\theta\right)\right),$$

$$\sqrt{n_2}\left(\hat{\theta}_2-\theta\right) \xrightarrow[n_2\to\infty]{D} N\left(0, {\theta_2}^2\left(1-\theta_2\right)\right),$$

and

$$\sqrt{n_3}\left(\hat{\theta}_3-\theta\right)\xrightarrow[n_3\to\infty]{D}N\left(0,{\theta_3}^2\left(1-\theta_3\right)\right).$$

Then

$$\sqrt{n_1}\left(\hat{\theta}_2-\theta\right)\xrightarrow[n_1\to\infty]{D}N\left(0,p{\theta_2}^2\left(1-\theta_2\right)\right),$$

$$\sqrt{n_1}\left(\hat{\theta}_3-\theta\right)\xrightarrow[n_1\to\infty]{D}N\left(0,q{\theta_3}^2\left(1-\theta_3\right)\right).$$

Hence the theorem is proved by the independence of $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$. □

Theorem 2. *Suppose that the ratios* $\frac{n_1}{n_2}$ *and* $\frac{n_1}{n_3}$ *converge to p and q, respectively, as* $n_1, n_2, n_3 \to \infty$, *then* $\sqrt{n_1}\left(\hat{R}-R\right)\xrightarrow{D}N(0,B)$, *where* $B = bJ(\theta_1,\ \theta_2,\ \theta_3)\,b'$ *and* $b=\left(\frac{\partial R}{\partial\theta_1}\ \ \frac{\partial R}{\partial\theta_2}\ \ \frac{\partial R}{\partial\theta_3}\right)$.

Proof. Using Theorem 1, one can readily prove this result. □

Now, the variance B can be estimated by using the empirical Fisher information matrix and the MLEs of θ_1, θ_2 and θ_3. Thus, we can construct an asymptotic confidence interval for R as

$$\left(\hat{R}-Z_\gamma\sqrt{\frac{\hat{B}}{n_1}},\ \hat{R}+Z_\gamma\sqrt{\frac{\hat{B}}{n_1}}\right),$$

where Z_γ is the $100\times(1-\gamma/2)^{th}$ percentile of a standard normal distribution and $\hat{B}$ is the plug in estimate of B.

5. Bootstrap confidence interval for R

The classical bootstrap methods based on resampling to construct approximate confidence intervals are a better choice when it is hard to figure out the exact sampling distribution of MLE. Following the concepts of [20,21], we propose the percentile bootstrap-p (boot-p) and bootstrap-t (boot-t) confidence intervals based on the bootstrapping method.

The algorithm can be described as follows:

(1) Generate samples $\underline{x}=(x_1,x_2,\ldots,x_{n_1})$, $\underline{y}=(y_1,y_2,\ldots,y_{n_2})$ and $\underline{z}=(z_1,z_2,\ldots,z_{n_3})$ of size n_1, n_2 and n_3 from the geometric distributions $Geo(\theta_1)$, $Geo(\theta_2)$ and $Geo(\theta_3)$, respectively.

(2) Generate bootstrap samples $\underline{x}^*=(x_1^*,x_2^*,\ldots,x_{n_1}^*)$, $\underline{y}^*=(y_1^*,y_2^*,\ldots,y_{n_2}^*)$, and $\underline{z}^*=(z_1^*,z_2^*,\ldots,z_{n_2}^*)$, and obtain bootstrap estimates $\hat{\theta}_1^*$, $\hat{\theta}_2^*$, $\hat{\theta}_3^*$ and $\hat{R}^*$ of θ_1, θ_2, θ_3 and R, respectively.

(3) Repeat step 2 B times and obtain the bootstrap estimates $\hat{\theta}_{1:\kappa}^*$, $\hat{\theta}_{2:\kappa}^*$, $\hat{\theta}_{3:\kappa}^*$ and $\hat{R}_\kappa^*$, $\kappa=1,2,\ldots,B$.

   (i) **Percentile bootstrap (boot-p) confidence interval:** Let $\left\{\hat{R}_{(1)}^*\leqslant\hat{R}_{(2)}^*\leqslant\ldots\leqslant\hat{R}_{(B)}^*\right\}$ denote the ordered values of estimates $\left\{\hat{R}_1^*,\ \hat{R}_2^*,\ldots,\hat{R}_B^*\right\}$. Also, let $\hat{R}^{*(\tau)}$ denote the τ^{th} percentile of the ordered values of estimates $\left\{\hat{R}_1^*,\ \hat{R}_2^*,\ldots,\hat{R}_B^*\right\}$. The $100\times(1-\gamma)\,\%$ boot-p confidence interval for R is given by

$$\left(\hat{R}^{*(\gamma/2)},\ \hat{R}^{*(1-\gamma/2)}\right)\ .$$

(ii) **Student t bootstrap (boot-t) confidence interval:** Let $\bar{\hat{R}}^*$ and $V\left(\hat{R}^*\right)$ be the mean and sample variance of $\left\{\hat{R}_1^*,\ \hat{R}_2^*, \ldots, \hat{R}_B^*\right\}$. Then, compute bootstrap pivots as $\hat{t}_\kappa^* = \frac{\hat{R}_\kappa^* - \bar{\hat{R}}^*}{\sqrt{V\left(\hat{R}^*\right)}}$, $\kappa = 1, 2, \ldots, B$. Let $\hat{t}^{*(\tau)}$ denote the τ^{th} percentile of the ordered values $\left\{\hat{t}_1^*,\ \hat{t}_2^*, \ldots, \hat{t}_B^*\right\}$. The $100 \times (1-\gamma)\,\%$ boot-t confidence interval for R is given by

$$\left(\hat{R} - \hat{t}^{*(\gamma/2)} \sqrt{V\left(\hat{R}^*\right)}, \hat{R} + \hat{t}^{*(\gamma/2)} \sqrt{V\left(\hat{R}^*\right)}\right).$$

6. Bayesian estimation

6.1. Methodology

The Bayesian methods permit us to incorporate and use information beyond that contained in experimental data. Given a statistical model for the experimental data, the Bayesian method mandates an additional probability model for all the involved unknown parameters. Our approach is to model this uncertainty about the parameters using a prior distribution. Informative priors are used if one has adequate information about the parameters; otherwise, it is preferable to use noninformative priors. We here consider both informative priors and non-informative Jeffreys prior.
Case 1: Let the unknown parameters θ_1, θ_2 and θ_3 have independent prior distributions, $Beta_I\left(a_1, b_1\right)$, $Beta_{II}\left(a_2, b_2\right)$ and $Beta_{III}\left(a_3, b_3\right)$, with the following probability density functions, respectively:

$$f_1\left(\theta_1, a_1, b_1\right) = \frac{1}{B\left(a_1, b_1\right)} {\theta_1}^{a_1-1}(1-\theta_1)^{b_1-1}; \quad a_1, b_1 > 0, \theta_1 \in [0, 1], \tag{6.1}$$

$$f_2\left(\theta_2, a_2, b_2\right) = \frac{1}{B\left(a_2, b_2\right)} {\theta_2}^{a_2-1}(1-\theta_2)^{b_2-1}; \quad a_2, b_2 > 0, \theta_2 \in [0, 1], \tag{6.2}$$

$$f_3\left(\theta_3, a_3, b_3\right) = \frac{1}{B\left(a_3, b_3\right)} {\theta_3}^{a_3-1}(1-\theta_3)^{b_3-1}; \quad a_3, b_3 > 0, \theta_3 \in [0, 1], \tag{6.3}$$

where $(a_1, b_1, a_2, b_2, a_3, b_3)$ are positive and known, and are called hyper-parameters, and $B(a, b)$ denotes the standard beta function. Then the joint prior distribution is given by

$$g\left(\theta_1, \theta_2, \theta_3\right) = \frac{1}{B\left(a_1, b_1\right) B\left(a_2, b_2\right) B\left(a_3, b_3\right)}.{\theta_1}^{a_1-1}(1-\theta_1)^{b_1-1}{\theta_2}^{a_2-1}(1-\theta_2)^{b_2-1}{\theta_3}^{a_3-1}(1-\theta_3)^{b_3-1}. \tag{6.4}$$

