AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 19892–19912.
DOI: 10.3934/math.20231014
Received: 09 April 2023
Revised: 29 May 2023
Accepted: 05 June 2023
Published: 14 June 2023

Research article

Controllability of fractional stochastic evolution inclusion via Hilfer derivative of fixed point theory

Abdelkader Moumen[1], Ammar Alsinai[2,*], Ramsha Shafqat[3], Nafisa A. Albasheir[4], Mohammed Alhagyan[5], Ameni Gargouri[5] and Mohammed M. A. Almazah[6]

[1] Department of Mathematics, Faculty of Sciences, University of Hail, Hail 55425, Saudi Arabia
[2] Department of mathematics, Kuvempu University, Shivamogga, Karnataka-577451, India
[3] Department of Mathematics and Statistics, The University of Lahore, Sargodha 40100, Pakistan
[4] Department of Mathematics, College of Science and Arts (Magardah), King Khalied University, Magardah 61421, Saudi Arabia
[5] Department of Mathematics, College of Humanities and Science in Al Aflaj, Prince Sattam Bin Abdulaziz University, Saudi Arabia, 11912
[6] Department of Mathematics, College of Sciences and Arts (Muhyil), King Khalied University, Muhyil 61421, Saudi Arabia

* **Correspondence:** Email: aliiammar1985@gmail.com.

Abstract: In this study, we use the Hilfer derivative to analyze the approximate controllability of fractional stochastic evolution inclusions (FSEIs) with nonlocal conditions. By assuming that the corresponding linear system is approximately controllable, we obtain a novel set of adequate requirements for the approximate controllability of nonlinear FSEIs in meticulous detail. The fixed-point theorem for multi-valued operators and fractional calculus are used to achieve the results. Finally, we use several instances to demonstrate our findings.

Keywords: controllability; fractional calculus; FSD-Inclusions; multi-valued maps; Hilfer derivative; fixed point theory
Mathematics Subject Classification: 34A07, 34A08, 60G22

1. Introduction

The details of the well-developed controllability theory for abstract linear and nonlinear control systems in finite and infinite dimensional spaces are provided in [1–3]. Furthermore, deterministic models frequently vary owing to ambient noise that appears to be random. As a result, in order to improve model performance, deterministic systems must be replaced with stochastic systems. Stochastic differential equations are widely used in the formulation and analysis of electrical, control engineering, physical sciences and mechanical [4, 5]. Several writers have explored the topic of exact and approximate controllability of nonlinear stochastic differential systems(see [6, 7] and references therein). It has been shown in [8] that an impulsive neutral stochastic functional differential system with state-dependent delay approximates controllability in Hilbert spaces (HS). On the other hand, due to its applications in numerous sectors of science and engineering over the past 20 years, fractional differential equations have captured the interest of many engineers, physicists and mathematicians. The memory and hereditary characteristics of several significant materials and processes can be described by differential equations of any order. We cite [9, 10] for the most recent developments in this sector for the reader's reference. Using a solution operator and the traditional fixed-point theorems, Shu et al. [11] demonstrated the existence of mild solutions for a class of impulsive fractional partial semi-linear differential equations. For a class of fractional neutral stochastic integro-differential equations with infinite delay, Cui and Yan [12] investigated the existence of mild solutions. Additionally, because FSD-Inclusions are used in the mathematical modeling of a number of issues in economics, optimal control, etc., the issue of whether fractional differential inclusion problems have solutions has been researched by a number of authors for a variety of dynamical systems [13]. More recently, Yan and Zhang [14] using the nonlinear alternate solution of Leray-Schauder type for multi-valued maps due to O'Regan and properties of the solution operator, derived a set of sufficient conditions for the existence of solutions of impulsive fractional partial neutral stochastic integro-differential inclusions. Furthermore, very few studies on the analysis of controllability issues for FSD-Inclusions have been recorded. Using the fixed-point theorem of the collapsing multi-valued map, Duan et al. [15] have examined the exact controllability of nonlinear stochastic impulsive evolution differential inclusions with indefinite delay. Additionally, investigations on the fractional differential system's controllability problem have drawn a lot of interest from various scholars [16, 17]. It should be noted that approximate controllability, as opposed to exact controllability, is more suitable for control systems. If a control process can be identified that is consumable and results in a solution that is relatively close to a specified square integrable end condition, then the control system is said to be roughly controllable for all beginning data and all finite time horizons with $b > 0$ in [18]. In the works by [19, 20], approximate controllability of fractional order systems has been reported. The approximate controllability of control systems governed by a class of partial fractional neutral integro-differential inclusions with state-dependent delay was established by Yan in his paper [21]. Ahmed et al. [22], Sathiyaraj et al. [23], Ma et al. [24], Lui et al. [25] and Shu et al. [26] worked on fractional stochastic differential equation and Guo et al. [27] studied optimal control system with random impulses. Byszewski et al. [28] studied the existence and uniqueness of a solution of a nonlocal abstract cauchy problem in a banach space. Alzabut et al. [29] worked on Nonlinear Discrete Fractional Pantograph Equations with Non-Local Initial Conditions. Abuasbeh et al. [30–33] worked on Time-Fractional Initial Boundary Value Problems and fractional differential equation. Mouman et al. [34, 35] studied simpson type inequalities and fractional integral pantograph differential equations. Ghafli et al. [36] studied topological structure of fractional control delay problem. Using fixed-point methods and fractional computations, Sakthivel et al. [37, 38] more recently investigated the results of approximation controllability for fractional deterministic and stochastic differential systems. To the best of our knowledge, no research has yet been done on the approximation controllability problem for fractional stochastic differential inclusions with nonlocal conditions and infinite delay. With nonlocal conditions of the kind, we investigate the roughly controllability of the following FSD-Inclusions:

$$\begin{cases} D_{0^+}^{\upsilon,\sigma;\varsigma} y(\rho) \in \Lambda y(\rho) + \mathcal{G}(\rho, y(\rho)) + \Delta(\rho, y(\rho)) \frac{dw(\rho)}{dt} + Bu(\rho), \rho \in [0, b], \\ I_{0^+}^{(1-\upsilon)(1-\sigma);\varsigma} y(0) = y_0 + g(y), \end{cases} \tag{1.1}$$

where$D_{0^+}^{\upsilon,\sigma,\varsigma}$ is the Hilfer fractional derivative (HFD) of order $1/2 < \upsilon \leq 1$. On a linear HS H, Λ is a distributional operator densely constructed with the inner product (.,.) and norm $\| \,.\, \|$. Let K be another separable HS with $(.,.)_K$ and $\| \,.\, \|_K$ as its inner products. Assuming a K-valued Brownian motion or Weiner process with a finite trace nuclear covariance operator $Q \geq 0$ described on a complete probability space. Consider $\{w(\rho)\}\rho \geq 0$ be that process, $\left(\upsilon, \mathcal{G}, \left\{\mathcal{G}_\rho\right\}_{\rho\geq 0}, \mathcal{P}\right)$. The control function is given as u(.), where U is a Hilbert space, B is a bounded linear operator into H and L(K,H) designates the set of all bounded linear operators from K into H. The random variable $y_0 \in H$ satisfies the equation $E\| y_0 \|^2 < \infty$; $\mathcal{G}$, Δ are multivalued mappings and g is a nonlinear function.

Stochastic processes, such as stochastic Brownian motion, have wide-ranging applications in various scientific disciplines. It is named after Robert Brown, who first observed the erratic movement of pollen grains suspended in a fluid. In the context of control theory and fractional stochastic evolution inclusions, the use of stochastic Brownian motion becomes particularly relevant. Fractional stochastic evolution inclusions are mathematical equations that involve both fractional calculus and stochastic processes. Controllability analysis involves considering nonlocal conditions, which were proposed by Byszewski et al. to generalize the study of the canonical initial problem. The motivation for introducing nonlocal conditions arises from physical science, where certain phenomena exhibit nonlocal behavior. To analyze the controllability of fractional stochastic evolution inclusions, researchers use the Wiener process and Hilfer derivative of fixed point theory. The Hilfer derivative is a fractional derivative operator introduced by R. Hilfer, which generalizes the concept of fractional derivatives and allows for a more comprehensive understanding of fractional calculus. The incorporation of nonlocal conditions and the utilization of the Wiener process, along with the Hilfer derivative, provide a mathematical framework to study the controllability properties of these complex systems.

2. Preliminaries

In this section, we provide a few definitions and early results which will be applied to this investigation.

Let $\left(\Psi, \mathcal{G}, \left\{\mathcal{G}_\rho\right\}_{\rho\geq 0}, \mathcal{P}\right)$ be filtered complete probability space satisfying the standard condition which suggests that the filtration may be right continuous increasing family and $\mathcal{G}_0$ contains all P-null sets. Let be a Q-Weiner process defined on $\left(\Psi, \mathcal{G}, \left\{\mathcal{G}_\rho\right\}_{\rho\geq 0}, \mathcal{P}\right)$ with the covariance operator Q such that $tr(Q) < \infty$. We assume that there exists a complete orthonormal system $e_k, k \geq 1$ in K, a bounded sequence of non-negative real numbers λ_k such that $Qe_k = \lambda_k e_k$, k= 1,2,... and a sequence of independent Brownian motions $\beta_k, k \geq 1$ such that $(W(\rho), e)_K = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (e_k, e) \beta_k(\rho), , , e \in K, , , \rho \geq 0$. Let $L_2^0 = L_2\left(Q^{1/2}K, H\right)$ be the space of all Hilbert-Schmidt operators from $\{\mathfrak{Q}^{1/2}\mathrm{K}\}$ to H with the inner product

$(\phi, \varphi) = Tr\left(\phi \mathfrak{Q} \varphi^*\right)$. Let be the Banach space of all F_b measurable square-integrable random variables in H. Denote C by $C\left([0, b], L_2\left(\upsilon, F, H\right)\right)$, the Banach space of all the continuous functions from [0,b] to $L_2\left(\upsilon, \mathcal{G}, H\right)$ satisfying the condition $\sup_{\rho \in J} E\|y(\rho)\|^2 < \infty$.

