Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20665–20678.
DOI: 10.3934/math.20231053
Received: 01 May 2023
Revised: 14 June 2023
Accepted: 16 June 2023
Published: 27 June 2023

Research article

Uniform boundedness results of solutions to mixed local and nonlocal elliptic operator

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Abstract: In this paper, by the Stampacchia method, we consider the boundedness of positive solutions to the following mixed local and nonlocal quasilinear elliptic operator

$$\begin{cases} -\Delta_p u + (-\Delta)^s_p u = f(x)u^\gamma, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $1 < p < N$, $f \in L^m(\Omega)$ with $m > \frac{Np}{p(s+p-1)-\gamma(N-sp)}$, $0 \leqslant \gamma < p^*_s - 1$, $p^*_s = \frac{Np}{N-sp}$ is the critical Sobolev exponent.

Keywords: mixed local and nonlocal operator; bioundedness; Stampacchia method
Mathematics Subject Classification: 35J67, 35R11

1. Introduction

The main goal of this paper is consider the boundedness result of solutions to the following mixed local and nonlocal quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + (-\Delta)^s_p u = f(x)u^\gamma, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $1 < p < N$, $s \in (0,1)$, $0 \leqslant \gamma < p^*_s - 1$, $p^*_s := \frac{Np}{N-sp}$ is the critical fractional Sobolev exponent. $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the classical p-Laplacian, $(-\Delta)^s_p$ is the fractional p-Laplacian defined as, up to a multiplicative constant,

$$(-\Delta)^s_p u(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

P.V. stands for the Cauchy principal value. For the nonlocal case,

$$\begin{cases} (-\Delta)^s u = f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.2}$$

Leonori et al. [19, Theorem 13] proved the boundedness of energy solutions to problem (1.2) if $f \in L^m(\Omega)$ with $m > \frac{N}{2s}$ by two different methods: the Moser method and Stampacchia method. Dipierro et al. [11, Theorem 2.3] established an L^∞ estimate for the solutions to the following problem with some general kind of growth assumptions:

$$(-\Delta)^s u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where

$$|f(x,t)| \leqslant \sum_{i=1}^{K} f_i(x)|t|^{\gamma_i}, \quad \gamma_1, \cdots, \gamma_K \in [0, 2_s^* - 1),$$
$$f_1, \ldots, f_K \in L^{m_i}\left(\mathbb{R}^N, [0, +\infty)\right), \quad m_i \in \left(\underline{m}_i, +\infty\right],$$

and

$$\underline{m}_i := \begin{cases} \frac{2_s^*}{2_s^*-2}, & \gamma_i \in [0,1], \\ \frac{2_s^*}{2_s^*-1-\gamma_i}, & \gamma_i \in (1, 2_s^*-1). \end{cases} \tag{1.4}$$

Servadei and Valdinoci [23, Proposition 9] used the argument a fractional version of the classical De Giorgi-Stampacchia iteration method, proved the boundedness of weak solutions to fractional boundary value problem

$$\begin{cases} (-\Delta)^s u = f, & x \in \Omega, \\ u = g, & x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

Biroud [7, Theorem 2.9] obtained the boundedness of unique solutions to problem

$$\begin{cases} (-\Delta)_p^s u = f, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases}$$

if $f \in L^m(\Omega)$ for some $m \geqslant 1$, $m > \frac{N}{ps}$. Moreover, there exists a constant $C := C(N, m, s) > 0$ such that,

$$\|w\|_{L^{m_s^{**}}(\Omega)} \leqslant C\|f\|_{L^m(\Omega)},$$

if

$$\frac{pN}{(p-1)N + ps} = (p_s^*)' \leqslant m < \frac{N}{ps},$$

where

$$m_s^{**} = \frac{(p-1)mN}{N - pms}.$$

Problems driven by mixed local and nonlocal have raised a certain interest in the last few years. When $p = 2$, problem (1.1) reduced to

$$\begin{cases} -\Delta u + (-\Delta)^s u = f(x)u^\gamma, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.5}$$

Biagi et al. [3, Theorem 4.7] obtained the boundedness of solutions to problem (1.5) with $\gamma = 0$ if $f \in L^m(\Omega)$ with $m > \frac{N}{2}$. This results was improved by LaMao [17, Theorem 1.1] for $m > \frac{N}{s+1}$. Su et al. [24, Theorem 1.1] showed the L^∞ boundedness of any weak solution (either not changing sign or sign-changing) to mixed local-nonlocal semilinear elliptic equations by the Moser iteration method. Arora and Rădulescu [2, Theorem 2.3], Huang and Hajaiej [16, Theorem 1.14] established the boundedness of solutions to problem (1.5) with $\gamma < 0$.