Now, using the Bayes Theorem, the joint posterior distribution of $(\theta_1, \theta_2, \theta_3)$ is

$$\pi_1\left(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}, \underline{z}\right) = \frac{L\left(\underline{x}, \underline{y}, \underline{z}; \theta_1, \theta_2, \theta_3\right).g\left(\theta_1, \theta_2, \theta_3\right)}{\int\limits_{\theta_1}\int\limits_{\theta_2}\int\limits_{\theta_3} L\left(\underline{x}, \underline{y}, \underline{z}; \theta_1, \theta_2, \theta_3\right).g\left(\theta_1, \theta_2, \theta_3\right) d\theta_1, d\theta_2, d\theta_3}.$$

Using Eqs (3.1) and (6.4), we get

$$\begin{aligned}\pi_1\left(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}, \underline{z}\right) =& K_1.{\theta_1}^{n_1+a_1-1}(1-\theta_1)^{b_1+n_1\bar{x}-n_1-1}{\theta_2}^{n_2+a_2-1}(1-\theta_2)^{b_2+n_2\bar{y}-n_2-1}{\theta_3}^{n_3+a_3-1}\\ &\times(1-\theta_3)^{b_3+n_3\bar{z}-n_3-1},\end{aligned} \tag{6.5}$$

where K_1 is the normalizing constant given by

$$K_1 = \frac{1}{B\left(n_1 + a_1, b_1 + n_1(\bar{x} - 1)\right) B\left(n_2 + a_2, b_2 + n_2(\bar{y} - 1)\right) B\left(n_3 + a_3, b_3 + n_3(\bar{z} - 1)\right)}.$$

Finally, the joint posterior distribution can be written as

$$\begin{aligned}\pi_1\left(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}, \underline{z}\right) \cong\ & Beta_I\left(n_1 + a_1, b_1 + n_1(\bar{x} - 1)\right) \times Beta_{II}\left(n_2 + a_2, b_2 + n_2(\bar{y} - 1)\right) \\ & \times Beta_{III}\left(n_3 + a_3, b_3 + n_3(\bar{z} - 1)\right).\end{aligned}$$

In Bayesian inference, a loss function plays a crucial role in selecting the optimal estimate of the parameters of interest from the posterior distribution. In this study, we consider the SELF, which accords equal weight to overestimation and underestimation. The Bayes estimates of θ_1, θ_2, θ_3 and R under the SELF are given by

$$\theta^*_{1,self} = \int_0^\infty \int_0^\infty \int_0^\infty \theta_1 . \pi_1\left(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}, \underline{z}\right) d\theta_1 d\theta_2 d\theta_3,$$

$$\theta^*_{2,self} = \int_0^\infty \int_0^\infty \int_0^\infty \theta_2 . \pi_1\left(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}, \underline{z}\right) d\theta_1 d\theta_2 d\theta_3,$$

$$\theta^*_{3,self} = \int_0^\infty \int_0^\infty \int_0^\infty \theta_3 . \pi_1\left(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}, \underline{z}\right) d\theta_1 d\theta_2 d\theta_3,$$

$$\theta^*_{R,self} = \int_0^\infty \int_0^\infty \int_0^\infty R . \pi_1\left(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}, \underline{z}\right) d\theta_1 d\theta_2 d\theta_3.$$

Case 2: When we don't have adequate prior knowledge about the parameters, we can execute a Bayesian inference using the non-informative prior distribution. Here, we take into account the Jeffreys prior, which describes the type of prior knowledge that would make the data as posterior as possible. We find the Jeffreys prior for θ_i by taking $J(\theta_i) \propto \sqrt{|I(\theta_i)|}$, where

$$I(\theta_i) = E\left(\frac{-\partial^2 \log L}{\partial \theta_i^2}\right) = \frac{n}{\theta_i^2(1 - \theta_i)}; i = 1, 2, 3.$$

Now, the Jeffreys prior for $(\theta_1,\ \theta_2,\ \theta_3)$ is given by $\xi(\theta_1,\ \theta_2,\ \theta_3) \propto \sqrt{|I(\theta_1,\ \theta_2,\ \theta_3)|}$, where

$$I(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \frac{n}{\theta_1{}^2(1-\theta_1)} & 0 & 0 \\ 0 & \frac{n}{\theta_2{}^2(1-\theta_2)} & 0 \\ 0 & 0 & \frac{n}{\theta_3{}^2(1-\theta_3)} \end{pmatrix}.$$

Finally, we have

$$\xi(\theta_1,\ \theta_2,\ \theta_3) \propto \theta_1^{-1} \theta_2^{-1} \theta_3^{-1} (1 - \theta_1)^{-\frac{1}{2}} (1 - \theta_2)^{-\frac{1}{2}} (1 - \theta_3)^{-\frac{1}{2}}\ ;\ \theta_1, \theta_2, \theta_3 \in [0, 1].$$

Now, under the Jeffreys prior, the joint posterior distribution of $(\theta_1, \theta_2, \theta_3)$ is given by

$$\pi_2(\theta_1, \theta_2, \theta_3|\underline{x}, \underline{y}, \underline{z}) = K_2.\theta_1{}^{n_1-1}(1-\theta_1)^{n_1\bar{x}-n_1-\frac{1}{2}}\theta_2{}^{n_2-1}(1-\theta_2)^{n_2\bar{y}-n_2-\frac{1}{2}}\theta_3{}^{n_3-1}(1-\theta_3)^{n_3\bar{z}-n_3-\frac{1}{2}}, \tag{6.6}$$

where K_2 is the normalizing constant given by

$$K_2 = \frac{1}{B\left(n_1, n_1(\bar{x}-1)+\frac{1}{2}\right)B\left(n_2+a_2, n_2(\bar{y}-1)+\frac{1}{2}\right)B\left(n_3+a_3, n_3(\bar{z}-1)+\frac{1}{2}\right)}.$$

The joint posterior distribution can be rewritten as

$$\pi_2(\theta_1, \theta_2, \theta_3|\underline{x}, \underline{y}, \underline{z}) \cong Beta_I\left(n_1, n_1(\bar{x}-1)+\frac{1}{2}\right).Beta_{II}\left(n_2, n_2(\bar{y}-1)+\frac{1}{2}\right).Beta_{III}\left(n_3, n_3(\bar{z}-1)+\frac{1}{2}\right).$$

Now, we can compute the Bayes estimates of θ_1, θ_2, θ_3 and R under the SELF in the same way we did in Case 1.

Since the Bayes estimates defined above in both Cases 1 and 2 cannot be obtained analytically, we adopt the advanced MCMC method and the ordinary Lindley approximation method for Bayesian computation. In the following subsections, these methods are discussed in detail.

6.2. Markov Chain Monte Carlo method

In this section, we discuss the MCMC method, which enables the simulation of direct draws from the complex posterior distribution of interest. One of the attractive methods for setting up an MCMC algorithm is Gibbs sampling, which was introduced by [22]. The Gibbs sampler is used in situations in which it is not possible to take a sample from a multivariate posterior, but it is possible to take a sample from a conditional distribution for each parameter.

Case 1: In the case of Beta informative priors, the full conditional posterior distributions of the parameters are

$$\begin{aligned}
\pi_{11}(\theta_1|\underline{x}) &\propto Beta(n_1+a_1, b_1+n_1(\bar{x}-1)),\\
\pi_{12}(\theta_2|\underline{y}) &\propto Beta(n_2+a_2, b_2+n_2(\bar{y}-1)),\\
\pi_{13}(\theta_3|\underline{z}) &\propto Beta(n_3+a_3, b_3+n_3(\bar{z}-1)).
\end{aligned}$$

We can see that the full conditional distributions are in standard distributional forms. Therefore, the Gibbs sampler algorithm consists of the following steps:

(1) Generate θ_1^1 from $\pi_{11}(\theta_1|\underline{x})$.

(2) Generate θ_2^1 from $\pi_{12}(\theta_2|\underline{y})$.

(3) Generate θ_3^1 from $\pi_{13}(\theta_3|\underline{z})$.

(4) Repeat steps 1–3, M times. We omit the first N burn-in draws and record the sequence $\left(\theta_1^{N+1}, \theta_2^{N+1}, \theta_3^{N+1}\right)$, $\left(\theta_1^{N+2}, \theta_2^{N+2}, \theta_3^{N+2}\right), \ldots, \left(\theta_1^{M}, \theta_2^{M}, \theta_3^{M}\right)$ to avoid the effects of the starting values of the parameters.

