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Research article

Local stabilization for a hyperchaotic finance system via time-delayed feedback based on discrete-time observations

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Abstract: This paper considers the local stabilization problem for a hyperchaotic finance system by using a time-delayed feedback controller based on discrete-time observations. The quadratic system theory is employed to represent the nonlinear finance system and a piecewise augmented discontinuous Lyapunov-Krasovskii functional is constructed to analyze the stability of the closed-loop system. By further incorporating some advanced integral inequalities, a stabilization criterion is proposed by means of the feasibility of a set of linear matrix inequalities under which the hyperchaotic finance system can be asymptotically stabilized for any initial condition satisfying certain constraint. As the by-product, a simplified criterion is also obtained for the case without time delay. Moreover, the optimization problems with respect to the domain of attraction are specially discussed, which are transformed into the minimization problems subject to linear matrix inequalities. Finally, numerical simulations are provided to illustrate the effectiveness of the derived results.

Keywords: local stabilization; hyperchaotic finance system; time-delayed feedback; discrete-time observations
Mathematics Subject Classification: 93C10, 93C43

Notations:

The superscript "T" refers to the matrix transpose. The matrix $\mathcal{P} > 0$ ($\mathcal{P} \geq 0$) means that $\mathcal{P}$ is positive definite (positive semi-definite). "$*$" is the symmetric terms in a matrix. $W[\alpha,\beta)$ is the space of vector functions φ defined over $[\alpha,\beta]$ that are absolutely continuous with a finite $\lim_{s\to\beta^-}\varphi(s)$ and square integrable derivatives. The norm of $W[\alpha,\beta)$ is defined as

$$\|\phi\|_W = \max_{s\in[\alpha,\beta)}\|\varphi(s)\| + [\int_{\alpha}^{\beta} \|\dot{\varphi}(s)\|\mathrm{d}s]^{1/2}.$$

1. Introduction

Over the past two decades, finance systems have been extensively studied due to their sophisticated dynamical behaviors such as the chaos and the bifurcation [1–6]. Note that the chaotic characteristics of finance systems could induce the potential uncertainties of the macroeconomic operation. Therefore, in the past decade or so, the stabilization and synchronization problems for the chaotic finance systems have become the most concerned research topics [7–11]. In particular, various control schemes have been adopted to achieve the effective stabilization and synchronization. For instance, in [12–15], the delayed control scheme has been utilized to stabilize the chaotic and hyperchaotic finance systems and, in [16, 17], the adaptive controllers have been proposed to realize desirable control and synchronization performance for the finance systems. In [18, 19], the adaptive sliding control strategy has been employed to stabilize the fractional-order finance systems. In [20], a resilient guaranteed cost controller has been designed to control the chaotic finance system. Also, the impulsive controller and intermittent controller have been designed in [21, 22], respectively.

In the literature [23], the control scheme based on discrete-time observations (DTOs) has been proposed to stabilize the continuous-time stochastic differential equations with Markov chain. Compared with the continuous-time control, such a control scheme costs less as the system state is only required to be observed at some discrete-time instants. The results in [23] have been further improved in [24] by using some new techniques and, in [25, 26], the time delay has been taken into account in the DTOs-based control scheme. Note that the control setting proposed in [19–22] is essentially the same as the sampled-data control encountered in engineering control systems [27, 28]. For the sampled-data control, it is worth pointing out that the discontinuous Lyapunov-Krasovskii (L-K) functionals and the Wirtinger's inequality have been developed in [28] to establish more effective stabilization conditions for *linear* control systems under a sampled-data controller with the transmission delay.

So far, most existing literature with respect to the control of finance systems have been based on continuous-time controllers. In addition, discontinuous control schemes have been proposed in [21,22] to address the synchronization problem for chaotic and hyperchaotic finance systems. However, to our knowledge, the DTOs-based control strategy has not been adopted to discuss the finance systems, not mention to the time delay is involved. In fact, the DTOs-based control is more realistic for finance systems since the financial policies are generally implemented for a period of time and then updated on the basis of the current economic situation. In addition, the time delay of policy implementation is often unavoidable. Thus, the time delay should be considered in designing controller. Unfortunately, the existing results concerning the DTOs-based time-delayed feedback can be only applicable for nonlinear control systems subject to the rigorous linear growth conditions [25, 26]. Note that the sampled-data control with transmission delay in [28] is similar to the DTOs-based time-delayed control. However, it is worth mentioning that the results in [28] are only concerned with linear systems, which are no longer applicable for chaotic finance systems due to the existence of nonlinear characteristics.

Inspired by the aforementioned discussions, the paper is devoted to considering the stabilization problem for a hyperchaotic finance system via a time-delayed feedback controller based on DTOs. By incorporating the quadratic system theory, a piecewise augmented discontinuous L-K functional, and some advanced inequalities, a local stabilization criterion is first established by means of linear matrix inequalities (LMIs). As the by-product, a simplified criterion is also provided in the case of no time delay. Moreover, the optimization problems are given to derive the larger domain of attraction (DOA). Finally, simulations show the availability of the derived results. The novelties of the paper are given as below:

1) The DTOs-based time-delayed control scheme is proposed, for the first time, to stabilize the hyperchaotic finance system.

2) A piecewise augmented discontinuous L-K functional is constructed under which a novel local stabilization criterion is obtained by means of LMIs.

3) The state evolution over the first time-interval is specifically considered in establishing the local stabilization criterion.

2. Problem formulation

In [1,2], the authors have proposed a finance system containing three variables and nine independent parameters. The finance system is composed of four sub-blocks (namely, labor force, production, stock and money) and can be simplified as follows:

$$\begin{cases} \dot{z}_1(t) = z_3(t) + (z_2(t) - a)z_1(t), \\ \dot{z}_2(t) = 1 - bz_2(t) - z_1^2(t), \\ \dot{z}_3(t) = -z_1(t) - cz_3(t), \end{cases} \tag{1}$$

where $z_1(t)$, $z_2(t)$ and $z_3(t)$ represent, respectively, the interest rate, the investment demand and the price index; a, b and c denote, respectively, the saving amount, the cost per investment and the demand elasticity of commercial markets.