Now, we introduce some basic definitions and results of multivalued maps. We use the notations P(H) for the family of all subsets of H and denote $P_{bd}(H) = Y \in P(H) : Y$ is bounded, $P_{cl}(H)$=$Y \in P(H)$: Y is closed, $P_{cv}(H)$=$Y \in P(H)$: Y is convex, $P_{cp}(H)$=$Y \in P(H)$: Y is compact. A multivalued map $\mathcal{G} : H \to P(H)$ is convex(closed) valued if $\mathcal{G}(y)$ is convex(closed) for all $y \in H$. $\mathcal{G}$ is bounded on bounded sets if $\mathcal{G}(C) = \bigcup_{y \in C} \mathcal{G}(y)$ is bounded in H, i.e $\sup_{y \in C}\left\{\sup\left\{\|z\| : z \in \mathcal{G}\left(y\right)\right\}\right\} < \infty$.

Definition 2.1. [13] $\mathcal{G}$ is called upper semicontinuous (u.s.c.) on H if for each $y_0 \in H$, the set $\mathcal{G}(y_0)$ is nonempty closed subset of H and if for each open set C of H containing $\mathcal{G}(y_0)$, there exists an open neighborhood V of y_0 such that $\mathcal{G}(V) \subseteq C$.

Definition 2.2. [13] $\mathcal{G}$ is called completely continuous if $\mathcal{G}(C)$ is relatively compact for every bounded subset C of H. If the multivalued map $\mathcal{G}$ is completely continuous with nonempty compact values, then $\mathcal{G}$ is u.s.c., if and only if $\mathcal{G}$ has a closed graph, i.e. $y^n \to y^*, z^n \to z^*, z^n \in \mathcal{G}y^n$ implies that $z^* \in \mathcal{G}y^*$. $\mathcal{G}$ has a fixed point if there is $y \in H$ such that $y \in \mathcal{G}(y)$.

Definition 2.3. [39] The multivalued map $\mathcal{G} : J \times H \to P_{bd,cl,cv}\left(H\right)$ is said to be $L^2 - Caratheodory$ if

(i) $\rho \mapsto \mathcal{G}(\rho, v)$ is measurable for each $v \in H$;
(ii) $\rho \mapsto \mathcal{G}(\rho, v)$ is upper semicontinuous for almost all $\rho \in J$;
(iii) for each $m > 0$, there exists $g_m \in L^1(J, R^+)$ such that $\|\mathcal{G}\left(\rho, v\right)\|^2 = \sup_{\mathcal{G} \in \mathcal{G}(\rho, v)} E\|\mathcal{G}\|^2 \leq g_m\left(\rho\right)$, for all $\|v\|_c^2 \leq m$ and for a.e. $\rho \in J$.

Lemma 2.1. *[39] Let J be a compact real interval and H be a HS. Let $\mathcal{F}$ be an $L^2 - Caratheodory$ multivalued map and for each $y \in C$, the set $R_{\mathcal{F},y} = \{h \in L^2(L(K, H)) : h(\rho) \in \mathcal{F}(\rho, y(\rho)),\ for\ a, e, \rho \in J\}$ is nonempty. Let ψ be a linear continuous mapping from $L^2(J, H)$ to $C(J, H)$, then the operator*

$$\Psi \circ R_{\mathcal{F}} : C\left(J, H\right) \to P_{cp,cv}\left(C\left(J, H\right)\right), y \mapsto \left(\Psi \circ R_{\mathcal{F}}\right)\left(y\right) = \Psi\left(R_{\mathcal{F},y}\right)$$

is closed graph operator in $C(J, H) \times C(J, H)$.

Definition 2.4. [40] The Caputo derivative of order δ for a function $\mathcal{G} : [0, \infty) \to R$ can be written as

$$D_\rho^\upsilon \mathcal{G}\left(\rho\right) = \frac{1}{\Gamma\left(n - \upsilon\right)} \int_0^\rho \left(\rho - \varpi\right)^{n-\upsilon-1} \mathcal{G}^{(n)}\left(\varpi\right) d\varpi = I^{n-\upsilon} \mathcal{G}^n\left(\rho\right),$$

for $n - 1 < \upsilon < n, n \in N$. If $0 < \upsilon \leq 1$, then

$$D_\rho^\upsilon \mathcal{G}\left(\rho\right) = \frac{1}{\Gamma\left(1 - \upsilon\right)} \int_0^\rho \left(\rho - \varpi\right)^{-\upsilon} \mathcal{G}^{(1)}\left(\varpi\right) d\varpi.$$

The Laplace Transform of Caputo derivative of order $\upsilon > 0$ is given as

$$L\left\{D_\rho^\upsilon \mathcal{G}\left(\rho\right) : \lambda\right\} = \lambda^\upsilon \mathcal{G}\left(\lambda\right) - \sum_{k=0}^{n-1} \lambda^{\upsilon-k-1} \mathcal{G}^{(k)}\left(0\right); n - 1 < \upsilon < n.$$

The two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{v,w}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(vk+w)}=\frac{1}{2\pi i}\int_C\frac{\mu^{v-w}e^{\mu}}{\mu^v-z}d\mu, v,w>0, z\in C,$$

where C is a contour that circles the disc $\|\mu\|\leq|z|^{1/2}$ counterclockwise and has an origin and destination at $-\infty$. The Mittag-Leffler function's most intriguing characteristics are connected to their Laplace integral

$$\int_0^{\infty}e^{-\lambda\rho}t^{w-1}E_{v,w}(\omega\rho^v)\,dt=\frac{\lambda^{v-w}}{\lambda^v-\omega}, Re\lambda>\omega^{1/v}, \omega>0.$$

Definition 2.5. [11] A closed and linear operator Λ is said to be sectorial if there are constants $\omega\in R, \theta\in[\pi/2,\pi], M>0$ such that the following two conditions are satisfied:

(1) $\rho(\Lambda)\subset\sum_{(\theta,\omega)}=\lambda\in C:\lambda\neq\omega, \left|\arg(\lambda-\omega)\right|<\theta\}$,
(2) $\|R(\lambda,\Lambda)\|_{L(X)}\leq\frac{M}{|\lambda-\omega|}, \lambda\in\sum_{(\theta,\omega)}$.

Definition 2.6. [41] Let $\zeta\in C^1([a,b])$ be an increasing function with $\zeta'(\rho)\neq 0\forall\rho\in[a,b]$ and Ψ be an integrable function defined on $[a,b]$. The ζ-Riemann-Liouville fractional integral operator of function Ψ of order $\upsilon>0$ is given by:

$$I_{a^+}^{\upsilon;\zeta}\Psi(\rho)=\frac{1}{\Gamma(\upsilon)}\int_a^{\rho}(\zeta(\rho)-\zeta(\varpi))^{\upsilon-1}\Psi(\varpi)\zeta'(\varpi)d\varpi.$$

Definition 2.7. [41] Let $n-1<\upsilon<n$ and $\zeta\in C^1([a,b])$ be an increasing function with $\zeta'(\rho)\neq 0\forall\rho\in[a,b]$. The ζ-Riemann-Liouville fractional derivative of order $\rho>0$ of an integrable function Ψ defined on $[a,b]$ is given by:

$$D_{a^+}^{\upsilon;\zeta}\Psi(\rho)=\left(\frac{1}{\zeta'(\rho)}\frac{d}{d\rho}\right)^n I_{a^+}^{n-\upsilon;\zeta}\Psi(\rho), n=[\upsilon]+1.$$

Definition 2.8. Let $0<\upsilon<1$ and $\zeta\in C^1([a,b])$ be such that $\zeta'(\rho)$is increasing and $\zeta'(\rho)\neq 0\ \forall\ \rho\in[a,b]$. The ζ-Hilfer fractional derivative of function $\Psi\in C^1([a,b])$ of order $0<\upsilon<1$ and type $0<\sigma\leq 1$ is defined as:

$$D_{a^+}^{\upsilon,\sigma;\zeta}\Psi(\rho)=I_{a^+}^{\sigma(1-\upsilon);\zeta}\Big(\frac{1}{\zeta'(\rho)}\frac{d}{d\rho}\Big)I_{a^+}^{(1-\sigma)(1-\upsilon);\zeta}\Psi(\rho), \rho>a.$$

It can also be expressed as

$$D_{a^+}^{\upsilon,\sigma;\zeta}\Psi(\rho)=\frac{1}{\Gamma(\gamma-\rho)}\int_a^t(\zeta(\rho)-\zeta(\varpi))^{\gamma-1}\Big(\frac{1}{\zeta'(\varpi)}\frac{d}{d\varpi}\Big)I_{a^+}^{(1-\upsilon)(1-\sigma);\zeta}\Psi(\varpi)d\varpi,$$

where $\gamma=\upsilon+\sigma-\upsilon\sigma$.

Lemma 2.2. *[11] If $\mathcal{G}$ meets the uniform Holder condition with the exponent $\upsilon \in (0, 1]$ and Λ is a sectorial operator, then the Cauchy problem has a single solution, which is*

$$D_{0^+}^{\upsilon,\sigma;\varsigma} y(\rho) = \Lambda y(\rho) + \mathcal{G}(\rho), 0 < \upsilon < 1, \; I_{0^+}^{(1-\upsilon)(1-\sigma);\varsigma} y(0) = y_0 \;, \tag{2.1}$$

is given by

$$y(\rho) = K_{\upsilon,\sigma}(\rho) y_0 + \int_0^\rho R_\upsilon (\varsigma(\rho) - \varsigma(s)) \mathcal{G}(\varpi) d\varpi, \tag{2.2}$$

where $K_{\upsilon,\sigma}(\rho) = I_{a^+}^{\upsilon(1-\sigma);\varsigma} K_\upsilon(\rho)$, $R_\upsilon(\rho) = \rho^{\upsilon-1} P_\upsilon(\rho)$, $P_\upsilon(\rho) = \int_0^\infty \upsilon\theta\Phi_\upsilon \Im(\rho^\upsilon\theta) d\theta$.

$\hat{B}_r$ denotes the Bromwich path, $R_\upsilon(\rho)$ is called the υ-resolvent family and $K_\upsilon(\rho)$ is the solution operator generated by δ.