(5) Put the generated values of θ_1, θ_2 and θ_3 in the expression of R given in Eq (2.1).

(6) The Bayes estimate of R, say, R^* under the SELF is $R^* = \frac{1}{M-N}\sum_{i=N+1}^{M} R_i$.

(7) Let $R_{(N+1)} < R_{(N+2)} < \ldots < R_{(M)}$ be the ordered values of the draws $R^{N+1}, R^{N+2}, \ldots, R^M$. Then, using the method proposed by [23], the $(1-\gamma)\times 100\%$ highest posterior density (HPD) interval for R can be constructed as $\left(R_{N+i^*}, R_{N+i^*+[(1-\gamma)(M-N)]}\right)$, where i^* is chosen so that

$$R_{N+i^*+[(1-\gamma)(M-N)]} - R_{N+i^*} = \min_{N\leqslant i\leqslant (M-N)-[(1-\gamma)(M-N)]} \left(R_{N+i+[(1-\gamma)(M-N)]} - R_{N+i}\right).$$

Case 2: Under the Jeffreys prior, the full conditional distributions of parameters are in standard distributional forms. Therefore, using a similar algorithm as we have done in Case 1, we can obtain the required Bayes estimate and HPD interval for R.

6.3. The Lindley approximation

In this section, we consider the Lindley approximation technique ([24]) to calculate the approximate Bayes estimate of R. Consider that the posterior expectation is expressible in the form of the ratio of integrals as given below:

$$R^*(\Theta|data) = \frac{\int_\Theta R(\Theta)\exp\left[\log L(\Theta) + \rho(\Theta)\right] d(\Theta)}{\int_\Theta \exp\left[\log L(\Theta) + \rho(\Theta)\right] d(\Theta)}, \tag{6.7}$$

where $\Theta = (\theta_1, \theta_2, \theta_3)$, $R(\Theta)$ is the parametric function of interest, $\log L(\Theta)$ is the log-likelihood function, and $\rho(\Theta)$ is the log of the joint prior distribution of Θ. Equation (6.7) can be approximated as

$$R^*(\Theta|data) \approx R + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(R_{ij} + 2R_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} L_{ijk}R_l\sigma_{ij}\sigma_{kl}. \tag{6.8}$$

The Bayes estimate of the stress-strength parameter R for the Beta informative priors under the SELF is

$$R^* \cong R + \frac{1}{2}\left[(R_{11} + 2R_1\rho_1)\sigma_{11} + (R_{22} + 2R_2\rho_2)\sigma_{22} + (R_{33} + 2R_3\rho_3)\sigma_{33}\right] + \frac{1}{2}\left(L_{111}R_1\sigma_{11}^1 + L_{222}R_2\sigma_{22}^2 + L_{333}R_3\sigma_{33}^2\right). \tag{6.9}$$

We have

$$R = \frac{\theta_1\theta_2(1-\theta_1)(1-\theta_3)^2}{(\theta_1 + \theta_3 - \theta_1\theta_3)(\theta_1 + \theta_3 - \theta_1\theta_3 + \theta_2(1-\theta_1)(1-\theta_3))} = \frac{\alpha}{\beta},$$

where

$$R_1 = \frac{\beta\alpha_1 - \alpha\beta_1}{\beta^2},\ R_2 = \frac{\beta\alpha_2 - \alpha\beta_2}{\beta^2},\ R_3 = \frac{\beta\alpha_3 - \alpha\beta_3}{\beta^2},\ R_{11} = \frac{\beta(\beta\alpha_{11} - \alpha\beta_{11}) - 2\beta_1(\beta\alpha_1 - \alpha\beta_1)}{\beta^3},$$

$$R_{22} = \frac{\beta(\beta\alpha_{22} - \alpha\beta_{22}) - 2\beta_1(\beta\alpha_2 - \alpha\beta_2)}{\beta^3},\ R_{33} = \frac{\beta(\beta\alpha_{33} - \alpha\beta_{33}) - 2\beta_1(\beta\alpha_3 - \alpha\beta_3)}{\beta^3},$$

$$\alpha_1 = \theta_2(1-2\theta_1)(1-\theta_3)^2,\ \alpha_2 = \theta_1(1-\theta_1)(1-\theta_3)^2,\ \alpha_3 = -2\theta_1\theta_2(1-\theta_1)(1-\theta_3),$$

$$\alpha_{11}=-2\theta_2(1-\theta_3)^2,\alpha_{22}=0,\ \alpha_{33}=2\theta_1\theta_2(1-\theta_1),\ \beta_1=(1-\theta_3)(\theta_2+2(\theta_1+\theta_3-\theta_1\theta_3)(1-\theta_2)),$$
$$\beta_{11}=2(1-\theta_2)(1-\theta_3)^2,\ \beta_2=(\theta_1+\theta_3-\theta_1\theta_3)(1-\theta_1)(1-\theta_3),\ \beta_{22}=0,$$
$$\beta_3=(1-\theta_1)(2(\theta_1+\theta_3-\theta_1\theta_3)(1-\theta_2)+\theta_2),\ \beta_{33}=-2\theta_1(1-\theta_1)(1-\theta_2),$$
$$\sigma_{11}=\frac{\partial^2 l}{\partial\theta_1^2}=-\frac{n_1}{\theta_1^2}-\frac{n_1(\bar{x}-1)}{(1-\theta_1)^2},\ L_{111}=\frac{\partial^3 l}{\partial\theta_1^3}=\frac{2n_1}{\theta_1^3}-\frac{2n_1(\bar{x}-1)}{(1-\theta_1)^3},\ \sigma_{22}=\frac{\partial^2 l}{\partial\theta_2^2}=-\frac{n_2}{\theta_2^2}-\frac{n_2(\bar{y}-1)}{(1-\theta_2)^2},$$
$$L_{222}=\frac{\partial^3 l}{\partial\theta_2^3}=\frac{2n_2}{\theta_2^3}-\frac{2n_2(\bar{y}-1)}{(1-\theta_2)^3},\ \sigma_{33}=\frac{\partial^2 l}{\partial\theta_3^2}=-\frac{n_3}{\theta_3^2}-\frac{n_3(\bar{z}-1)}{(1-\theta_3)^2},\ L_{333}=\frac{\partial^3 l}{\partial\theta_3^3}=\frac{2n_3}{\theta_3^3}-\frac{2n_3(\bar{z}-1)}{(1-\theta_3)^3}.$$

From Eq (6.5), the log of the joint prior distribution of Θ is given by

$$\rho(\Theta)=(a_1-1)\log\theta_1+(b_1-1)\log(1-\theta_1)+(a_2-1)\log\theta_2+(b_2-1)\log(1-\theta_2)+(a_3-1)\log\theta_3+(b_3-1)\log(1-\theta_3),$$

$\rho_1=\frac{a_1-1}{\theta_1}-\frac{b_1-1}{1-\theta_1};\rho_2=\frac{a_2-1}{\theta_2}-\frac{b_2-1}{1-\theta_2}.$

For the Jeffreys prior, we obtain the same results as with the informative case, by taking $a_i=0$ and $b_i=\frac{1}{2}$; $i=1,2,3$. Therefore, the Lindley approximation of R, given Jeffreys prior under the SELF, is

$$R'^{*}\cong R+\frac{1}{2}\left((R_{11}+2R_1\rho_1')\sigma_{11}+(R_{22}+2R_2\rho_2')\sigma_{22}+(R_{33}+2R_3\rho_3')\sigma_{33}\right)+\frac{1}{2}\left(L_{111}R_1\sigma_{11}^1+L_{222}R_2\sigma_{22}^2+L_{333}R_3\sigma_{33}^2\right),\quad(6.10)$$