In the literature [9], by introducing an additional variable $z_4(t)$ representing the average profit margin, the system (1) has been modified as below:

$$\begin{cases} \dot{z}_1(t) = z_3(t) + (z_2(t) - a)z_1(t) + z_4(t), \\ \dot{z}_2(t) = 1 - bz_2(t) - z_1^2(t), \\ \dot{z}_3(t) = -z_1(t) - cz_3(t), \\ \dot{z}_4(t) = -lz_1(t)z_2(t) - mz_4(t), \end{cases} \tag{2}$$

where l and m are scalars. In [9], it has been shown that the finance system (2) displays the complicated hyperchaotic phenomenon for the case that $a = 0.9$, $b = 0.2$, $c = 1.5$, $l = 0.2$ and $m = 0.17$. Under the assumption that

$$\Delta \triangleq (abcm + bm + cl - cm)/(cl - cm) > 0,$$

it is easy to verify that the hyperchaotic system (2) has the following equilibrium points:

$$\Big(0, \frac{1}{b}, 0, 0\Big), \quad \Big(\pm\sqrt{\Delta}, \frac{acm + m}{cm - cl}, \mp\frac{\sqrt{\Delta}}{c}, \frac{\sqrt{\Delta}(ac + 1)l}{cl - cm}\Big). \tag{3}$$

Denoting $z(t) \triangleq [z_1(t)\ \ z_2(t)\ \ z_3(t)\ \ z_4(t)]^T$ and adding the feedback control $u(t)$ into (2), we have

$$\dot{z}(t) = Az(t) + f(z(t)) + u(t), \tag{4}$$

where

$$A = \begin{bmatrix} -a & 0 & 1 & 1 \\ 0 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & 0 & -m \end{bmatrix}, \quad f(z(t)) = \begin{bmatrix} z_1(t)z_2(t) \\ 1 - z_1^2(t) \\ 0 \\ -lz_1(t)z_2(t) \end{bmatrix}.$$

In order to stabilize the continuous-time stochastic hybrid dynamical systems, in [23,24], Mao et al. have proposed the following DTOs-based feedback controller:

$$u(t) = u(z([t/h]h), t), \tag{5}$$

where $h > 0$ refers to the duration between two consecutive observations, $[t/h]$ is the maximum integer less than or equal to t/h. Considering that the time delay is often inevitable in data transmission, in [25,26], the controller (5) has been modified as

$$u(t) = u(z([t/h]h - \eta), t),$$

where $\eta > 0$ is a time delay.

Remark 1. Under the DTOs-based control scheme, it is seen from (5) that the only the state at the discrete instants $0, h, 2h, \cdots$ are needed in designing the controller. Compared with the feedback control using the continuous-time state, it is clear that the DTOs-based scheme costs less. The DTOs-based control is essentially the same as the sampled-data control in engineering systems [27,28].

In the paper, we will design the DTOs-based time-delayed feedback controller. Let $s_k \triangleq kh$ ($k = 0, 1, 2, \cdots$) be the state observation instants and $t_k \triangleq kh + \eta$ ($k = 0, 1, 2, \cdots$) be the updating instants of control signals. Then, the DTOs-based time-delayed controller can be described as

$$u(t) = K(z(t_k - \eta) - z^*),\ t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots. \tag{6}$$

Remark 2. In reality, the financial policies are generally implemented over a period of time and then modified on the basis of the current economic situation. Moreover, the time delay of policy implementation is often unavoidable. Compared with the continuous-time delayed feedback [9,12–15], the DTOs-based delayed feedback might be more realistic in controlling the unstable finance systems. Literature survey shows that, this paper is the first time to address the stabilization problem for the unstable finance systems under a DTOs-based delayed control scheme.

Let $z^* = [z_1^*\ \ z_2^*\ \ z_3^*\ \ z_4^*]^T$ be the unstable equilibrium point of the system (2). Then, it follows that

$$Az^* + f(z^*) = 0. \tag{7}$$

Moreover, using (4), (6) and (7), one has the closed-loop system

$$\dot{e}(t) = \bar{A}e(t) + Ke(t_k - \eta) + \bar{f}(e(t)),\ t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots, \tag{8}$$

where $e(t) \triangleq z(t) - z^*$, $\bar{A} \triangleq A + F_0$ and

$$F_0 \triangleq \begin{bmatrix} x_2^* & x_1^* & 0 & 0 \\ -2x_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -lx_2^* & -lx_1^* & 0 & 0 \end{bmatrix}, \quad \bar{f}(e(t)) \triangleq \begin{bmatrix} e_1(t)e_2(t) \\ -e_1^2(t) \\ 0 \\ -le_1(t)e_2(t) \end{bmatrix}.$$

In particular, it is seen that the nonlinearity $\bar{f}(e)$ can be explicitly formulated as

$$\bar{f}(e) = \begin{bmatrix} e^T F_1 \\ e^T F_2 \\ e^T F_3 \\ e^T F_4 \end{bmatrix} e \triangleq F(e)e, \tag{9}$$

where $F_3 = 0_{4\times4}$, $F_4 = -dF_1$ and

$$F_1 = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using (9), the system (8) can be modified as the following form:

$$\dot{e}(t) =[\bar{A} + F(e)]e(t) + Ke(t_k - \eta), \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots. \tag{10}$$

Denoting

$$\tau(k) \triangleq t - t_k + \eta \ (t_k \le t < t_{k+1}),$$

it is obvious that $\eta \le \tau(t) \le h+\eta$ with $\dot{\tau}(t) = 1$ for $t \ne t_k$. Furthermore, the system (10) can be described as the time-varying delay system

$$\dot{e}(t) =[\bar{A} + F(e)]e(t) + Ke(t - \tau(t)), \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots. \tag{11}$$

Due to the existence of time delay, the control signals will be updated only when $t \ge t_0 = \eta$. In this case, the system (4) should be specially handled within the time interval $[0, \eta)$ in the framework of local stabilization [29]. Here, we set $u(t) = 0$ within $[0, \eta)$. Then, one has the open-loop system

$$\dot{e}(t) =[\bar{A} + F(e)]e(t), \quad t \in [0, \eta). \tag{12}$$

The paper aims to design the DTOs-based time-delayed controller (6) such that the resulting closed-loop system (10) is locally asymptotically stable and has a larger estimate of the DOA.

