AIMS Mathematics
http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20561–20575.
DOI: 10.3934/math.20231048
Received: 27 April 2023
Revised: 06 June 2023
Accepted: 09 June 2023
Published: 26 June 2023

Research article

On a two-dimensional nonlinear system of difference equations close to the bilinear system

Stevo Stević[1,2,*] **and Durhasan Turgut Tollu**[3]

[1] Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, Beograd 11000, Serbia
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[3] Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computer Sciences, Konya, Turkey

* **Correspondence:** Email: sscite1@gmail.com, sstevic@ptt.rs.

Abstract: We consider the two-dimensional nonlinear system of difference equations

$$x_n = x_{n-k}\frac{ay_{n-l} + by_{n-(k+l)}}{cy_{n-l} + dy_{n-(k+l)}}, \quad y_n = y_{n-k}\frac{\alpha x_{n-l} + \beta x_{n-(k+l)}}{\gamma x_{n-l} + \delta x_{n-(k+l)}},$$

for $n \in \mathbb{N}_0$, where the delays k and l are two natural numbers, and the initial values x_{-j}, y_{-j}, $1 \le j \le k+l$, and the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$ are real numbers. We show that the system of difference equations is solvable by presenting a method for finding its general solution in detail. Bearing in mind that the system of equations is a natural generalization of the corresponding one-dimensional difference equation, whose special cases appear in the literature from time to time, our main result presented here also generalizes many results therein.

Keywords: nonlinear system of difference equations; solvable system; closed-form formula
Mathematics Subject Classification: 39A20

1. Introduction

1.1. Some notation

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of natural, integer, real and complex numbers, respectively, whereas by $\mathbb{N}_k$, where k is a fixed whole number, we denote the set $\{n \in \mathbb{Z} : n \ge k\}$. If $s, t \in \mathbb{Z}$ are such that $s \le t$, then we use the notation $r = \overline{s,t}$ instead of writing the phrase $s \le r \le t$ for $r \in \mathbb{Z}$. For any sequence of numbers $(s_k)_{k\in I}$, where I is a subset of $\mathbb{Z}$, we use the standard convention

$$\prod_{k=m}^{m-1} s_k = 1, \quad m \in I.$$

1.2. A bit on the history of difference equations used here

Finding closed-form formulas for difference equations and systems of difference equations is a very old topic. Some of the basic methods and ideas for solving the equations and systems belong, among others, to de Moivre, Daniel Bernoulli, Euler, Lagrange and Laplace (see, e.g., [5,8,9,15,17,18] and the related references therein), and some of them can be found in many later books on difference equations and systems of difference equations, as well as in many books on some related topics, such as calculus of finite differences, numerical mathematics, combinatorics, economics, etc. (see, e.g., [6, 10, 16, 19, 20, 29, 30]).

Solvable difference equations and systems have re-attracted some recent attention within the mathematical community, and the topic has been growing gradually and constantly (see, e.g., [11, 12, 28, 39–49, 51] and the related references therein). One of the reasons for the renewed interest is a recent increasing use of computers, especially the use of some symbolic algebra packages, which help in various kinds of calculations. This is a reasonably good way for starting an investigation in the topic, but it is far from being sufficient for some thorough investigations of solvability of difference equations and systems of difference equations (see, e.g., our detailed analyses and many comments on various kinds of problems given in [11, 41–45, 47] and the related references therein).

The solvability of difference equations and systems of difference equations is a topic of interest since many of the equations and systems are not solvable, practically or theoretically. This is why some authors try to find, at least, their invariants, which can also be useful in investigating and describing the long-term behaviour of solutions to the equations and systems of equations (see, e.g., [22, 24, 27, 31, 32]). Various methods for studying the long-term behaviour of solutions to the equations and systems can be found, e.g., in [1, 4, 7, 12–14, 21–28, 33, 35–37, 39, 40, 48, 49].

The bilinear difference equation

$$x_n = \frac{ax_{n-1} + b}{cx_{n-1} + d}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $a, b, c, d, x_{-1} \in \mathbb{R}$ (or $\mathbb{C}$), and

$$c^2 + d^2 \neq 0, \tag{1.2}$$

is a classical nonlinear equation which can be solved in a closed form. To avoid the trivial difference equation

$$x_n = 0, \quad n \in \mathbb{N},$$

we may also assume that

$$a^2 + b^2 \neq 0. \tag{1.3}$$

The literature shows that Laplace essentially knew how to solve Eq (1.1). What is certainly known is that the solvability of the equation was known to the mathematicians of the late eighteenth century (see, e.g., [18, 41]). The equation can be solved by using several different methods (see, e.g., [1, 7, 16, 19, 41]). An investigation on the periodicity of the equation can be found in [6]. Long-term behaviour of solutions to the equation was investigated, e.g., in [1, 7]. For some later results on the equation, as well as the corresponding systems of difference equations of two or more variables and their applications, see, e.g., [41, 44, 45, 47]. Many interesting historical facts and detailed theoretical explanations can be found in [41].

Aside from the natural restrictions posed in (1.2) and (1.3) there are some other cases which are much easier than the general one. Namely, if $ad = bc$, then Eq (1.1) reduces to the trivial equation

$$x_n = \frac{a}{c} \quad \text{or} \quad x_n = \frac{b}{d}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

for all $x_{-1} \in \mathbb{R}$, except for $x_{-1} = -d/c$, when the corresponding solution is not defined. Note that if the condition $cd \neq 0$ holds, then both equations in (1.4) are the same.

If $c = 0$, then from the condition in (1.2), it follows that $d \neq 0$, and Eq (1.1) reduces to the nonhomogeneous linear difference equation with constant coefficients of the first order

$$x_n = \frac{a}{d}x_{n-1} + \frac{b}{d}, \quad n \in \mathbb{N}_0,$$

which was solved for the first time by Lagrange in [17] (see also [41]). Hence, this case is also less interesting than the general. Recall that the first-order linear difference equation with nonconstant coefficients was solved by Lagrange and Laplace in several ways [15, 18, 41].