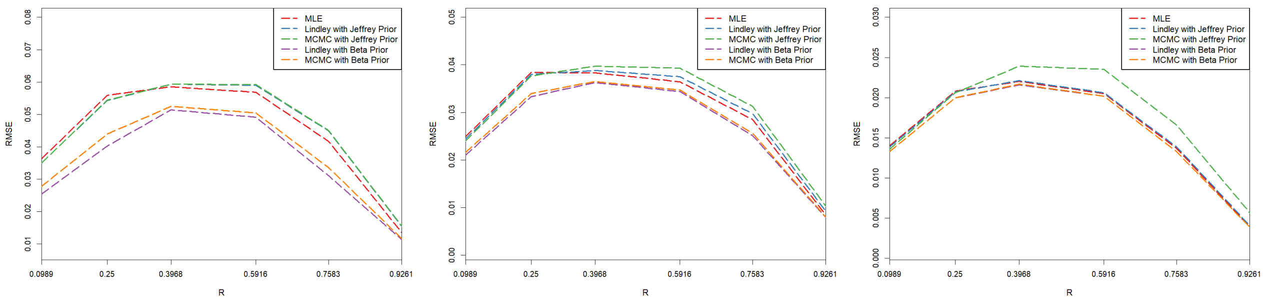
where

$$\rho_1'=-\frac{1}{\theta_1}+\frac{1}{2(1-\theta_1)},\quad \rho_2'=-\frac{1}{\theta_2}+\frac{1}{2(1-\theta_2)},\quad \rho_3'=-\frac{1}{\theta_3}+\frac{1}{2(1-\theta_3)}.$$

7. Simulation experiments

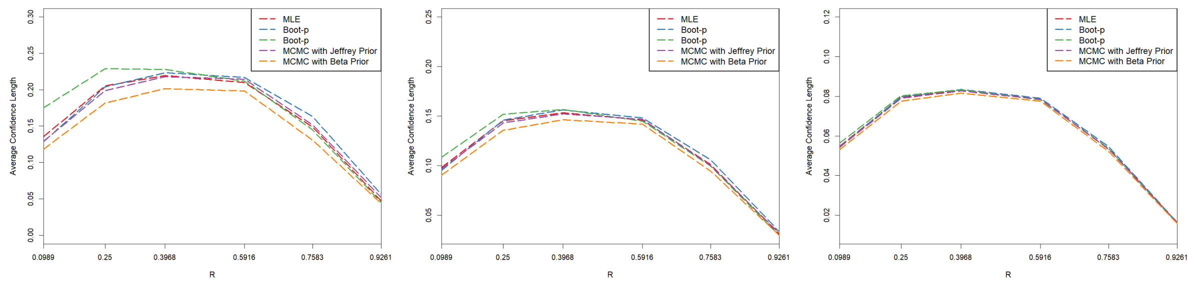
In this section, we conduct a vast simulation study to compare the performance of different estimation procedures. For this purpose, we consider nine pairs of sample sizes, which are $(n_1,n_2,n_3)\in$ {(15, 15, 15), (15, 15, 30), (15, 30, 15), (30, 15, 15), (30, 30, 15), (30, 15, 30), (15, 30, 30), (30, 30, 30), (100, 100, 100)} with six different sets of parametric values, which are $(\theta_1,\theta_2,\theta_3)\in$ {(0.5, 0.7, 0.4), (0.5, 0.9, 0.2), (0.3, 0.7, 0.1), (0.1, 0.5, 0.03), (0.15, 0.8, 0.01), (0.04, 0.8, 0.001)}. These different sets of parametric values are taken such that R takes a small, moderate, and high value. We simulate 1000 samples drawn from the following geometric distributions: $Geo(\theta_1)$, $Geo(\theta_2)$, and $Geo(\theta_3)$ for each combination of θ_1, θ_2, and θ_3. Then, for efficiency analysis of the classical and Bayesian estimation procedures, we compute the average estimate (AE) and root mean square error (RMSE) for point estimates, whereas for interval estimates, we compute the average lower confidence limit (ALCL), average upper confidence limit (AUCL), and coverage probability (CP).

In the classical estimation, we compute the MLEs, 95% ACI, boot-p, and boot-t intervals of the stress-strength reliability parameter R. Under bootstrapping intervals, it is worthwhile to mention that we use 3000 bootstrap samples. In the Bayesian paradigm, for the Beta informative prior, we define the prior distributions by solving the prior moments equations, where the prior means are assumed to be the true values of the parameters and the prior variance is 1. Thus, we obtain the Bayes estimates and 95% HPD credible intervals of R under the Jeffreys and Beta priors. These Bayes estimates are determined with the SELF through the Lindley approximation and MCMC approaches. By considering the Bayes estimates with the MCMC method, we generate a Markov chain of length 11,000 parametric draws using the Gibbs algorithm discussed in the previous section. The first 1000 values are discarded as burn-in-samples and, after convergence testing, these values are used to obtain Bayes estimates and credible intervals with associated CP. All the calculations are done with the R software. In Figures 4 and 5, we present the results for the RMSE and average confidence length (ACL) for different values of R at sample sizes of $(n_1, n_2, n_3) = (15, 15, 15)$, $(30, 30, 30)$, and $(100, 100, 100)$ for the purpose of comparison of considered point and interval estimates.



(a) (b) (c)

Figure 4. RMSE of the estimates for varying value of R at (a) $(n_1, n_2, n_3) = (15, 15, 15)$, (b)$(n_1, n_2, n_3) = (30, 30, 30)$, (c) $(n_1, n_2, n_3) = (100, 100, 100)$.



(a) (b) (c)

Figure 5. ACL of the estimates for varying value of R at (a) $(n_1, n_2, n_3) = (15, 15, 15)$, (b)$(n_1, n_2, n_3) = (30, 30, 30)$, (c) $(n_1, n_2, n_3) = (100, 100, 100)$.

However, due to space constraints, the plots for the other combinations of (n_1, n_2, n_3) are not included. The outcomes of the simulation study are given in Tables 1 and 2. From these tables, the following important conclusions can be drawn:

- The RMSE of all the estimates decreases with increasing sample size. This validates the consistency property of the estimates.
- On the overall comparison, the Bayes estimate using the Lindley approximation method under Beta priors demonstrates superior performance in terms of RMSE compared to all other estimation methods. The second preferred estimation method is Bayesian estimation using the MCMC method with Beta priors.
- For $0 < R < 0.25$, the Bayes estimate obtained using the MCMC method under Jeffreys prior outperforms the estimates obtained using the Lindley approximation under the Jeffreys

prior and MLE.

- For 0.25 < R <1, the MLE works quite satisfactorily, better than the Bayes estimates with the Lindley approximation and MCMC methods under Jeffreys prior.
- From Table 2, we may simply conclude that all confidence intervals perform well. Even with small sizes, most of the confidence intervals computed here are able to sustain nominal levels.
- The HPD credible interval with Beta prior outperforms all other classical and credible intervals in terms of the length of the intervals.
- The ACI, boot-p, and boot-t intervals show more or less the same behavior in terms of confidence width, but the HPD credible interval with the Jeffreys prior performs superior to these classical intervals.

Table 1. Average estimates (AEs) and RMSE for different methods with various sample sizes and true R.