As in [29], it is assumed that the initial conditions of (12) is denoted by $e(t) = e_0$, $t \in [-\eta, 0]$. Note that the finance system (2) is a typical quadratic system [30–32]. For convenience of the subsequent analysis, as in [15, 30], the following box is introduced:

$$\Omega \triangleq [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] \times [-\varepsilon_3, \varepsilon_3] \times [-\varepsilon_4, \varepsilon_4], \tag{13}$$

where $\varepsilon_j > 0$ $(j = 1, 2, 3, 4)$ are some given scalars. The box Ω can be rewritten as follows:

$$\Omega = \{e : |\gamma_j e| \le \varepsilon_j, j = 1, 2, 3, 4\} = \mathrm{Co}\{\upsilon_i, 1 \le i \le 16\}, \tag{14}$$

where γ_j is the row vector whose i-th element is 1 and others are zero, "Co" is the convex hull and

$$\begin{aligned}
\upsilon_1 &= [-\varepsilon_1, -\varepsilon_2, -\varepsilon_3, -\varepsilon_4]^T, & \upsilon_2 &= [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]^T,\\
\upsilon_3 &= [-\varepsilon_1, -\varepsilon_2, -\varepsilon_3, \varepsilon_4]^T, & \upsilon_4 &= [\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4]^T,\\
\upsilon_5 &= [-\varepsilon_1, -\varepsilon_2, \varepsilon_3, -\varepsilon_4]^T, & \upsilon_6 &= [\varepsilon_1, \varepsilon_2, -\varepsilon_3, \varepsilon_4]^T,\\
\upsilon_7 &= [-\varepsilon_1, \varepsilon_2, -\varepsilon_3, -\varepsilon_4]^T, & \upsilon_8 &= [\varepsilon_1, -\varepsilon_2, \varepsilon_3, \varepsilon_4]^T,\\
\upsilon_9 &= [\varepsilon_1, -\varepsilon_2, -\varepsilon_3, -\varepsilon_4]^T, & \upsilon_{10} &= [-\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]^T,\\
\upsilon_{11} &= [-\varepsilon_1, -\varepsilon_2, \varepsilon_3, \varepsilon_4]^T, & \upsilon_{12} &= [\varepsilon_1, \varepsilon_2, -\varepsilon_3, -\varepsilon_4]^T,\\
\upsilon_{13} &= [-\varepsilon_1, \varepsilon_2, -\varepsilon_3, \varepsilon_4]^T, & \upsilon_{14} &= [\varepsilon_1, -\varepsilon_2, \varepsilon_3, -\varepsilon_4]^T,\\
\upsilon_{15} &= [-\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4]^T, & \upsilon_{16} &= [\varepsilon_1, -\varepsilon_2, -\varepsilon_3, \varepsilon_4]^T.
\end{aligned}$$

In addition, we define the ellipsoid $\mathscr{E}(P, \rho)$ described as

$$\mathscr{E}(P, \rho) \triangleq \{x : x^T P x \le \rho, P > 0, \rho > 0\}. \tag{15}$$

Next, we will introduce two inequalities, which are of vital importance in establishing our results.

Lemma 1. [28] (Wirtinger inequality) Let $x(t) \in W[a, b)$ and $x(a) = 0$. Then, for any $n \times n$ matrix $Z > 0$, the following integral inequality holds:

$$\int_a^b x^T(s) Z x(s)\,\mathrm{d}s \le \frac{4(b-a)^2}{\pi^2} \int_a^b \dot{x}^T(s) Z \dot{x}(s)\,\mathrm{d}s.$$

Lemma 2. [33] (Wirtinger-based inequality) Let the differentiable vector function $x(t)$, and the scalars a and b $(b > a)$ be given. Then, for any $n \times n$ matrix $Z > 0$, the following inequalities are true:

$$(b-a)\int_a^b x^T(s) Z x(s)\,\mathrm{d}s \ge \Big(\int_a^b x(s)\,\mathrm{d}s\Big)^T Z \Big(\int_a^b x(s)\,\mathrm{d}s\Big) + 3\Upsilon^T Z \Upsilon,$$

where

$$\Upsilon = \int_a^b x(s)\,\mathrm{d}s - \frac{2}{b-a}\int_a^b \int_\theta^b x(s)\,\mathrm{d}s\mathrm{d}\theta.$$

3. Main results

Here, we will consider the stabilization problem for the hyperchaotic finance system (2) in a local framework by using a piecewise discontinuous L-K functional and the quadratic system theory.

Theorem 1. Let the scalars $h > 0$, $\eta > 0$, $\varepsilon_j > 0$ $(j = 1, 2, 3, 4)$, $\alpha > 0$ and $\delta \ne 0$ be given. The hyperchaotic finance system (2) is asymptotically stabilized for all initial conditions e_0 satisfying the constraint $V(0) \leq \mathbf{e}^{-\alpha\eta}$ via the controller (6) with the gain $K = X^{-1}Y$, if there exist 8×8 matrix $P = (P_{ij})_{2\times 2} > 0$, and 4×4 matrices $X, Y, Q > 0, R > 0, S > 0, Z > 0$, such that the LMIs

$$\begin{bmatrix} \Phi_{11}^{\upsilon_i} & \Phi_{12} & \Phi_{13} & Y & \Phi_{15}^{\upsilon_i} \\ * & \Phi_{22} & \Phi_{23} & (\pi^2/4)R & 0 \\ * & * & \Phi_{33} & 0 & P_{12}^T \\ * & * & * & -(\pi^2/4)R & \delta Y^T \\ * & * & * & * & \Phi_{55} \end{bmatrix} < 0, \qquad i = 1, 2, \ldots, 16, \tag{16}$$

$$\begin{bmatrix} \hat{\Phi}_{11}^{\upsilon_i} & \Phi_{12} & \hat{\Phi}_{13} & 0 & \Phi_{15}^{\upsilon_i} \\ * & \hat{\Phi}_{22} & \Phi_{23} & 0 & 0 \\ * & * & \hat{\Phi}_{33} & 0 & P_{12}^T \\ * & * & * & -h^2 R & 0 \\ * & * & * & * & \Phi_{55} \end{bmatrix} \triangleq \hat{\Phi}(\upsilon_i) < 0, \quad i = 1, 2, \ldots, 16, \tag{17}$$

$$\begin{bmatrix} P_{11} + 2\eta Z - S & P_{12} - 2Z \\ * & P_{22} + Q/\eta + (2/\eta)Z \end{bmatrix} \geq 0, \tag{18}$$

$$\gamma_j^T \gamma_j \leq \varepsilon_j^2 S, \; j = 1, 2, 3, 4, \tag{19}$$

are satisfied, where

$$\begin{aligned}
\Phi_{11}^{\upsilon_i} =& X[\bar{A} + F(\upsilon_i)] + [\bar{A} + F(\upsilon_i)]^T X^T + P_{12} + P_{12}^T + Q - 4Z, \; \Phi_{12} = -P_{12} - 2Z, \\
\Phi_{13} =& P_{22} + (6/\eta)Z, \; \Phi_{15}^{\upsilon_i} = P_{11} - X + \delta[\bar{A} + F(\upsilon_i)]^T X^T, \\
\Phi_{22} =& -Q - 4Z - (\pi^2/4)R, \; \Phi_{23} = (6/\eta)Z - P_{22}, \\
\Phi_{33} =& -(12/\eta^2)Z, \; \Phi_{55} = \eta^2 Z + h^2 R - \delta(X + X^T), \\
\hat{\Phi}_{11}^{\upsilon_i} =& -\alpha P_{11} + \Phi_{11}^{\upsilon_i}, \; \hat{\Phi}_{13} = -\alpha P_{12} + \Phi_{13}, \\
\hat{\Phi}_{22} =& -Q - 4Z, \; \hat{\Phi}_{33} = -\alpha P_{22} + \Phi_{33}.
\end{aligned}$$

