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Research article

Certain new iteration of hybrid operators with contractive M-dynamic relations

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Abstract: This article investigates Wardowski's contraction in the setting of extended distance spaces known as M-metric spaces using hybrid operators based an M-dynamic iterative process. The main purpose is to observe new set-valued Chatterjea type common fixed point theorems for hybrid operators with respect to an M-dynamic iterative process, i.e., $\check{D}(\Psi_1, \Psi_2, s_0)$. We realize an application: the existence of a solution for a multistage system and integral equation. Also, we give a graphical interpretation of our obtained theorems. The main results are explicated with the help of a relevant example. Some important corollaries are extracted from the main theorems as well.

Keywords: hybrid operators; fixed point; M-dynamic iterative process M-metric space; multistage system; integral equation
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1. Introduction and preliminaries

After the Banach fixed point theorem, many interesting generalizations have been established by various authors [1, 3–6, 8, 9]. The generalizations were presented either by changing the axioms of the metric space, or by modifying the contractive condition. However, a new debate was instigated when using the idea of the Pompeiu-Hausdorff metric. Namely, Nadler [20] discussed the Banach fixed point theorem for multivalued mappings. In continuation to this, Patle et al. [21] presented the idea of an $\widetilde{m}$-Pompeiu-Hausdorff metric, which was further promoted as an $\widetilde{m}$-metric, i.e., let (Ω, d_m) be an $\widetilde{m}$-metric space. For $s_1 \in \Omega$ and $\Phi_1 \subseteq \Omega$,

$$d_m(s_1, \Phi_1) = \inf\{d_m(s_1, s_2) : s_2 \in \Phi_1\}.$$

Define the Pompeiu-Hausdorff metric $\mathcal{H}$ induced by d_m on $CB(\Omega)$ as follows:

$$\mathcal{H}(\Phi_1, B) = \max\{\sup_{s_1\in\Phi_1} d_m(s_1, B), \sup_{s_2\in B} d_m(s_2, \Phi_1)\},$$

for all $\Phi_1, \Phi_2 \in CB(\Omega)$, where

$$d_m(s_1, \Phi_2) = \inf_{s_2\in\Phi_2} d_m(s_1, s_2).$$

An element $s \in \Omega$ is known as a fixed point of a set-valued mapping ρ: $\Omega \to CB(\Omega)$ such that $s \in \rho(s)$. Furthermore, let ρ_1: $\Omega \to \Omega$ and ρ_2: $\Omega \to CB(\Omega)$, and a point $s \in \Omega$ is called a coincidence point of ρ_1 and ρ_2 if $\rho_1 s \in \rho_2 s$. The set of all such elements is denoted by $\hat{C}(\rho_1, \rho_2)$. If for some element $s \in \Omega$, we have $s = \rho_1 s \in \rho_2 s$, then an element s is called a common fixed point of ρ_1 and ρ_2. A mapping ρ: $\Omega \to CB(\Omega)$ is known as continuous at point $c \in \Omega$, if for any sequence $\{s_n\}$ in Ω with

$$\lim_{n\to\infty} d_m(s_n, c) = 0,$$

we have

$$\lim_{n\to\infty} \mathcal{H}(\rho_1 s_n, \rho_2 c) = 0.$$

This article is divided into three sections: Section 1 deals with the fundamental preliminaries and results that pertain to our main work. In section 2, we present some theorems dealing with Chatterjea type F-contractions for hybrid operators based on an M-dynamic iterative process in the setting of $\widetilde{m}$-metric spaces. An example and some corollaries are developed as consequences of the obtained theorems. This portion also has graphs that best illustrate our results for the better understanding of readers. Section 3 gives an application of our results in finding a solution of a multistage system. The pivotal role of functional equations in a dynamic system related to a multistage process is stated. Another application is also stipulated discussing the solution of integral equations. At last, a summary of the article is described in the conclusion section. Throughout our work, denote by $N(\Omega)$, $CL(\Omega)$, $CB(\Omega)$ and $K(\Omega)$ the collections of all the following non-empty: subsets of Ω, closed subsets of Ω, bounded closed subsets of Ω and compact subsets of Ω, respectively.

The multivalued version of the Banach fixed point theorem was initiated by Nadler [20], and some lemmas are useful for the rest: Let (Ω, d) be a complete metric space, and Γ: $\Omega \to CB(\Omega)$ is known as a Nadler contraction if there is $\delta \in [0, 1)$ such that

$$\mathcal{H}(\Gamma s_1, \Gamma s_2) \le \delta d(s_1, s_2) \text{ for all } s_1, s_2 \in \Omega.$$

Then, Γ has a fixed point.

Lemma 1.1. *[20] Let (Ω, d) be a metric space, $\beta \in CB(\Omega)$ and $s \in \Omega$. Then, for each $\epsilon > 0$, there is $v \in \beta$ such that*

$$d(s, v) \le d(s, \beta) + \epsilon.$$

Lemma 1.2. *[25] Let (Ω, d) be a metric space and $\beta, \beta^* \in CB(\Omega)$ with $\mathcal{H}(\beta, \beta^*) > 0$. Then, for every $h > 1$ and $s \in \beta$, there is $v = v(s) \in \beta^*$ such that*

$$d(s, v) < h\mathcal{H}(\beta, \beta^*).$$

Later, many papers dealing with multivalued mappings and hybrid operators appeared. For more details, see [2, 9–11, 13, 14, 17–24]. In 1995, Matthews [16] established the theory of a partial metric space and improved the Banach fixed point theorem in the context of partial metric spaces. After, based on the result of Matthews [16], Asadi et al. [8] introduced a new idea of an $\widetilde{m}$-metric space and studied its topological behavior. They also established fixed point results, which are generalizations of Banach and Kannan types fixed point theorems.

Now, we present some basic definitions and results as follows:

Definition 1.3. [8] An $\widetilde{m}$-metric space on a non-empty set Ω is a function d_m: $\Omega \times \Omega \to R^+ \cup \{0\}$, such that for all $s_1, s_2, s_3 \in \Omega$, the following conditions hold:

(m_ι) $s_1 = s_2$ if and only if $d_m(s_1, s_1) = d_m(s_2, s_2) = d_m(s_1, s_2)$;

$(m_{\iota\iota})$ $m_{s_1 s_2} \leq d_m(s_1, s_2)$;

$(m_{\iota\iota\iota})$ $d_m(s_1, s_2) = d_m(s_2, s_1)$;

$(m_{\iota v})$ $d_m(s_1, s_2) - m_{s_1 s_2} \leq [(d_m(s_1, s_3) - m_{s_1 s 3}) + (d_m(s_3, s_2) - m_{s_3 s_2})]$.

The pair (Ω, d_m) is known as an $\widetilde{m}$-metric space. Note that herein we define $m_{s_1 s_2}$ and $M_{s_1 s_2}$ by

$$\begin{cases} m_{s_1 s_2} = \min\{d_m(s_1, s_1), d_m(s_2, s_2)\}, \\ M_{s_1 s_2} = \max\{d_m(s_1, s_1), d_m(s_2, s_2)\}. \end{cases}$$

Remark 1.4. [8] Every partial metric is an $\widetilde{m}$-metric, but the converse may not be hold true. For example, the following setting is an $\widetilde{m}$-metric, but it is not a partial metric. Let $\Omega = \{1, 2, 3\}$ and take $d_m(1, 1) = 1$, $d_m(2, 2) = 9$, $d_m(3, 3) = 5$, $d_m(1, 2) = d_m(2, 1) = 10$, $d_m(1, 3) = d_m(3, 1) = 7$ and $d_m(2, 3) = d_m(3, 2) = 7$.

Now, we discuss some basic concepts: convergence, Cauchyness, completeness and open balls of the $\widetilde{m}$-metric space $(\Omega\ , d_m)$.

A sequence $\{s(\iota)\}$ in $(\Omega\ , d_m)$ is known as $\widetilde{m}$-convergent to $s \in \Omega$ iff

$$\lim_{\iota\to\infty}(d_m(s_\iota, s) - m_{s_\iota s}) = 0$$

and an $\widetilde{m}$-Cauchy sequence if $\lim_{\iota, j\to\infty}\left(d_m(s_\iota, s_j) - m_{s_\iota s_j}\right)$ exists and is finite. (Ω, d_m) is $\widetilde{m}$-complete if every $\widetilde{m}$-Cauchy sequence is $\widetilde{m}$-convergent to an element $s \in \Omega$, and then each $\widetilde{m}$-metric space on Ω generates a T_0 topology denoted by $\tau(\widetilde{m})$. The base of topology $\tau(\widetilde{m})$ on Ω is the class of open b^m-balls:

$$\{B_{d_m}(s_1, r) : s_1 \in \Omega, r > 0\},$$

where

$$B_{d_m}(s_1, r) = \{s_2 \in \Omega : d_m(s_1, s_2) < m_{s_1 s_2} + r\}$$

for all $s_1 \in \Omega$ and $r > 0$.