Parameter	Sample Size	MLE		Lindley with Jeffreys Prior		MCMC with Jeffreys Prior		Lindley with Beta Prior		MCMC with Beta Prior	
	(n1, n2, n3)	AE	RMSE	AE	RMSE	AE	RMSE	AE	RMSE	AE	RMSE
(0.5, 0.7, 0.4) True R = 0.0989	(15, 15, 15)	0.0945	0.0364	0.0973	0.035	0.0971	0.035	0.0986	0.0254	0.0974	0.0278
	(15, 15, 30)	0.0958	0.0269	0.0956	0.026	0.0984	0.026	0.097	0.0212	0.0968	0.0223
	(15, 30, 15)	0.0915	0.0347	0.0954	0.0333	0.0942	0.0331	0.0972	0.0246	0.0955	0.0266
	(30, 15, 15)	0.0931	0.0347	0.0966	0.0333	0.0958	0.0332	0.0972	0.0247	0.0963	0.0268
	(30, 30, 15)	0.0945	0.0331	0.0989	0.0321	0.097	0.0317	0.0988	0.0243	0.0976	0.026
	(30, 15, 30)	0.096	0.0255	0.0964	0.0248	0.0986	0.0245	0.0967	0.0205	0.0969	0.0214
	(15, 30, 30)	0.0949	0.0276	0.0957	0.0267	0.0975	0.0264	0.0971	0.022	0.0964	0.023
	(30, 30, 30)	0.0965	0.0249	0.0979	0.0244	0.099	0.024	0.0979	0.021	0.0977	0.0216
	(100, 100, 100)	0.0981	0.014	0.0985	0.0139	0.1006	0.0136	0.0983	0.0133	0.0983	0.0133
(0.5, 0.9, 0.2) True R = 0.25	(15, 15, 15)	0.2315	0.0559	0.2299	0.0543	0.2298	0.0544	0.2396	0.0402	0.237	0.0439
	(15, 15, 30)	0.2321	0.0487	0.2286	0.0479	0.2304	0.0474	0.2384	0.0345	0.2362	0.0383
	(15, 30, 15)	0.2318	0.0555	0.2325	0.0534	0.2301	0.054	0.239	0.0401	0.2368	0.0437
	(30, 15, 15)	0.2373	0.0477	0.2357	0.0468	0.2352	0.0467	0.2422	0.0384	0.2411	0.0399
	(30, 30, 15)	0.2362	0.0469	0.2371	0.0456	0.2341	0.0461	0.2403	0.0385	0.2395	0.0398
	(30, 15, 30)	0.2392	0.04	0.2357	0.0401	0.237	0.0393	0.2417	0.0338	0.2411	0.0349
	(15, 30, 30)	0.2364	0.0473	0.2352	0.0457	0.2345	0.0458	0.2406	0.0344	0.2391	0.0377
	(30, 30, 30)	0.2392	0.0384	0.2382	0.038	0.2371	0.0377	0.2407	0.0333	0.2405	0.034
	(100, 100, 100)	0.2473	0.0208	0.2469	0.0207	0.2445	0.0206	0.2471	0.02	0.2472	0.02
(0.3, 0.7, 0.1) True R = 0.3968	(15, 15, 15)	0.3776	0.0585	0.3714	0.0594	0.3713	0.0594	0.3724	0.0514	0.3737	0.0525
	(15, 15, 30)	0.3861	0.048	0.378	0.0491	0.3795	0.0486	0.3779	0.0426	0.3798	0.0434
	(15, 30, 15)	0.38	0.0571	0.3767	0.0568	0.3736	0.0578	0.3765	0.0511	0.377	0.0519
	(30, 15, 15)	0.3866	0.0552	0.3819	0.0551	0.3798	0.0555	0.3829	0.0473	0.3839	0.0487
	(30, 30, 15)	0.3881	0.0511	0.3864	0.0505	0.3813	0.0515	0.3865	0.0459	0.3866	0.0465
	(30, 15, 30)	0.3903	0.0443	0.3839	0.045	0.3834	0.0448	0.3843	0.0383	0.3858	0.0395
	(15, 30, 30)	0.3852	0.0443	0.3801	0.0447	0.3786	0.0453	0.3793	0.0411	0.3802	0.0415
	(30, 30, 30)	0.3876	0.0383	0.3842	0.0388	0.3808	0.0397	0.3843	0.0363	0.3846	0.0365
	(100, 100, 100)	0.3927	0.022	0.3917	0.0221	0.3855	0.0239	0.3915	0.0216	0.3916	0.0217
(0.1, 0.5, 0.03) True R = 0.5916	(15, 15, 15)	0.5787	0.0568	0.5691	0.059	0.5689	0.0592	0.5733	0.0491	0.5738	0.0505
	(15, 15, 30)	0.5815	0.0425	0.5703	0.0464	0.5716	0.0456	0.5733	0.0398	0.5744	0.0403
	(15, 30, 15)	0.5791	0.05	0.5722	0.0515	0.5692	0.0527	0.5755	0.0447	0.5754	0.0456
	(30, 15, 15)	0.5799	0.0521	0.5739	0.0531	0.57	0.0547	0.5776	0.0444	0.5778	0.0457
	(30, 30, 15)	0.5783	0.0515	0.5751	0.0516	0.5684	0.0543	0.5781	0.0449	0.5777	0.046
	(30, 15, 30)	0.5848	0.043	0.5771	0.0448	0.5747	0.0454	0.5791	0.0384	0.5801	0.0391
	(15, 30, 30)	0.5818	0.0401	0.5734	0.0428	0.5718	0.0432	0.5756	0.039	0.5759	0.0391
	(30, 30, 30)	0.5846	0.0364	0.5796	0.0375	0.5745	0.0393	0.5812	0.0344	0.5813	0.0347
	(100, 100, 100)	0.5902	0.0205	0.5886	0.0206	0.5798	0.0235	0.589	0.0202	0.589	0.0202
(0.15, 0.8, 0.01) True R = 0.7583	(15, 15, 15)	0.7435	0.0417	0.7362	0.0449	0.7359	0.045	0.7446	0.031	0.7443	0.0336
	(15, 15, 30)	0.7457	0.0404	0.7383	0.0431	0.7382	0.0432	0.7462	0.0285	0.7459	0.0316
	(15, 30, 15)	0.743	0.0405	0.7384	0.0421	0.7354	0.0438	0.7448	0.031	0.7441	0.0332
	(30, 15, 15)	0.7491	0.0321	0.7428	0.0349	0.7414	0.0354	0.7481	0.0269	0.7485	0.0278
	(30, 30, 15)	0.7504	0.0294	0.7468	0.0306	0.7427	0.0325	0.7498	0.026	0.7499	0.0264
	(30, 15, 30)	0.7495	0.0297	0.7431	0.0326	0.7419	0.0329	0.7481	0.0245	0.7485	0.0256
	(15, 30, 30)	0.7474	0.0386	0.7427	0.0395	0.7398	0.0411	0.7482	0.0279	0.7476	0.0306
	(30, 30, 30)	0.7511	0.0285	0.7474	0.0297	0.7434	0.0313	0.7502	0.0251	0.7503	0.0256
	(100, 100, 100)	0.7565	0.0137	0.7554	0.0139	0.7489	0.0166	0.756	0.0133	0.756	0.0133
(0.04, 0.8, 0.001) True R = 0.9261	(15, 15, 15)	0.9213	0.0135	0.9179	0.0154	0.9177	0.0155	0.9204	0.0113	0.9205	0.0117
	(15, 15, 30)	0.9219	0.0121	0.9185	0.0139	0.9183	0.014	0.9211	0.0098	0.9211	0.0103
	(15, 30, 15)	0.9215	0.0125	0.919	0.0137	0.9179	0.0145	0.9208	0.0109	0.9208	0.0112
	(30, 15, 15)	0.9227	0.0108	0.9201	0.0122	0.9191	0.0127	0.922	0.0095	0.9221	0.0097
	(30, 30, 15)	0.9232	0.0102	0.9216	0.011	0.9197	0.0121	0.9226	0.0094	0.9227	0.0095
	(30, 15, 30)	0.9234	0.0093	0.9208	0.0106	0.9199	0.0111	0.9226	0.008	0.9227	0.0083
	(15, 30, 30)	0.9224	0.0113	0.92	0.0124	0.9189	0.0131	0.9216	0.0096	0.9216	0.01
	(30, 30, 30)	0.9235	0.0084	0.9219	0.0092	0.92	0.0103	0.9228	0.0079	0.9228	0.008
	(100, 100, 10)	0.9255	0.0039	0.925	0.0041	0.9221	0.0057	0.9252	0.0039	0.9252	0.0039

Table 2. Various interval estimates for *R* under a different set of parameters.