If $\upsilon \in (0, 1)$ and $\Lambda \in \Lambda^\upsilon(\theta_0, \upsilon_0)$, then for any $y \in C$ and $\rho > 0$, we have $\|K_\upsilon(\rho)\| \leq Me^{\upsilon\rho}$ and $\|R_\upsilon(\rho)\| \leq Ce^{\upsilon\rho}(1 + \rho^{\upsilon-1})$, $\rho > 0$, $\upsilon > \upsilon_0$. Let

$$\tilde{M}_T = \sup_{0\leq\rho\leq b} Me^{\upsilon\rho}, \; \tilde{M}_S = \sup_{0\leq\rho\leq b} Ce^{\upsilon\rho}\left(1 + \rho^{1-\upsilon}\right).$$

Hence, we have $\|K_\upsilon(\rho)\| \leq \tilde{M}_T$ and $\|R_\upsilon(\rho)\| \leq \rho^{\upsilon-1}\tilde{M}_R$.

Definition 2.9. A Stochastic process $y \in C$ is called the mild solution of system 1.1 if $y_0, g \in L_2^0(\upsilon, C)$, $u(.) \in L_F^2(J, U)$ and there exists a function $\mathcal{G}(\rho) \in \mathcal{G}(\rho, y(\rho))$ and $\Delta(\rho) \in \Delta(\rho, y(\rho))$ such that

$$\begin{aligned} y(\rho) = {} & K_{\upsilon,\sigma}(\rho)(y_0 - g(y)) + \int_0^\rho R_\upsilon(\varsigma(\rho) - \varsigma(s)) Bu(\varpi) d\varpi + \int_0^\rho R_\upsilon(\varsigma(\rho) - \varsigma(s)) \mathcal{G}(\varpi) d\varpi \\ & + \int_0^\rho R_\upsilon(\varsigma(\rho) - \varsigma(\varpi)) \Delta(\varpi) dw(\varpi). \end{aligned} \tag{2.3}$$

Let $y_b(y_0; u)$ be the state value of 1.1 terminal time b corresponding to the control u and the initial value y_0. Introduce the set $\mathfrak{R}(b, y_0) = \{y_b(y_0; u)(0) : u(.) \in L_F^2(J, U)\}$ which is called reachable set of system 1.1 at terminal time b and its closure in H is denoted by $\overline{\mathfrak{R}}(b, y_0)$.

Definition 2.10. The system 1.1 is said to be approximate controllable control, first we take into account the approximate controllability of its linear part

$$\begin{aligned} & D_\rho^{\upsilon,\sigma;\varsigma} y(\rho) \in \Lambda y(\rho) + Bu(\rho), \; \rho \in J, \\ & I_{0^+}^{(1-\upsilon)(1-\sigma);\varsigma} y(0) = y_0 - g(y). \end{aligned} \tag{2.4}$$

To accomplish this, we must implement the operator linked with 2.4 as

$$\Psi_0^b = \int_0^b R_\upsilon(b - \varpi) BB^* R_\upsilon^*(b - \varpi) d\varpi.$$

In this case, B^* represents the adjoint of B and $R_\upsilon^*(\rho)$ represents the adjoint of $R_\upsilon(\rho)$. The fact that the operator Ψ_0^b is a linear bounded operator is obvious. Let $R(\alpha, \Psi_0^b) = (\alpha I + \Psi_0^b)^{-1}$.

Lemma 2.3. *[7] The linear control system 2.4 is approximately controllable on J if* $\alpha(\alpha I+\Psi_0^b)^{-1}$ *tends to zero as* $\alpha \to 0^+$ *in the strong operator topology.*

We will now discuss the fixed-point theorems.

Lemma 2.4. *[42] Let* $B(0,r)$ *and* $B[0,r]$ *denote the open and closed balls respectively in HS H centered at the origin and of radius r and let* $\Phi : B[0,r] \to P_{cp,cv}(H)$ *is an upper semicontinuous and completely continuous . Then, either*

(1) Φ *has a solution, or*

(2) there exists $y \in H$ *with* $\|y\| = r$ *such that* $\lambda y \in \phi y$ *for some* $\lambda > 1$.

3. Definition of mild solution

We first consider the following fractional stochastic evolution system

$$\begin{cases} D_0^{\upsilon,\sigma;\varsigma} y(\rho) = \Lambda y(\rho) + \mathcal{G}(\rho, y(\rho)) + \Delta(\rho, y(\rho)) \frac{dw(\rho)}{d\rho} + Bu(\rho), \ \rho \in [0,T], \\ I_{0^+}^{(1-\upsilon)(1-\sigma);\varsigma} y(0) = y_0 - g(y), \end{cases} \tag{3.1}$$

where $\mathcal{D}_0^{\upsilon,\sigma;\varsigma}$ is the Hilfer fractional derivative of order $\frac{1}{2} < \upsilon \leq 1$.

The above system is equivalent to the below integral equation

$$\begin{aligned} y(\rho) = {} & \frac{(\varsigma(\rho)-\varsigma(0))^{\upsilon-1}}{\Gamma(\upsilon)}(y_0 - g(y)) + \frac{1}{\Gamma(\upsilon)} \int_0^\rho (\varsigma(\rho) - \varsigma(s))^{\upsilon-1} \Lambda y(\varpi)\, d\varpi + \frac{1}{\Gamma(\upsilon)} \int_0^\rho (\varsigma(\rho) - \varsigma(\varpi))^{\upsilon-1} \\ & [\mathcal{G}(\varpi, y(\varpi)) + Bu(\varpi)]\, d\varpi + \int_0^\rho (\varsigma(\rho) - \varsigma(s))^{\upsilon-1} \Delta(\varpi, y(\varpi)) dW(\rho). \end{aligned} \tag{3.2}$$

If (3.2) holds, then

$$\begin{aligned} y(\rho) = {} & K_{\upsilon,\sigma}(\rho)(y_0 - g(y)) + \int_0^\rho R_\upsilon(\varsigma(\rho) - \varsigma(s))[\mathcal{G}(\varpi, y(\varpi) + \mathfrak{K}u(\varpi))]\varsigma'(\varpi) d\varpi \\ & + \int_0^\rho R_\upsilon(\varsigma(\rho) - \varsigma(\varpi))\Delta(\varpi)\varsigma'(\varpi) dw(\varpi) \end{aligned} \tag{3.3}$$

where $K_{\upsilon,\sigma}(\rho) = I_{a^+}^{\upsilon(1-\sigma);\varsigma} K_\upsilon(\rho)$, $R_\upsilon(\rho) = \rho^{\upsilon-1} P_\upsilon(\rho)$, $P_\upsilon(\rho) = \int_0^\infty \upsilon\theta\Phi_\upsilon \mathfrak{I}(\rho^\upsilon\theta) d\theta$. Let $\lambda > 0$. Taking laplace transform on both sides of equation 2.1, we get

$$\begin{aligned} \chi(r) &= (L^{-1}(\lambda^{\upsilon(\mu-1)} * K_\mu(r))\chi_0 + \int_0^r K_\mu(r-\zeta)[\mathcal{F}(\zeta, \chi(\zeta) + \mathfrak{D}u(\zeta))] d\zeta + \int_0^r K_\mu(r-\zeta)\varpi(\zeta, \chi(\zeta))\, dW(\zeta) \\ &= (I_{0^+}^{\upsilon(1-\mu)} K_\mu(r))\chi_0 + \int_0^r K_\mu(r-\zeta)[\mathcal{F}(\zeta, \chi(\zeta) + \mathfrak{D}u(\zeta))] d\zeta + \int_0^r K_\mu(r-\zeta)\varpi(\zeta, \chi(\zeta))\, dW(\zeta) \\ &= \mathfrak{S}_{\upsilon,\mu}(r)\chi_0 + \int_0^r K_\mu(r-\zeta)[\mathcal{F}(\zeta, \chi(s) + \mathfrak{D}u(\zeta))] d\zeta + \int_0^r K_\mu(r-\zeta)\varpi(\zeta, \chi(\zeta))\, dW(\zeta). \end{aligned} \tag{3.4}$$

4. Main results

Here, we first show that the system 1.1 has exact solutions. On the supposition that the controllability operator has an enforced inverse on a quotient space, we specifically convert the controllability issue into a fixed-point problem. Furthermore, we demonstrate whether under specific conditions, it is possible to infer the approximate controllability of 1.1 from the approximate controllability of the associated linear system. The preceding presumptions are required in order to demonstrate the finding:

(A1) The operators $K_{\upsilon}(\rho)$ and $R_{\upsilon}(\rho)$ are compact.

(A2) The multivalued map $\mathcal{G} : J \times H \to P_{bd,cl,cv}(H)$ is an $L^2 - Caratheodory$ function satisfies the below conditions:

(i) For each $\rho \in J$, the function $F(\rho, .) : H \to P_{bd,cl,cv}(H)$ is upper semicontinuous and for each $y \in H$, the function $\mathcal{F}(., y)$ is measurable. Also, for each fixed $y \in C$, the set

$$R_{\mathcal{G},y} = \{\mathcal{G} \in L^2(\upsilon, H) : \mathcal{G}(\rho) \in \mathcal{G}(\rho, y), fora.e.\rho \in J\}$$

is nonempty.

(ii) For each positive $h > 0$, there exists a positive function $M_{\mathcal{G}}(h)$ independent on h such that

$$\sup_{E\|y\|^2 \le h} \|F(\rho, y)\|^2 \le M_{\mathcal{G}}(h), \ fora.e. \ \rho \in J$$

where $\|\mathcal{G}(\rho, y)\|^2 = \sup_{\mathcal{G} \in F(\rho,y)} E\|\mathcal{G}\|^2$.

(A3) The multivalued map $\mathcal{F} : J \times H \to P_{bd,cl,cv}(L(K, H))$ is a $L^2 - Caratheodory$ function satisfies the following conditions:

(i) For each $\rho \in J$, the function $\mathcal{F}(\rho, .) : H \to P_{bd,cl,cv}(L(K, H))$ is upper semicontinuous and for each $y \in H$ the function $\mathcal{F}(., y)$ is measurable. Additionally, for each fixed $X \in C$, the set

$$R_{\mathcal{F},y} = \{G \in L^2(L(K, H)) : G(\rho) \in \mathcal{F}(\rho, y), \ \ for \ a.e. \ \rho \in J\}$$

is nonempty.