1.3. Our motivations for the present study

Special cases of the nonlinear difference equation

$$x_n = x_{n-k}\frac{ax_{n-l} + bx_{n-(k+l)}}{cx_{n-l} + dx_{n-(k+l)}}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $k, l \in \mathbb{N}$, $a, b, c, d \in \mathbb{R}$, and the conditions posed in (1.2) and (1.3) hold, with the real initial values x_{-i}, $i = \overline{1, k+l}$, appear in the literature from time to time (see, e.g., [2, 50]).

It is not difficult to see that Eq (1.5) is closely related to Eq (1.1). Namely, for each solution $(x_n)_{n\geq -(k+l)}$ to Eq (1.5) such that

$$x_n \neq 0, \quad \text{for} \quad n \geq -(k+l),$$

by using the change of variables

$$z_n = \frac{x_n}{x_{n-k}}, \quad n \in \mathbb{N}_{-l}, \tag{1.6}$$

Eq (1.5) is transformed to the equation

$$z_n = \frac{az_{n-l} + b}{cz_{n-l} + d}, \quad n \in \mathbb{N}_0. \tag{1.7}$$

Employing this change of variables, S. Stević solved the problem in [50].

Now note that Eq (1.7) is a difference equation with interlacing indices. For the notion and some detailed explanations related to it see, e.g., [44]. Therefore, the sequences

$$z_m^{(j)} := z_{ml+j}, \quad m \in \mathbb{N}_{-1},\ j = \overline{0, l-1},$$

are l solutions to Eq (1.1). On the other hand, since Eq (1.6), as a simple product-type difference equation, is also solvable, we have that the solvability of Eq (1.5) is a direct consequence of the solvability of Eq (1.1) and Eq (1.6).

A majority of the papers on solvability of difference equations and systems of difference equations employ a similar idea. Namely, behind the solvability of a difference equation or a system of difference equations is usually found one or several changes of variables which transform it into a well known solvable difference equation or system, which was also the case in [11, 12, 28, 39–47, 49, 51]. The basic problem is how to find such changes of variables, since they could be quite complex. Some methods, ideas and tricks for finding the changes of variables can be found in the abovementioned papers on solvability of difference equations and systems of difference equations.

During the last fifteen years, more than a dozen papers have appeared dealing with very special cases of Eq (1.5), which have been usually written in the following form:

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k} x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where k and l are concrete natural numbers, the parameters α, β, γ, δ are concrete real numbers such that $\beta^2 + \gamma^2 \neq 0$, and the initial values x_{-i}, $i = \overline{1, k+l}$, are real numbers.

Although [11] showed the solvability of Eq (1.8) and consequently the simplicity of investigating the special cases of the equation (by giving the abovementioned explanation on the connection between Eqs (1.1) and (1.5)), papers which consider some special cases of Eq (1.8) continued to appear. Moreover, there are some serious problems with many results in these papers; see, e.g., the conducted analyses and given comments in a recent paper [45]. [49] presented a pretty detailed analysis of the solvability of Eq (1.8), which was essentially based on the above described method.

During the second half of the nineties, Papaschinopulos and Schinas started studying concrete symmetric and close-to-symmetric systems of difference equations (see, e.g., [21–26, 31, 32]), which motivated us to study the solvability of such systems, the long-term behaviour of their solutions, and their generalizations, such as the cyclic ones (see, e.g., [12, 33, 38–42, 44, 46–49, 51] and the related references therein).

1.4. Our aim in the present investigation

Motivated by the abovementioned investigations on solvability, and symmetric and close-to-symmetric systems of difference equations, here we consider the following two-dimensional system of difference equations:

$$x_n = x_{n-k}\frac{a y_{n-l} + b y_{n-(k+l)}}{c y_{n-l} + d y_{n-(k+l)}}, \quad y_n = y_{n-k}\frac{\alpha x_{n-l} + \beta x_{n-(k+l)}}{\gamma x_{n-l} + \delta x_{n-(k+l)}}, \tag{1.9}$$

for $n \in \mathbb{N}_0$, where $k, l \in \mathbb{N}$, whereas the initial values x_{-j}, y_{-j}, $j = \overline{1, k+l}$, and the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$ are real numbers.

Our aim is to show that the system is solvable. To do this we use and modify the methods in, e.g., [1,11,41,47,49]. Bearing in mind that system (1.9) is a natural generalization of the corresponding one-dimensional difference equation, i.e., Eq (1.5), whose special cases from time to time appear in the literature, our main result also generalizes many results therein.

2. Main result

In this section, we present a detailed analysis of the solvability of system (1.9), which will lead to our main result. At the end of the section, we will give a concrete example.

2.1. Solvability of system (1.9)

First, we identify and remove the cases that prevent solutions from being well-defined.

If there is $n_0 \in \mathbb{N}_{-k}$ such that $x_{n_0} = 0$, then from the first equation in (1.9), we have $x_{n_0+k} = 0$, and consequently, y_{n_0+k+l} is not defined. If there is $n_1 \in \mathbb{N}_{-k}$ such that $y_{n_1} = 0$, then from the second equation in (1.9), we have $y_{n_1+k} = 0$, and consequently, x_{n_1+k+l} is not defined. Hence, we may assume that $x_n \neq 0 \neq y_n$ for every $n \in \mathbb{N}_{-k}$. We may also assume that

$$x_n \neq 0 \neq y_n, \quad \text{for} \quad n \in \mathbb{N}_{-(k+l)}, \tag{2.1}$$

since otherwise we can consider system (1.9) with indices shifted for l.

So, if a solution $(x_n, y_n)_{n\geq -(k+l)}$ to system (1.9) satisfies condition (2.1), then we can use the change of variables

$$z_n = \frac{x_n}{x_{n-k}}, \quad w_n = \frac{y_n}{y_{n-k}}, \quad n \in \mathbb{N}_{-l}, \tag{2.2}$$

in (1.9) and obtain the system

$$z_n = \frac{aw_{n-l} + b}{cw_{n-l} + d}, \quad w_n = \frac{\alpha z_{n-l} + \beta}{\gamma z_{n-l} + \delta}, \tag{2.3}$$

for $n \in \mathbb{N}_0$.