Lemma 1.5. *[8] Let $\{s(j)\}$ be a sequence in an $\widetilde{m}$-metric space. Let there be $\ell^m \in [0, 1)$ so that $d_m(s_{\iota+1}, s_\iota) \leq \ell_m d^m(s_\iota, s_{\iota-1})$ for every $\iota \geq 1$. Then,*

(L1) $\lim_{n\to\infty} d_m(s_\iota, s_{\iota-1}) = 0$;

(L2) $\lim_{n\to\infty} d_m(s_\iota, s_\iota) = 0$;

(L3) $\lim_{n\to\infty} m_{s_j s_\iota} = 0$;

(L4) s_ι is an $\widetilde{m}$-Cauchy sequence.

Herein, we start by recognizing some basic approaches of a dynamic iterative process in the context of $\widetilde{m}$-metric spaces. Consider a function $\Psi\colon \Omega \to CB(\Omega)$ such that

$$\check{D}(\Psi, s_0) = \{(s_j)_{j\in N\cup\{0\}} \subset \Omega : \left(s_j\right)_{j\in N} \in \Psi\left(s_{j-1}\right)\}$$

for each $j \geq 1$. The term $\check{D}(\Psi, s_0)$ is an M-dynamic iterative process of the mapping Ψ having the starting point s_0. The M-dynamic iterative process. $\check{D}(\Psi, s_0)$ is usually written as s_j, and let $\Psi_1\colon \Omega \to \Omega$ and $\Psi_2\colon \Omega \to CB(\Omega)$ be such that

$$\check{D}(\Psi_1, \Psi_2, s_0) = \{(s_j)_{j\in N\cup\{0\}} : s_{j+1} = \Psi_1\left(s_j\right) \in \Psi_2\left(s_{j-1}\right)\}$$

for every $j \geq 1$. The set $\check{D}(\Psi_1, \Psi_2, s_0)$ is said to be an M-dynamic iterative process of hybrid (Ψ_1, Ψ_2) having the starting point s_0. The M-dynamic iterative process. $\check{D}(\Psi_1, \Psi_2, s_0)$ is usually written as $\Psi_1\left(s_j\right)$. For more details, see [7, 12, 15].

Example 1.6. Let $\Omega = [0, \infty)$. Define $\Psi_1\colon \Omega \to \Omega$ and $\Psi_2\colon \Omega \to CB(\Omega)$ by $\Psi_1(s) = \frac{s}{2}$, $\Psi_2(s) = [0, \frac{s}{2}]$, respectively. A sequence $\{s_i\}$ is given as $s_i = s_0 g^{i-1}$ for all $n \in N$ with $s_0 = 2$ and $g = \frac{1}{2}$. Then, we have Table 1.

Table 1. M-dynamic system, for $i \geq 2$.

$i \geq 2$	$s_i = s_0 g^{i-1}$	$f_1(s_i) = \frac{s}{2}$	$f_2(s_i) = [0, \frac{s}{2}]$
$s_{i=2}$	1	$f_1 s_{i=1} = 1$	$f_1 s_{i=1} = [0, 1]$
$s_{i=3}$	$\frac{1}{2}$	$f_1 s_{i=2} = \frac{1}{2}$	$f_2 s_{i=2} = [0, \frac{1}{2}]$
$s_{i=4}$	$\frac{1}{4}$	$f_1 s_{i=3} = \frac{1}{4}$	$f_2 s_{i=3} = [0, \frac{1}{4}]$
$s_{i=5}$	$\frac{1}{8}$	$f_1 s_{i=4} = \frac{1}{8}$	$f_2 s_{i=4} = [0, \frac{1}{8}]$

In light of the above table, one asserts that

$$\check{D}(\Psi_1, \Psi_2, s_0) = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$$

is an M-dynamic iterative process of hybrid operator Ψ_1 and Ψ_2 starting from $\varepsilon_0 = 2$.

In 2012, Wardowski [26] introduced the concept of F-contractions as follows:

Definition 1.7. [26] Let $K\colon (0, \infty) \longrightarrow (-\infty, +\infty)$ be such that:
(K_ι) K is strictly increasing, for every $s_1, s_2 \in (0, \infty)$ such that $s_1 < s_2$ implies $K(s_1) < K(s_2)$.
$(K_{\iota\iota})$ For each positive sequence $\{s(j)\}$,

$$\lim_{j\longrightarrow\infty} s(j) = 0 \text{ if and only if} \lim_{j\longrightarrow\infty} K(s(j)) = -\infty.$$

$(K_{\iota\iota\iota})$ There is $k \in (0, 1)$ such that $\lim_{c\to 0} c^k K(c) = 0$.

Herein, ∇ is the set of functions verifying (K_ι)–$(K_{\iota\iota\iota})$.

Let (Ω, d) be a metric space. $\Gamma_2\colon \Omega \to \Omega$ is known as a F-contraction if there is $\tau > 0$ such that

$$d(\Gamma_2(s_1), \Gamma_2(s_2)) > 0 \Rightarrow \tau + K(d(\Gamma_2(s_1), \Gamma_2(s_2)) \leq K(d(s_1, s_2))$$

for each $s_1, s_2 \in \Omega$.

Definition 1.8. [7] Consider $\Gamma_1\colon\ \Omega \to \Omega$ and $\Gamma_2\colon\ \Omega \to CB(\Omega)$. The pair (Γ_1, Γ_2) is called weakly compatible if $\Gamma_1(n) = \Gamma_2(n)$ for some $n \in \Omega$, and then $\Gamma_1\Gamma_2(n) = \Gamma_2\Gamma_1(n)$.

In this manuscript, we will generalize the following result with respect to M-dynamic iterative process $\check{D}(\Psi_1, \Psi_2, s_0)$ for hybrid operators: Let (Ω, d_m) be a complete $\widetilde{m}$-metric space and let $\Gamma\colon\ \Omega \to CB(\Omega)$ be a multivalued mapping. Suppose there is $\ell_m \in (0, \frac{1}{2})$ such that

$$d_m(\Gamma(s), \Gamma(v))) \leq \ell_m [d_m(s, \Gamma(s)) + d_m(v, \Gamma(v)]$$

for all $s, v \in \Omega$. Then, f possesses a fixed point [21]. In particular, we will consider Chatterjea type contractions based on M-dynamic iterative process: $\check{D}(\Psi_1, \Psi_2, s_0)$ and its graphical interpretations. The main approach is to find new common fixed points for hybrid operators.

2. Main results

First, in order to give our new generalized definition, we have the following:

Definition 2.1. Let (Ω, d_m) be an $\widetilde{m}$-metric space and let $\Psi_1\colon\ \Omega \to \Omega$. The mapping $\Psi_2\colon\ \Omega \to CB(\Omega)$ is called a set-valued F-Chatterjea type contraction with respect to the M-dynamic iterative process: $\check{D}(\Psi_1, \Psi_2, s_0)$, where $s_0 \in \Omega$, if there are $K \in \nabla$ and $\tau\colon (0, \infty) \to (0, \infty)$ such that

$$\tau(\Lambda(s_{\iota-1}, s_\iota)) + K(d_m(\Psi_1(s_\iota), \Psi_1(s_{\iota+1}))) \leq K(\Lambda(s_{\iota-1}, s_\iota)), \tag{2.1}$$

where

$$\Lambda(s_{\iota-1}, s_\iota) = \ell_m [d_m(\Psi_1(s_{\iota-1}), \Psi_2(s_\iota)) + d_m(\Psi_1(s_\iota), \Psi_2(s_{\iota-1})]$$

for all $s_\iota, s_{\iota+1} \in \check{D}(\Psi_1, \Psi_2, s_0)$, $d_m(\Psi_1(s_\iota), \Psi_1(s_{\iota+1})) > 0$ and $\ell_m \in \left[0, \frac{1}{2}\right)$.

Remark 2.2. Observe that we will only consider the M-dynamic iterative process. $s_\iota \in \check{D}(\Psi_1, \Psi_2, s_0)$ that satisfies the following:

$$d_m(\Psi_1(s_{\iota-1}), \Psi_1(s_\iota)) > 0 \text{ implies that } d_m(\Psi_1(s_{\iota-2}), \Psi_1(s_{\iota-1})) > 0 \tag{2.2}$$

for each integer $\iota \geq 2$. If the process does not satisfy (2.2), then there exists some $\iota_0 \in N$ such that

$$d_m(\Psi_1(s_{\iota_0-1}), \Psi_1(s_{\iota_0})) > 0 \text{ and } d_m(\Psi_1(s_{\iota_0-2}), \Psi_1(s_{\iota_{0-1}})) = 0 \tag{2.3}$$

imply that

$$\Psi_1(s_{\iota_{0-1}}) = \Psi_1(s_{\iota_0}) \in \Psi_2(s_{\iota_{0-1}}),$$

which implies the existence of a common fixed point.