(ii) For each positive $h > 0$, there exists a positive function $M_F(h)$ independent on h such that

$$\sup_{E\|y\|^2 \le h} \|\mathcal{F}(\rho, y)\|^2 \le M_{\mathcal{F}}(h), \ for \ a.e. \ \rho \in J.$$

(A4) The function $g \in C \to H$ is completely continuous and there exists positive constant μ_1 and μ_2 such that

$$\|g(y)\|^2 \le \mu_1 \|y\|_C^2 + \mu_2, \ \ \forall \ \ y \in C.$$

(A5) There exists a real number r such that

$$\frac{L_1 + 4\tilde{M}_R^2 \frac{b^{2\upsilon-1}}{2\upsilon-1}[bM_{\mathcal{G}}(r) + Tr(Q)\, M_{\Delta}(r)]\left(1 + \frac{4}{\alpha^2} M_B^4 \tilde{M}_R^4 \frac{b^{4\upsilon-2}}{4\upsilon-3}\right)}{1 - L_2} < r, \tag{4.1}$$

where

$$L_1 = 8\tilde{M}_K^2\left(E\|y_0\|^2+\mu_2\right)+\frac{4}{\alpha^2}M_B^4\tilde{M}_R^4\frac{b^{4\upsilon-2}}{4\upsilon-3}\left\{8\|E\tilde{y}_b\|^2+8\int\limits_0^b E\left\|\tilde{\phi}(\varpi)\right\|_{L_2^0}^2 d\varpi+8\tilde{M}_K^2\left(E\|y_0\|^2+\mu_2\right)\right\},$$
$$L_2 = 8\tilde{M}_K^2\mu_1\left(1+\frac{4}{\alpha^2}M_B^4\tilde{M}_R^4\frac{b^{4\upsilon-2}}{4\upsilon-3}\right).$$

The control function must be defined by the following lemma.

Lemma 4.1. *[7] For any $\bar{y}_b \in L^2(\mathcal{G}_b, H)$ there exists $\bar{\phi} \in L^2_{\mathcal{G}}\left(\upsilon, L^2\left([0,b], L_2^0\right)\right)$ such that $\bar{y}_b = Ey_b+\int\limits_0^b \bar{\phi}dw(\varpi)$. Then, we define the control function for every value of $\alpha > 0$ and $\bar{y}_b \in L^2(\mathcal{G}_b, H)$:*

$$u_y^\alpha(\rho) = B^*R_\upsilon^*(b-\rho)\left[\left(\alpha I+\Psi_0^b\right)^{-1}\left[E\bar{y}_b-K_\upsilon(b)(y_0-g(y))\right]+\int\limits_0^b\left(\alpha I+\Psi_\varpi^b\right)^{-1}\bar{\phi}(\varpi)dw(\varpi)\right]$$
$$-B^*R_\upsilon^*(b-\rho)\int\limits_0^b\left(\alpha I+\Psi_\varpi^b\right)^{-1}R_\upsilon(b-\varpi)\mathcal{F}(\varpi)d\varpi-B^*R_\upsilon^*(b-\rho)\int\limits_0^b\left(\alpha I+\Psi_\varpi^b\right)^{-1}R_\upsilon(b-\varpi)\Delta(\varpi)dw(\varpi).$$

Theorem 4.2. *Assume that (A1)–(A5) are fulfilled. The moderate solution on J is therefore provided by the fractional control system 1.1.*

Proof. For any $\alpha > 0$, we define the multivalued operator $\Phi : C \to P(C)$ by

$$\Phi(y) = \left\{z \in C : z(\rho) = K_\upsilon(\rho)(y_0-g(y))+\int\limits_0^\rho R_\upsilon(\rho-\varpi)Bu_y^\alpha(\varpi)d\varpi\right.$$
$$\left.+\int\limits_0^\rho R_\upsilon(\rho-\varpi)\mathcal{G}(\varpi)d\varpi+\int\limits_0^\rho R_\upsilon(\rho-\varpi)\Delta(\varpi)dw(\varpi),\ \mathcal{G}\in R_{\mathcal{G},y}, \Delta\in R_{\mathcal{F},y}\right\}.$$

A fixed point for the operator Φ will now be demonstrated. There will be numerous steps in the evidence.

Step 1. For $\alpha > 0$, $\Phi(y)$ is convex for each $y \in C$. In fact, if $z_1, z_2 \in \Phi(y)$, then there exists $\mathcal{G}_1, \mathcal{G}_2 \in R_{\mathcal{G},y}$ and $\Delta_1, \Delta_2 \in R_{\mathcal{F},y}$ such that for each $\rho \in J$,

$$\begin{aligned} z_i(\rho) = {} & K_{\upsilon,\sigma}(\rho)(y_0-g(y))+\int_0^\rho R_\upsilon(\varsigma(\rho)-\varsigma(\varpi))\mathfrak{F}_i(\varpi)d\varpi+\int_0^\rho R_\upsilon(\varsigma(\rho)-\varsigma(\varpi))\Delta_i(\varpi)dw(\varpi) \\ & +\int_0^\rho R_\upsilon(\rho-\xi)B\times B^*R_\upsilon^*(b-\xi)[(\alpha I+\Psi_0)^{b^{-1}}[E\bar{y}_b-K_{\upsilon,\sigma}(b)(y_0-g(y))] \\ & +\int_0^b(\alpha I+\Psi_\varpi^b)^{-1}\bar{\phi}(\varpi)dw(\varpi)]-B^*R_{\upsilon,\sigma}^*(b-\xi)\int_0^b(\alpha I+\Psi_\varpi^b)^{-1}R_\upsilon(\varsigma(b)-\varsigma(\varpi))\mathcal{G}_i(\varpi)d\varpi \\ & -B^*R_{\upsilon,\sigma}^*(b-\xi)\int_0^b(\alpha I+\Psi_\varpi^b)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\Delta_i(\varpi)dw(\varpi)d\xi,\ i=1,2. \end{aligned}$$

Let $\lambda \in [0,1]$, then for each $\rho \in J$, we have

$$\lambda z_1(\rho) + (1-\lambda) z_2(\rho) = K_{\upsilon,\sigma}(\rho)(y_0 - g(y)) + \int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho) - \varsigma(\varpi))\left[\lambda \mathcal{G}_1(\varpi) + (1-\lambda)\mathcal{G}_2(\varpi)\right] d\varpi$$
$$+ \int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho) - \varsigma(\varpi))\left[\lambda \Delta_1(\varpi) + (1-\lambda)\Delta_2(\varpi)\right] dw(\varpi) + \int_0^\rho R_{\upsilon,\sigma}(\rho - \xi) B$$
$$\times \left\{ B^* R^*_{\upsilon,\sigma}(b-\xi)\left[\left(\alpha I + \Psi_0^b\right)^{-1}\left[E\tilde{y}_b - K_{\upsilon,\sigma}(b)(y_0 - g(y))\right] + \int_0^b \left(\alpha I + \Psi_\varpi^b\right)^{-1} \tilde{\phi}(\varpi) dw(\varpi)\right]\right.$$
$$-B^* R^*_{\upsilon,\sigma}(b-\xi)\int_0^b \left(\alpha I + \Psi_\varpi^b\right)^{-1} R_{\upsilon,\sigma}(\varsigma(\rho) - \varsigma(\varpi))\left[\lambda \mathcal{G}_1(\varpi) + (1-\lambda)\mathcal{G}_2(\varpi)\right] d\varpi$$
$$\left. -B^* R^*_{\upsilon,\sigma}(b-\xi)\int_0^b \left(\alpha I + \Psi_\varpi^b\right)^{-1} R_{\upsilon,\sigma}(\varsigma(\rho) - \varsigma(\varpi))\left[\lambda \Delta_1(\varpi) + (1-\lambda)\Delta_2(\varpi)\right] dw(\varpi)\right\} d\xi.$$

Since $R_{\mathcal{G},y}$ and $R_{\mathcal{F},y}$ are convex , $\lambda\mathcal{G}_1 + (1-\lambda)\mathcal{G}_2 \in R_{\mathcal{G},y}$, $\lambda\Delta_1 + (1-\lambda)\Delta_2 \in R_{\mathcal{F},y}$. Thus $\lambda z_1 + (1-\lambda) z_2 \in \Phi(y)$.

Step 2. Φ_2 maps bounded sets into bounded sets in C. Take into consideration a set $B_h = \{y \in C : \|y\|_C^2 \le h, 0 \le \rho \le b\}$, where h is a positive constant. It is clear that set C is bounded, closed and convex in B_h. In fact, it is sufficient to establish the existence of a positive constant $\hat{L}$ such that one obtains $E\|z(\rho)\|^2 \le \hat{L}$ for any $z \in \Phi(y), y \in B_h$.