Since system (2.3) is with interlacing indices, we have that the two-dimensional sequences

$$z_m^{(j)} := z_{ml+j}, \quad w_m^{(j)} := w_{ml+j}, \quad m \in \mathbb{N}_{-1}, \; j = \overline{0, l-1}, \tag{2.4}$$

are l solutions to the bilinear system of difference equations

$$u_m = \frac{av_{m-1} + b}{cv_{m-1} + d}, \quad v_m = \frac{\alpha u_{m-1} + \beta}{\gamma u_{m-1} + \delta}, \tag{2.5}$$

for $m \in \mathbb{N}_0$.

In other words, the sequences $(z_m^{(j)})_{m\in\mathbb{N}_{-1}}$ and $(w_m^{(j)})_{m\in\mathbb{N}_{-1}}$, $j = \overline{0, l-1}$, can be obtained from the general solution of system (2.5). Here, we present a method for solving system (2.5).

To do this, we must consider some restrictions on the system parameters. If $c = 0$, then the first equation of (2.5) can be used in its second one, yielding an equation with interlacing indices that stems from a bilinear one, so it is solvable. The case $\gamma = 0$ is dual. So, we will assume that $c \neq 0 \neq \gamma$.

Assume that $c \neq 0 \neq \gamma$. Then, in this case, we can rewrite system (2.5) in the following form:

$$\gamma u_m + \delta = \frac{a\gamma + c\delta}{c} + \frac{\frac{\gamma}{c}(bc - ad)}{cv_{m-1} + d}, \quad cv_m + d = \frac{\alpha c + d\gamma}{\gamma} + \frac{\frac{c}{\gamma}(\beta\gamma - \alpha\delta)}{\gamma u_{m-1} + \delta}, \tag{2.6}$$

for $m \in \mathbb{N}_0$.

In order not to deal with not well defined solutions to system (2.5), we also assume that

$$\gamma u_m + \delta \neq 0 \quad \text{and} \quad cv_m + d \neq 0$$

for all $m \in \mathbb{N}_{-1}$.

Now, we decompose system (2.6) with respect to the terms with odd and even indices as follows:

$$\gamma u_{2m} + \delta = \frac{a\gamma + c\delta}{c} + \frac{\frac{\gamma}{c}(bc - ad)}{cv_{2m-1} + d}, \tag{2.7}$$

$$cv_{2m} + d = \frac{\alpha c + d\gamma}{\gamma} + \frac{\frac{c}{\gamma}(\beta\gamma - \alpha\delta)}{\gamma u_{2m-1} + \delta}, \tag{2.8}$$

$$\gamma u_{2m+1} + \delta = \frac{a\gamma + c\delta}{c} + \frac{\frac{\gamma}{c}(bc - ad)}{cv_{2m} + d}, \tag{2.9}$$

$$cv_{2m+1} + d = \frac{\alpha c + d\gamma}{\gamma} + \frac{\frac{c}{\gamma}(\beta\gamma - \alpha\delta)}{\gamma u_{2m} + \delta}, \tag{2.10}$$

for $m \in \mathbb{N}_0$.

Multiplying the equations in (2.7)–(2.10) by the products

$$r_m = \prod_{k=0}^{m-1} (\gamma u_{2k} + \delta) \prod_{k=1}^{m} (cv_{2k-1} + d), \tag{2.11}$$

$$s_m = \prod_{k=1}^{m} (\gamma u_{2k-1} + \delta) \prod_{k=0}^{m-1} (cv_{2k} + d), \tag{2.12}$$

$$p_m = \prod_{k=1}^{m} (\gamma u_{2k-1} + \delta) \prod_{k=0}^{m} (cv_{2k} + d), \tag{2.13}$$

$$q_m = \prod_{k=0}^{m} (\gamma u_{2k} + \delta) \prod_{k=1}^{m} (cv_{2k-1} + d), \tag{2.14}$$

respectively, we obtain

$$q_m = \frac{a\gamma + c\delta}{c} r_m + \frac{\gamma}{c}(bc - ad)q_{m-1}, \tag{2.15}$$

$$p_m = \frac{\alpha c + d\gamma}{\gamma} s_m + \frac{c}{\gamma}(\beta\gamma - \alpha\delta)p_{m-1}, \tag{2.16}$$

$$s_{m+1} = \frac{a\gamma + c\delta}{c} p_m + \frac{\gamma}{c}(bc - ad)s_m, \tag{2.17}$$

$$r_{m+1} = \frac{\alpha c + d\gamma}{\gamma} q_m + \frac{c}{\gamma}(\beta\gamma - \alpha\delta)r_m. \tag{2.18}$$

From (2.15) and (2.18), we easily obtain

$$r_{m+1} - (a\alpha + b\gamma + c\beta + d\delta)\, r_m + (bc - ad)(\beta\gamma - \alpha\delta)r_{m-1} = 0, \tag{2.19}$$

$$q_{m+1} - (a\alpha + b\gamma + c\beta + d\delta)\, q_m + (bc - ad)(\beta\gamma - \alpha\delta)q_{m-1} = 0. \tag{2.20}$$

Similarly, from (2.16) and (2.17), it follows that

$$s_{m+1} - (a\alpha + b\gamma + c\beta + d\delta)\, s_m + (bc - ad)(\beta\gamma - \alpha\delta)s_{m-1} = 0, \tag{2.21}$$

$$p_{m+1} - (a\alpha + b\gamma + c\beta + d\delta)\, p_m + (bc - ad)(\beta\gamma - \alpha\delta)p_{m-1} = 0, \tag{2.22}$$

respectively.