Proof. Construct a piecewise augmented discontinuous L-K functional

$$V(t) = \begin{cases} V_0(t), \; t \in [0, t_0) \; (t_0 = \eta), \\ V_0(t) + V_R(t), \; t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \cdots, \end{cases} \tag{20}$$

where

$$\begin{aligned}
V_0(t) =& \vartheta^T(t) P \vartheta(t) + \int_{t-\eta}^t e^T(s) Q e(s) \mathrm{d}s + \eta \int_{-\eta}^0 \int_{t+\theta}^t \dot{e}^T(s) Z \dot{e}(s) \mathrm{d}s \mathrm{d}\theta + h^2 \int_{t-\eta}^t \dot{e}^T(s) R \dot{e}(s) \mathrm{d}s, \\
V_R(t) =& h^2 \int_{t_k-\eta}^{t-\eta} \dot{e}^T(s) R \dot{e}(s) \mathrm{d}s - \frac{\pi^2}{4} \int_{t_k-\eta}^{t-\eta} \beta^T(s, t_k) R \beta(s, t_k) \mathrm{d}s, \quad t \in [t_k, t_{k+1}), \; k = 0, 1, 2, \cdots,
\end{aligned}$$

with

$$\vartheta(t) = [e^T(t) \int_{t-\eta}^t e^T(s) \mathrm{d}s]^T, \; \beta(s, t_k) = x(s) - x(t_k - \eta) \; \text{ and } P > 0, \; Q > 0, \; R > 0, \; Z > 0.$$

Using Lemma 1 (Wirtinger inequality), it can be seen that $V_R(t) \geq 0$. Moreover, noting $V_R(t_k) = 0$, it follows that $\lim_{t \to t_k^-} V(t) \geq V(t_k)$. In addition, one can see that $V(t)$ is continuous at $t = t_0$.

By some calculations, we have

$$\dot{V}(t) = 2\vartheta^T(t)P\dot{\vartheta}(t) + e^T(t)Qe(t) - e^T(t-\eta)Qe(t-\eta) + \dot{e}^T(t)(\eta^2 Z + h^2 R)\dot{e}(t) - \eta\int_{t-\eta}^{t} \dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s - (\pi^2/4) \times \beta^T(t-\eta, t_k)R\beta(t-\eta, t_k), \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots . \tag{21}$$

Using Lemma 2 and denoting

$$\varpi(t) \triangleq e(t) + e(t-\eta) - (2/\eta)\int_{t-\eta}^{t} e(s)\mathrm{d}s,$$

it follows that

$$-\eta\int_{t-\eta}^{t} \dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s \le -[e(t) - e(t-\eta)]^T Z[e(t) - e(t-\eta)] - 3\varpi^T(t)Z\varpi(t). \tag{22}$$

Using the closed-loop system (10), it is seen that

$$(\bar{A} + F(e))e(t) + Ke(t_k - \eta) - \dot{e}(t) = 0, \quad t \in [t_k, t_{k+1}).$$

Then, for any scalar $\delta \neq 0$, we have the following zero equation:

$$2[e^T(t) + \delta\dot{e}^T(t)]X[(\bar{A} + F(e))e(t) + Ke(t_k - \eta) - \dot{e}(t)] = 0, \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots . \tag{23}$$

Adding the left-hand side of (23) to $\dot{V}(t)$ and using (22), one obtains

$$\begin{aligned}\dot{V}(t) \le\ & 2\vartheta^T(t)P\dot{\vartheta}(t) + e^T(t)Qe(t) - e^T(t-\eta)Qe(t-\eta) + \dot{e}^T(t)(\eta^2 Z + h^2 R)\dot{e}(t) \\ & - (\pi^2/4)\beta^T(t-\eta, t_k)R\beta(t-\eta, t_k) - [e(t) - e(t-\eta)]^T Z[e(t) - e(t-\eta)] \\ & - 3\varpi^T(t)Z\varpi(t) + 2[e^T(t) + \delta\dot{e}^T(t)]X[(\bar{A} + F(e))e(t) + Ke(t_k - \eta) - \dot{e}(t)] \\ =\ & \xi^T(t)\Phi(e)\xi(t), \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots ,\end{aligned} \tag{24}$$

where

$$\xi(t) = [e^T(t) e^T(t-\eta) \int_{t-\eta}^{t} e^T(s)\mathrm{d}s e^T(t_k - \eta) \dot{e}^T(t)]^T$$

and

$$\Phi(e) = \begin{bmatrix} \Phi_{11}(e) & \Phi_{12} & \Phi_{13} & XK & \Phi_{15}(e) \\ * & \Phi_{22} & \Phi_{23} & (\pi^2/4)R & 0 \\ * & * & \Phi_{33} & 0 & P_{12}^T \\ * & * & * & -(\pi^2/4)R & \delta(XK)^T \\ * & * & * & * & \Phi_{55} \end{bmatrix},$$

with

$$\begin{aligned}\Phi_{11}(e) =& P_{12} + P_{12}^T + X[\bar{A} + F(e)] + [\bar{A} + F(e)]^T X^T + Q - 4Z, \\ \Phi_{15}(e) =& P_{11} - X + \delta[\bar{A} + F(e)]^T X^T.\end{aligned}$$

Denote $Y \triangleq XK$ and notice that $\Phi(e)$ is affine about the states e_j, $j = 1, 2, 3, 4$. Then, it can be seen that, if the LMIs (16) holds, the relation $\Phi(e) < 0$ is ensured on Ω. Moreover, on the box Ω, we get

$$\dot{V}(t) < 0, \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots . \tag{25}$$

Using the open-system system (12), we have

$$2[e^T(t) + \delta\dot{e}^T(t)]X[(\bar{A} + F(e))e(t) - \dot{e}(t)] = 0, \quad t \in [0, \eta).$$