Let $z \in \Phi_2(y)$ and $y \in B_h$. Then, there exists $\mathcal{G} \in R_{(\mathcal{G},y)}$ and $\Delta \in R_{\mathcal{F},y}$ such that for each $\rho \in J$,

$$z(\rho) = K_{\upsilon,\sigma}(\rho)(y_0 - g(y)) + \int_0^\rho R_\upsilon(\varsigma(\rho) - \varsigma(\varpi)) B u_y^\alpha(\varpi) d\varpi + \int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho) - \varsigma(\varpi))\mathcal{G}(\varpi) d\varpi$$
$$+ \int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho) - \varsigma(\varpi))\Delta(\varpi) dw(\varpi).$$

For each $\rho \in J$,

$$E\left\|u_y^\alpha\right\|^2 \le \frac{1}{\alpha^2} M_B^2 \tilde{M}_R^2 (b-\rho)^{2\upsilon-2}\left\{4\left\|E\tilde{y}_b + \int_0^b \tilde{\phi}(\varpi) dw(\varpi)\right\|^2 + 4E\left\|K_{\upsilon,\sigma}(b)(y_0 - g(y))\right\|^2\right.$$
$$\left. +4E\left\|\int_0^b R_{\upsilon,\sigma}(\varsigma(b) - \varsigma(\varpi))\mathcal{G}(\varpi) d\varpi\right\|^2 + 4E\left\|\int_0^b R_{\upsilon,\sigma}(\varsigma(b) - \varsigma(\varpi))\Delta(\varpi) dw(\varpi)\right\|^2\right\}$$
$$\le \frac{4}{\alpha^2} M_B^2 \tilde{M}_R^2 (b-\rho)^{2\delta-2}\left\{2\|E\tilde{y}_b\|^2 + 2\int_0^b E\left\|\tilde{\phi}(\varpi)\right\|_{L_2^0}^2 d\varpi + \tilde{M}_T^2\left(2E\|y_0\|^2 + 2E\|g(y)\|^2\right)\right.$$
$$\left. +bE\int_0^b \left\|R_{\upsilon,\sigma}(\varsigma(b) - \varsigma(\varpi))\mathcal{G}(\varpi)\right\|^2 d\varpi + Tr(Q) E\left\|\int_0^b R_{\upsilon,\sigma}(\varsigma(b) - \varsigma(\varpi))\Delta(\varpi)\right\|^2 dw(\varpi)\right\}$$

$$
\begin{aligned}
&\leq \frac{4}{\alpha^2} M_B^2 \tilde{M}_R^2 (b-\rho)^{2\upsilon-2} \left\{ 2\|E\tilde{y}_b\|^2 + 2\int_0^b E\left\|\tilde{\phi}(\varpi)\right\|_{L_2^0}^2 d\varpi + 2\tilde{M}_T^2 \left(E\|y_0\|^2 + \mu_1 h + \mu_2\right) \right. \\
&\quad \left. + b\tilde{M}_R^2 \frac{b^{2\upsilon-1}}{2\upsilon-1} M_{\mathcal{G}}(h) + Tr(Q)\,\tilde{M}_R^2 \frac{b^{2\upsilon-1}}{2\upsilon-1} M_\Delta(h) \right\} \\
&\leq (b-\rho)^{2\upsilon-2} M_U,
\end{aligned}
$$

where

$$
\begin{aligned}
M_U = \frac{4}{\alpha^2} M_B^2 \tilde{M}_R^2 &\left\{ 2\|E\tilde{y}_b\|^2 + 2\int_0^b E\left\|\tilde{\phi}(\varpi)\right\|_{L_2^0}^2 d\varpi + 2\tilde{M}_T^2 \left(E\|y_0\|^2 + \mu_1 h + \mu_2\right) \right. \\
&\quad \left. + b\tilde{M}_R^2 \frac{b^{2\upsilon-1}}{2\upsilon-1} M_{\mathcal{G}}(h) + Tr(Q)\,\tilde{M}_R^2 \frac{b^{2\upsilon-1}}{2\upsilon-1} M_\Delta(h) \right\}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
E\|z(\rho)\|^2 &\leq 4E\left\|K_{\upsilon,\sigma}(b)(y_0 - g(y))\right\|^2 + 4E\left\|\int_0^b R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\,Bu_y^\alpha(\varpi)\,d\varpi\right\|^2 \\
&\quad + 4E\left\|\int_0^b R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\,\mathcal{G}(\varpi)\,d\varpi\right\|^2 + 4E\left\|\int_0^b R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\,\Delta(\varpi)\,dw(\varpi)\right\|^2 \\
&\leq 8\tilde{M}_K^2\left(E\|y_0\|^2 + \mu_1 h + \mu_2\right) + 4bM_B^2\tilde{M}_R^2 \frac{b^{4\upsilon-3}}{4\upsilon-3} M_U + 4b\tilde{M}_R^2 \frac{b^{2\upsilon-1}}{2\upsilon-1} M_{\mathcal{G}}(h) \\
&\quad + 4Tr(Q)\,\tilde{M}_R^2 \frac{b^{2\upsilon-1}}{2\upsilon-1} M_\Delta(h) \\
&= \hat{L}.
\end{aligned}
$$

Hence, Φ maps bounded sets into bounded set in C.

Step 3. Φ maps bounded sets into equicontinuous sets of C. Let $0 < \rho_1 < \rho_2 \leq b$ and $\epsilon > o$. For each $z \in \Phi(y)$ and $y \in B_h$, there exists $\mathcal{G} \in R_{\mathcal{G},y}$ and $\Delta \in R_{\mathcal{F},y}$ such that

$$
\begin{aligned}
z(\rho) &= K_{\upsilon,\sigma}(\rho)(y_0 - g(y)) + \int_0^\rho R_{\upsilon,\varsigma}(\varsigma(\rho)-\varsigma(\varpi))\,Bu_y^\alpha(\varpi)\,d\varpi + \int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\,\mathcal{G}(\varpi)\,d\varpi \\
&\quad + \int_0^\rho R_{\upsilon,\sigma}(\varsigma\rho - \varsigma\varpi)\,\Delta(\varpi)\,dw(\varpi) E\|z(\rho_2) - z(\rho_1)\|^2 \\
&\leq 4E\left\|[K_{\upsilon,\sigma}(\rho_2) - K_\delta(\rho_1)](y_0 - g(y))\right\|^2 + 4E\|\int_0^{\rho_2} R_{\upsilon,\sigma}(\varsigma(\rho_2)-\varsigma(\varpi))\,Bu_y^\alpha(\varpi)\,d\varpi \\
&\quad - \int_0^{\rho_1} R_{\upsilon,\sigma}(\varsigma(\rho_1)-\varsigma(\varpi))\,Bu_y^\alpha(\varpi)\,d\varpi\|^2 + 4E\|\int_0^{\rho_2} R_{\upsilon,\sigma}(\varsigma(\rho_2)-(\varpi))\,\mathcal{G}(\varpi)\,d\varpi
\end{aligned}
$$

$$
\begin{aligned}
& -\int_0^{\rho_1} R_{\upsilon,\sigma}\left(\varsigma(\rho_1)-\varsigma(\varpi)\right)\mathcal{G}(\varpi)d\varpi\|^2+4E\|\int_0^{\rho_2} R_{\upsilon,\sigma}(\varsigma(\rho_2)-\varsigma(\varpi))\Delta(\varpi)dw(\varpi) \\
& -\int_0^{\rho_1} R_{\upsilon,\sigma}(\varsigma(\rho_1)-\varsigma(\varpi))\Delta(\varpi)dw(\varpi)\|^2 \\
\leq\ & 10E\left\|\left[K_{\upsilon,\sigma}(\rho_2)-K_{\upsilon,\sigma}(\rho_1)\right](y_0-g(y))\right\|^2+10bM_B^2M_U\int_0^{\rho_1-\varepsilon}(\varsigma(b)-\varsigma(\varpi))^{2\upsilon-2} \\
& \|R_{\upsilon,\sigma}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)-R_{\upsilon,\varsigma}\left(\varsigma(\rho_1)-\varsigma(\varpi)\right)\|^2d\varpi+10\varepsilon M_B^2M_U\int_{\rho_1-\varepsilon}^{\rho_1}(\varsigma(b)-\varsigma(\varpi))^{2\upsilon-2} \\
& \|R_{\upsilon}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)-R_{\upsilon,\sigma}\left(\varsigma(\rho_1)-\varsigma(\varpi)\right)\|^2d\varpi+10\left(\rho_2-\rho_1\right)M_B^2M_U\int_{\rho_1}^{\rho_2}(\varsigma(b)-\varsigma(\varpi))^{2\upsilon-2} \\
& \left\|R_{\upsilon,\sigma}\left(\upsilon(\rho_2)-\upsilon(\varpi)\right)\right\|^2d\varpi+10bM_{\mathcal{G}}(h)\int_0^{\rho_1-\varepsilon}\left\|R_{\upsilon,\sigma}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)-R_{\upsilon,\sigma}\left(\varsigma(\rho_1)-\varsigma(\varpi)\right)\right\|^2d\varpi \\
& +10\varepsilon M_{\mathcal{G}}(h)\int_{\rho_1-\varepsilon}^{\rho_1}\left\|R_{\upsilon,\sigma}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)-R_{\upsilon,\sigma}\left(\varsigma(\rho_1)-\varsigma(\varpi)\right)\right\|^2d\varpi+10\left(\rho_2-\rho_1\right)M_{\mathcal{G}}(h) \\
& \int_{\rho_1}^{\rho_2}\left\|R_{\upsilon,\sigma}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)\right\|^2d\varpi+10Tr(Q)M_{\Delta}(h)\int_0^{\rho_1-\varepsilon}\left\|R_{\upsilon,\sigma}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)-R_{\upsilon,\sigma}\left(\varsigma(\rho_1)-\varsigma(\varpi)\right)\right\|^2d\varpi \\
& +10Tr(Q)M_{\Delta}(h)\int_{\rho_1-\varepsilon}^{\rho_1}\left\|R_{\upsilon,\sigma}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)-R_{\upsilon,\sigma}\left(\varsigma(\rho_1)-\varsigma(\varpi)\right)\right\|^2d\varpi+10Tr(Q)M_{\Delta}(h) \\
& \int_{\rho_1}^{\rho_2}\left\|R_{\upsilon,\sigma}\left(\varsigma(\rho_2)-\varsigma(\varpi)\right)\right\|^2d\varpi.
\end{aligned}
$$

If ϵ is sufficiently small, the right side of the inequality given reduces to zero as $\rho_1 \to \rho_2$. As $K_\upsilon(\rho)$ and $R_\upsilon(\rho)$ are compact, they imply that the uniform operator topology is continuous for both $K_\upsilon(\rho)$ and $R_\upsilon(\rho)$, accordingly. Φ is therefore equicontinuous.