Garain and Ukhlov [15, Theorem 2.16] obtained the boundedness of solutions to problem (1.1) with $\gamma = 0$ and $f \in L^m(\Omega)$, where $m > \frac{N}{p}$. Biagi et al. [5, Theorem 4.1 and Remark 4.2] obtained the boundedness of weak solutions to problem (1.1) provided the nonlinear term satisfies some suitable growth assumptions. Filippis and Mingione [12, Proposition 2.1] obtained the boundedness of the minimizers of the following functionals

$$\mathcal{F}(w) := \int_\Omega [F(Dw) - fw]\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi(w(x) - w(y))K(x, y)\mathrm{d}x\, \mathrm{d}y, \tag{1.6}$$

provided

$$\begin{cases} q > \frac{N}{p}, & \text{if } p \leqslant N, \\ q = 1, & \text{if } p > N, \end{cases}$$

where the integrand F: $\mathbb{R}^N \to \mathbb{R}$ is assumed to be $C^2\left(\mathbb{R}^N \backslash \{0\}\right) \cap C^1\left(\mathbb{R}^N\right)$-regular and to satisfy the following standard p-growth and coercivity assumptions

$$\begin{cases} \Lambda^{-1}\left(|z|^2 + \mu^2\right)^{p/2} \leqslant F(z) \leqslant \Lambda\left(|z|^2 + \mu^2\right)^{p/2}, \\ |\partial_z F(z)| + \left(|z|^2 + \mu^2\right)^{1/2} |\partial_{zz} F(z)| \leqslant \Lambda\left(|z|^2 + \mu^2\right)^{(p-1)/2}, \\ \Lambda^{-1}\left(|z|^2 + \mu^2\right)^{(p-2)/2} |\xi|^2 \leqslant \partial_{zz} F(z)\xi \cdot \xi, \end{cases}$$

for all $z \in \mathbb{R}^N \backslash \{0\}$, $\xi \in \mathbb{R}^N$, where $\mu \in [0, 1]$ and $\Lambda \geqslant 1$ are fixed constants. The function Φ: $\mathbb{R} \to \mathbb{R}$ is assumed to satisfy

$$\begin{cases} \Phi(\cdot) \in C^1(\mathbb{R}), & t \mapsto \Phi(t) \text{is convex}, \\ \Lambda^{-1}|t|^\gamma \leqslant \Phi(t) \leqslant \Lambda|t|^\gamma, & \Lambda^{-1}|t|^\gamma \leqslant \Phi'(t)t \leqslant \Lambda|t|^\gamma, \end{cases}$$

for all $t \in \mathbb{R}$. The kernel K: $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies

$$\frac{k}{\Lambda|x - y|^{N+s\gamma}} \leqslant K(x, y) \leqslant \frac{\Lambda k}{|x - y|^{N+s\gamma}}, \quad \text{where } k \in (0, 1]$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$ and $p \geqslant s\gamma$. Some other related results about mixed local and nonlocal elliptic operator see [4, 6, 8–10, 13, 14, 18] and references therein.

Motivated by the results of the above cited papers, especially [11, 17], the main purpose of this paper is to establishes the boundedness of solutions to problem (1.1).

Theorem 1.1. *Assume that $u \in \mathbb{X}_p(\Omega)$ is a weak solution to problem* (1.1) *(the definition of $\mathbb{X}_p(\Omega)$ see* (2.1) *below). Then, $u \in L^\infty(\mathbb{R}^N)$ provided $0 \leqslant \gamma < p_s^* - 1$ and $f \in L^m(\Omega)$ with $m > m_p^*$, where*

$$m_p^* = \frac{p_s^* p^*}{p_s^* p^* - p_s^*(p-1) - p^*(1+\gamma)} = \frac{Np}{p(s+p-1) - \gamma(N-sp)}. \tag{1.7}$$

Remark 1.2. According to Theorem 1.1, we know that, the weak solutions to problem (1.1) with $\gamma = 0$ are bounded provided $f \in L^m(\Omega)$ with $m > \frac{N}{p-1+s}$. This generalizes [17, Theorem 1.1] to mixed local and nonlocal elliptic operator $-\Delta_p + (-\Delta)_p^s$.