Similarly, within the first time-interval $[0, \eta)$, we can obtain

$$\begin{aligned}\dot{V}(t) \leq\ & 2\vartheta^T(t)P\dot{\vartheta}(t) + e^T(t)Qe(t) - e^T(t-\eta)Qe(t-\eta) + \dot{e}^T(t)(\eta^2 Z + h^2 R)\dot{e}(t) \\ & - h^2\dot{e}^T(t-\eta)R\dot{e}(t-\eta) - [e(t) - e(t-\eta)]^T Z[e(t) - e(t-\eta)] \\ & - 3\varpi^T(t)Z\varpi(t) + 2[e^T(t) + \delta\dot{e}^T(t)]X[(\bar{A} + F(e))e(t) - \dot{e}(t)] \\ & - \alpha\vartheta^T(t)P\vartheta(t) + \alpha\vartheta^T(t)P\vartheta(t) \\ =\ & \hat{\xi}^T(t)\hat{\Phi}(e)\hat{\xi}(t) + \alpha\vartheta^T(t)P\vartheta(t), \quad t \in [0, \eta),\end{aligned} \tag{26}$$

where

$$\hat{\xi}(t) = [e^T(t)e^T(t-\eta)\int_{t-\eta}^{t} e^T(s)\mathrm{d}s\dot{e}^T(t-\eta)\dot{e}^T(t)]^T$$

and

$$\hat{\Phi}(e) = \begin{bmatrix} \hat{\Phi}_{11}(e) & \Phi_{12} & \hat{\Phi}_{13} & 0 & \Phi_{15}(e) \\ * & \hat{\Phi}_{22} & \Phi_{23} & 0 & 0 \\ * & * & \hat{\Phi}_{33} & 0 & P_{12}^T \\ * & * & * & -h^2R & 0 \\ * & * & * & * & \Phi_{55} \end{bmatrix},$$

with

$$\hat{\Phi}_{11}(e) = -\alpha P_{11} + \Phi_{11}(e),\ \hat{\Phi}_{13} = -\alpha P_{12} + \Phi_{13},$$

$$\hat{\Phi}_{22} = -Q - 4Z,\ \hat{\Phi}_{33} = -\alpha P_{22} + \Phi_{33}.$$

If the LMIs (17) are satisfied, the matrix inequality $\hat{\Phi}(e) < 0$ can be guaranteed on the box Ω. Then, one can obtain from (26) that

$$\dot{V}(t) \leq \alpha\vartheta^T(t)P\vartheta(t) \leq \alpha V(t), \quad t \in [0, \eta).$$

Moreover, it follows that

$$V(t) \leq \mathbf{e}^{\alpha t}V(0) \leq \mathbf{e}^{\alpha\eta}V(0), \quad t \in [0, \eta). \tag{27}$$

Noting that $\lim_{t\to t_k^-} V(t) \geq V(t_k)$, and using (25) and (27), it can be seen that

$$V(t) \leq V(\eta) \leq \mathbf{e}^{\alpha\eta}V(0), \quad t \geq \eta. \tag{28}$$

On the other hand, noting (20) and using Jensen integral inequalities [34], we have

$$\begin{aligned}V(t) \geq& \vartheta^T(t)P\vartheta(t) + \int_{t-\eta}^{t} e^T(s)Qe(s)\mathrm{d}s + \eta\int_{-\eta}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s\mathrm{d}\theta \\ \geq& \vartheta^T(t)P\vartheta(t) + \frac{1}{\eta}\Big(\int_{t-\eta}^{t} e(s)\mathrm{d}s\Big)^T Q\Big(\int_{t-\eta}^{t} e(s)\mathrm{d}s\Big) + \frac{2}{\eta}\Big(\int_{-\eta}^{0}\int_{t+\theta}^{t} \dot{e}(s)\mathrm{d}s\mathrm{d}\theta\Big)^T Z\Big(\int_{-\eta}^{0}\int_{t+\theta}^{t} \dot{e}(s)\mathrm{d}s\mathrm{d}\theta\Big) \\ \geq& \vartheta^T(t)(P+\Psi)\vartheta(t), \quad t \geq 0, \end{aligned} \tag{29}$$

where the relation

$$\int_{-\eta}^{0}\int_{t+\theta}^{t} \dot{e}(s)\mathrm{d}s\mathrm{d}\theta = \eta e(t) - \int_{t-\eta}^{t} e(s)\mathrm{d}s$$

is utilized, and

$$\Psi = \begin{bmatrix} 2\eta Z & -2Z \\ * & Q/\eta + (2/\eta)Z \end{bmatrix}.$$

If the LMI (18) is true, then we can get from (29) that

$$V(t) \geq e^T(t)S\,e(t), \quad t \geq 0. \tag{30}$$

In addition, it is seen from the LMIs (19) that

$$e^T\gamma^T\gamma e \leq \varepsilon_j^2 e^T S\,e, \quad j = 1,2,3,4.$$

For any $e \in \mathscr{E}(S,1)$, we have

$$e^T\gamma^T\gamma e \leq \varepsilon_j^2 \ (i.e., |\gamma_j e| \leq \varepsilon_j), \quad j = 1,2,3,4,$$

which implies that the following relation is true:

$$\mathscr{E}(S,1) \subseteq \Omega. \tag{31}$$

For any initial condition e_0 satisfying the constraint $V(0) \leq \mathbf{e}^{-\alpha\eta}$, from (27), (28) and (30), we have

$$e^T(t)S\,e(t) \leq V(t) \leq 1, \quad t \geq 0,$$

which means that the system state $e(t)$ is evolved in the ellipsoid $\mathscr{E}(S,1)$. Moreover, using (31), it is seen that the system state $e(t)$ will be evolved in the box Ω.

Then, noting (25) and using the relation $\lim_{t\to t_k^-} V(t) \geq V(t_k)$, one can conclude that the closed-loop system (10) is asymptotically stable for all e_0 satisfying the constraint $V(0) \leq \mathbf{e}^{-\alpha\eta}$. The proof is completed. □

Remark 3. The proposed L-K functional (20) is continuous at the instant $t = t_0$ and discontinuous at the instants t_k, $k = 1,2,\cdots$ [28]. Moreover, the functional (20) is piecewise. Using the functional (20), the local stability of the closed-loop system (10) can be rigorously analyzed by sufficiently considering the evolution of the open-loop system (12) within the first time-interval $[0,\eta)$.