		ACI			Boot-p Intervals			Boot-t Intervals			MCMC Jeffreys Prior			MCMC Beta Prior		
Parameters	(n1, n2, n3)	ALCL	AUCL	CP	ALCL	AUCL	CP	ALCL	AUCL	CP	ALCL	AUCL	CP	ALCL	AUCL	CP
	(15, 15, 15)	0.0268	0.1626	0.886	0.0303	0.1591	0.911	0.0426	0.2178	0.974	0.0342	0.1632	0.899	0.0417	0.1598	0.951
	(15, 15, 30)	0.0423	0.1494	0.929	0.0425	0.1468	0.924	0.0537	0.1762	0.948	0.046	0.1497	0.93	0.0511	0.1465	0.95
	(15, 30, 15)	0.0083	0.1804	0.947	0.0308	0.1574	0.914	0.0437	0.2149	0.978	0.0352	0.1621	0.913	0.041	0.1563	0.957
(0.5, 0.7, 0.4)	(30, 15, 15)	0.0435	0.1428	0.811	0.0339	0.16	0.906	0.0431	0.209	0.976	0.0382	0.1672	0.917	0.0418	0.1569	0.952
	(30, 30, 15)	0.03	0.1592	0.912	0.034	0.1575	0.916	0.0438	0.2054	0.978	0.0395	0.1655	0.923	0.0435	0.1576	0.964
True R = 0.0989	(30, 15, 30)	0.0559	0.1362	0.861	0.048	0.1471	0.929	0.0551	0.1691	0.944	0.0497	0.1495	0.927	0.0524	0.145	0.952
	(15, 30, 30)	0.0274	0.1626	0.949	0.0415	0.1421	0.912	0.0532	0.1706	0.974	0.0465	0.1459	0.935	0.0518	0.1447	0.941
	(30, 30, 30)	0.0473	0.1457	0.929	0.0478	0.143	0.918	0.0555	0.1642	0.96	0.0524	0.1492	0.931	0.0542	0.1446	0.955
	(100, 100, 100)	0.0707	0.1255	0.943	0.0715	0.126	0.954	0.0745	0.1309	0.957	0.0719	0.1261	0.933	0.0724	0.1255	0.956
	(15, 15, 15)	0.1289	0.3341	0.934	0.1114	0.3151	0.891	0.1529	0.382	0.965	0.1308	0.3296	0.914	0.148	0.3297	0.949
	(15, 15, 30)	0.1439	0.3203	0.924	0.1267	0.3044	0.902	0.1639	0.3514	0.953	0.1449	0.3158	0.914	0.1595	0.3145	0.955
	(15, 30, 15)	0.1062	0.3574	0.954	0.1142	0.3186	0.901	0.1523	0.3795	0.959	0.1403	0.3381	0.925	0.1486	0.3287	0.956
(0.5, 0.9, 0.2)	(30, 15, 15)	0.1625	0.312	0.868	0.1298	0.3086	0.891	0.1617	0.3583	0.958	0.1452	0.3246	0.926	0.159	0.3265	0.949
True R = 0.25	(30, 30, 15)	0.1463	0.3261	0.931	0.1357	0.3152	0.889	0.1643	0.3588	0.964	0.1531	0.3293	0.933	0.1585	0.3239	0.952
	(30, 15, 30)	0.1747	0.3037	0.873	0.1533	0.3002	0.897	0.1788	0.3331	0.96	0.1626	0.311	0.942	0.173	0.311	0.953
	(15, 30, 30)	0.1346	0.3383	0.955	0.132	0.3093	0.914	0.166	0.3504	0.954	0.1543	0.3212	0.933	0.1633	0.3165	0.941
	(30, 30, 30)	0.1665	0.3119	0.938	0.1577	0.3038	0.89	0.179	0.331	0.944	0.1695	0.3126	0.93	0.1736	0.3093	0.943
	(100, 100, 100)	0.2075	0.2871	0.94	0.2042	0.284	0.936	0.2104	0.2908	0.947	0.2072	0.2863	0.951	0.2086	0.2862	0.95
	(15, 15, 15)	0.2678	0.4874	0.934	0.2467	0.4704	0.908	0.286	0.5141	0.945	0.2627	0.4809	0.936	0.2735	0.4751	0.943
	(15, 15, 30)	0.2936	0.4785	0.947	0.27	0.4597	0.927	0.2999	0.4877	0.945	0.2812	0.4653	0.937	0.2942	0.4643	0.939
	(15, 30, 15)	0.2467	0.5133	0.975	0.2498	0.4665	0.906	0.2923	0.5096	0.957	0.2728	0.4801	0.937	0.2802	0.4753	0.948
(0.3, 0.7, 0.1)	(30, 15, 15)	0.3031	0.47	0.855	0.2647	0.4717	0.921	0.2925	0.5054	0.954	0.2781	0.4822	0.938	0.2899	0.4791	0.937
True R = 0.3968	(30, 30, 15)	0.2915	0.4847	0.931	0.2695	0.4674	0.93	0.2996	0.5002	0.959	0.286	0.478	0.93	0.296	0.4783	0.954
	(30, 15, 30)	0.3185	0.4622	0.898	0.2951	0.4625	0.936	0.3106	0.4816	0.943	0.3012	0.4694	0.942	0.3081	0.4634	0.937
	(15, 30, 30)	0.281	0.4895	0.985	0.2771	0.4565	0.935	0.3105	0.4835	0.957	0.2935	0.4646	0.953	0.2993	0.4606	0.951
	(30, 30, 30)	0.311	0.4643	0.959	0.2974	0.4537	0.937	0.3164	0.4731	0.962	0.3099	0.4624	0.936	0.3116	0.4581	0.947
	(100,100, 100)	0.3512	0.4343	0.949	0.3493	0.4327	0.94	0.3551	0.4384	0.953	0.3516	0.4343	0.943	0.3508	0.4325	0.955
	(15, 15, 15)	0.4738	0.6835	0.943	0.4449	0.6619	0.927	0.4764	0.6885	0.956	0.46	0.6744	0.94	0.4731	0.6716	0.948
	(15, 15, 30)	0.4972	0.6659	0.96	0.4793	0.6521	0.94	0.4979	0.6649	0.941	0.4818	0.6576	0.958	0.4913	0.6541	0.961
	(15, 30, 15)	0.4475	0.7106	0.994	0.4476	0.6536	0.919	0.4839	0.6821	0.949	0.4672	0.669	0.949	0.4795	0.6687	0.968
(0.1, 0.5, 0.03)	(30, 15, 15)	0.4986	0.6612	0.895	0.4542	0.6658	0.947	0.479	0.6891	0.965	0.4677	0.6741	0.93	0.4812	0.6726	0.971
True R = 0.5916	(30, 30, 15)	0.4814	0.6752	0.947	0.4591	0.6581	0.916	0.4886	0.6831	0.941	0.4801	0.6717	0.943	0.4854	0.6683	0.976
	(30, 15, 30)	0.5173	0.6522	0.889	0.4929	0.656	0.951	0.504	0.6666	0.954	0.496	0.6599	0.962	0.5024	0.6556	0.959
	(15, 30, 30)	0.4837	0.68	0.993	0.4843	0.6439	0.937	0.5083	0.6592	0.97	0.4907	0.6507	0.953	0.4988	0.6501	0.952
	(30, 30, 30)	0.5115	0.6577	0.954	0.499	0.6471	0.946	0.515	0.6598	0.959	0.5068	0.6536	0.948	0.5095	0.6516	0.965
	(100,100, 100)	0.5509	0.6294	0.942	0.5466	0.6257	0.941	0.5513	0.6297	0.946	0.5486	0.6271	0.947	0.5499	0.6275	0.961
	(15, 15, 15)	0.6692	0.8179	0.957	0.631	0.7943	0.878	0.6701	0.8146	0.948	0.6565	0.8082	0.96	0.6767	0.8075	0.944
	(15, 15, 30)	0.6754	0.816	0.957	0.6395	0.7936	0.916	0.6713	0.8103	0.92	0.665	0.8053	0.964	0.6823	0.8039	0.967
	(15, 30, 15)	0.6651	0.8209	0.974	0.6398	0.7955	0.923	0.6772	0.8141	0.946	0.6732	0.8097	0.956	0.6792	0.8054	0.973
(0.15, 0.8, 0.01)	(30, 15, 15)	0.6934	0.8048	0.926	0.6701	0.7953	0.928	0.693	0.8091	0.942	0.6784	0.8036	0.952	0.692	0.8023	0.961
True R = 0.7583	(30, 30, 15)	0.6954	0.8054	0.959	0.6765	0.7936	0.924	0.6982	0.8064	0.949	0.6918	0.8028	0.95	0.6972	0.8009	0.963
	(30, 15, 30)	0.6957	0.8033	0.947	0.6768	0.7914	0.942	0.693	0.8013	0.927	0.6854	0.8004	0.956	0.696	0.7975	0.965
	(15, 30, 30)	0.6783	0.8165	0.972	0.6454	0.7926	0.928	0.6761	0.8078	0.947	0.675	0.8037	0.964	0.6878	0.8026	0.971
	(30, 30, 30)	0.7005	0.8016	0.948	0.6848	0.7906	0.941	0.6999	0.7995	0.938	0.6975	0.7977	0.949	0.7018	0.7964	0.969
	(100,100, 100)	0.7297	0.7833	0.954	0.7248	0.7794	0.945	0.7285	0.7821	0.943	0.7283	0.7817	0.95	0.7295	0.7818	0.958
	(15, 15, 15)	0.8978	0.9448	0.969	0.8805	0.9366	0.901	0.8972	0.9419	0.934	0.8903	0.9416	0.959	0.8979	0.9414	0.961
	(15, 15, 30)	0.9006	0.9432	0.979	0.8865	0.9361	0.929	0.9	0.94	0.927	0.8945	0.9394	0.969	0.9008	0.9393	0.975
	(15, 30, 15)	0.8953	0.9476	0.984	0.8833	0.9363	0.898	0.8991	0.9413	0.931	0.8944	0.9411	0.964	0.8991	0.941	0.977
	(30, 15, 15)	0.9049	0.9404	0.927	0.893	0.9371	0.924	0.9029	0.9412	0.927	0.8976	0.9405	0.946	0.9028	0.9403	0.971
(0.04, 0.8, 0.001)	(30, 30, 15)	0.9049	0.9415	0.947	0.8962	0.9369	0.913	0.9056	0.9408	0.941	0.9014	0.9396	0.956	0.9046	0.9399	0.969
True R = 0.9261	(30, 15, 30)	0.9073	0.9395	0.93	0.8987	0.936	0.934	0.9056	0.9387	0.913	0.9005	0.9383	0.966	0.9059	0.9383	0.953
	(15, 30, 30)	0.9007	0.9441	0.982	0.889	0.9355	0.912	0.9018	0.9391	0.926	0.8982	0.9386	0.965	0.9026	0.9387	0.967
	(30, 30, 30)	0.9079	0.9391	0.963	0.902	0.9356	0.929	0.9084	0.9382	0.929	0.9055	0.9376	0.953	0.9074	0.9374	0.966
	(100,100, 100)	0.9175	0.9335	0.962	0.9161	0.9325	0.956	0.9175	0.9333	0.961	0.9165	0.9328	0.939	0.9172	0.933	0.965