Note that from (2.11)–(2.14), we have

$$r_0 = 1, \tag{2.23}$$
$$r_1 = (\gamma u_0 + \delta)(cv_1 + d), \tag{2.24}$$
$$s_0 = 1, \tag{2.25}$$
$$s_1 = (\gamma u_1 + \delta)(cv_0 + d), \tag{2.26}$$
$$p_0 = cv_0 + d, \tag{2.27}$$
$$p_1 = (\gamma u_1 + \delta)(cv_0 + d)(cv_2 + d), \tag{2.28}$$
$$q_0 = \gamma u_0 + \delta, \tag{2.29}$$
$$q_1 = (\gamma u_0 + \delta)(\gamma u_2 + \delta)(cv_1 + d). \tag{2.30}$$

The characteristic equation associated with each of the linear equations in (2.19)–(2.22) is

$$p_2(\lambda) := \lambda^2 - (a\alpha + b\gamma + c\beta + d\delta)\, \lambda + (bc - ad)(\beta\gamma - \alpha\delta). \tag{2.31}$$

If the roots of p_2 are λ_1 and λ_2, then we can express the general solutions of (2.19)–(2.22) as follows:

$$r_m = \frac{(r_1 - \lambda_2 r_0)\lambda_1^m - (r_1 - \lambda_1 r_0)\lambda_2^m}{\lambda_1 - \lambda_2}, \tag{2.32}$$

$$q_m = \frac{(q_1 - \lambda_2 q_0)\lambda_1^m - (q_1 - \lambda_1 q_0)\lambda_2^m}{\lambda_1 - \lambda_2}, \tag{2.33}$$

$$s_m = \frac{(s_1 - \lambda_2 s_0)\lambda_1^m - (s_1 - \lambda_1 s_0)\lambda_2^m}{\lambda_1 - \lambda_2}, \tag{2.34}$$

$$p_m = \frac{(p_1 - \lambda_2 p_0)\lambda_1^m - (p_1 - \lambda_1 p_0)\lambda_2^m}{\lambda_1 - \lambda_2}, \tag{2.35}$$

if $\lambda_1 \neq \lambda_2$, and

$$r_m = (r_0\lambda_1 + (r_1 - \lambda_1 r_0)\, m)\, \lambda_1^{m-1}, \tag{2.36}$$

$$q_m = (q_0\lambda_1 + (q_1 - \lambda_1 q_0)\, m)\, \lambda_1^{m-1}, \tag{2.37}$$

$$s_m = (s_0\lambda_1 + (s_1 - \lambda_1 s_0)\, m)\, \lambda_1^{m-1}, \tag{2.38}$$

$$p_m = (p_0\lambda_1 + (p_1 - \lambda_1 p_0)\, m)\, \lambda_1^{m-1}, \tag{2.39}$$

if

$$\lambda_1 = \lambda_2 = \frac{a\alpha + b\gamma + c\beta + d\delta}{2}.$$

Now, it should be noticed that there are some relations between the products in (2.11) and (2.14) that give some representations for the sequences u_{2m} and v_{2m+1}, as well as some relations between the products in (2.12) and (2.13) that give some representations for the sequences v_{2m} and u_{2m+1}. Indeed, by employing some elementary calculations, it is easy to see that the following relations hold:

$$\gamma u_{2m} + \delta = \frac{q_m}{r_m}, \tag{2.40}$$

$$\gamma u_{2m+1} + \delta = \frac{s_{m+1}}{p_m}, \tag{2.41}$$

$$c v_{2m} + d = \frac{p_m}{s_m}, \tag{2.42}$$

$$c v_{2m+1} + d = \frac{r_{m+1}}{q_m}, \tag{2.43}$$

for $m \in \mathbb{N}_0$.

By using the closed-form formulas (2.32)–(2.35) in relations (2.40)–(2.43), as well as the values of p_j, q_j, r_j and s_j, $j=0,1$, given in (2.23)–(2.30), it is easy to see that the following formulas hold:

$$\gamma u_{2m}+\delta=(\gamma u_0+\delta)\frac{((\gamma u_2+\delta)(cv_1+d)-\lambda_2)\lambda_1^m-((\gamma u_2+\delta)(cv_1+d)-\lambda_1)\lambda_2^m}{((\gamma u_0+\delta)(cv_1+d)-\lambda_2)\lambda_1^m-((\gamma u_0+\delta)(cv_1+d)-\lambda_1)\lambda_2^m}, \tag{2.44}$$

$$\gamma u_{2m+1}+\delta=\frac{1}{cv_0+d}\frac{((\gamma u_1+\delta)(cv_0+d)-\lambda_2)\lambda_1^{m+1}-((\gamma u_1+\delta)(cv_0+d)-\lambda_1)\lambda_2^{m+1}}{((\gamma u_1+\delta)(cv_2+d)-\lambda_2)\lambda_1^m-((\gamma u_1+\delta)(cv_2+d)-\lambda_1)\lambda_2^m}, \tag{2.45}$$

$$cv_{2m}+d=(cv_0+d)\frac{((\gamma u_1+\delta)(cv_2+d)-\lambda_2)\lambda_1^m-((\gamma u_1+\delta)(cv_2+d)-\lambda_1)\lambda_2^m}{((\gamma u_1+\delta)(cv_0+d)-\lambda_2)\lambda_1^m-((\gamma u_1+\delta)(cv_0+d)-\lambda_1)\lambda_2^m}, \tag{2.46}$$

$$cv_{2m+1}+d=\frac{1}{\gamma u_0+\delta}\frac{((\gamma u_0+\delta)(cv_1+d)-\lambda_2)\lambda_1^{m+1}-((\gamma u_0+\delta)(cv_1+d)-\lambda_1)\lambda_2^{m+1}}{((\gamma u_2+\delta)(cv_1+d)-\lambda_2)\lambda_1^m-((\gamma u_2+\delta)(cv_1+d)-\lambda_1)\lambda_2^m}, \tag{2.47}$$

if $\lambda_1\neq\lambda_2$.