Corollary 1.3. *Let $u \in \mathbb{X}_p(\Omega)$ be a weak solution to problem*

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = f(x, u(x)), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $|f(x, u(x))| \leqslant \sum_{i=1}^K f_i(x)|u|^{\gamma_i}$, $\gamma_i \in [0, p_s^* - 1)$, $f_i \in L^{m_i}(\Omega)$ with $m_i > m_{pi}^*$, $i = 1, 2, \cdots, K$, where

$$m_{pi}^* = \frac{p_s^* p^*}{p_s^* p^* - p_s^*(p-1) - p^*(1+\gamma_i)} = \frac{Np}{p(s+p-1) - \gamma_i(N-sp)}. \tag{1.8}$$

Then, $u \in L^\infty(\Omega)$.

Remark 1.4. When $p = 2$, (1.8) reduces to

$$m_{2i}^* = \frac{2_s^* 2^*}{2_s^* 2^* - 2_s^* - 2^*(1+\gamma_i)} = \frac{2N}{2(s+1) - \gamma_i(N-2s)}. \tag{1.9}$$

Obviously, $m_{2i}^* > \underline{m}_i$, where $\underline{m}_i$ is defined by (1.4). Therefore, Theorem 1.1 extends the corresponding results of [11, Theorem 2.3].

Corollary 1.5. *Let $u \in \mathbb{X}_p(\Omega)$ be a weak solution to*

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = |x|^\alpha |u|^\gamma, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.10}$$

where $\gamma \in [0, p_s^* - 1)$ and

$$\alpha > -\frac{N p_s^* p^* - p_s^*(p-1) - p^*(1+\gamma)}{p_s^* p^*} = -\frac{p(s+p-1) - \gamma(N-sp)}{p} = \gamma\left(\frac{N}{p} - s\right) - (s+p-1).$$

Then, $u \in L^\infty(\mathbb{R}^N)$.

Remark 1.6. Salort and Vecchi [22, Corollary 2.5] showed that the weak solution u to problem (1.10) belongs to L^∞ if

$$\alpha > \max\left\{0, \gamma\left(\frac{N}{p} - 1\right) - p\right\}.$$

By Corollary 1.5, we find $u \in L^\infty(\Omega)$ holds also for some $\alpha < 0$.

This paper is organized as follows: In Section 2, we give some preliminary lemmas. Finally, we prove Theorem 1.1 in Section 3.

2. Preparations

In this section, we collect some notation and preliminary results which will be used in the rest of the paper. Firstly, we introduce the proper function spaces for problem (1.1).

For $p \in (1, +\infty)$. Let $\Omega \subset \mathbb{R}^N$ be a connected and bounded open set with C^1-smooth boundary. Define

$$\mathbb{X}_p = \{u \in W^{1,p}(\mathbb{R}^N) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\}, \tag{2.1}$$

which is Banach space equipped with the norm

$$\|u\|_{\mathbb{X}_p} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Give a fractional parameter $s \in (0, 1)$ and $p > 1$, the mixed local–nonlocal elliptic operator

$$\mathcal{L}u = -\Delta_p u + (-\Delta)^s_p u$$

is well define between $\mathbb{X}_p$ and its dual space $\mathbb{X}_p^*$ and the following representation formula holds:

$$\begin{aligned}\langle \mathcal{L}u, v\rangle = &\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx \\ &+ \iint_{\mathcal{D}(\Omega)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dxdy, \quad v \in \mathbb{X}_p,\end{aligned}$$

where

$$\mathcal{D}(\Omega) = \mathbb{R}^N \times \mathbb{R}^N \setminus (C\Omega \times C\Omega), \;\; C\Omega = \mathbb{R}^N \setminus \Omega.$$

Definition 2.1. We say that $u \in \mathbb{X}_p$ is a weak solution to problem (1.1) if

$$\langle \mathcal{L}u, v\rangle = \int_\Omega f(x)|u|^\gamma v dx$$

for all $v \in \mathbb{X}_p$.