8. Applications

For the purpose of illustration in this part, we analyze two real datasets. The first dataset contains the observed lifetimes of steel specimens tested at 14 different stress levels. [25] originally presented this dataset, which was later analyzed by [26]. To demonstrate our theoretical results, we use lifetimes under 33 and 32 stress levels as the stress variables Z and Y, respectively, and lifetimes at 32.5 stress levels as the strength variable X. The datasets are represented in Table 3. The second dataset is related to post-weld treatments for improving the fatigue life of welded joints, originally studied by [27]. The post-weld treatments involve burr grinding (BG) and TIG dressing, represented by the variables X and Z, respectively. The main objective of the study is to determine a suitable post-weld treatment for mass production of crane components. The dataset allows us to determine whether BG and TIG dressing treatments led to improved fatigue strength compared to the as-welded (AW) condition, denoted by Y, and to identify the order of these improvements, if any. The fatigue strength data, indicating the number of cycles to failure for each test specimen under AW, BG, and TIG conditions, is given in Table 3. Similar to the assumption made by [11] , the data points for stresses below 500 MPa are discarded in the case of the AW. This ensures fair comparisons by considering only high stress ranges. Additionally, for the second dataset, the values are discretized to the nearest integer, as the models used are discrete.

Table 3. Values of the datasets I and II.

Dataset I	Stress Level 32 (Y)	1144, 231, 523, 474, 4510, 3107, 815, 6297, 1580, 605, 1786, 206, 1943, 935, 283, 1336, 727, 370, 1056, 413, 619, 2214, 1826, 597
	Stress Level 32.5 (X)	4257, 879, 799, 1388, 271, 308, 2073, 227, 347, 669, 1154, 393, 250, 196, 548, 475, 1705, 2211, 975, 2925
	Stress Level 33 (Z)	184, 241, 273, 1842, 371, 830, 683, 1306, 562, 166, 981, 1867, 493, 418, 2978, 1463, 2220, 312, 251, 760
Dataset II	AW (Y)	12, 15, 19, 20, 26, 28, 35, 43, 48, 58, 78, 96
	BG (X)	25, 25, 39, 44, 100, 72, 102, 74, 76, 144, 172
	TIG (Z)	84, 93, 156, 352, 666, 91, 112, 179, 136, 36, 94, 48, 44

First, we want to see whether a geometric distribution can be used to fit each data set separately. For this purpose, we use the Kolmogorov-Smirnov (K-S) statistic with its associated p-value. The K-S goodness-of-fit statistics and MLE-based parameter estimates of the models' parameters are shown on Table 4. The high p-values associated with the K-S test show that the geometric distribution is a reasonable fit for these datasets. This is further supported by the fitted versus empirical CDF plots shown in Figures 6 and 7.

Table 4. K-S statistics with associated p-values for datasets I and II.

Dataset	Variables	Model	Estimate	K-S statistic	p-value
Dataset I	Stress Level 32 (Y)	Geometric (θ_2)	$\theta_2 = 0.0011$	0.1368	0.7092
	Stress Level 32.5 (X)	Geometric (θ_1)	$\theta_1 = 0.0009$	0.1629	0.6062
	Stress Level 33 (Z)	Geometric (θ_3)	$\theta_3 = 0.0007$	0.1668	0.577
Dataset II	AW (Y)	Geometric (θ_2)	$\theta_2 = 0.0251$	0.2629	0.3196
	BG (X)	Geometric (θ_1)	$\theta_1 = 0.0126$	0.2716	0.3913
	TIG (Z)	Geometric (θ_3)	$\theta_3 = 0.0062$	0.2011	0.6001

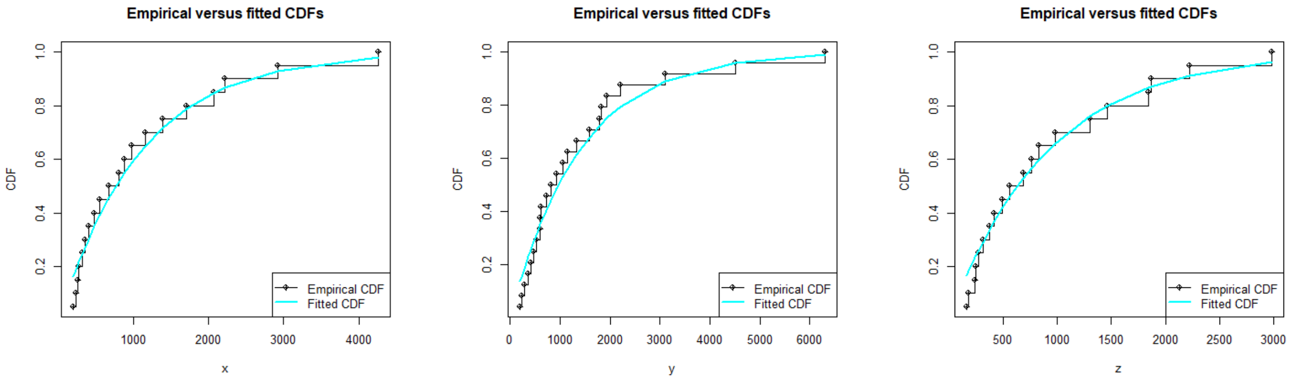


Figure 6. Empirical versus fitted CDFs plots for dataset I.