Similarly, employing the closed-form formulas (2.36)–(2.39) in relations (2.40)–(2.43), as well as the values of p_j, q_j, r_j and s_j, $j=0,1$, given in (2.23)–(2.30), we obtain the formulas

$$\gamma u_{2m}+\delta=(\gamma u_0+\delta)\frac{\lambda_1+((\gamma u_2+\delta)(cv_1+d)-\lambda_1)m}{\lambda_1+((\gamma u_0+\delta)(cv_1+d)-\lambda_1)m}, \tag{2.48}$$

$$\gamma u_{2m+1}+\delta=\frac{\lambda_1}{cv_0+d}\frac{\lambda_1+((\gamma u_1+\delta)(cv_0+d)-\lambda_1)(m+1)}{\lambda_1+((\gamma u_1+\delta)(cv_2+d)-\lambda_1)m}, \tag{2.49}$$

$$cv_{2m}+d=(cv_0+d)\frac{\lambda_1+((\gamma u_1+\delta)(cv_2+d)-\lambda_1)m}{\lambda_1+((\gamma u_1+\delta)(cv_0+d)-\lambda_1)m}, \tag{2.50}$$

$$cv_{2m+1}+d=\frac{\lambda_1}{\gamma u_0+\delta}\frac{\lambda_1+((\gamma u_0+\delta)(cv_1+d)-\lambda_1)(m+1)}{\lambda_1+((\gamma u_2+\delta)(cv_1+d)-\lambda_1)m}, \tag{2.51}$$

if $\lambda_1=\lambda_2$.

The general solution to system (2.5) is obtained from the closed-form formulas in (2.44)–(2.47), when $\lambda_1\neq\lambda_2$, and from the closed-form formulas (2.48)–(2.51), when $\lambda_1=\lambda_2$.

Employing the relations in (2.4), the closed-form formulas to the solution of system (2.3) are

$$z_{2m}^{(j)}=-\frac{\delta}{\gamma}+\frac{(\gamma z_0^{(j)}+\delta)(((\gamma z_2^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_2)\lambda_1^m-((\gamma z_2^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_1)\lambda_2^m)}{\gamma(((\gamma z_0^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_2)\lambda_1^m-((\gamma z_0^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_1)\lambda_2^m)}, \tag{2.52}$$

$$z_{2m+1}^{(j)}=-\frac{\delta}{\gamma}+\frac{((\gamma z_1^{(j)}+\delta)(cw_0^{(j)}+d)-\lambda_2)\lambda_1^{m+1}-((\gamma z_1^{(j)}+\delta)(cw_0^{(j)}+d)-\lambda_1)\lambda_2^{m+1}}{\gamma(cw_0^{(j)}+d)(((\gamma z_1^{(j)}+\delta)(cw_2^{(j)}+d)-\lambda_2)\lambda_1^m-((\gamma z_1^{(j)}+\delta)(cw_2^{(j)}+d)-\lambda_1)\lambda_2^m)}, \tag{2.53}$$

$$w_{2m}^{(j)}=-\frac{d}{c}+\frac{(cw_0^{(j)}+d)(((\gamma z_1^{(j)}+\delta)(cw_2^{(j)}+d)-\lambda_2)\lambda_1^m-((\gamma z_1^{(j)}+\delta)(cw_2^{(j)}+d)-\lambda_1)\lambda_2^m)}{c(((\gamma z_1^{(j)}+\delta)(cw_0^{(j)}+d)-\lambda_2)\lambda_1^m-((\gamma z_1^{(j)}+\delta)(cw_0^{(j)}+d)-\lambda_1)\lambda_2^m)}, \tag{2.54}$$

$$w_{2m+1}^{(j)}=-\frac{d}{c}+\frac{((\gamma z_0^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_2)\lambda_1^{m+1}-((\gamma z_0^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_1)\lambda_2^{m+1}}{c(\gamma z_0^{(j)}+\delta)(((\gamma z_2^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_2)\lambda_1^m-((\gamma z_2^{(j)}+\delta)(cw_1^{(j)}+d)-\lambda_1)\lambda_2^m)}, \tag{2.55}$$

if $\lambda_1 \neq \lambda_2$, and

$$z^{(j)}_{2m} = -\frac{\delta}{\gamma} + \frac{(\gamma z^{(j)}_0 + \delta)(\lambda_1 + ((\gamma z^{(j)}_2 + \delta)(cw^{(j)}_1 + d) - \lambda_1)m)}{\gamma(\lambda_1 + ((\gamma z^{(j)}_0 + \delta)(cw^{(j)}_1 + d) - \lambda_1)m)}, \tag{2.56}$$

$$z^{(j)}_{2m+1} = -\frac{\delta}{\gamma} + \frac{\lambda_1(\lambda_1 + ((\gamma z^{(j)}_1 + \delta)(cw^{(j)}_0 + d) - \lambda_1)(m+1))}{\gamma(cw^{(j)}_0 + d)(\lambda_1 + ((\gamma z^{(j)}_1 + \delta)(cw^{(j)}_2 + d) - \lambda_1)m)}, \tag{2.57}$$

$$w^{(j)}_{2m} = -\frac{d}{c} + \frac{(cw^{(j)}_0 + d)(\lambda_1 + ((\gamma z^{(j)}_1 + \delta)(cw^{(j)}_2 + d) - \lambda_1)m)}{c(\lambda_1 + ((\gamma z^{(j)}_1 + \delta)(cw^{(j)}_0 + d) - \lambda_1)m)}, \tag{2.58}$$

$$w^{(j)}_{2m+1} = -\frac{d}{c} + \frac{\lambda_1(\lambda_1 + ((\gamma z^{(j)}_0 + \delta)(cw^{(j)}_1 + d) - \lambda_1)(m+1))}{c(\gamma z^{(j)}_0 + \delta)(\lambda_1 + ((\gamma z^{(j)}_2 + \delta)(cw^{(j)}_1 + d) - \lambda_1)m)}, \tag{2.59}$$

for $j = \overline{0, l-1}$, if $\lambda_1 = \lambda_2$.

Using (2.2), we have

$$x_n = z_n x_{n-k}, \quad y_n = w_n y_{n-k}, \quad n \in \mathbb{N}_{-l}, \tag{2.60}$$

from which it follows that

$$x_{km_1-i} = z_{km_1-i} x_{k(m_1-1)-i}, \quad y_{km_1-i} = w_{km_1-i} y_{k(m_1-1)-i}, \tag{2.61}$$

for $i = \overline{l-k+1, l}$ and $m_1 \in \mathbb{N}_0$.

The equations in (2.61) are independent of each other, and their solutions are given by

$$x_{km_1-i} = x_{-k-i} \prod_{s=0}^{m_1} z_{ks-i}, \tag{2.62}$$

$$y_{km_1-i} = y_{-k-i} \prod_{s=0}^{m_1} w_{ks-i}, \tag{2.63}$$

for $i = \overline{l-k+1, l}$ and $m_1 \in \mathbb{N}_0$.