Lemma 2.2. *[21, Lemma 4.1] Let ψ: $\mathbb{R}^+ \to \mathbb{R}$ be a non-increasing function such that*

$$\psi(h) \leqslant \frac{M\psi(k)^\delta}{(h - k)^\gamma}, \qquad \forall h > k > 0,$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where

$$d^\gamma = M\psi(0)^{\delta-1} 2^{\frac{\delta\gamma}{\delta-1}}.$$

Lemma 2.3. *[1, Lemma 2.5] For any $a, b \in \mathbb{R}$ and $k \geqslant 0$, $p \geqslant 1$, define*

$$T_k(a) = \begin{cases} a, & \text{if } |a| \leqslant k, \\ k\frac{a}{|a|}, & \text{if } |a| > k, \end{cases}$$

and

$$G_k(a) = a - T_k(a).$$

We have the algebraic inequalities

$$|a-b|^{p-2}(a-b)\left(G_k(a) - G_k(b)\right) \geqslant |G_k(a) - G_k(b)|^p .$$

Lemma 2.4. *[20, Theorem 6.5] Let $s \in (0,1)$, $p \in [1,+\infty)$ be such that $ps < N$. Then there exists a positive constant $C = C(N,p,s)$ such that, for any measurable and compactly supported function u: $\mathbb{R}^N \to \mathbb{R}$,*

$$||u||^p_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy.$$

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1 by two methods

Proof of Theorem 1.1: the first method. Note that in this paper, we consider the positive solutions to problem (1.1). Therefore we can decompose $\mathbb{R}^N$ as $\mathbb{R}^N = A_k \cup A_k^c$, where

$$\begin{aligned} A_k &= \{x \in \mathbb{R}^N : u(x) \geqslant k\}, \\ A_k^c &= \{x \in \mathbb{R}^N : 0 < u(x) < k\}. \end{aligned} \tag{3.1}$$

Clearly, $G_k(u(x)) = u(x) - k$ for $x \in A_k$ and $G_k(u(x)) = 0$ for $x \in A_k^c$.

For any $k > 0$, taking $G_k(u)$ as test function in the definition of weak solution to problem (1.1), we have

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla G_k(u(x))dx + \iint_{\mathcal{D}(\Omega)} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][G_k(u(x)) - G_k(u(y))]}{|x-y|^{N+ps}}dxdy \\ &= \int_\Omega f(x)G_k(u(x))u^\gamma dx, \end{aligned} \tag{3.2}$$

where $\mathcal{D}(\Omega) = \mathbb{R}^N \times \mathbb{R}^N \setminus (C\Omega \times C\Omega)$.

Obviously,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla G_k(u(x))dx \geqslant 0,$$

which, together with (3.2), implies that

$$\iint_{\mathcal{D}(\Omega)} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][G_k(u(x)) - G_k(u(y))]}{|x-y|^{N+ps}}dxdy \leqslant \int_\Omega f(x)G_k(u(x))u^\gamma dx. \tag{3.3}$$

According to Lemma 2.3, we have

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[G_k(u(x))-G_k(u(y))\right] \geqslant |G_k(u(x))-G_k(u(y))|^p, \quad (x,y)\in\mathcal{D}(\Omega), \tag{3.4}$$

which, together with (3.3), imply that

$$\iint_{\mathcal{D}(\Omega)} \frac{|G_k(u(x))-G_k(u(y))|^p}{|x-y|^{N+ps}}dxdy \leqslant \int_\Omega f(x)G_k(u(x))u^\gamma dx. \tag{3.5}$$

This fact, combined with Sobolev theorem (see Lemma 2.4) and Hölder inequality, leads to

$$\begin{aligned}\|G_k(u)\|^p_{L^{p_s^*}(\Omega)} \leqslant & C\iint_{D(\Omega)} \frac{|G_k(u(x))-G_k(u(y))|^p}{|x-y|^{N+ps}}dxdy\\ \leqslant & C\int_\Omega f(x)G_k(u(x))u^\gamma dx\\ \leqslant & C\|f\|_{L^m(\Omega)}\|G_k(u)\|_{L^{p_s^*}(\Omega)}\|u\|^\gamma_{L^{p_s^*}(\Omega)}|A_k|^{1-\frac{1}{m}-\frac{1+\gamma}{p_s^*}},\end{aligned} \tag{3.6}$$

where $p_s^*=\frac{Np}{N-sp}$. Here we have used the fact that $G_k(u(x))=0$ for $x\in A_k^c$, A_k and A_k^c are given by (3.1). Therefore,

$$\|G_k(u)\|_{L^{p_s^*}(\Omega)} \leqslant \mathcal{S}\|f\|^{\frac{1}{p-1}}_{L^m(\Omega)}\|u\|^{\frac{\gamma}{p-1}}_{L^{p_s^*}}|A_k|^{\frac{1-\frac{1}{m}-\frac{1+\gamma}{p_s^*}}{p-1}}. \tag{3.7}$$