Our main result is given as follows:

Theorem 2.3. *Let (Ω, d_m) be a complete $\widetilde{m}$-metric space and $\Psi_1\colon\ \Omega \to \Omega$. Assume $\Psi_2\colon\ \Omega \to CB(\Omega)$ is a set-valued F-Chatterjea type contraction with respect to the M-dynamic iterative process: $\check{D}(\Psi_1, \Psi_2, s_0)$. Then, the hybrid pair (Ψ_1, Ψ_2) possesses a common fixed point, say, $c \in \Omega$.*

Proof. Let $s_0 \in \Omega$ be an arbitrary element. In the presence of an M-dynamic iterative process, we obtain the following:

$$\check{D}(\Psi_1, \Psi_2, s_0) = \left\{(s_\imath)_{\imath \in N \cup \{0\}} : s_{\imath+1} = \Psi_1 (s_\imath) \in \Psi_2 (s_{\imath-1}), \text{ for each } \imath \in N\right\}.$$

If $s_{\imath_0} = s_{\imath_{0+1}}$ for some $\imath_0 \in N$, the proof is done. Now, let us take for $\imath \in \mathbb{N}$,

$$\zeta (\imath) = d_m(s_{\imath,} \Psi_1 (s_\imath)) = d_m(s_{\imath,} s_{\imath+1}).$$

If we let $s_\imath \neq s_{\imath+1}$ for every $\imath \in \mathbb{N}$, then owing to (2.1),

$$\begin{aligned} K(\zeta(\imath + 1)) &= K(d_m(s_{\imath+1}, s_{\imath+2}) = K [d_m(\Psi_1 (s_\imath), \Psi_1 (s_{\imath+1})] \\ &\leq K [\ell_m (d_m(\Psi_1 (s_{\imath-1}), \Psi_2 (s_\imath))) + d_m(\Psi_1 (s_\imath), \Psi_2 (s_{\imath-1})))] - \tau (\Lambda(s_{\imath-1}, s_\imath)) \\ &\leq K [\ell_m (d_m(\Psi_1 (s_{\imath-1}), \Psi_1 (s_{\imath+1})) + d_m(\Psi_1 (s_\imath), \Psi_1 (s_\imath)))] - \tau (\Lambda(s_{\imath-1}, s_\imath)). \end{aligned} \tag{2.4}$$

Next, we have to examine the following:

$$\zeta(\imath + 1) < \zeta(\imath) \tag{2.5}$$

for all $\imath \in \mathbb{N}$.

Assume, on the contrary, that there exists $\sigma \in \mathbb{N}$ such that $\zeta(\sigma + 1) \geq \zeta(\sigma)$. By (2.4), one writes

$$\begin{aligned} K(\zeta(\sigma + 1)) &= K(d_m(s_{\sigma+1}, s_{\sigma+2}) = K [d_m(\Psi_1 (s_\sigma), \Psi_1 (s_{\sigma+1}))] \\ &\leq K [\ell_m (d_m(\Psi_1 (s_{\sigma-1}), \Psi_2 (s_\sigma))) + d_m(\Psi_1 (s_\sigma), \Psi_2 (s_{\sigma-1})))] - \tau (\Lambda(s_{\sigma-1}, s_\sigma)) \\ &\leq K [\ell_m (d_m(\Psi_1 (s_{\sigma-1}), \Psi_1 (s_{\sigma+1}))) + d_m(\Psi_1 (s_\sigma), \Psi_1 (s_\sigma)))] - \tau (\Lambda(s_{\sigma-1}, s_\sigma)) \\ &\leq K [\ell_m (d_m(\Psi_1 (s_{\sigma-1}), \Psi_1 (s_{\sigma+1}))) + d_m(\Psi_1 (s_\sigma), \Psi_1 (s_\sigma)))] - \tau (\Lambda(s_{\sigma-1}, s_\sigma)) \\ &\leq K[\ell_m (d_m(\Psi_1 (s_{\sigma-1}), \Psi_1 (s_\sigma)) - m_{s_{\sigma-1}, s\sigma} + d_m(\Psi_1 (s_\sigma), \Psi_1 (s_{\sigma+1}))) \\ &\quad -m_{s_\sigma, s_{\sigma+1}} + m_{s_{\sigma-1}, s_{\sigma+1}} + d_m(\Psi_1 (s_\sigma), \Psi_1 (s_\sigma)))] - \tau (\Lambda(s_{\sigma-1}, s_\sigma)) \\ &= K[\ell_m d_m(\Psi_1 (s_{\sigma-1}), \Psi_1 (s_\sigma)) - \ell_m d_m(\Psi_1 (s_{\sigma-1}), \Psi_1 (s_{\sigma-1})) \\ &\quad +\ell_m d_m(\Psi_1 (s_\sigma), \Psi_1 (s_{\sigma+1})) - \ell_m d_m(\Psi_1 (s_\sigma), \Psi_1 (s_\sigma)) \\ &\quad +\ell_m d_m(\Psi_1 (s_{\sigma-1}), \Psi_1 (s_{\sigma-1})) + \ell_m d_m(\Psi_1 (s_\sigma), \Psi_1 (s_\sigma)))] - \tau (\Omega(s_{\sigma-1}, s_\sigma)) \\ &\leq K[\ell_m d_m(\Psi_1 (s_{\sigma-1}), \Psi_1 (s_\sigma)) + \ell_m d_m(\Psi_1 (s_\sigma), \Psi_1 (s_{\sigma+1}))] - \tau\Omega(s_{\sigma-1}, s_\sigma) \\ &\leq K[\ell_m \zeta(\sigma) + \ell_m \zeta(\sigma + 1)]. \end{aligned}$$

Using $(\Psi_\imath)$, one writes

$$\zeta(\sigma + 1) \leq \ell_m \zeta(\sigma) + \ell_m \zeta(\sigma + 1).$$

That is,

$$(1 - \ell_m)\, \zeta(\sigma + 1) \leq \ell_m \zeta(\sigma).$$

This implies that

$$\zeta(\sigma + 1) \leq \frac{\ell_m}{1 - \ell_m} \zeta(\sigma).$$

Take

$$\vartheta_m = \frac{\ell_m}{1 - \ell_m}.$$

Since $\ell_m \in [0, \frac{1}{2})$, we have $0 \leq \vartheta_m < 1$. Therefore, $\zeta(\sigma + 1) < \zeta(\sigma)$, which is a contradiction. Hence, (2.5) holds. Thus, the real sequence

$$\{\zeta(\iota)\} = \{d_m(s_\iota, s_{\iota+1})\}$$

is decreasing, so there is $\Upsilon \geq 0$ so that

$$\Upsilon = \lim_{\iota\to\infty} \zeta(\iota) = \inf\{\zeta(\iota) : \iota \in N\}. \tag{2.6}$$

Next, we have to prove that $\Upsilon = 0$. Suppose, on the contrary, that $\Upsilon > 0$, and for every $\varepsilon > 0$, there is $v \in N$ such that

$$\zeta(v) < \Upsilon + \varepsilon. \tag{2.7}$$

Using (K_ι), we have

$$K(\zeta(v)) < K(\Upsilon + \varepsilon). \tag{2.8}$$

Since K is a strictly increasing-mapping w.r.t. an M-dynamic iterative process, one writes

$$\begin{aligned}
K(\zeta(v+1)) &= K(\zeta(s_{v+1}, s_{v+2}) \\
&= K[d_m(\Psi_1(s_v), \Psi_1(s_{v+1}))] \\
&\leq K[\ell_m(d_m(\Psi_1(s_{v-1}), \Psi_2(s_v))) + d_m(\Psi_1(s_v), \Psi_2(s_{v-1})))] \\
&\quad -\tau[\ell_m(d_m(\Psi_1(s_{v-1}), \Psi_2(s_v))) + d_m(\Psi_1(s_v), \Psi_2(s_{v-1})))] \\
&\leq K[\ell_m(d_m(\Psi_1(s_{v-1}), \Psi_1(s_{v+1}))) + d_m(\Psi_1(s_v), \Psi_1(s_v)))] \\
&\quad -\tau[\ell_m(d_m(\Psi_1(s_{v-1}), \Psi_1(s_{v+1}))) + d_m(\Psi_1(s_v), \Psi_1(s_v)))] \\
&\leq K[\ell_m(d_m(\Psi_1(s_{v-1}), \Psi_1(s_v)) - m_{s_{v-1},sv} + d_m(\Psi_1(s_v), \Psi_1(s_{v+1}))) \\
&\quad -m_{s_v,s_{v+1}} + m_{s_{v-1},s_{v+1}} + d_m(\Psi_1(s_v), \Psi_1(s_v)))] \\
&\quad -\tau[\ell_m(d_m(\Psi_1(s_{v-1}), \Psi_1(s_v)) - m_{s_{v-1},sv} + d_m(\Psi_1(s_v), \Psi_1(s_{v+1}))) \\
&\quad -m_{s_v,s_{v+1}} + m_{s_{v-1},s_{v+1}} + d_m(\Psi_1(s_v), \Psi_1(s_v)))] \\
&= K[\ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_v)) - \ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_{v-1})) \\
&\quad +\ell_m d_m(\Psi_1(s_v), \Psi_1(s_{v+1})) - \ell_m d_m(\Psi_1(s_v), \Psi_1(s_v)) \\
&\quad +\ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_{v-1})) + \ell_m d_m(\Psi_1(s_v), \Psi_1(s_v)))] \\
&\quad -\tau[\ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_v)) - \ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_{v-1})) \\
&\quad +\ell_m d_m(\Psi_1(s_v), \Psi_1(s_{v+1})) - \ell_m d_m(\Psi_1(s_v), \Psi_1(s_v)) \\
&\quad +\ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_{v-1})) + \ell_m d_m(\Psi_1(s_v), \Psi_1(s_v)))] \\
&\leq \Psi[\ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_v)) + \ell_m d_m(\Psi_1(s_v), \Psi_1(s_{v+1})) \\
&\quad -\tau[\ell_m d_m(\Psi_1(s_{v-1}), \Psi_1(s_v)) + \ell_m d_m(\Psi_1(s_v), \Psi_1(s_{v+1}))] \\
&\leq \Psi[\ell_m\zeta(v) + \ell_m\zeta(v+1)] - \tau[\ell_m\zeta(v) + \ell_m\zeta(v+1)] \\
&\leq \Psi[\ell_m\zeta(v) + \ell_m\zeta(v+1)].
\end{aligned}$$