The closed-form formulas for solutions to system (1.9) are obtained by employing (2.52)–(2.55) in (2.62) and (2.63) when $\lambda_1 \neq \lambda_2$ and (2.56)–(2.59) in (2.62) and (2.63) when $\lambda_1 = \lambda_2$.

From the above conducted detailed analysis, we see that system (1.9) is solvable in a theoretical way.

Remark 1. When

$$ad = bc \quad \text{or} \quad \alpha\delta = \beta\gamma,$$

from (2.6), we obtain several simple cases.

Namely, if $ad = bc$, then we obtain

$$u_m = \frac{a}{c}, \quad v_m = \frac{a\alpha + c\beta}{a\gamma + c\delta}, \tag{2.64}$$

for $m \in \mathbb{N}$, if $c \neq 0$, or

$$u_m = \frac{b}{d}, \quad v_m = \frac{b\alpha + d\beta}{b\gamma + d\delta}, \tag{2.65}$$

for $m \in \mathbb{N}$, if $d \neq 0$. Meanwhile, if $\beta\gamma = \alpha\delta$, then we obtain

$$u_m = \frac{a\alpha + b\gamma}{c\alpha + d\gamma}, \quad v_m = \frac{\alpha}{\gamma}, \tag{2.66}$$

for $m \in \mathbb{N}$, if $\gamma \neq 0$, or

$$u_m = \frac{a\beta + b\delta}{c\beta + d\delta}, \quad v_m = \frac{\beta}{\delta}, \tag{2.67}$$

for $m \in \mathbb{N}$, if $\delta \neq 0$.

Since in these cases $(z_m^{(j)})_{m\geq -1}$ and $(w_m^{(j)})_{m\geq -1}$, $j = \overline{0, l-1}$, are l eventually constant sequences, the formulas for the general solution to system (1.9) can be easily obtained by using formulas (2.62) and (2.63).

Since we now have a theoretical method for solving the system of difference equations in (1.9), it is a natural question if there are some concrete cases of delays k and l for which it is possible to present the general solutions of the corresponding systems in the form of several concrete closed-form formulas. Note that to do this, we should apply formulas (2.52)–(2.59) in (2.62) and (2.63). In the example which follows, we present one such case.

Example 1. Let $k = 2$ and $l = 1$. Then, we have the following system of difference equations:

$$x_n = x_{n-2}\frac{ay_{n-1} + by_{n-3}}{cy_{n-1} + dy_{n-3}}, \quad y_n = y_{n-2}\frac{\alpha x_{n-1} + \beta x_{n-3}}{\gamma x_{n-1} + \delta x_{n-3}}, \tag{2.68}$$

for $n \in \mathbb{N}_0$, where the initial values x_{-j}, y_{-j}, $j = \overline{1, 3}$, and the parameters a, b, c, d, $\alpha, \beta, \gamma, \delta$ are real numbers.

We have that in (2.62) and (2.63) $i = 0, 1$, and in (2.4) $j = 0$. Accordingly, using (2.52)–(2.59), we get the closed-form formulas for solutions to system (2.68) and obtain the following theorem.

Theorem 2.1. *Let* $k = 2$, $l = 1$, $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$, $(a^2 + b^2)(\alpha^2 + \beta^2) \neq 0$, $c\gamma \neq 0$, *and* λ_1 *and* λ_2 *be the roots of polynomial* (2.31). *Then, system* (1.9) *is solvable in closed form, and its general solution is given by the formulas*

$$\begin{aligned}
x_{2m} =& x_{-2}\prod_{s=0}^{m}\left(-\frac{\delta}{\gamma} + \frac{(\gamma\frac{x_0}{x_{-2}} + \delta)(((\gamma\frac{x_2}{x_0} + \delta)(c\frac{y_1}{y_{-1}} + d) - \lambda_2)\lambda_1^s - ((\gamma\frac{x_2}{x_0} + \delta)(c\frac{y_1}{y_{-1}} + d) - \lambda_1)\lambda_2^s)}{\gamma(((\gamma\frac{x_0}{x_{-2}} + \delta)(c\frac{y_1}{y_{-1}} + d) - \lambda_2)\lambda_1^s - ((\gamma\frac{x_0}{x_{-2}} + \delta)(c\frac{y_1}{y_{-1}} + d) - \lambda_1)\lambda_2^s)}\right),\\
x_{2m+1} =& x_{-1}\prod_{s=0}^{m}\left(-\frac{\delta}{\gamma} + \frac{((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_0}{y_{-2}} + d) - \lambda_2)\lambda_1^{s+1} - ((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_0}{y_{-2}} + d) - \lambda_1)\lambda_2^{s+1}}{\gamma(c\frac{y_0}{y_{-2}} + d)(((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_2}{y_0} + d) - \lambda_2)\lambda_1^s - ((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_2}{y_0} + d) - \lambda_1)\lambda_2^s)}\right),\\
y_{2m} =& y_{-2}\prod_{s=0}^{m}\left(-\frac{d}{c} + \frac{(c\frac{y_0}{y_{-2}} + d)(((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_2}{y_0} + d) - \lambda_2)\lambda_1^s - ((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_2}{y_0} + d) - \lambda_1)\lambda_2^s)}{c(((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_0}{y_{-2}} + d) - \lambda_2)\lambda_1^s - ((\gamma\frac{x_1}{x_{-1}} + \delta)(c\frac{y_0}{y_{-2}} + d) - \lambda_1)\lambda_2^s)}\right),
\end{aligned}$$

$$y_{2m+1}=y_{-1}\prod_{s=0}^{m}\left(-\frac{d}{c}+\frac{((\gamma\frac{x_0}{x_{-2}}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_2)\lambda_1^{s+1}-((\gamma\frac{x_0}{x_{-2}}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_1)\lambda_2^{s+1}}{c(\gamma\frac{x_0}{x_{-2}}+\delta)(((\gamma\frac{x_2}{x_0}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_2)\lambda_1^{s}-((\gamma\frac{x_2}{x_0}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_1)\lambda_2^{s})}\right),$$