Using $u(x)=T_k(u(x))+G_k(u(x))$, we get

$$\begin{aligned}&|u(x)-u(y)|^{p-2}(u(x)-u(y))\left(G_k(u(x))-G_k(u(y))\right)\\ =&\begin{cases}|u(x)-u(y)|^p, & (x,y)\in A_k\times A_k,\\ (u(x)-u(y))^{p-1}G_k(u(x)), & (x,y)\in A_k\times A_k^c,\\ (u(y)-u(x))^{p-1}G_k(u(y)), & (x,y)\in A_k^c\times A_k,\\ 0, & (x,y)\in A_k^c\times A_k^c,\end{cases} \geqslant 0.\end{aligned} \tag{3.8}$$

On the other hand, by $\nabla u(x)=\nabla G_k(u(x))$ for $x\in A_k$ and $\nabla u(x)=0$ for $x\in A_k^c$, we find

$$\int_\Omega \nabla u^{p-2}\nabla u\cdot\nabla G_k(u)dx = \int_{A_k}|\nabla G_k(u)|^p dx. \tag{3.9}$$

This fact, together with (3.2) and (3.8), lead to

$$\int_{A_k}|\nabla G_k(u)|^p dx \leqslant \int_\Omega f(x)G_k(u(x))u^\gamma dx. \tag{3.10}$$

Thus, taking into account (3.7) and (3.10), we obtain

$$\begin{aligned}\|G_k(u)\|^p_{L^{p^*}(A_k)} \leqslant & \int_{A_k}|\nabla G_k(u)|^p dx\\ \leqslant & \int_{A_k} f(x)G_k(u(x))u^\gamma dx\end{aligned}$$

$$\leqslant \|f\|_{L^m(\Omega)} \|G_k(u)\|_{L^{p_s^*}(A_k)} \|u\|^{\gamma}_{L^{p_s^*}(\Omega)} |A_k|^{1-\frac{1}{m}-\frac{1+\gamma}{p_s^*}}$$
$$\leqslant \mathcal{S}\|f\|^{\frac{p}{p-1}}_{L^m(\Omega)} \|u\|^{\frac{p\gamma}{p-1}}_{L^{p_s^*}(\Omega)} |A_k|^{\frac{p\left(1-\frac{1}{m}-\frac{1+\gamma}{p_s^*}\right)}{p-1}}. \tag{3.11}$$

For every $h > k$ we know that $A_h \subset A_k$ and $|G_k(u(x))|\chi_{A_h(x)} \geqslant (h-k)$ in Ω. Therefore

$$(h-k)|A_h|^{\frac{1}{p^*}} \leqslant \left(\int_{A_h} |G_k(u)|^{p^*} \right)^{\frac{1}{p^*}}$$
$$\leqslant \|f\|^{\frac{1}{p-1}}_{L^m(A_k)} \|u\|^{\frac{\gamma}{(p-1)}}_{L^{p_s^*}(A_k)} |A_k|^{\frac{1-\frac{1}{m}-\frac{1+\gamma}{p_s^*}}{p-1}}. \tag{3.12}$$

Therefore

$$|A_h| \leqslant \frac{\|f\|^{\frac{p^*}{p-1}}_{L^m(A_k)} \|u\|^{\frac{p^*\gamma}{(p-1)}}_{L^{p_s^*}(A_k)} |A_k|^{\frac{p^*\left(1-\frac{1}{m}-\frac{1+\gamma}{p_s^*}\right)}{p-1}}}{(h-k)^{p^*}}. \tag{3.13}$$

Note that

$$\frac{p^*\left(1-\frac{1}{m}-\frac{1+\gamma}{p_s^*}\right)}{p-1} > 1, \tag{3.14}$$

if

$$m > \frac{p_s^* p^*}{p_s^* p^* - p_s^*(p-1) - p^*(1+\gamma)}$$
$$= \frac{Np}{p(s+p-1) - \gamma(N-sp)}.$$

Finally, by Lemma 2.2 with the choice $\psi(u) = |A_u|$, hence there exists k_0 such that $\psi(k) \equiv 0$ for any $k \geqslant k_0$. Therefore $\text{esssup}_{\Omega} u \leqslant k_0$. □

Proof of Theorem 1.1: the second method. In order to prove the desired bounded of u, we use a similarly argument of Stampacchia. We can certainly assume that u does not vanish identical else there is nothing to prove.