For the case that $\eta = 0$, one can select the simplified discontinuous L-K functional

$$\check{V}(t) = e^T(t)Pe(t) + h^2 \int_{t_k}^{t} \dot{e}^T(s)R\dot{e}(s)\mathrm{d}s \\ - \frac{\pi^2}{4} \int_{t_k}^{t} [x(s) - x(t_k)]^T R[x(s) - x(t_k)]\mathrm{d}s, \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots. \tag{32}$$

Then, a simplified stabilization criterion is readily obtained as follows:

Corollary 1. Let the scalars $h > 0$, $\varepsilon_j > 0$ $(j = 1, 2, 3, 4)$ and $\delta \neq 0$ be given. The conclusion of Theorem 1 holds for the case $\eta = 0$, if there exist matrices X, Y, $P > 0$, $R > 0$, such that the LMIs

$$\begin{bmatrix} \Pi_{11}^{\upsilon_i} & \Pi_{12} & \Pi_{13}^{\upsilon_i} \\ * & -(\pi^2/4)R & \delta Y^T \\ * & * & \Pi_{33} \end{bmatrix} < 0, \quad i = 1, 2, \cdots, 16, \tag{33}$$

$$\gamma_j^T \gamma_j \leq \varepsilon_j^2 P, \quad j = 1, 2, 3, 4, \tag{34}$$

are satisfied, where

$$\begin{aligned} \Pi_{11}^{\upsilon_i} =& X[\bar{A} + F(\upsilon_i)] + [\bar{A} + F(\upsilon_i)]^T X^T - (\pi^2/4)R, \\ \Pi_{13}^{\upsilon_i} =& -X + \delta[\bar{A} + F(\upsilon_i)]^T X^T + P, \\ \Pi_{12} =& Y + (\pi^2/4)R,\ \Pi_{33} = h^2 R - \delta(X + X^T). \end{aligned}$$

Proof. Along the proof of Theorem 1, it is seen that

$$\begin{aligned} \dot{\check{V}}(t) =& 2e^T(t)P\dot{e}(t) + h^2\dot{e}^T(t)R\dot{e}(t) - (\pi^2/4)[x(t) - x(t_k)]^T \\ & \times R[x(t) - x(t_k)] + 2[e^T(t) + \delta\dot{e}^T(t)]X \\ & \times [(\bar{A} + F(e))e(t) + Ke(t_k) - \dot{e}(t)] \\ =& \zeta^T(t)\Pi(e)\zeta(t), \quad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \cdots, \end{aligned} \tag{35}$$

where

$$\zeta(t) = [e^T(t)\ \ e^T(t_k)\ \ \dot{e}^T(t)]^T$$

and

$$\Pi(e) = \begin{bmatrix} \Pi_{11}(e) & \Pi_{12} & \Pi_{13}(e) \\ * & -(\pi^2/4)R & Y^T \\ * & * & \Pi_{33} \end{bmatrix},$$

with

$$\begin{aligned} \Pi_{11}(e) =& -(\pi^2/4)R + X[\bar{A} + F(e)] + [\bar{A} + F(e)]^T X^T, \\ \Pi_{13}(e) =& P - X + \delta[\bar{A} + F(e)]^T X^T. \end{aligned}$$

From the LMIs (33), it is seen that $\Pi(e) < 0$ is ensured on the box Ω. Moreover, from (35), we have

$$\dot{\check{V}}(t) < 0,\ t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \cdots. \tag{36}$$

In addition, it is inferred from the LMIs (34) that

$$\mathscr{E}(P, 1) \subseteq \Omega. \tag{37}$$

For any e_0 satisfying $\check{V}(0) = e_0^T P e_0 \leq 1$, using (36) and (37), and noting $\check{V}(t) \geq e^T(t)Pe(t)$, it is inferred that the trajectory $e(t)$ is contained in the box Ω. Moreover, using (36), it can be concluded that the closed-loop system (10) is locally asymptotically stable, and this completes the proof. □

In the sequel, we will address the estimate of the DOA. Here, we employ the ellipsoid $\mathscr{E}(\mathcal{P}, \mathbf{e}^{-\alpha\eta})$ as the the estimate of the DOA [35, 36]. Note that

$$e(t) = e_0 = 0, \quad t \in [-\eta, 0].$$

Then, we have

$$V(0) = \vartheta^T(0)P\vartheta(0) + \int_{-\eta}^{0} e^T(s)Qe(s)\mathrm{d}s = e_0^T \mathcal{P} e_0, \tag{38}$$

where

$$\mathcal{P} \triangleq P_{11} + \eta P_{12} + \eta P_{12}^T + \eta^2 P_{22} + \eta Q.$$

For any initial condition e_0 belongs to $\mathscr{E}(\mathcal{P}, \mathbf{e}^{-\alpha\eta})$, it is clear that the relation $V(0) \leq \mathbf{e}^{-\alpha\eta}$ is guaranteed. Let us introduce the following LMI:

$$\mathbf{e}^{\alpha\eta}\mathcal{P} \leq pI \; (p > 0). \tag{39}$$

Then, the optimization with respect to the ellipsoid $\mathscr{E}(\mathcal{P}, \mathbf{e}^{-\alpha\eta})$ in Theorem 1 can be described as

$$\textit{Problem (1)} \min_{P>0,\ Q>0,\ Z>0,\ R>0,\ S>0,\ X,\ Y,\ p>0} p, \ \textit{s.t.},$$
$$\text{LMIs(16)–(19) and (39) hold.}$$

For the case that $\eta = 0$, we introduce the following matrix inequality:

$$P \leq pI \; (p > 0). \tag{40}$$

The optimization problem about the estimate of the DOA (i.e., the ellipsoid $\mathscr{E}(P, 1)$) is given as

$$\textit{Problem (2)} \min_{P>0,\ R>0,\ X,\ Y,\ p>0} p, \ \textit{s.t.},$$
$$\text{LMIs(33)–(34) and (40) hold.}$$

Remark 4. In [25, 26], the DTOs-based time-delayed feedback has been proposed to stabilize the stochastic hybrid differential equations. However, the nonlinearities in [25, 26] are assumed to satisfy the rigorous *linear growth conditions*. It is obvious the results in [25, 26] cannot be applicable for the finance system (2). In [27, 28], the similar control scheme has been utilized to stabilize *linear* systems.

Again, it is seen that the results in [28] are no longer applicable for nonlinear system (2). Moreover, different from the existing results, our obtained stabilization criteria are in a local framework. In particular, the state evolution within $[0, \eta)$ is specifically taken into account. It is obvious that the proposed results in this paper are essentially the significant supplements of some existing ones.

Remark 5. Over the past two decades, the fractional-order systems have become an extremely active research field [37–39]. Different from the integer-order systems, the fractional-order systems can possess memory. Note that some financial variables often possess very long memory. Therefore, it has been identified that the fractional-order models should be more appropriate to describe the dynamical behaviors in financial systems [3, 10, 18, 19]. As the further research topic, we would like to address the local stabilization problem for fractional-order financial systems.

Remark 6. Over the past a decade or so, the event-triggered mechanisms have been extensively employed in network-based control systems [40–44]. Under the event-triggered mechanisms, the necessary data are released only when certain triggering condition is satisfied, thereby significantly decreasing the usage of communication resources. It is obvious that the event-triggered mechanisms can also be applicable to the finance system (2) under which the number of state observation and control implementation can be greatly reduced, which is our further research topic.