By (Ψ_ι), we have

$$\zeta(v+1) \leq \ell_m\zeta(v) + \ell_m\zeta(v+1).$$

That is,

$$(1-\ell_m)\zeta(v+1) \leq \ell_m \zeta(v). \tag{2.9}$$

Hence,

$$\zeta(v+1) \leq \frac{\ell_m}{1-\ell_m}\zeta(v).$$

Again, $(\Psi_\imath)$ and $0 \leq \vartheta_m = \frac{\ell_m}{1-\ell_m} < 1$. Therefore,

$$K(\zeta(v+1) \leq K(\vartheta_m \zeta(v)) - \frac{1}{1-\ell_m}\tau\left[\ell_m\left(\zeta(v)+\ell_m\zeta(v+1)\right)\right] \tag{2.10}$$

for all $v \in N$.

Next, by given hypothesis on τ there is $\kappa > 0$ and $v \in N$ such that

$$\tau\left[\ell_m\left(\zeta(v)+\ell_m\zeta(v+1)\right)\right] > \kappa$$

for all $v > v_0$. Thus, we obtain for all $v > v_0$ with setting of $\delta = \frac{1}{1-\ell_m}$the following inequalities:

$$\begin{aligned}
K(\zeta(v+\imath)) &\leq K(\vartheta_m\zeta(v+(\imath-1)) - \delta\tau\left[\ell_m\zeta(v+(\imath-1)+\ell_m\zeta(v+\imath)\right] \\
&\leq K((\vartheta_m)^2\zeta(v+(\imath-2)) - \delta\tau\left[\ell_m\zeta(v+(\imath-2)+\ell_m\zeta(v+(\imath-1)\right] \\
&\quad -\delta\tau\left[\ell_m\zeta(v+(\imath-1)+\ell_m\zeta(v+\imath)\right] \\
&\leq K((\vartheta_m)^3\zeta(v+(\imath-3)) - \delta\tau\left[\ell_m\zeta(v+(\imath-3)+\ell_m\zeta(v+(\imath-2)\right] \\
&\quad -\delta\tau\left[\ell_m\zeta(v+(\imath-2)+\ell_m\zeta(v+(\imath-1)\right] \\
&\quad -\delta\tau\left[\ell_m\zeta(v+(\imath-1)+\ell_m\zeta(v+\imath)\right] \\
&= K((\vartheta_m)^3\zeta(v+(\imath-3)) - \delta\tau(\left[\ell_m\zeta(v+(\imath-3)+\ell_m\zeta(v+(\imath-2)\right] \\
&\quad +\left[\ell_m\zeta(v+(\imath-2)+\ell_m\zeta(v+(\imath-1)\right] \\
&\quad +\left[\ell_m\zeta(v+(\imath-1)+\ell_m\zeta(v+\imath)\right]) \\
&\quad \vdots \\
&\leq K((\vartheta_m)^n\zeta(v_0)) - \delta(v-v_0)\tau\left[(\ell_m)^n\zeta(v_0)\right] \\
&< K(\Gamma+\varepsilon) - \delta(v-v_0)\kappa.
\end{aligned} \tag{2.11}$$

Letting $\imath \to \infty$ along with (Ψ_{ii}), we have $\lim_{\imath\to\infty}\Psi(\zeta(v+\imath)) = -\infty$ in such a way that

$$\lim_{\imath\to\infty}\Psi(\zeta(v+\imath)) = 0. \tag{2.12}$$

Then, there exists $\imath_1 \in N$ such that $\zeta(v+\imath) < \Gamma$ for all $\imath > \imath_1$. This contradicts the definition of Γ. Thus,

$$\lim_{\imath\to\infty}\zeta(\imath) = 0 = \Gamma. \tag{2.13}$$

From $(K_{\imath\imath\imath})$, there is $k \in (0,1)$ such that

$$\lim_{\imath\to\infty}\left[\zeta(\imath)\right]^k K\left[\zeta(\imath)\right] = 0.$$

By (2.11), the following holds:

$$\lim_{\imath\to\infty}\left[d_m(\imath)\right]^k\left[\zeta(\imath)\right] - \left[\zeta(\imath)\right]^k K\left[\zeta(\imath_o)\right] \;\leq\; \left[\zeta(\imath)\right]^k \delta(v-v_0)\kappa \leq 0.$$

Taking the limit as $\imath \to \infty$ in (2.14), we have

$$\lim_{\imath\to\infty} \imath \left[\zeta(\imath)\right]^k = 0. \tag{2.14}$$

From (2.14), there is $\imath_1 \in N$ such that $n\left[\zeta(\imath)\right]^k \leq 1$ for all $\imath \geq \imath_1$. We have

$$\zeta(\imath) \leq \frac{1}{\imath^{\frac{1}{k}}}. \tag{2.15}$$

Next is to prove that $\{s_\imath\}$ is an $\widetilde{m}$-Cauchy sequence. For this, we consider $j_1, j_2 \in \mathbb{N}$ such that $j_1 > j_2 \geq \imath_1$. Using the triangular inequality of (m_{iv}) and from (2.15), we have

$$\begin{aligned} d_m\left(s_{j_1}, s_{j_2}\right) - m_{s_{j_1},s_{j_2}} &\leq \left[d_m\left(s_{j_1}, s_{j_{1+1}}\right) - m_{s_{j_1},s_{j_{1+1}}}\right] + ... + \left[d_m\left(s_{j_{2-1}}, s_{j_2}\right) - m_{s_{j_{2-1}},s_{j_2}}\right] \\ &\leq d_m\left(s_{j_1}, s_{j_{1+1}}\right) + d_m\left(s_{j_{1+1}}, s_{j_{1+2}}\right) + ... + d_m\left(s_{j_{2-1}}, s_{j_2}\right) \\ &= \sum_{l=j_1}^{j_2-1} d_m\left(s_l, s_{l+1}\right) \leq \sum_{l=j_1}^{\infty} d_m\left(s_l, s_{l+1}\right) \leq \sum_{l=j_1}^{\infty} \frac{1}{l^{\frac{1}{k}}}. \end{aligned}$$

Due to the convergence of the series $\sum_{l=j_1}^{\infty} \frac{1}{l^{\frac{1}{k}}}$, letting $\imath \to \infty$, we get

$$d_m\left(s_{j_1}, s_{j_2}\right) - m_{s_{j_1},s_{j_2}} \to 0.$$

Hence, $\{s_\imath\}$ is $\widetilde{m}$-Cauchy in (Ω, d_m). Owing to the completeness of Ω, we have $s_\imath \to s$ as $\imath \to \infty$ for some $s \in \Omega$. So, we have

$$d_m\left(s_\imath, s\right) - m_{s_\imath,s} \to 0 \text{ as } \imath \to \infty \tag{2.16}$$

and

$$M_{s_\imath s} - m_{s_\imath s} \to 0 \text{ as } \imath \to \infty. \tag{2.17}$$

Due to (L_2), we obtain $d_m\left(s_\imath, s_\imath\right) \to 0$ as $\imath \to \infty$, so

$$m_{s_\imath s} = \min\left\{d_m\left(s_\imath, s_\imath\right), d_m\left(s, s\right)\right\} \to 0 \text{ as } \imath \to \infty, \tag{2.18}$$

and

$$m_{s_\imath}\Psi_1\left(s\right) = \min\left\{d_m\left(s_\imath, s_\imath\right), d_m\left(\Psi_1(s), \Psi_1(s)\right)\right\} \to 0 \text{ as } \imath \to \infty. \tag{2.19}$$

By (2.16)–(2.18), we obtain

$$d_m\left(s_\imath, s\right) \to 0 \text{ as } \imath \to \infty \tag{2.20}$$

and

$$M_{s_\imath s} \to 0 \text{ as } \imath \to \infty. \tag{2.21}$$

Hence,

$$M_{s_\imath s} + m_{s_\imath s} = d_m\left(s_\imath, s_\imath\right) + d_m\left(s, s\right) \tag{2.22}$$

for all $\imath \in \mathbb{N}$.