Now, let $f \in L^m(\Omega)$ and $\delta > 0$ be a positive constant which be conveniently choose later. Define

$$\tilde{u}(x) = K\delta^{\frac{1}{p-1}} u(x), \tag{3.15}$$

where

$$K = \frac{1}{\|u\|_{L^{p_s^*}(\Omega)} + \|f\|_{L^m(\Omega)} + \|u\|_{L^{p^*}(\Omega)}}.$$

According to (1.1) and (3.15), we know that $\tilde{u}(x)$ satisfies

$$\begin{cases} -\Delta_p \tilde{u} + (-\Delta)^s_p \tilde{u} = \tilde{f}(x)\tilde{u}^{\gamma}, & x \in \Omega, \\ \tilde{u} = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.16}$$

and

$$\|\tilde{u}\|_{L^{p_s^*}(\Omega)} \leqslant \delta^{\frac{1}{p-1}}, \quad \|\tilde{u}\|_{L^{p^*}(\Omega)} \leqslant \delta^{\frac{1}{p-1}}, \tag{3.17}$$

where

$$\tilde{f}(x) = K^{p-1-\gamma}\delta^{1-\frac{\gamma}{p-1}} f(x). \tag{3.18}$$

For every $k \in \mathbb{N}$, define $B_k = 1 - 2^{-k}$ and

$$w_k(x) := (\tilde{u}(x) - B_k)^+ = \max\{0, \tilde{u}(x) - B_k\}, \quad x \in \mathbb{R}^N,\ U_k = \|w_k\|^{p_s^*}_{L^{p_s^*}(\Omega)}.$$

It is easy to see that $w_k \in X^*_\beta$ and

$$w_{k+1}(x) \leqslant w_k(x), \text{ a.e. } x \in \mathbb{R}^N. \tag{3.19}$$

Moreover,

$$w_k(x) = (\tilde{u}(x) - B_k)^+ = \left(\tilde{u}(x) - B_{k+1} + \frac{1}{2^{k+1}}\right)^+ = \left(w_{k+1} + \frac{1}{2^{k+1}}\right)^+.$$

By the definition of w_k, we find that

$$\{w_k > 0\} \subseteq \left\{w_{k-1} > \frac{1}{2^k}\right\} \tag{3.20}$$

and

$$0 < \tilde{u}(x) < 2^{k+1} w_k(x),\ \forall x \in \{w_{k+1} > 0\}. \tag{3.21}$$

Obviously, (3.21) implies that

$$\delta^{\frac{1}{p-1}} u < K^{-1} 2^{k+1} w_k,\ \forall x \in \{w_{k+1} > 0\}. \tag{3.22}$$

Taking w_k as a test function in (3.16), we obtain that

$$\begin{aligned}\int_{\Omega\cap\{\tilde{u}>B_k\}} |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\cdot\nabla w_k dx + \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-2}(\tilde{u}(x)-\tilde{u}(y))\,(w_k(x)-w_k(y))}{|x-y|^{N+ps}} dxdy \\ = \int_{\Omega\cap\{\tilde{u}>B_k\}} w_k(x)\tilde{f}(x)\tilde{u}^\gamma dx \\ = K^{p-1}\delta \int_{\Omega\cap\{\tilde{u}>B_k\}} w_k(x) f(x) u^\gamma dx. \end{aligned} \tag{3.23}$$

Note that, for

$$x \in \Omega \cap \{\tilde{u} > B_k\}, \quad \tilde{u}(x) = w_k(x) + B_k$$

and

$$|\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\cdot\nabla w_k = |\nabla w_k|^p \geqslant 0,$$

it is easily seen that

$$\begin{aligned}|w_k(x) - w_k(y)|^2 &= \left|(\tilde{u}(x) - B_k)^+ - (\tilde{u}(y) - B_k)^+)\right|^2 \\ &\leqslant (\tilde{u}(x) - B_k)^+ - (\tilde{u}(y) - B_k)^+)(\tilde{u}(x) - \tilde{u}(y))\end{aligned}$$

$$= (w_k(x) - w_k(y))\,(\tilde{u}(x) - \tilde{u}(y)). \tag{3.24}$$

This fact, together with (3.23), implies that

$$\begin{aligned}
[w_k]_{s,p}^p &= \iint_{\mathbb{R}^{2N}} \frac{|w_k(x) - w_k(y)|^p}{|x-y|^{N+sp}} dxdy \\
&\leqslant \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2}(\tilde{u}(x) - \tilde{u}(y))\,(w_k(x) - w_k(y))}{|x-y|^{N+ps}} dxdy \\
&\leqslant K^{p-1}\delta \int_{\Omega\cap\{\tilde{u}>B_k\}} w_k f u^\gamma dx.
\end{aligned} \tag{3.25}$$