Step 4. We show that $V(\rho) = [z(\rho) : z \in \Phi(B_h)]$ is relatively compact for $\rho \in J$. The case $\rho = 0$ is trivial. Let $\rho \in (0, b]$ be fixed. For each $\varepsilon \in (0, \rho)$. For each $z \in \Phi(y)$ and $y \in B_h$, there exists $\mathcal{G} \in R_{\mathcal{G},y}$ and $\Delta \in R_{\mathcal{F},y}$ such that

$$
\begin{aligned}
z(\rho) &= K_{\upsilon,\sigma}(\rho)(y_0-g(y))+\int_0^{\rho} R_{\upsilon,\sigma}\left(\varsigma(\rho)-\varsigma(\varpi)\right)Bu_y^{\alpha}(\varpi)d\varpi+\int_0^{\rho} R_{\upsilon,\sigma}\left(\varsigma(\rho)-\varsigma(\varpi)\right)\mathcal{G}(\varpi)d\varpi \\
&\quad +\int_0^{\rho} R_{\upsilon,\sigma}\left(\varsigma(\rho)-\varsigma(\varpi)\right)\Delta(\varpi)dw(\varpi).
\end{aligned}
$$

Define

$$z_{\varepsilon}(\rho) = K_{\upsilon,\sigma}(\rho)(y_0 - g(y)) + R_{\upsilon,\sigma}(\varepsilon)\int_0^{\rho-\varepsilon} R_{\upsilon,\sigma}(\varsigma(\rho-\varepsilon)-\varsigma(\varpi))\,Bu_y^{\alpha}(\varpi)\,d\varpi$$
$$+R_{\upsilon,\sigma}(\varepsilon)\int_0^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho-\varepsilon)-\varsigma(\varpi))\,\mathcal{G}(\varpi)\,d\varpi + R_{\upsilon,\sigma}(\varepsilon)\int_0^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho-\varepsilon)-\varsigma(\varpi))\,\Delta(\varpi)\,dw(\varpi).$$

Since $R_{\upsilon}(\rho)$ is compact, the set $V_{\varepsilon}(\rho) = \{z_{\varepsilon}(\rho) : z \in \Phi(B_h)\}$ is relatively compact in H. On the other hand, we have

$$\begin{aligned}
&E\|z(\rho) - z_{\varepsilon}(\rho)\|^2\\
\leq\ & 4E\left\|K_{\upsilon,\sigma}(\rho)(y_0 - g(y)) - K_{\upsilon,\sigma}(\rho)(y_0 - g(y))\right\|^2\\
&+4E\left\|\int_0^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(s))\,Bu_y^{\alpha}(\varpi)\,d\varpi - R_{\upsilon,\sigma}(\varepsilon)\int_0^{\rho-\varepsilon} R_{\upsilon,\sigma}(\varsigma(\rho-\varepsilon)-\varsigma(\varpi))\,Bu_y^{\alpha}(\varpi)\,d\varpi\right\|^2\\
&+4E\left\|\int_0^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\,\mathcal{G}(\varpi)\,d\varpi - R_{\upsilon,\sigma}(\varepsilon)\int_0^{\rho-\varepsilon} R_{\upsilon,\sigma}(\varsigma(\rho-\varepsilon)-\varsigma(\varpi))\,\mathcal{G}(\varpi)\,d\varpi\right\|^2\\
&+4E\left\|\int_0^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(s))\,\Delta(\varpi)\,dw(\varpi) - R_{\upsilon,\sigma}(\varepsilon)\int_0^{\rho-\varepsilon} R_{\upsilon,\sigma}(\varsigma(\rho-\varepsilon)-\varsigma(\varpi))\,\Delta(\varpi)\,dw(\varpi)\right\|^2\\
\leq\ & 3E\left\|\int_{\rho-\varepsilon}^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(s))\,Bu_y^{\alpha}(\varpi)\,d\varpi\right\|^2 + 3E\left\|\int_{\rho-\varepsilon}^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\,\mathcal{G}(\varpi)\,d\varpi\right\|^2\\
&+3E\left\|\int_{\rho-\varepsilon}^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\,\Delta(\varpi)\,dw(\varpi)\right\|^2\\
\leq\ & 3bM_B^2\tilde{M}_R^2\frac{\varepsilon^{4\upsilon-3}}{4\upsilon-3}M_U + 3b\tilde{M}_R^2\frac{\varepsilon^{2\upsilon-1}}{2\upsilon-1}M_{\mathcal{G}}(h) + 3Tr(Q)\,\tilde{M}_R^2\frac{\varepsilon^{2\upsilon-1}}{2\upsilon-1}M_{\Delta}(h).
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we can see that there are relatively compact sets arbitrarily close to $V(\rho)$ for each $\rho \in (0, b]$. Thus, $V(\rho)$ is relatively compact in X, for all $\rho \in (0, b]$. Hence, $V(\rho)$ is relatively compact in X for all $\rho \in J$.

Step 5. Φ has a closed graph.

Let $y^n \to y^*$ and $z^n \to z^*$ as $n \to \infty$. We shall prove that $z^* \in \Phi(y^*)$. Since $z^n \in \Phi(y^n)$, there exists $\mathcal{G}^n \in R_{\mathcal{G},y^n}$ and $\Delta^n \in R_{\mathcal{F},y^n}$ such that for each $\rho \in J$,

$$z^n(\rho) = K_{\upsilon,\sigma}(\rho)(y_0 - g(y^n)) + \int_0^{\rho} R_{\upsilon,\sigma}(\varsigma\rho - \varsigma\varpi)\mathcal{G}^n(\varpi)d\varpi + \int_0^{\rho} R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(s))\Delta^n(\varpi)dw(\varpi)$$
$$+\int_0^{\rho} R_{\upsilon,\sigma}(\rho-\xi)B \times B^* R^*_{\upsilon,\sigma}(b-\xi)(\alpha I + \Psi_0^b)^{-1}[E\bar{y}_b - K_{\upsilon,\sigma}(b)(y_0 - g(y^n))]$$

$$+\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}\tilde{\phi}(\varpi)dw(\varpi)-B^*R_{\upsilon,\sigma}^*(b-\xi)\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}R_{\delta}(\varsigma(b)-\varsigma(\varpi))\mathcal{G}^n(\varpi)d\varpi$$

$$-B^*R_{\upsilon,\sigma}^*(b-\xi)\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\Delta^n(\varpi)dw(\varpi)d\xi.$$

We must prove that there exists $\mathcal{G}^*\in R_{\mathcal{G},y^*}$ and $\Delta^*\in R_{\mathcal{F},y^*}$, such that for each $\rho\in J$,

$$\begin{aligned} z^*(\rho) &= K_{\upsilon,\sigma}(\rho)(y_0-g(y^*))+\int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\mathcal{G}^*(\varpi)d\varpi+\int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\Delta^*(\varpi)dw(\varpi) \\ &\quad+\int_0^\rho R_{\upsilon,\sigma}(\rho-\xi)B\times B^*R_{\upsilon,\sigma}^*(b-\xi)[\alpha I+\Psi_0^{b^{-1}}[E\bar{y}_b-K_{\upsilon}(b)(y_0-g(y^*))] \\ &\quad+\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}\tilde{\phi}(\varpi\upsilon)dw(\varpi)]-B^*R_{\upsilon,\sigma}^*(b-\xi)\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}R_{\upsilon}(\varsigma(b)-\varsigma(\varpi))\mathcal{G}^*(s\varpi)d\varpi \\ &\quad-B^*R_{\upsilon,\sigma}^*(b-\xi)\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}R_{\upsilon}(\varsigma(b)-\varsigma(\varpi))\Delta^*(\varpi)dw(\varpi)d\xi. \end{aligned}$$

Since g is continuous, we obtain that

$$\begin{aligned} &\|(z^n(\rho)-K_{\upsilon,\sigma}(\rho)(y_0-g(y^n))+\int_0^\rho R_{\upsilon,\sigma}(\rho-\xi)BB^*R_{\upsilon,\sigma}^*(b-\xi) \\ &\times[(\alpha I+\Psi_0^b)^{-1}[E\tilde{y}_b-K_{\upsilon,\sigma}(b)(y_0-g(y^n))]+\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}\tilde{\phi}(\varpi)dw(s)]d\xi) \\ &-(z^*(\rho)-K_{\upsilon,\sigma}(\rho)(y_0-g(y^*))+\int_0^\rho R_{\upsilon,\sigma}(\rho-\xi)BB^*R_{\upsilon,\sigma}^*(b-\xi)) \\ &\times[(\alpha I+\Psi_0^b)^{-1}[E\tilde{y}_b-K_{\upsilon,\sigma}(b)(y_0-g(y^*))]+\int_0^b (\alpha I+\Psi_{\varpi}^b)^{-1}\tilde{\phi}(\varpi)dw(\varpi)]d\xi\|^2\to 0,\ n\to\infty. \end{aligned}$$

Consider the linear continuous operator $\Psi: L^2(\upsilon,H)\times L^2(L(K,H))\to C(J,H)$,

$$\begin{aligned} (\mathcal{G},\Delta)\to\Psi(\mathcal{G},\Delta)(\rho) &= \int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\Big[\mathcal{G}(\varpi)+BB^*R_{\upsilon,\sigma}^*(\varsigma(b)-\varsigma(\varpi))\times\Big(\int_0^b\Big(\alpha I+\Psi_{\varpi}^b\Big)^{-1} \\ & R_{\upsilon,\sigma}(b-\rho)\mathcal{G}(\rho)d\rho\Big)\Big]d\varpi+\int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\Big[\Delta(\varpi)+BB^*R_{\upsilon,\sigma}^*(\upsilon(b)-\upsilon(\varpi))\times \\ & \Big(\int_0^b(\alpha I+\Psi_{\varpi}^b)^{-1}R_{\upsilon,\sigma}(b-\rho)\Delta(\rho)dw(\rho)\Big)\Big]dw(\varpi). \end{aligned}$$

Clearly, it follows from Lemma 2.3 that $\Psi\circ R_{\mathcal{G},\mathcal{F}}$ is closed graph operator, where $R_{\mathcal{G},\mathcal{F}}=\{\mathcal{G}\in\mathcal{G}(\rho,y(\rho))\}\times\{\Delta\in\mathcal{G}(\rho,y(\rho))\}$. Also from the definition of Ψ, we have

$$(z^n(\rho)-K_{\upsilon,\sigma}(\rho)(y_0-g(y^n))+\int_0^\rho R_{\upsilon,\sigma}(\rho-\xi)BB^*R_{\upsilon,\sigma}^*(b-\xi)$$

$$\times\left[\left(\alpha I+\Psi_0^b\right)^{-1}\left[E\tilde{y}_b-K_{\upsilon,\sigma}(b)\left(y_0-h\left(y^n\right)\right)\right]+\int_0^b\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}\tilde{\phi}(\varpi)\,dw(\varpi)\right]d\xi)\in\Psi\left(R_{\mathcal{G},\mathcal{G},y^n}\right).$$

Since $y^n\to y^*$, it follows from Lemma 2.4 that

$$(z^*(\rho)-K_{\upsilon,\sigma}(\rho)(y_0-g(y^*))+\int_0^\rho R_{\upsilon,\sigma}(\rho-\xi)\,BB^*R^*_{\upsilon,\sigma}(b-\xi)$$

$$\times\left[\left(\alpha I+\Psi_0^b\right)^{-1}\left[E\tilde{y}_b-K_{\upsilon,\sigma}(b)\left(y_0-g\left(y^*\right)\right)\right]+\int_0^b\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}\tilde{\phi}(\varpi)\,dw(\varpi)\right]d\xi)\in\Psi\left(R_{\mathcal{G},\mathcal{G},y^*}\right).$$

This shows that $z^*\in\Phi(y^*)$. Therefor, Φ has a closed graph.