Taking the limit as $\imath \to \infty$ in the above equation and using (2.18), (2.21) and (L_2), we have

$$d_m\left(s, s\right) = 0. \tag{2.23}$$

This yields

$$m_{s\Psi_1(s)} = \min\{d_m(s,s), d^m(\Psi_1(s),\Psi_1(s))\} = 0. \tag{2.24}$$

We shall show that $d_m(s,\Psi_1(s)) = 0$. Due to $(m_{\iota v})$, we have

$$\begin{aligned} d_m(s,\Psi_1(s)) &= d_m(s,\Psi_1(s)) - m_{s\Psi_1(s)} \\ &\leq d_m(s,s_\iota) - m_{ss_\iota} + d_m(s_\iota,\Psi_1(s)) - m_{s_\iota\Psi_1(s)} \end{aligned} \tag{2.25}$$

for all $\iota \in \mathbb{N}$. Taking the superior limit as $\iota \to \infty$ in (2.25) and using (2.16)–(2.18) and (2.24), we have

$$\begin{aligned} d_m(s,\Psi_1(s)) &\leq \limsup_{\iota\to\infty}[d_m(s,s_\iota) - m_{ss_\iota} + d_m(s_\iota,\Psi_1(s)) - m_{s_\iota\Psi_1(s)}] \\ &\leq \limsup_{\iota\to\infty}[d_m(s,s_\iota) - m_{ss_\iota} + d_m(s_\iota,\Psi_1(s))] \\ &\leq \limsup_{\iota\to\infty}[d_m(s,s_\iota) - m_{ss_\iota}] + \limsup_{\iota\to\infty}[d_m(s_\iota,\Psi_1(s))] \\ &= \limsup_{\iota\to\infty}[d_m(s_\iota,\Psi_1(s))] \\ &\leq \limsup_{\iota\to\infty}[\ell_m(d_m(s_{\iota-1},\Psi_1(s))) + d_m(\Psi_1(s_{\iota-1}),s)] \\ &\leq \ell_m\limsup_{\iota\to\infty}[(d_m(s_{\iota-1},\Psi_1(s))) + \limsup_{\iota\to\infty} d_m(s_\iota,s)] \\ &= \ell_m[\limsup_{\iota\to\infty}(d_m(s_{\iota-1},\Psi_1(s)))] \\ &\leq \ell_m\limsup_{\iota\to\infty}[d_m s_{\iota-1},s) - m_{s_{\iota-1}s}] \\ &\quad +\ell_m[d_m(s,\Psi_1(s)) - m_{s\Psi_1(s)} + m_{s_{\iota-1}\Psi_1(s)}] \\ &\leq \ell_m[d_m(s,\Psi_1(s))]. \end{aligned}$$

We get that

$$d_m(s,\Psi_1(s)) = 0. \tag{2.26}$$

By (2.1), we have

$$d_m(\Psi_1(s),\Psi_1(s)) \leq 2\ell_m d_m(s,\Psi_1(s)) = 0. \tag{2.27}$$

That is,

$$d_m(\Psi_1(s),\Psi_1(s)) = 0.$$

From (2.18), (2.26) and (2.27), we have

$$d_m(s,s) = d_m(\Psi_1(s),\Psi_1(s)) = d^m(s,\Psi_1(s)).$$

Owing to (m_ι), we get

$$s = \Psi_1(s),$$

so we have $s = \Psi_1(s) \in \Psi_2(s)$. Next, let s^* be another fixed point of Ψ_1 such that

$$\begin{aligned} d_m(s,s^*) &= d_m(\Psi_1(s),\Psi_1(s^*)) \leq \ell_m[d_m(s,\Psi_1(s^*)) + d_m(s^*,\Psi_1(s))] \\ &= \ell_m[d_m(s,s^*) + d_m(s^*,s)] \\ &\leq 2\ell_m[d_m(s,s^*)] \\ &< d_m(s,s^*), \end{aligned}$$

which is a contradiction. Hence, the hybrid pair (Ψ_1,Ψ_2) has a common fixed point. □

Further, some corollaries are developed as consequences of the obtained theorems of our endeavor.

Corollary 2.4. *Let* (Ω, d_m) *be a complete* $\widetilde{m}$*-metric space and* $\Psi_1\colon \Omega \to \Omega$. *Assume* $\Psi_2\colon \Omega \to CB(\Omega)$ *is a set-valued* F*-Kannan type contraction, that is,*

$$\tau(\Lambda(s_{\imath-1}, s_\imath)) + K(d_m(\Psi_1(s_\imath), \Psi_1(s_{\imath+1}))) \leq K(\Lambda(s_{\imath-1}, s_\imath)),$$

where

$$d(s_{\imath-1}, s_\imath) = \ell_m [d_m(\Psi_1(s_{\imath-1}), \Psi_2(s_{\imath-1})) + d_m(\Psi_1(s_\imath), \Psi_2(s_\imath)]$$

with respect to M*-dynamic iterative process.* $\check{D}(\Psi_1, \Psi_2, s_0)$, $d_m(\Psi_1(s_\imath), \Psi_1(s_{\imath+1})) > 0$ *and* $\ell_m \in \left[0, \frac{1}{2}\right)$. *Then, the hybrid pair* (Ψ_1, Ψ_2) *possesses a common fixed point.*

Corollary 2.5. *Let* (Ω, d_m) *be a complete* $\widetilde{m}$*-metric space and* $\Psi_1\colon \Omega \to \Omega$. *Assume* $\Psi_2\colon \Omega \to CB(\Omega)$ *is a set-valued* F*-Banach contraction, and*

$$\tau(d_m(s_{\imath-1}, s_\imath)) + K(d_m(\Psi_1(s_\imath), \Psi_1(s_{\imath+1}))) \leq K(d_m(s_{\imath-1}, s_\imath))$$

with respect to M*-dynamic iterative process* $\check{D}(\Psi_1, \Psi_2, s_0)$, $d_m(\Psi_1(s_\imath), \Psi_1(s_{\imath+1})) > 0$ *and* $\ell_m \in (0, 1)$. *Then, the pair* (Ψ_1, Ψ_2) *possesses a common fixed point.*

Corollary 2.6. *Let* (Ω, d_m) *be a complete* $\widetilde{m}$*-metric space and* $\Psi_1\colon \Omega \to \Omega$. *Assume* $\Psi_2\colon \Omega \to CB(\Omega)$ *is a set-valued* F*-Banach contractive. If there is* $\Delta\colon \kappa \to \kappa$, *a non-negative Lebesgue integrable operator which is summable on each compact subset of* κ *such that*

$$d_m(\Psi_1 s_\imath, \Psi_1 s_{\imath+1}) > 0 \;\Rightarrow\; \tau(d_m(s_{\imath-1}, s_\imath)) + K\Big(\int_0^{d_m(\Psi_1 s_\imath, \Psi_1 s_{\imath+1})} \Delta(s)\,\delta s\Big) \leq K\Big(\int_0^{d_m(s_{\imath-1}, s_\imath)} \Delta(s)\,\delta s\Big)$$

for all $s_\imath \in \check{D}(\Psi_1, \Psi_2, s_0)$ *and for all given* $\epsilon > 0$ *so that* $\int_0^\epsilon \Delta(s)\,\delta s > 0$, *then the pair* (Ψ_1, Ψ_2) *possesses a common fixed point.*

Theorem 2.7. *Let* (Ω, d_m) *be an* $\widetilde{m}$*-metric space and let* $\Psi_1\colon \Omega \to \Omega$ *be continuous. The mapping* $\Psi_2\colon \Omega \to CB(\Omega)$ *is called a set-valued* F*-Chatterjea type contraction with respect to the* M*-dynamic iterative process.* $\check{D}(\Psi_1, \Psi_2, s_0)$, *where* $s_0 \in \Omega$, *if* $K \in \nabla$, Ψ *is right continuous at* $\mathcal{H}(\Psi_2(s_1), \Psi_2(s_2))$, *and* $\tau\colon (0, \infty) \to (0, \infty)$ *such that*

$$2\tau(\Lambda(s_1, s_2)) + K(\mathcal{H}(\Psi_2(s_1), \Psi_2(s_2))) \leq K(\Lambda(s_1, s_2)), \tag{2.28}$$

where

$$\Lambda(s_1, s_2) = \ell_m [d_m(\Psi_1(s_1), \Psi_2(s_2)) + d_m(\Psi_1(s_2), \Psi_2(s_1)]$$

for all $s_1, s_2 \in \check{D}(\Psi_1, \Psi_2, s_0)$, $\mathcal{H}(\Psi_2(s_1), \Psi_2(s_2)) > 0$ *and* $\ell_m \in \left[0, \frac{1}{2}\right)$. *Then, the hybrid pair* (Ψ_1, Ψ_2) *has a common fixed point, say,* $c \in \Omega$.