For the right hand of (3.25), using the Hölder inequality with exponents

$$\left(m,\ \ p_s^*,\ \ \frac{p_s^*}{\gamma},\ \ \frac{1}{\xi}\right),$$

where

$$\xi = 1 - \frac{1}{m} - \frac{1+\gamma}{p_s^*} \in (0,1),$$

we get

$$\begin{aligned}
\int_{\Omega\cap\{\tilde{u}>B_k\}} f w_k u^\gamma dx &\leqslant \left[\int_{\Omega\cap\{\tilde{u}>B_k\}} f^m dx\right]^{\frac{1}{m}} \left[\int_{\Omega\cap\{\tilde{u}>B_k\}} w_k^{p_s^*} dx\right]^{\frac{1}{p_s^*}} \left[\int_{\Omega\cap\{\tilde{u}>B_k\}} |u|^{p_s^*} dx\right]^{\frac{\gamma}{p_s^*}} \left[\int_{\Omega\cap\{\tilde{u}>B_k\}} 1 dx\right]^{\xi} \\
&\leqslant \|f\|_{L^m(\Omega)}\, \|u\|^\gamma_{L^{p_s^*}(\Omega)} U_k^{\frac{1}{p_s^*}}\, |\{w_k > 0\}|^\xi\,.
\end{aligned} \tag{3.26}$$

By Lemma 2.4, (3.25) and (3.26), we get

$$U_k^{\frac{p}{p_s^*}} \leqslant [w_k]_{s,p}^p \leqslant K^{p-1}\delta\, \|f\|_{L^m(\Omega)}\, \|u\|^\gamma_{L^{p_s^*}(\Omega)} U_k^{\frac{1}{p_s^*}}\, |\{w_k > 0\}|^\xi\,,$$

that is

$$U_k \leqslant \widetilde{K}\, |\{w_k > 0\}|^{\frac{p_s^*\xi}{p-1}}\,, \tag{3.27}$$

where

$$\widetilde{K} = \left(K^{p-1}\delta\, \|f\|_{L^m(\Omega)}\, \|u\|^\gamma_{L^{p_s^*}(\Omega)}\right)^{\frac{p_s^*}{p-1}}\,.$$

On the other hand,

$$\begin{aligned}
\int_\Omega |\nabla\tilde{u}(x)|^{p-2}\, \langle\nabla\tilde{u}(x), \nabla w_k(x)\rangle\, dx &= \int_{\Omega\cap\{\tilde{u}>B_k\}} |\nabla\tilde{u}(x)|^{p-2}\, \langle\nabla\tilde{u}(x), \nabla w_k(x)\rangle\, dx \\
&\quad + \int_{\Omega\cap\{\tilde{u}<B_k\}} |\nabla\tilde{u}(x)|^{p-2}\, \langle\nabla\tilde{u}(x), \nabla w_k(x)\rangle\, dx \\
&= \int_{\Omega\cap\{\tilde{u}>B_k\}} |\nabla w_k(x)|^p\, dx,
\end{aligned} \tag{3.28}$$

here we used the fact that $\nabla w_k(x) = 0$ for any $x \in \Omega \cap \{\tilde{u} < B_k\}$.