4. Numerical simulations

In this section, we will demonstrate the feasibility of the obtained results via numerical simulations. Here, we choose $a = 0.9$, $b = 0.2$, $c = 1.5$, $l = 0.2$ and $m = 0.17$. In Figure 1, we plot the phase portraits of the finance system (2).

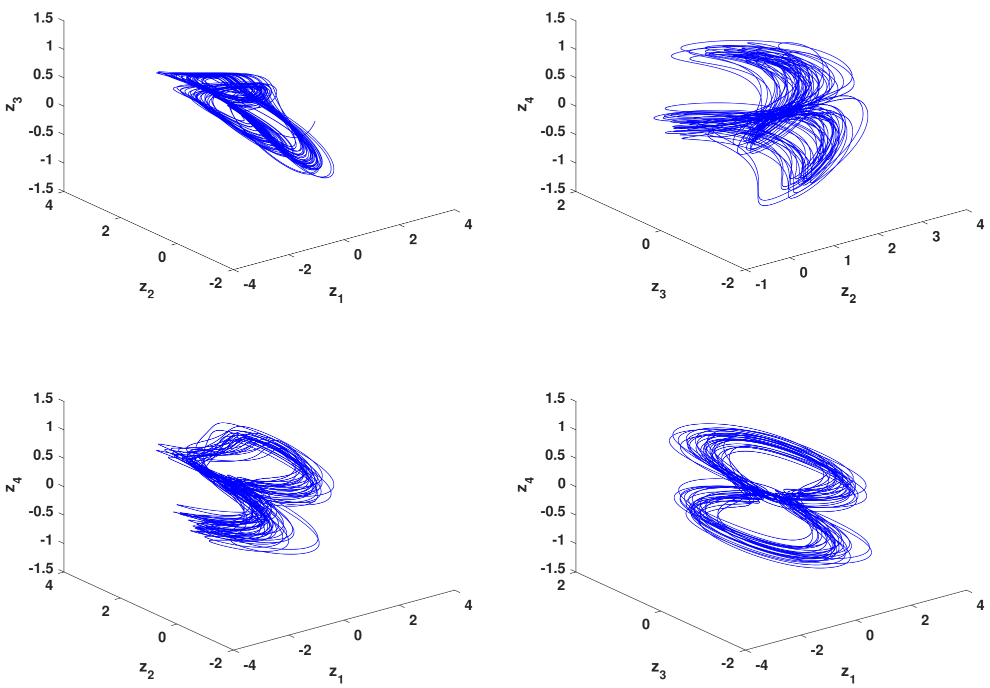

Figure 1. Phase portraits of the finance system (2) ($z_1(0) = 1$, $z_2(0) = 2$, $z_3(0) = z_4(0) = 0.5$, $a = 0.9$, $b = 0.2$, $c = 1.5$, $l = 0.2$, $m = 0.17$).

Figure 1 shows the hyperchaotic behaviour. From Figure 1, one can see that the finance system

(2) displays the sophisticated hyperchaotic behaviour. Then, it is verified that the system (2) has three unstable equilibrium points

$$\mathbb{P}_1^* \triangleq (0, 5, 0, 0),\ \mathbb{P}_2^* \triangleq (1.6660, -8.8778, -1.1107, 17.4004),$$
$$\mathbb{P}_3^* \triangleq (-1.6660, -8.8778, 1.1107, 17.4004).$$

First, we will be concerned with the case without time delay. For the equilibrium points $\mathbb{P}_2^*$ and $\mathbb{P}_3^*$, letting $\varepsilon_1 = 7.3$, $\varepsilon_2 = 13$, $\varepsilon_3 = 10$, $\varepsilon_4 = 12$, $h = 1$ and $\delta = 0.05$, and solving problem (2), one obtains

$$P = \begin{bmatrix} 0.0189 & -0.0006 & -0.0003 & 0.0008 \\ -0.0006 & 0.0069 & 0.0002 & 0.0002 \\ -0.0003 & 0.0002 & 0.0105 & 0.0020 \\ 0.0008 & 0.0002 & 0.0020 & 0.0085 \end{bmatrix} (\mathbb{P}_2^*),$$

$$K = \begin{bmatrix} -0.0163 & -0.0447 & -0.1080 & -0.0848 \\ -0.0169 & -0.1038 & -0.0014 & 0.0137 \\ -0.0712 & -0.1684 & -0.8149 & -0.5191 \\ -0.0167 & 0.0573 & 0.1448 & -0.1120 \end{bmatrix} (\mathbb{P}_2^*),$$

$$P = \begin{bmatrix} 0.0189 & 0.0005 & -0.0004 & 0.0008 \\ 0.0005 & 0.0070 & -0.0006 & -0.0003 \\ -0.0004 & -0.0006 & 0.0106 & 0.0021 \\ 0.0008 & -0.0003 & 0.0021 & 0.0082 \end{bmatrix} (\mathbb{P}_3^*),$$

$$K = \begin{bmatrix} -0.0168 & -0.0192 & -0.0983 & -0.0949 \\ 0.0007 & -0.1022 & -0.0647 & -0.0467 \\ -0.0847 & -0.2126 & -0.7814 & -0.5060 \\ -0.0158 & 0.0288 & 0.1614 & -0.0970 \end{bmatrix} (\mathbb{P}_3^*).$$

Using the above parameters, we plot the state evolutions of the error system (8) for the case $\eta = 0$. In the simulation, we choose $e_0 = [6, 5, 3, 1]^T \in \mathscr{E}(P, 1)$. From Figures 2 and 3, it is seen that our proposed DTOs-based time-delayed feedback scheme can stabilize the unstable hyperchaotic finance system. Figure 2 shows that the error state converges to the origin. Figure 3 shows that the error state converges to the origin.

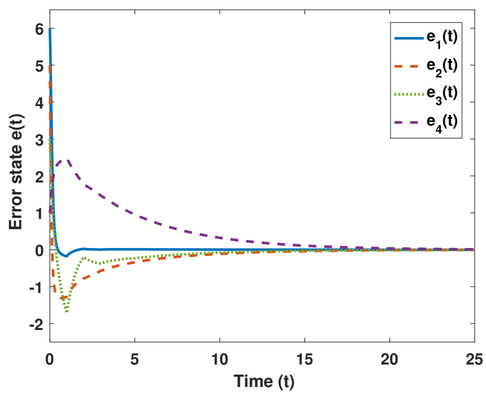

Figure 2. State evolutions of the error system (8) for P_2^* ($\eta = 0$, $e_0 = [6, 5, 3, 1]^T$, $a = 0.9$, $b = 0.2$, $c = 1.5$, $l = 0.2$, $m = 0.17$).