Proof. Choose $s_0 \in \Omega$ as an arbitrary point. Based on the M-dynamic iterative process, we have $\Psi_1(s_1) \in \Psi_2(s_0)$. Assume that $\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1)) > 0$. Since, by hypothesis, Ψ is right continuous at $\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1))$, there is $h > 1$ such that

$$K[h(\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1)))] < K(\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1))) + \tau(\Lambda(s_0, s_1)).$$

Since $\Psi_1(s_1) \in \Psi_2(s_0)$, we derive $d_m(\Psi_1(s_1), \Psi_2(s_1)) \leq \mathcal{H}(\Psi_2(s_0), \Psi_2(s_1))$, and thus there is $s^* \in \Psi_2(s_1)$ such that

$$d_m(\Psi_1(s_1), s^*) < h\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1)). \tag{2.29}$$

Setting an element $s_2 \in \Omega$ such that $\Psi_1(s_2) = s^*$, (2.29) becomes

$$d_m(\Psi_1(s_1), \Psi_1(s_2)) < h\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1)). \tag{2.30}$$

In the case that

$$d_m(\Psi_1(s_1), \Psi_1(s_2)) = 0,$$

$\Psi_1(s_1) \in \Psi_2(s_1)$. Hence, the proof is complete. Assume that

$$d_m(\Psi_1(s_1), \Psi_1(s_2)) > 0.$$

Since Ψ is strictly increasing, we have

$$\begin{aligned} K[d_m(\Psi_1(s_1), \Psi_1(s_2))] &< K[h(\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1)))] \\ &< K(\mathcal{H}(\Psi_2(s_0), \Psi_2(s_1))) + \tau(\Lambda(s_0, s_1)). \end{aligned} \tag{2.31}$$

Owing to (2.31) and applying Theorem (2.3), we easily derive that the pair (Ψ_1, Ψ_2) has a common fixed point. □

Corollary 2.8. *Let (Ω, d_m) be a complete $\widetilde{m}$-metric space and $\Psi_1\colon \Omega \to \Omega$. Assume $\Psi_2\colon \Omega \to CB(\Omega)$ is a set-valued F-Kannan type contraction, that is,*

$$2\tau(\Omega(s_1, s_2)) + K(\mathcal{H}(\Psi_2(s_1), \Psi_2(s_2))) \leq K(\Lambda(s_1, s_2)),$$

where

$$\Lambda(s_1, s_2) = \ell_m [d_m(\Psi_1(s_1), \Psi_2(s_2)) + d_m(\Psi_1(s2), \Psi_2(s_1)]$$

with respect to M-dynamic iterative process. $\check{D}(\Psi_1, \Psi_2, s_0)$, $\mathcal{H}(\Psi_2(s_1), \Psi_2(s_2)) > 0$ and $\ell_m \in \left[0, \frac{1}{2}\right)$. Then, the hybrid pair (Ψ_1, Ψ_2) possesses a common fixed point.

Corollary 2.9. *Let (Ω, d_m) be a complete $\widetilde{m}$-metric space and $\Psi_1\colon \Omega \to \Omega$. Assume $\Psi_2\colon \Omega \to CB(\Omega)$ is a set-valued F-Banach contraction, and*

$$2\tau(d_m(s_1, s_2)) + K(\mathcal{H}(\Psi_2(s_1), \Psi_2(s_2))) \leq K(d_m(s_1, s_2))$$

with respect to M-dynamic iterative process $\check{D}(\Psi_1, \Psi_2, s_0)$, $\mathcal{H}(\Psi_2(s_1), \Psi_2(s_2)) > 0$ and $\ell_m \in (0, 1)$. Then, the pair (Ψ_1, Ψ_2) possesses a common fixed point.

Corollary 2.10. *Let (Ω, d_m) be a complete $\widetilde{m}$-metric space and $\Psi_1\colon \Omega \to \Omega$. Assume $\Psi_2\colon \Omega \to CB(\Omega)$ is a set-valued F-Banach contractive. If there is $\Delta\colon \kappa \to \kappa$ a non-negative Lebesgue integrable operator which is summable on each compact subset of κ such that*

$$H(\Psi_2 s_\iota, \Psi_2 s_{\iota+1}) > 0 \Rightarrow 2\tau(d_m(s_{\iota-1}, s_\iota)) + K(\int_0^{H(\Psi_2 s_\iota, \Psi_2 s_{\iota+1})} \Delta(s)\,\delta s) \leq K(\int_0^{d_m(s_{\iota-1}, s_\iota)} \Delta(s)\,\delta s)$$

for all $s_\iota \in \check{D}(\Psi_1, \Psi_2, s_0)$ and for all given $\epsilon > 0$ so that $\int_0^\epsilon \Delta(s)\,\delta s > 0$, then the pair (Ψ_1, Ψ_2) possesses a common fixed point.

Corollary 2.11. *Let (Ω, d_m) be a complete $\widetilde{m}$-metric space, and $\Psi_1\colon \Omega \to \Omega$ is continuous. Assume $\Psi_2\colon \Omega \to CB(\Omega)$ is a set-valued F-Banach contractive. If there is $\Delta\colon \kappa \to \kappa$ a non-negative Lebesgue integrable operator which is summable on each compact subset of κ such that*

$$H(\Psi_2 s_\iota, \Psi_2 s_{\iota+1}) > 0 \;\Rightarrow\; 2\tau\left(d_m\left(s_{\iota-1}, s_\iota\right)\right) + K\left(\int_0^{H(\Psi_2 s_\iota, \Psi_2 s_{\iota+1})} \Delta\left(s\right)\delta s\right) \leq K\left(\int_0^{d_m(s_{\iota-1}, s_\iota)} \Delta\left(s\right)\delta s\right)$$

for all $s_\iota \in \check{D}(\Psi_1, \Psi_2, s_0)$ and for all given $\epsilon > 0$ so that $\int_0^{\epsilon} \Delta\left(s\right)\delta s > 0$, then the pair (Ψ_1, Ψ_2) possesses a common fixed point.

Next, we present an example to show the benefits of our endeavor.

Example 2.12. Let $\Omega = [0, \infty)$, and let $d_m : \Omega \times \Omega \to [0, \infty)$ be defined by

$$d_m\left(s_1, s_2\right) = |s_1 - s_2| + \frac{s_1 + s_2}{2} \text{ for all } s_1, s_2 \in \Omega.$$

Clearly, (d_m, Ω) is an $\widetilde{m}$-complete metric space. The mappings $\Psi_1\colon \Omega \to \Omega$, $\Psi_2\colon \Omega \to CB\left(\Omega\right)$, $\Psi\colon (0, \infty) \to (-\infty, +\infty)$ and $\tau\colon (0, \infty) \to (0, \infty)$ are given as

$$\Psi_1\left(s\right) = \frac{s^2}{4},\; \Psi_2\left(s\right) = [\frac{s^2}{4}, 0],\; K(s) = \ln(s)$$

and

$$\tau(\alpha) = \alpha \ln(\frac{101}{100})$$

for $\alpha \in (0, \infty)$.

A sequence $\{s_n\}$ can be defined by $s_i = s_0 g^{i-1}$ for all $i \in N$ with $s_0 = 2$ and $g = \frac{1}{2}$. Then,

$$\check{D}(\Psi_1, \Psi_2, 2) = \left\{1, \frac{1}{4}, \frac{1}{16}, ...\right\}$$

is an M-dynamic iterative process of a hybrid pair of mappings (Ψ_1, Ψ_2), starting at the point $s_0 = 2$. Next, we choose $s_1 = 1$, $s_2 = \frac{1}{4}$ and $\ell_m = \frac{1}{3}$. Take the M-Hausdorff induced by an $\widetilde{m}$-metric under an M-dynamic iterative process, defined by

$$\mathcal{H}_m\left[\Psi_2\left(s_{\iota-1}\right), \Psi_2\left(s_\iota\right)\right] = \left[\frac{\left(s_{\iota-1} + s_\iota\right)}{4} d_m(s_{\iota-1}, s_\iota)\right],$$

such that

$$2\tau\left(\frac{11}{24}\right) + \ln\left(\frac{11}{240}\right) \leq \ln\left(\frac{11}{24}\right).$$

Equivalently, one writes

$$\tau\left(\frac{11}{24}\right) + \ln\left(\frac{11}{240}\right) \leq \ln\left(\frac{11}{24}\right).$$

Owing to F-contraction, (2.29) is written as $\tau\left(\alpha\right) \leq \Xi\left(\iota\right)$, where

$$\Xi\left(\iota\right) = K\left[\ell_m\left(d_m(s_{\iota-1}, s_\iota)\right)\right] - K\left[\mathcal{H}_m\left(\Psi_2\left(s_{\iota-1}\right), \Psi_2\left(s_\iota\right)\right)\right].$$

Hence, by Table 2 and Figure 1, the required hypotheses of Corollary (2.8), regarding $\tau(\alpha) \leq \Xi(\iota)$ are satisfied. Here, $0 = \Psi_1(0) \in \Psi_2(0)$ is a common fixed point of Ψ_1 and Ψ_2.