Define

$$V_{k-1} = \|w_{k-1}\|^{p^*}_{L^{p^*}(\Omega)}.$$

By (3.20), we have

$$\begin{aligned} V_{k-1} &= \|w_{k-1}\|^{p^*}_{L^{p^*}(\Omega)} \\ &\geqslant \int_{\left\{w_{k-1} > \frac{1}{2^k}\right\}} w_{k-1}^{p^*} dx \\ &\geqslant \frac{1}{2^{kp^*}} \left| \left\{ w_{k-1} > \frac{1}{2^k} \right\} \right| \\ &\geqslant \frac{1}{2^{kp^*}} \left| \{w_k > 0\} \right|, \end{aligned}$$

which leads to

$$|\{w_k > 0\}|^{\xi} \leqslant \left(2^{kp^*} V_{k-1}\right)^{\xi} \leqslant 2^{kp^*\xi} V_{k-1}^{\xi}. \tag{3.29}$$

Using the Sobolev inequality and (3.23), we find

$$\begin{aligned} V_k^{\frac{p}{p^*}} &\leqslant C \int_{\Omega} |\nabla w_k|^p \, dx \\ &\leqslant C K^{p-1} \delta \int_{\Omega \cap \{w_k > 0\}} w_k f u^{\gamma} dx \\ &\leqslant C K^{p-1} \delta \, \|f\|_{L^m(\Omega)} \, \|u\|^{\gamma}_{L^{p_s^*}(\Omega)} U_k^{\frac{1}{p_s^*}} \, |\{w_k > 0\}|^{\xi} \\ &= \widetilde{T} U_k^{\frac{1}{p_s^*}} \, |\{w_k > 0\}|^{\xi}, \end{aligned}$$

where

$$\widetilde{T} = C K^{p-1} \delta \, \|f\|_{L^m(\Omega)} \, \|u\|^{\gamma}_{L^{p^*}(\Omega)}.$$

According to (3.27) and (3.29), we get

$$\begin{aligned} V_k^{\frac{p}{p^*}} &\leqslant \widetilde{T} U_k^{\frac{1}{p_s^*}} \, |\{w_k > 0\}|^{\xi} \\ &\leqslant \widetilde{T} \widetilde{K}^{\frac{1}{p_s^*}} \left(|\{w_k > 0\}|^{\frac{p_s^* \xi}{p-1}} \right)^{\frac{1}{p_s^*}} |\{w_k > 0\}|^{\xi} \\ &\leqslant \widetilde{T} \widetilde{K}^{\frac{1}{p_s^*}} 2^{kp^*(\frac{\xi}{p-1} + \xi)} V_{k-1}^{\frac{\xi}{p-1} + \xi} \\ &= \widetilde{H} V_{k-1}^{\frac{p\xi}{p-1}}, \end{aligned} \tag{3.30}$$

Notice that $m > m_p^*$ implies that

$$\frac{p}{p^*} < \frac{p\xi}{p-1}, \tag{3.31}$$

where m_p^* is defined as (1.7) and

$$\widetilde{H} = \widetilde{T}\widetilde{K}^{\frac{1}{p_s^*}} 2^{kp^*(\frac{\xi}{p-1}+\xi)}.$$

We observe that

$$V_0 \leqslant \delta^{\frac{p^*}{p-1}}. \tag{3.32}$$

As a result, according to (3.32) and keeping in mind that $\delta > 0$ can be taken sufficiently small, we conclude that

$$\lim_{k\to+\infty} V_k = 0.$$

Moreover, since $0 \leqslant w_k \leqslant |\tilde{u}| \in L^{p^*}(\Omega)$ for any $k \in \mathbb{N}$ and $\lim_{k\to\infty} w_k = (\tilde{u}-1)^+$ a.e. in $\mathbb{R}^N$, by the dominated convergence theorem we get

$$\lim_{k\to\infty} V_k = \left\|(\tilde{u}-1)^+\right\|_{L^{p^*}(\Omega)}^{p^*} = 0$$

and therefore $\tilde{u} \leqslant 1$. a.e in $\mathbb{R}^N$. As a consequence, recalling (3.15), we conclude that

$$u(x) \leqslant \frac{\|u\|_{L^{p_s^*}(\Omega)} + \|f\|_{L^m(\Omega)} + \|u\|_{L^{p^*}(\Omega)}}{\delta}$$

with $\delta \in (0,1)$.

The proof of Theorem 1.1 is now complete. □

4. Conclusions

In this paper , we study the boundedness of positive solutions of the mixed local and nonlocal elliptic equation (Theorem1.1). To obtain this results, two different methods are used. The first one based on choosing appropriate test functions and the second one using an argument of Stampacchia. This results generalizes and complements the existing results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This works was partially supported by Fundamental Research Funds for the Central Universities (No. 31920220067), Innovation Team Project of Northwest Minzu University (No. 1110130131) and First-Rate Discipline of Northwest Minzu University(No. 2019XJYLZY-02).

Conflict of interest

The authors declare no competing interests.

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