for $m\in\mathbb{N}_{-1}$, *if* $\lambda_1\neq\lambda_2$, *and*

$$x_{2m}=x_{-2}\prod_{s=0}^{m}\left(-\frac{\delta}{\gamma}+\frac{(\gamma\frac{x_0}{x_{-2}}+\delta)(\lambda_1+((\gamma\frac{x_2}{x_0}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_1)s)}{\gamma(\lambda_1+((\gamma\frac{x_0}{x_{-2}}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_1)s)}\right),$$

$$x_{2m+1}=x_{-1}\prod_{s=0}^{m}\left(-\frac{\delta}{\gamma}+\frac{\lambda_1(\lambda_1+((\gamma\frac{x_1}{x_{-1}}+\delta)(c\frac{y_0}{y_{-2}}+d)-\lambda_1)(s+1))}{\gamma(c\frac{y_0}{y_{-2}}+d)(\lambda_1+((\gamma\frac{x_1}{x_{-1}}+\delta)(c\frac{y_2}{y_0}+d)-\lambda_1)s)}\right),$$

$$y_{2m}=y_{-2}\prod_{s=0}^{m}\left(-\frac{d}{c}+\frac{(c\frac{y_0}{y_{-2}}+d)(\lambda_1+((\gamma\frac{x_1}{x_{-1}}+\delta)(c\frac{y_2}{y_0}+d)-\lambda_1)s)}{c(\lambda_1+((\gamma\frac{x_1}{x_{-1}}+\delta)(c\frac{y_0}{y_{-2}}+d)-\lambda_1)s)}\right),$$

$$y_{2m+1}=y_{-1}\prod_{s=0}^{m}\left(-\frac{d}{c}+\frac{\lambda_1(\lambda_1+((\gamma\frac{x_0}{x_{-2}}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_1)(s+1))}{c(\gamma\frac{x_0}{x_{-2}}+\delta)(\lambda_1+((\gamma\frac{x_2}{x_0}+\delta)(c\frac{y_1}{y_{-1}}+d)-\lambda_1)s)}\right),$$

for $m\in\mathbb{N}_{-1}$, *if* $\lambda_1=\lambda_2$.

Remark 2. From (2.31) we have that $(a\alpha+b\gamma+c\beta+d\delta)^2\neq 4(bc-ad)(\beta\gamma-\alpha\delta)$ is a necessary and sufficient condition for $\lambda_1\neq\lambda_2$.

Remark 3. The closed-form formulas for the general solution to system (2.68) given in Theorem 1 can be used in describing the long-term behaviour of the solutions to the system, by using some standard methods in the above cited references, such as [3,4,34,37,40,42,43,49]. This pretty standard problem we leave to the interested reader as a simple exercise.

Remark 4. Formulas (2.52)–(2.59), (2.62) and (2.63) can also be used for finding closed-form formulas for the general solution to system (1.9) in the case of some other values of delays k and l. We also leave this simple problem to the interested reader.

3. Conclusions

We give some detailed theoretical explanations for solving a two-dimensional nonlinear system of difference equations close to the bilinear system of some interest. Since the system is a natural generalization of the corresponding one-dimensional difference equation, whose special cases appear in the literature from time to time, our main result generalizes many results therein. As an application, we present closed-form formulas for the general solution to the system for a concrete choice of the delays. The methods given in the paper can be used in many related situations and should be useful to a wide audience.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. D. Adamović, Solution to problem 194, *Mat. Vesnik*, **23** (1971), 236–242.
2. M. Bataille, Problem 11559, *Amer. Math. Monthly*, **118** (2011), 275.
3. K. S. Berenhaut, S. Stević, The behaviour of the positive solutions of the difference equation $x_n = A + (x_{n-2}/x_{n-1})^p$, *J. Differ. Equ. Appl.*, **12** (2006), 909–918. https://doi.org/10.1080/10236190600836377
4. L. Berg, S. Stević, On the asymptotics of the difference equation $y_n(1 + y_{n-1} \cdots y_{n-k+1}) = y_{n-k}$, *J. Differ. Equ. Appl.*, **17** (2011), 577–586. https://doi.org/10.1080/10236190903203820
5. D. Bernoulli, Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur (in Latin), *Commentarii Acad. Petropol. III*, **1728** (1732), 85–100.
6. G. Boole, *A treatise on the calculus of finite differences*, 3 Eds., Macmillan and Co., London, 1880.
7. L. Brand, A sequence defined by a difference equation, *Am. Math. Mon.*, **62** (1955), 489–492. https://doi.org/10.2307/2307362
8. A. de Moivre, *Miscellanea analytica de seriebus et quadraturis (in Latin)*, J. Tonson & J. Watts, Londini, 1730.
9. L. Euler, *Introductio in analysin infinitorum, tomus primus (in Latin)*, Lausannae, 1748.
10. C. Jordan, *Calculus of finite differences*, New York: Chelsea Publishing Company, 1965.
11. B. Iričanin, S. Stević, On some rational difference equations, *Ars Comb.*, **92** (2009), 67–72.
12. M. Kara, Y. Yazlik, D. T. Tollu, Solvability of a system of higher order nonlinear difference equations, *Hacet. J. Math. Stat.*, **49** (2020), 1566–1593. https://doi.org/10.15672/hujms.474649
13. G. Karakostas, Asymptotic 2-periodic difference equations with diagonally self-invertible responces, *J. Differ. Equ. Appl.*, **6** (2000), 329–335. https://doi.org/10.1080/10236190008808232
14. G. L. Karakostas, Asymptotic behavior of the solutions of the difference equation $x_{n+1} = x_n^2 f(x_{n-1})$, *J. Differ. Equ. Appl.*, **9** (2003), 599–602. https://doi.org/10.1080/1023619021000056329
15. S. F. Lacroix, *Traité des differénces et des séries (in French)*, J. B. M. Duprat, Paris, 1800.
16. V. A. Krechmar, *A problem book in algebra*, Mir Publishers, Moscow, 1974.
17. J. L. Lagrange, Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes (in French), *Miscellanea Taurinensia*, 1759, 33–42.
18. P. S. Laplace, Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la théorie des hasards (in French), *Mém. Acad. R. Sci. Paris*, **VII** (1776).
19. H. Levy, F. Lessman, *Finite difference equations*, New York: Macmillan, 1961.