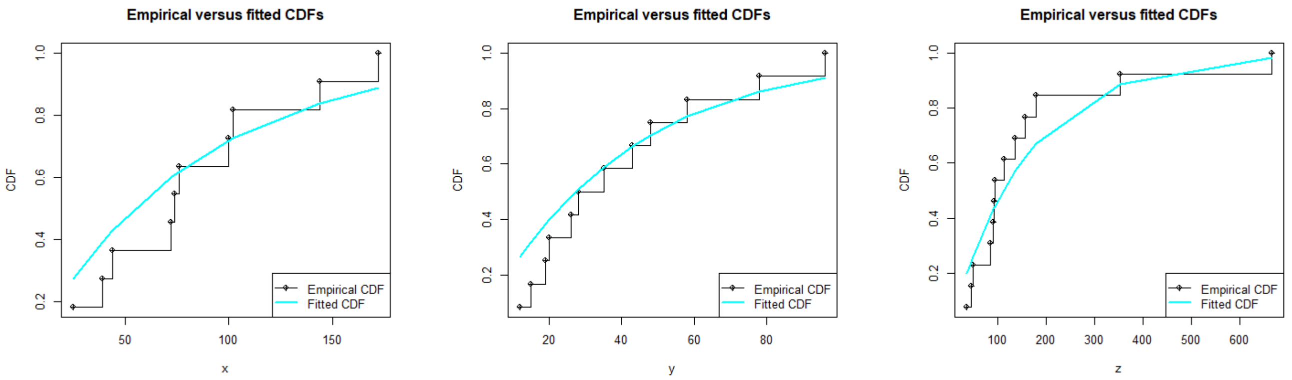


Figure 7. Empirical versus fitted CDFs plots for dataset II.

For both of the considered datasets, under the classical viewpoint, we compute the MLE along with standard error (SE), 95% ACI, boot-p, and boot-t confidence intervals for stress-strength reliability $R = P[Y < X < Z]$. In the Bayesian study, since we need to specify the prior distributions and we have no prior information about the unknown parameters for the given data, we assume the Jeffreys prior for the parameters involved in the model. Using the Jeffreys prior, we obtain the Bayes estimates of R using the Lindley approximation with SELF. Under MCMC methods, we generate a Markov chain with $M = 11,000$ observations. The first 1,000 observations are discarded to remove the effect of starting values. To determine the convergence of the generated chain of R, we plot the MCMC iteration, posterior density, and auto-correlation plots in Figures 8 and 9 for datasets I and II, respectively. These figures indicate that the MCMC chain converges on its stationary distribution. Then, using the posterior samples, we obtain the Bayes estimates of R under the SELF. We also provide the 95% HPD credible interval for R. The various point estimates of R are reported in Table 5, whereas the confidence intervals are given in Table 6.

Table 5. Point estimates of R under datasets I and II.

Dataset	Classical		Bayesian		
	MLE	SE	Lindley Estimate	MCMC	
				Estimate	PSE
Dataset I	0.1185	0.0334	0.1210	0.1201	0.0314
Dataset II	0.3795	0.0703	0.3702	0.3693	0.0686

Table 6. Various confidence intervals for R under datasets I and II.

Dataset	Method	Confidence interval	Width
Dataset I	ACI	[0.0530, 0.1840]	0.1309
	Boot-p	[0.0587, 0.1801]	0.1214
	Boot-t	[0.0689, 0.2096]	0.1406
	HPD	[0.0617, 0.1830]	0.1212
Dataset II	ACI	[0.2416, 0.5173]	0.2757
	Boot-p	[0.2220, 0.5048]	0.2828
	Boot-t	[0.2509, 0.5504]	0.2995
	HPD	[0.2322, 0.5002]	0.2680

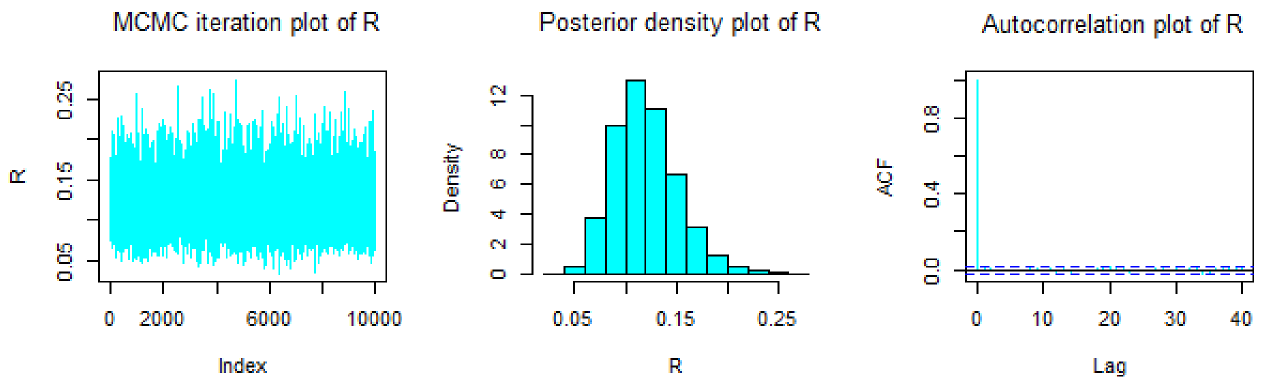


Figure 8. MCMC diagnostic plots for R under dataset I.

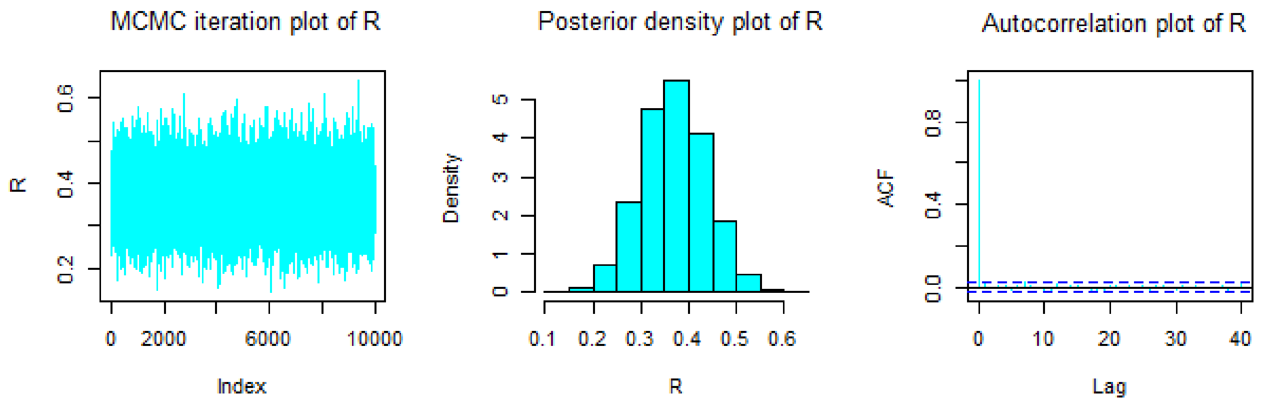


Figure 9. MCMC diagnostic plots for R under dataset II.

As seen in Tables 5 and 6, all the estimation procedures perform pretty effectively. Additionally, the Bayesian estimation with Jeffreys prior through the MCMC method works better than other estimates in terms of standard error for both datasets. Under interval estimation, the HPD interval for R has the smallest length among the ACI, boot-p, and boot-t intervals for both datasets. For dataset II, the results suggest that there was some improvement in fatigue strength from the AW treatment to the BG and TIG treatments, with the TIG treatment being the strongest. This is consistent with the findings in [27].

9. Conclusions

In this article, we analyzed a stress-strength model of the type $P[Y < X < Z]$, where X is the discrete strength of the subjacent system and Y and Z are the two discrete stresses that are applied to it. The analysis was carried out by assuming that X, Y and Z are from the geometric distribution. The methods of maximum likelihood estimation and bootstrapping were used in the classical setup. The Bayes estimates were obtained under the consideration of the squared error loss function, assuming Beta informative and Jeffreys non-informative priors. The Bayes estimates and HPD intervals were computed using MCMC and the Lindley approximation method. An extensive simulation study was presented to compare the performance of different estimates. Based on a simulation study, it is recommended to use the Bayes estimate with the Lindley approximation method under the Beta priors for point estimation and the Bayes credible interval with the MCMC method under the Beta priors for interval estimation of R in practice. Finally, the fitting capability of the proposed model was analyzed using two real datasets. The proposed stress-strength system model may be used to examine various types of censored data and different discrete distributions as part of a future course of study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Dr. Chesneau is the Guest Editor of special issue "Modern advances in statistical modeling" for AIMS Mathematics. Dr. Chesneau was not involved in the editorial review and the decision to publish this article.

All authors declare no conflicts of interest in this paper.

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