The Arzela-Ascoli theorem and steps 1–5 together result in Φ being a compact multivalued map with convex closed values.

Step 6. The operator Φ has a solution.

Create an open ball with the definition $B(0,r)\in C$, where r fulfills the inequality 4.1. As a result of the aforementioned procedures, we are aware that Φ fulfills all of the requirements of Lemma 2.4. Since the system 1.1 has at least one mild solution, we may demonstrate that the second assumption of lemma 2.4 is false. Let $y\in C$ be possible solution for $\lambda y\in\Phi y$ for some $\lambda>1$ with $E\|y\|_\upsilon^2=r$. Then, we have

$$\begin{aligned} y(\rho) &= \lambda^{-1}K_{\upsilon,\sigma}(\rho)(y_0-g(y))+\lambda^{-1}\int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\,Bu_y^\alpha(\varpi)\,d\varpi \\ &\quad +\lambda^{-1}\int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\,\mathcal{G}(\varpi)\,d\varpi+\lambda^{-1}\int_0^\rho R_{\upsilon,\sigma}(\varsigma(\rho)-\varsigma(\varpi))\,\Delta(\varpi)\,dw(\varpi). \end{aligned}$$

Then, by the assumptions, we get

$$\begin{aligned} E\|y(\rho)\|^2 &\le 8\tilde{M}_K^2\left(E\|y_0\|^2+\mu_1E\|y\|^2+\mu_2\right)+\frac{4}{\alpha^2}bM_B^4\tilde{M}_R^4\int_0^\rho(b-\xi)^{4\upsilon-4}\left\{8\|E\tilde{y}_b\|^2+8\int_0^b E\left\|\tilde{\phi}(\varpi)\right\|^2_{L_2^0}d\varpi\right. \\ &\quad +8\tilde{M}_K^2(E\|y_0\|^2+\mu_1E\|y\|^2+\mu_2)+4b\tilde{M}_R^2\int_0^\rho(\varsigma(b)-\varsigma(\varpi))^{2\upsilon-2}M_{\mathcal{G}}(E\|y\|^2)d\varpi \\ &\quad +4Tr(Q)\,\tilde{M}_R^2\int_0^\rho(\varsigma(b)-\varsigma(\varpi))^{2\upsilon-2}M_\Delta\left(E\|y\|^2\right)d\varpi\}d\xi+4b\tilde{M}_R^2\int_0^\rho(\varsigma(\rho)-\varsigma(\varpi))^{2\upsilon-2} \\ &\quad M_{\mathcal{G}}\left(E\|y\|^2\right)d\varpi+4Tr(Q)\,\tilde{M}_R^2\int_0^\rho(\varsigma(\rho)-\varsigma(s))^{2\upsilon-2}M_\Delta\left(E\|y\|^2\right)d\varpi. \end{aligned}$$

When we apply the supremum to ρ, we get

$$E\|y(\rho)\|^2 \le 8\tilde{M}_K^2\left(E\|y_0\|^2+\mu_1E\|y\|^2+\mu_2\right)+\frac{4}{\alpha^2}bM_B^4\tilde{M}_R^4\int_0^\rho(b-\xi)^{4\upsilon-4}\left\{8\|E\tilde{y}_b\|^2+8\int_0^b E\left\|\tilde{\phi}(\varpi)\right\|^2_{L_2^0}d\varpi\right.$$

$$+8\tilde{M}_K^2\left(E\|y_0\|^2+\mu_1E\|y\|^2+\mu_2\right)+4b\tilde{M}_R^2\int\limits_0^\rho(\varsigma(\rho)-\varsigma(s))^{2\upsilon-2}M_{\mathcal{G}}\left(E\|y\|^2\right)d\varpi$$

$$+4Tr(Q)\tilde{M}_R^2\int_0^\rho(\varsigma(\rho)-\varsigma(\varpi))^{2\upsilon-2}M_\Delta(E\|y\|^2)ds\}d\xi+4b\tilde{M}_R^2\int\limits_0^\rho(\varsigma(\rho)-\varsigma(\varpi))^{2\upsilon}$$

$$-2M_{\mathcal{G}}(E\|y\|^2)d\varpi+4Tr(Q)\tilde{M}_R^2\int\limits_0^\rho(\varsigma(\rho)-\varsigma(\varpi))^{2\upsilon-2}M_\Delta(E\|y\|^2)d\varpi.$$

Substituting $E\|y\|_\upsilon^2=r$,

$$\begin{aligned}
r &\le 8\tilde{M}_K^2\left(E\|y_0\|^2+\mu_1r+\mu_2\right)+\frac{4}{\alpha^2}bM_B^4\tilde{M}_R^4\frac{b^{4\upsilon-3}}{4\upsilon-3}\left\{8\|E\tilde{y}_b\|^2+8\int\limits_0^b E\left\|\tilde{\phi}(\varpi)\right\|_{L_2^0}^2d\varpi\right.\\
&\left.+8\tilde{M}_K^2\left(E\|y_0\|^2+\mu_1E\|y\|^2+\mu_2\right)+4b\tilde{M}_R^2\frac{b^{2\upsilon-1}}{2\upsilon-1}M_{\mathcal{G}}(r)+4Tr(Q)\tilde{M}_R^2\frac{b^{2\upsilon-1}}{2\upsilon-1}M_\Delta(r)\right\}\\
&+4b\tilde{M}_R^2\frac{b^{2\upsilon-1}}{2\upsilon-1}M_{\mathcal{G}}(r)+4Tr(Q)\tilde{M}_R^2\frac{b^{2\upsilon-1}}{2\upsilon-1}M_\Delta(r)\\
r &\le \frac{L_1+4\tilde{M}_R^2\frac{b^{2\upsilon-1}}{2\upsilon-1}\left[bM_{\mathcal{G}}(r)+Tr(Q)M_\Delta(r)\right]\left(1+\frac{4}{\alpha^2}bM_B^4\tilde{M}_R^4\frac{b^{4\upsilon-3}}{4\upsilon-3}\right)}{1-L_2}
\end{aligned}$$

which is a contradiction to 4.1. Thus, the operator inclusions $y\in\Phi y$ has a solution in B[0,r]. Hence, the fractional stochastic evolution inclusion 1.1 has atleast one mild solution on J. □

Theorem 4.3. *Suppose that (A1)–(A5) are true and that $\mathcal{G}$ and $\mathcal{F}$ are both uniformly confined on their respective domains. Moreover, the fractional stochastic system 1.1 is roughly controllable on J if the fractional linear differential inclusions 2.4 are approximately controllable.*

Proof. Suppose that y^α is a fixed point on Φ. It is clear that via the stochastic Fubini theorem

$$y^\alpha(b)=\tilde{y}_b-\alpha\left(\alpha I+\Psi_0^b\right)^{-1}\left[E\tilde{y}_b-K_\upsilon(b)(y_0-g(y^\alpha))\right]-\alpha\int\limits_0^b\left(\alpha I+\Psi_\varpi^b\right)^{-1}\tilde{\phi}(\varpi)dw(\varpi)$$

$$+\alpha\int\limits_0^b\left(\alpha I+\Psi_\varpi^b\right)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\mathcal{G}^\alpha(\varpi)d\varpi+\alpha\int\limits_0^b\left(\alpha I+\Psi_\varpi^b\right)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\Delta^\alpha(\varpi)dw(\varpi),$$

where

$$\mathcal{G}^\alpha\in R_{\mathcal{G},y^\alpha}=\left\{\mathcal{G}^\alpha\in L^2(\upsilon,H).\mathcal{G}^\alpha(\rho)\in\mathcal{G}(\rho,y^\alpha(\rho)),\ for\ \rho\in J\right\}$$

and

$$\Delta^\alpha\in R_{\mathcal{F},y^\alpha}=\left\{\Delta^\alpha\in L^2(\upsilon,H).\Delta^\alpha(\rho)\in\mathcal{F}(\rho,y^\alpha(\rho))\ \ for\ a.e.\ \rho\in J\right\}.$$