20. A. A. Markoff, *Differenzenrechnung (in German)*, Teubner, Leipzig, 1896.
21. G. Papaschinopoulos, C. J. Schinas, On a system of two nonlinear difference equations, *J. Math. Anal. Appl.*, **219** (1998), 415–426. https://doi.org/10.1006/jmaa.1997.5829
22. G. Papaschinopoulos, C. J. Schinas, Invariants for systems of two nonlinear difference equations, *Differ. Equ. Dyn. Syst.*, **7** (1999), 181–196.
23. G. Papaschinopoulos, C. J. Schinas, Stability of a class of nonlinear difference equations, *J. Math. Anal. Appl.*, **230** (1999), 211–222. https://doi.org/10.1006/jmaa.1998.6194
24. G. Papaschinopoulos, C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal.: Theory Methods Appl.*, **7** (2001), 967–978.
25. G. Papaschinopoulos, C. J. Schinas, Oscillation and asymptotic stability of two systems of difference equations of rational form, *J. Differ. Equ. Appl.*, **7** (2001), 601–617. https://doi.org/10.1080/10236190108808290
26. G. Papaschinopoulos, C. J. Schinas, On the system of two difference equations $x_{n+1} = \sum_{i=0}^{k} A_i/y_{n-i}^{p_i}, y_{n+1} = \sum_{i=0}^{k} B_i/x_{n-i}^{q_i}$, *J. Math. Anal. Appl.*, **273** (2002), 294–309. https://doi.org/10.1016/S0022-247X(02)00223-8
27. G. Papaschinopoulos, C. J. Schinas, G. Stefanidou, On a k-order system of Lyness-type difference equations, *Adv. Differ. Equ.*, **2007** (2007), 1–13.
28. G. Papaschinopoulos, G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, *Int. J. Differ. Equ.*, **5** (2010), 233–249.
29. Y. N. Raffoul, *Qualitative theory of Volterra difference equations*, Springer, 2018.
30. J. Riordan, *Combinatorial identities*, John Wiley & Sons Inc., New York-London-Sydney, 1968.
31. C. J. Schinas, Invariants for difference equations and systems of difference equations of rational form, *J. Math. Anal. Appl.*, **216** (1997), 164–179. https://doi.org/10.1006/jmaa.1997.5667
32. C. J. Schinas, Invariants for some difference equations, *J. Math. Anal. Appl.*, **212** (1997), 281–291. https://doi.org/10.1006/jmaa.1997.5499
33. S. Stević, A global convergence results with applications to periodic solutions, *Indian J. Pure Appl. Math.*, **33** (2002), 45–53.
34. S. Stević, On the recursive sequence $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$, *Taiwan. J. Math.*, **7** (2003), 249–259.
35. S. Stević, On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ II, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **10** (2003), 911–916.
36. S. Stević, Boundedness character of a class of difference equations, *Nonlinear Anal.: Theory Methods Appl.*, **70** (2009), 839–848. https://doi.org/10.1016/j.na.2008.01.014
37. S. Stević, Global stability of a difference equation with maximum, *Appl. Math. Comput.*, **210** (2009), 525–529. https://doi.org/10.1016/j.amc.2009.01.050
38. S. Stević, On some periodic systems of max-type difference equations, *Appl. Math. Comput.*, **218** (2012), 11483–11487. https://doi.org/10.1016/j.amc.2012.04.077
39. S. Stević, Solutions of a max-type system of difference equations, *Appl. Math. Comput.*, **218** (2012), 9825–9830. https://doi.org/10.1016/j.amc.2012.03.057

40. S. Stević, On the system of difference equations $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$, *Appl. Math. Comput.*, **219** (2013), 4755–4764. https://doi.org/10.1016/j.amc.2012.10.092

41. S. Stević, Representations of solutions to linear and bilinear difference equations and systems of bilinear difference equations, *Adv. Differ. Equ.*, **2018** (2018), 1–21. https://doi.org/10.1186/s13662-018-1930-2

42. S. Stević, J. Diblik, B. Iričanin, Z. Šmarda, On a solvable system of rational difference equations, *J. Differ. Equ. Appl.*, **20** (2014), 811–825. https://doi.org/10.1080/10236198.2013.817573

43. S. Stević, J. Diblik, B. Iričanin, Z. Šmarda, Solvability of nonlinear difference equations of fourth order, *Electron. J. Differ. Equ.*, **2014** (2014), 1–14.

44. S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Note on the bilinear difference equation with a delay, *Math. Methods Appl. Sci.*, **41** (2018), 9349–9360. https://doi.org/10.1002/mma.5293

45. S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, On a solvable class of nonlinear difference equations of fourth order, *Electron. J. Qual. Theory Differ. Equ.*, **2022** (2022), 1–17.

46. S. Stević, B. Iričanin, Z. Šmarda, On a product-type system of difference equations of second order solvable in closed form, *J. Inequal. Appl.*, **2015** (2015), 1–15. https://doi.org/10.1186/s13660-015-0835-9

47. S. Stević, B. Iričanin, Z. Šmarda, On a symmetric bilinear system of difference equations, *Appl. Math. Lett.*, **89** (2019), 15–21. https://doi.org/10.1016/j.aml.2018.09.006

48. N. Taskara, D. T. Tollu, N. Touafek, Y. Yazlik, A solvable system of difference equations, *Commun. Korean Math. Soc.*, **35** (2020), 301–319.

49. D. T. Tollu, Y. Yazlik, N. Taskara, On a solvable nonlinear difference equation of higher order, *Turk. J. Math.*, **42** (2018), 1765–1778. https://doi.org/10.3906/mat-1705-33

50. Western Maryland College Problems Group, Problem 1572, *Math. Mag.*, **72** (1999), 149. https://doi.org/10.2307/2690603

51. Y. Yazlik, D. T. Tollu, N. Taskara, On the solutions of a three-dimensional system of difference equations, *Kuwait J. Sci.*, **43** (2016), 95–111.