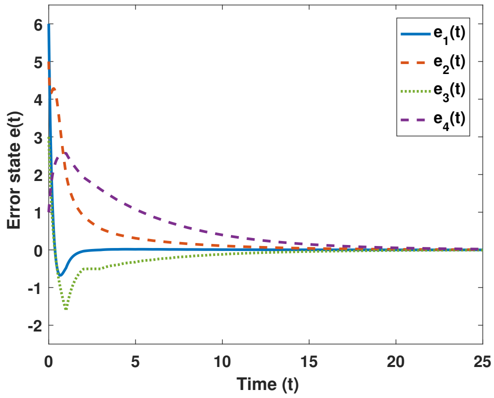

Figure 3. State evolutions of the error system (8) for P_3^* ($\eta = 0$, $e_0 = [6, 5, 3, 1]^T$, $a = 0.9$, $b = 0.2$, $c = 1.5$, $l = 0.2$, $m = 0.17$).

Next, we will consider the case with time delay. Letting $\varepsilon_1 = 7.2$, $\varepsilon_2 = 12$, $\varepsilon_3 = 17$, $\varepsilon_4 = 100$, $\eta = 0.5$, $h = 1$, $\alpha = 0.16$ and $\delta = 0.05$, and solving the optimization problem (1), we have

$$\mathcal{P} = \begin{bmatrix} 0.0198 & -0.0005 & -0.0004 & 0.0022 \\ -0.0005 & 0.0070 & 0.0000 & 0.0002 \\ -0.0004 & 0.0000 & 0.0041 & 0.0026 \\ 0.0022 & 0.0002 & 0.0026 & 0.0145 \end{bmatrix} (\mathbb{P}_2^*),$$

$$K = \begin{bmatrix} -0.0014 & -0.0423 & 0.0167 & 0.0017 \\ 0.0013 & -0.0449 & 0.0195 & 0.0162 \\ -0.1249 & -0.1160 & -0.0299 & -0.6415 \\ 0.0043 & 0.0354 & -0.0119 & 0.0154 \end{bmatrix} (\mathbb{P}_2^*),$$

$$\mathcal{P} = \begin{bmatrix} 0.0198 & 0.0005 & -0.0003 & 0.0022 \\ 0.0005 & 0.0070 & -0.0003 & -0.0003 \\ -0.0003 & -0.0003 & 0.0040 & 0.0027 \\ 0.0022 & -0.0003 & 0.0027 & 0.0143 \end{bmatrix} (\mathbb{P}_3^*),$$

$$K = \begin{bmatrix} 0.0002 & 0.0186 & 0.0140 & 0.0058 \\ -0.0049 & -0.0374 & -0.0290 & -0.0350 \\ -0.1064 & -0.1831 & -0.1534 & -0.6036 \\ 0.0015 & 0.0191 & 0.0161 & 0.0131 \end{bmatrix} (\mathbb{P}_3^*).$$

Using the above obtained parameters, the state evolutions of the error system (8) are plotted in Figures 4 and 5, where the initial condition is selected as $e_0 = [6, 4, 3, 1]^T \in \mathscr{E}(\mathcal{P}, \mathbf{e}^{-\alpha\eta})$. Figures 4 and 5 show again that our proposed control scheme can effectively stabilize the unstable hyperchaotic finance system. However, compared with the case without time delay, it is seen that from Figures 4 and 5 that the convergence rate of the error system (8) becomes slower due to the existence of time delay. Figure 4 shows that the error state converges to the origin. Figure 5 shows that the error state converges to the origin.

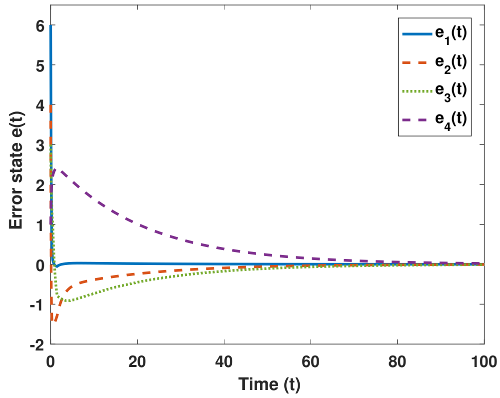

Figure 4. State evolutions of the error system (8) for P_2^* ($\eta = 0.5$, $e_0 = [6, 4, 3, 1]^T$, $a = 0.9$, $b = 0.2$, $c = 1.5$, $l = 0.2$, $m = 0.17$).

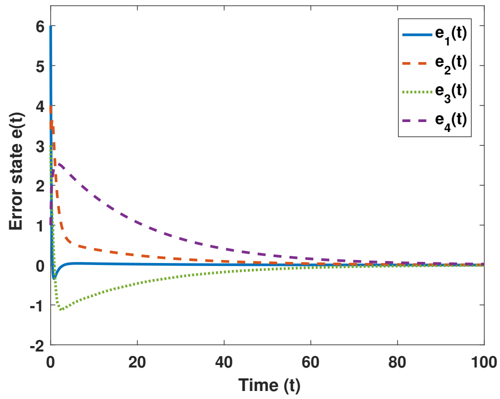

Figure 5. State evolutions of the error system (8) for P_3^* ($\eta = 0.5$, $e_0 = [6, 4, 3, 1]^T$, $a = 0.9$, $b = 0.2$, $c = 1.5$, $l = 0.2$, $m = 0.17$).

In solving problems (1) and (2), we employ the "mincx" solver involved in LMI toolbox in MATLAB to numerically solve the minimization problem of a linear objective function subject to LMI constraints [45]. In the simulation, we utilize the Euler method, where the step size is selected as 0.01.

5. Conclusions

In the paper, we have investigated the local stabilization design for a hyperchaotic finance system via the time-delayed feedback based on DTOs. By incorporating quadratic system theory, a piecewise augmented discontinuous L-K functional, and two advanced inequalities, a local stabilization criterion has been obtained in the framework of LMIs. In the case of no time delay, the corresponding result is also proposed. Then, the optimization problems have been provided to estimate the DOA as large as possible. The feasibility of proposed results has been illustrated by simulation results. The proposed techniques in this paper can be extended to the synchronization control problem [8, 10, 46].

However, it is worth mentioning that the obtained results in this paper are conservative to a certain extent. As the further improvement direction, we can employ the more effective Bessel-Legendre inequality to deal with the time delay [47]. In addition, we can the design the nonlinear feedback controller to reduce the potential conservatism [32]. On the other hand, the time delay might be time-varying [48,49]. Moreover, the external disturbances might be inevitable in the finance system [15,50]. As the further research topic, it is also interesting to address the local stabilization problem for the hyperchaotic finance system subject to external disturbances and time-varying delay.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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