Table 2. Corresponding values of $\tau(\alpha)$ & $\Xi(\iota)$.

s_ι	$s_{\iota-1}$	$\tau(\alpha)$	$\Xi(\iota)$
1	0.25	0.004975165	0.064539
0.0625	0.0156	0.009950331	2.837127243
0.0039	0.00097	0.014925497	5.609715966
0.00024	$6.103e^{-5}$	0.019900662	8.382304725
$1.52e^{-5}$	$3.814e^{-6}$	0.024875827	11.15489427
$9.54e^{-7}$	$2.38e^{-7}$	0.029850992	13.92748213
$5.96e^{-8}$	$1.49e^{-8}$	0.034826158	16.70007085
$3.73e^{-9}$	$9.31e^{-10}$	0.039801323	19.47265958
$2.33e^{-10}$	$5.82e^{-11}$	0.044776488	22.24524831
$1.46e^{-11}$	$3.64e^{-12}$	0.049751654	25.01783703

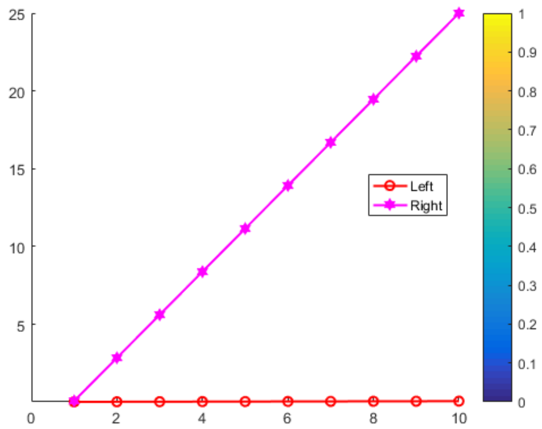

Figure 1. $\tau(\alpha) \leq \Xi(\iota)$.

3. Applications

Herein, this section deals with applications of our obtained results. Two important applications are given below:

3.1. An application to multistage systems

A study of a decision space and a state space makes up two fundamental parts of M-dynamic iterative programming problems. A state space is a family of states regarding initial states, action states and transitional states. So, a state space is the collection of parameters that show different type states. A decision space is the set of possible actions that can be taken to solve the problem. These natures allow us to formulate many problems in mathematical optimization and computer programming. In particular, the problem of M-dynamic iterative programming problems related to multistage process reduces to the problem of solving functional equations.

$$\xi(r_1) = \sup_{\wp_2 \in H} \{h(\wp_{1,}\wp_2) + D(\wp_1, \wp_2, \xi(l(\wp_{1,}\wp_2)))\} \tag{3.1}$$

for $\wp_1 \in s$,

$$\xi^{'}(\wp_1) = \sup_{\wp_2 \in H} \left\{h^{'}(\wp_{1,}\wp_2) + D^{'}\left(\wp_1, \wp_2, \xi^{'}(l(\wp_{1,}\wp_2))\right)\right\} \tag{3.2}$$

for $\wp_1 \in s$, where

$$D, D^{'} : S \times H \times R \to R,$$
$$h, h^{'} : S \times H \to R,$$
$$l : S \times H \to S.$$

Suppose X_1 and X_2 are Banach spaces. $S \subset X_1$ is a state space, and $H \subset X_2$ is a decision space. For more, see [4]. Let $\sigma \in B(S)$ be an collection of all bounded real valued functions on S, and defined by $\|\sigma\| = \sup_{\wp \in S} |\sigma(\wp)|$. $(B(s), \|.\|)$ which endowed with $\widetilde{m}$-metric, we have

$$d_m(\sigma, \vartheta) = \sup_{\wp \in s} \left|\frac{\sigma(\wp) + \vartheta(\wp)}{2}\right| \tag{3.3}$$

for all $\sigma, \vartheta \in B(S)$. Define X_1, X_2: $B(S) \to B(S)$ by

$$X_1(\varpi)(\wp) = \sup_{t \in H} \{\Psi_1(\wp, t) + D_1(\wp, t, \varpi(l(\wp, t)))\}, \tag{3.4}$$

$$X_2(\varpi)(\wp) = \sup_{t \in H} \{\Psi_2(\wp, t) + D_2(\wp, t, \varpi(l(\wp, t)))\} \tag{3.5}$$

for all $\varpi \in B(S)$ and $\wp \in S$. In addition, assume that τ: $(0, \infty) \to (0, \infty)$ such that

$$|D_1(\wp, t, \sigma(\wp)) + D_1(\wp, t, \vartheta(\wp))| \leq e^{-\tau(\wp)}\mathcal{H}(\sigma(\wp), \vartheta(\wp)) \tag{3.6}$$

for all $\sigma, \vartheta \in B(S)$, where $\wp \in S$ and $t \in D$.

Theorem 3.1. *Let semi-continuous mappings X_1, X_2: $B(s) \to B(s)$ as defined in (3.4) and (3.5) such that:*

($\imath$) D_1, D_2, Ψ_1 *and* Ψ_2 *are continuous and bounded.*

($\imath\imath$) *For all $\varpi \in B(S)$ $\exists$ $t \in B(S)$ so that*

$$X_1(\varpi)(\wp) = X_2(s)(\wp).$$

($\imath\imath\imath$) *Assume there is $\varpi \in B(s)$ such that (X_1 and X_2 are called weakly compatible),*

$$X_1(\varpi)(\wp) = X_2(\varpi)(\wp) \Rightarrow X_1X_2(\varpi)(\wp) = X_2X_1(\varpi)(\wp).$$

Then, the functional equations defined by (3.1) and (3.2) possess a bounded solution.

Proof. Let $(B(S), d_m)$ be a complete $\widetilde{m}$-metric space, where $d_m(\sigma, \vartheta)$ is the $\widetilde{m}$-metric, as defined by (3.3). Consider an arbitrary $\kappa > 0$ and ϖ_1, $\varpi_2 \in B(S)$, so there are $\wp \in s$ and $t_1, t_2 \in D$ such that

$$X_1(\varpi_1) < \Psi_1(\wp, t_1) + D_1(\wp, t_1, \varpi_1(l(\wp, t_1))) + \kappa, \tag{3.7}$$

$$X_1(\varpi_2) < \Psi_1(\wp, t_2) + D_2(\wp, t_2, \varpi_2(l(\wp, t_2))) + \kappa, \tag{3.8}$$

$$X_1(\varpi_1) \geq \Psi_1(\wp, t_2) + D_1(\wp, t_2, \varpi_1(l(\wp, t_2))), \tag{3.9}$$

$$X_1(\varpi_2) \geq \Psi_1(\wp, t_1) + D_2(\wp, t_1, \varpi_2(l(\wp, t_1))). \tag{3.10}$$

Using (3.7) and (3.10), one writes

$$\begin{aligned} d_m(X_1(\varpi_1)(\wp) + X_1(\varpi_2)(\wp))) &= \left|\frac{X_1(\varpi_1)(\wp) + X_1(\varpi_2)(\wp)}{2}\right| \\ &= \left|\frac{D_1(\wp, t_1, \varpi_1(l(\wp, t_1))) + D_2(\wp, t_1, \varpi_2(l(\wp, t_1)))}{2}\right| \\ &\leq e^{-\tau(\wp)}(\mathcal{H}(\sigma(\wp), \vartheta(\wp))), \end{aligned}$$

which implies that

$$X_1(\varpi_1)(\wp) + X_1(\varpi_2)(\wp) \leq 2e^{-\tau(\wp)}(\mathcal{H}(\sigma(\wp), \vartheta(\wp))). \tag{3.11}$$

In the same manner, by (3.8) and (3.9),

$$X_1(\varpi_2)(\wp) + X_1(\varpi_1)(\wp) \leq 2e^{-\tau(\wp)}(\mathcal{H}(\sigma(\wp), \vartheta(\wp))).$$

This implies

$$d_m(X_1(\varpi_1)(\wp), X_1(\varpi_2)(\wp)) \leq e^{-\tau(\wp)}(\mathcal{H}(\sigma(\wp), \vartheta(\wp))) \tag{3.12}$$

for every $\wp \in s$. It follows that

$$\ln[d_m(X_1(\varpi_1)(\wp), X_1(\varpi_2)(\wp))] \leq \ln[e^{-\tau(\wp)}(\mathcal{H}(\sigma(\wp), \vartheta(\wp)))].$$

This leads to

$$\tau + \ln[d_m(X_1(\varpi_1)(\wp), X_1(\varpi_2)(\wp))] \leq \ln[(\hat{H}(\sigma(\wp), \vartheta(\wp)))].$$

As a result, all conditions of Corollary (2.8) are fulfilled. Hence, a fixed point $\varpi^* \in B(s)$, that is, ϖ^* is a bounded solution of (3.4) and (3.5). □

3.2. An application to integral equations

In this section, we consider the following integral equation:

$$\varpi_1(\wp) = \sigma_1(\wp) + \int_a^{\wp} D_1(\wp, t, \varpi_1(\wp^*))\, d\wp^*, \tag{3.13}$$

$$\varpi_2(\wp) = \sigma_2(\wp) + \int_a^{\wp} D_2(\wp, \wp^*, \varpi_2(\wp^*))\, d\wp^*, \tag{3.14}$$

$\wp \in [\eta_1, \eta_2]$. Herein, D_ι: $[\eta_1, \eta_2]^2 \times R \to P^{cv}(R)$ (set of nonempty compact and convex subsets of R), $\iota = 1, 2$, the operator $D_\iota(\wp, \wp^*, \varpi_\iota(\wp^*))$ is lower semicontinuous for all $\varpi \in C([\eta_1, \eta_2], \mathbb{R})$, and σ_ι:

$[\eta_1, \eta_2] \to R$ is continuous, $\iota = 1, 2$. Let $X = C([\eta_1, \eta_2], R)$ be endowed with the $\widetilde{m}$-metric defined by

$$d_m(\vartheta_1, \vartheta_2) = \sup_{\wp \in s} \left| \frac{\sigma(\wp) + \vartheta(\wp)}{2} \right|. \tag{3.15}$$

Define a set-valued operator Ψ: $C([\eta_1, \eta_2], R) \to CL(C([\eta_1, \eta_2], R))$ by

$$\Psi(\varpi_\iota)(\wp^*) = \left\{ \varkappa_1 \in C([\eta_1, \eta_2], R) : \varkappa_1 \in \sigma_\iota(\wp) + \int_a^\wp D_\iota(\wp, \wp^*, \varpi_\iota(\wp^*))\, d\wp^* \right\},$$

where $\wp \in [\eta_1, \eta_2]$, $\iota = 1, 2$. Let $\varpi_\iota \in C([\eta_1, \eta_2], R)$ and

$$D_\iota = d_{\iota m}(\wp, \wp^*, \varpi_\iota(\wp^*)), \ \wp, \wp^* \in [\eta_1, \eta_2].$$

Consider, for D_ι: $[\eta_1, \eta_2]^2 \to P^{cv}(R)$, by Michael's selection theorem, there is a continuous function $d_{\varpi\iota m}$: $[\eta_1, \eta_2]^2 \to R$ defined as follows: $d_{\varpi\iota m}(\wp, \wp^*) \in d_{\iota m}(\wp, \wp^*)$, $\wp, \wp^* \in [\eta_1, \eta_2]$ and

$$\sigma_\iota(\wp) + \int_a^\wp d_\iota(\wp, \wp^*, \varpi_\iota(\wp^*))\, d\wp^* \in \Psi(\varpi_\iota)(\wp^*). \tag{3.16}$$

Precisely, the operator $\Psi(\varpi_\iota) \neq \emptyset$ is closed.

Theorem 3.2. *Let*

$$X = C([\eta_1, \eta_2], R)$$

and the set-valued operator Ψ, T*:* $C([\eta_1, \eta_2], R) \to CL(C([\eta_1, \eta_2], R))$ *as given by*

$$\Psi(\varpi_\iota)(\wp^*) = \left\{ \varkappa_1 \in C([\eta_1, \eta_2], R) : \varkappa_1 \in \sigma_\iota(\wp) + \int_a^\wp D_\iota(\wp, \wp^*, \varpi_\iota(\wp^*))\, d\wp^*, \ \wp \in [\eta_1, \eta_2] \right\},$$

$\iota = 1, 2$*, be such that the following hold:*
(*i*) *There exists a continuous and bounded function* $\Omega : X \to R^+ \cup \{0\}$ *such that*

$$|D_1(\wp, \wp^*, \sigma(\wp)) + D_1(\wp, \wp^*, \vartheta(\wp))| \leq \Omega(\wp^*)\, e^{-\tau(\wp)} \mathcal{H}(\varpi_1(\wp^*), \varpi_2(\wp^*)),$$

for every $\wp, \wp^* \in [\eta_1, \eta_2]$*,* $\varpi_1, \varpi_2 \in X$ *and*

$$\frac{1}{2}\left(\int_a^\wp \Omega(\wp^*)\, e^{-\tau(\wp^*)} d\wp^* \right) \leq e^{-\tau(\wp^*)},$$

for each $\wp^* \in [\eta_1, \eta_2]$.
(*ii*) *There exists*

$$\varkappa_1 \in C([\eta_1, \eta_2], R),$$

such that

$$\Psi(\varkappa_1)(\wp) = T(\varkappa_1)(\wp)$$

implies that

$$\Psi T(\varkappa_1)(\wp) = T\Psi(\varkappa_1)(\wp).$$

Then, the above Eqs (3.13) and (3.14) possess a bounded solution.

Proof. Let $\varpi_1, \varpi_2 \in X$ be such that $\varkappa_1 \in \Psi(\varpi_1)$. It follows that

$$d_{\varpi_1 m}(\wp, \wp^*) \in d_{\varpi_1 m}(\wp, \wp^*),\ \wp, \wp^* \in [\eta_1, \eta_2],$$

such that

$$\varkappa_1(\wp) = \sigma_1(\wp) + \int_a^{\wp} d_{\varpi_1}(\wp, \wp^*)\, d\wp^*,\ \wp \in [\eta_1, \eta_2].$$

By(ι), there exists $\varpi_2(\wp, \wp^*) \in d_{\varpi_2 m}(\wp, \wp^*)$ such that

$$\left|\frac{d_{\varpi_1}(\wp, \wp^*) + \varpi_2(\wp, \wp^*)}{2}\right| \leq \Upsilon(\wp^*)\, e^{-\tau(\wp^*)} \left|\frac{\varpi_1(\wp^*) + \varpi_2(\wp^*)}{2}\right|,\ \wp, \wp^* \in [\eta_1, \eta_2].$$

Consider

$$T(\wp, \wp^*) = D_{\varpi_2}(\wp, \wp^*) \cap \left\{\omega \in R : \left|\frac{d_{\varpi_1}(\wp, \wp^*) + \omega}{2}\right| \leq \Upsilon(\wp^*)\, e^{-\tau(\wp^*)} \frac{\varpi_1(\wp^*) + \varpi_2(\wp^*)}{2}\right\},$$

$\wp, \wp^* \in [\eta_1, \eta_2]$. Since the operator is lower semi-continuous, there exists ϖ_2: $[\eta_1, \eta_2]^2 \times R \to R$, such that $d_{\varpi_2 m}(\wp, \wp^*) \in T(\wp, \wp^*)$, for all $\wp, \wp^* \in [\eta_1, \eta_2]$. Thus,

$$\varkappa_2(\wp) = \sigma_2(\wp) + \int_a^{\wp} d\varpi_2(\wp, \wp^*)\, d\wp^* \in \sigma_2(\wp) + \int_a^{\wp} D_2(\wp, \wp^*, \varpi_2(\wp^*))\, d\wp^*$$

for all $\wp \in [\eta_1, \eta_2]$.

Now, we have

$$\begin{aligned}
d_m(\varpi_1(\wp), \varpi_2(\wp)) &= \left|\frac{\varkappa_1(\wp) + \varkappa_2(\wp)}{2}\right| \\
&= \left|\int_a^{\wp} \frac{(d_{\varpi_1}(\wp, \wp^*) + d_{\varpi_2}(\wp, \wp^*))}{2} d\wp^*\right| \\
&\leq \frac{1}{2}\int_a^{\wp} \left|d_{\varpi_1}(\wp, \wp^*) + d_{\varpi_2}(\wp, \wp^*))\right| d\wp^* \\
&\leq \frac{1}{2}\int_a^{\wp} \Upsilon(\wp^*)\, e^{-\tau(\wp^*)} |\varpi_1(\wp^*) + \varpi_2(\wp^*)|\, d\wp^* \\
&\leq \frac{1}{2}|\varpi_1(\wp^*) + \varpi_2(\wp^*)| \left(\int_a^{\wp} \Upsilon(\wp^*)\, e^{-\tau(\wp^*)} d\wp^*\right) \\
&\leq \mathcal{H}(\varpi_1(\wp^*), \varpi_2(\wp^*)) \left(\int_a^{\wp} \Upsilon(\wp^*)\, e^{-\tau(\wp^*)} d\wp^*\right) \\
&\leq \mathcal{H}(\varpi_1(\wp^*), \varpi_2(\wp^*)) e^{-\tau(\wp^*)}.
\end{aligned}$$

Hence, we have

$$d_m(\varpi_1(\wp), \varpi_2(\wp)) \leq e^{-\tau(\wp^*)} d_m(\varpi_1(\wp), \varpi_2(\wp)).$$

Thus,

$$d_m(\Psi\varpi_1(\wp), \Psi\varpi_2(\wp)) \leq e^{-\tau(\wp^*)} d_m(\varpi_1(\wp), \varpi_2(\wp)).$$

This yields, with $l_m = 1$ and $\Psi(\ell) = \ln(\ell)$,

$$\tau + \ln[d_m(\Psi\varpi_1(\wp), \Psi\varpi_2(\wp))] \leq \ln[d_m(\varpi_1(\wp), \varpi_2(\wp))].$$

As a result, all conditions of Corollary (2.8) are fulfilled, so the system of integral equations has a bounded solution. □

4. Conclusions

In this manuscript, we have identified fixed points of a hybrid operators using the tool of M-dynamic processes. A contraction of the F-Chatterjea type is examined within the category of M-metric spaces, accompanied by an illustrative example. The obtained results are illustrated with graphical structures, and some corollaries are acquired consequently. At last, we illustrate our results by some applications by solving multistage systems that are primarily useful in dynamic systems. Along with that, we ensure the existence of a solution of integral equations. In the future, these extended results can be furthered to acquire fixed point theorems for Reich and Hardy Rogers type F-contractions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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