It follows from the assumptions on $\mathcal{G}$ and $\mathcal{F}$ that there exist D such that $\|\mathcal{G}^{\alpha}(\varpi)\|^2 + \|\Delta^{\alpha}(\varpi)\|^2 \leq D$. Then, there is a subsequence denoted by $\{\mathcal{G}^{\alpha}(\varpi), \Delta^{\alpha}(\varpi)\}$ weakly converging to say $\{\mathcal{G}(\varpi), \Delta(\varpi)\}$. Now, the compactness of $R_{\upsilon}(\rho)$ implies that, $R_{\upsilon}(b-\varpi)\mathcal{G}^{\alpha}(s) \to R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\mathcal{G}(\varpi), R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\Delta^{\alpha}(s) \to R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\Delta(\varpi)$. The result of the equation before is

$$\begin{aligned}
E\|y^{\alpha}(b)-\tilde{y}_b\|^2 \leq\ & 6\left\|\alpha\left(\alpha I+\Psi_0^b\right)^{-1}[E\tilde{y}_b - K_{\upsilon,\sigma}(b)(y_0 - g(y^{\alpha}))]\right\|^2 \\
&+6E\left(\int_0^b \left\|\alpha\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}\tilde{\phi}(\varpi)\right\|_{L_2^0}^2 d\varpi\right) \\
&+6E\left(\int_0^b \left\|\alpha\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))[\mathcal{G}^{\alpha}(\varpi)-\mathcal{G}(\varpi)]\right\|d\varpi\right)^2 \\
&+6E\left(\int_0^b \left\|\alpha\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\mathcal{G}(\varpi)\right\|d\varpi\right)^2 \\
&+6E\left(\int_0^b \left\|\alpha\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))[\Delta^{\alpha}(\varpi)-\Delta(\varpi)]\right\|_{L_2^0}^2 d\varpi\right) \\
&+6E\left(\int_0^b \left\|\alpha\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}R_{\upsilon,\sigma}(\varsigma(b)-\varsigma(\varpi))\Delta(\varpi)\right\|_{L_2^0}^2 d\varpi\right).
\end{aligned}$$

On the other hand, by Lemma 2.3, for all $0 \leq \varpi \leq b$, the operator $\alpha\left(\alpha I+\Psi_{\varpi}^b\right)^{-1} \to 0$ strongly as $\alpha \to 0^+$, and moreover, $\|\alpha\left(\alpha I+\Psi_{\varpi}^b\right)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem, we obtain $E\|y^{\alpha}(b)-\tilde{y}_b\|^2 \to 0$ as $\alpha \to 0^+$.

This provides an approximate idea of the control system 1.1's controllability. □

5. Example

Assume the subsequent fractional stochastic partial differential equations with the given nonlocal conditions:

$$\begin{cases}
\frac{\partial^{\frac{2}{3}}}{\partial\rho^{\frac{2}{3}}}y(\rho,z) = y_{zz}(\rho,z) + \mu(\rho,z) + K_1(\rho,y(\rho,z)) + K_2(\rho,y(\rho,z))\frac{dW(\rho)}{d\rho}, \ \rho \in J = [0,1], \\
y(\rho,0) = y(\rho,0) = 0, \\
y(0,z) + \sum_{i=1}^{m} c_i y(\rho_i,z) = y_0(z), \ 0 \leq z \leq 1,
\end{cases} \tag{5.1}$$

where $W(\rho)$ denotes a standard cylindrical Weiner process on $(\upsilon, \mathcal{G}, \mathcal{G}_{\rho}\}, P)$ and $y_0 \in L^2(0,1)$; $\mu : [0,1]\times(0,1) \to (0,1)$ is continuous in ρ; $K_1, K_2 : R \to P(R)$ is continuous and $c_i > 0$. Let $H = U = L^2(0,1)$ and define the operator $\Lambda : H \to H$ by $\Lambda, z = z''$ with domain $D(\Lambda) = \{z \in H, z, z'$ are absolutely continuous, $z'' \in H, z(0) = z(1) = 0\}$. Then, Λ generates an analytic semigroup $T(\rho)$ and it is given by

$$T(\rho)z = \sum_{n=1}^{\infty} e^{-n^2\rho}(z, e_n)e_n, \ z \in H,$$

where $e_n(z) = \sqrt{2}\sin(ny)$, $n = 1, 2, ...$ is complete orthonormal set of eigenvectors of Λ. From these expression, it follows that $\{T(\rho), \rho > 0\}$ is uniformly bounded compact semigroup, so that, $R(\lambda, \Lambda) = (\lambda I - \Lambda)^{-1}$ is compact operator for $\lambda \in \rho(\Lambda)$,

$$\Lambda z = \sum_{n=1}^{\infty} n^2(z, e_n)e_n, \ z \in H.$$

Let $y(\rho)(z) = y(\rho, z)$ and define the bounded linear operator $B : U \rightarrow H$ by $Bu(\rho)(z) = \mu(\rho, z), 0 \leq z \leq 1$. Further, define $\mathcal{G}(\rho, y(\rho))(z) = K_1(\rho, y(\rho, z)) = \frac{e^{-\rho}}{1+e^{-\rho}}\sin(y(\rho, z))$, $\Delta(\rho, y(\rho))(z) = K_2(\rho, y(\rho, z)) = \frac{e^{-\rho}}{1+e^{-\rho}}\sin(y(\rho, z))$ and $g(y)(z) = \sum_{i=1}^{\infty} c_i y(\rho_i, z)$. Then, the condition (A2)-(A4) are satisfied. On the other hand, the linear system corresponding to 5.1 is approximate controllable. Thus, with the above choices of Λ, $\mathfrak{F}$, Δ and B, system 5.1 can be written into the abstract form of 1.1. Hence, by the theorem 4.3, the stochastic control system 5.1 is approximate controllable on J.

6. Conclusions

This work investigates the Hilfer derivative to analyze the approximate controllability of FSEIs with nonlocal conditions. By assuming that the corresponding linear system is approximately controllable, we obtain a novel set of adequate requirements for the approximate controllability of nonlinear FSEIs in meticulous detail. The fixed-point theorem for multi-valued operators and fractional calculus are used to achieve the results. Additionally, our future study will focus on the regularity of mild solutions for FSEEs with nonlocal beginning conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

Acknowledgement: Their authors extend Their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Larg Groups Research Project under grant number (RGP.2/44/44) and this study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. Y. K. Chang, J. J. Nieto, W. S. Li, Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces, *J. Optim. Theory Appl.*, **142** (2009), 267–273. https://doi.org/10.1007/s10957-009-9535-2
2. L. Górniewicz, S. K. Ntouyas, D. O'Regan, Existence and controllability results for first-and second-order functional semilinear differential inclusions with nonlocal conditions, *Numer. Funct. Anal. Optim.*, **28** (2007), 53–82. https://doi.org/10.1080/01630560600883093
3. L. Górniewicz, S. K. Ntouyas, D. O'Regan, Controllability results for first and second order evolution inclusions with nonlocal conditions, *Ann. Pol. Math.*, **89** (2007), 65–101. https://doi.org/10.4064/ap89-1-5
4. N. U. Ahmed, Nonlinear stochastic differential inclusions on balance space, *Stoch. Anal. Appl.*, **12** (1994), 1–10. https://doi.org/10.1080/07362999408809334
5. G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge: Cambridge university press, 2014.
6. X. Fu, Approximate controllability for neutral impulsive differential inclusions with nonlocal conditions, *J. Dyn. Control Syst.*, **17** (2011), 359–386. https://doi.org/10.1007/s10883-011-9126-z
7. N. I. Mahmudov, Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces, *SIAM J. Control Optim.*, **42** (2003), 1604–1622. https://doi.org/10.1137/S0363012901391688
8. P. Muthukumar, C. Rajivganthi, Approximate controllability of impulsive neutral stochastic functional differential system with state-dependent delay in Hilbert spaces, *J. Control Theory Appl.*, **11** (2013), 351–358. https://doi.org/10.1007/s11768-013-2061-7
9. R. P. Agarwal, B. de Andrade, G. Siracusa, On fractional integro-differential equations with state-dependent delay, *Comput. Math. Appl.*, **62** (2011), 1143–1149. https://doi.org/10.1016/j.camwa.2011.02.033
10. G. M. Mophou, G. M. N'Guérékata, Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay, *Appl. Math. Comput.*, **216** (2010), 61–69. https://doi.org/10.1016/j.amc.2009.12.062
11. X. B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal. Theory Method. Appl.*, **74** (2011), 2003–2011. https://doi.org/10.1016/j.na.2010.11.007
12. J. Cui, L. Yan, Existence result for fractional neutral stochastic integro-differential equations with infinite delay, *J. Phys. A Math. Theor.*, **44** (2011), 335201. https://doi.org/10.1088/1751-8113/44/33/335201
13. J. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal. Real World Appl.*, **12** (2011), 3642–3653. https://doi.org/10.1016/j.nonrwa.2011.06.021
14. Z. Yan, H. Zhang, Existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay, *Electron. J. Differ. Equ.*, **2013** (2013), 1–21.
15. S. Duan, J. Hu, Y. Li, Exact controllability of nonlinear stochastic impulsive evolution differential inclusions with infinite delay in Hilbert spaces, *Int. J. Nonlinear Sci. Numer. Simul.*, **12** (2011), 23–33. https://doi.org/10.1515/ijnsns.2011.023

16. A. Debbouche, D. Baleanu, Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems, *Comput. Math. Appl.*, **62** (2011), 1442–1450. https://doi.org/10.1016/j.camwa.2011.03.075
17. A. Debbouche, D. Baleanu, Exact null controllability for fractional nonlocal integrodifferential equations via implicit evolution system, *J. Appl. Math.*, **2012** (2012), 931975. https://doi.org/10.1155/2012/931975
18. A. Debbouche, D. F. Torres, Approximate controllability of fractional nonlocal delay semilinear systems in Hilbert spaces, *Int. J. Control*, **86** (2013), 1577–1585. https://doi.org/10.1080/00207179.2013.791927
19. S. Kumar, N. Sukavanam, Approximate controllability of fractional order semilinear systems with bounded delay, *J. Differ. Equ.*, **252** (2012), 6163–6174. https://doi.org/10.1016/j.jde.2012.02.014
20. N. I. Mahmudov, Approximate controllability of fractional neutral evolution equations in Banach spaces, *Abstr. Appl. Anal.*, **2013** (2013), 531894. https://doi.org/10.1155/2013/531894
21. Z. Yan, Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces, *IMA J. Math. Control Informa.*, **30** (2013), 443–462. https://doi.org/10.1093/imamci/dns033
22. H. M. Ahmed, M. M. El-Borai, W. El-Sayed, A. Elbadrawi, Null controllability of Hilfer fractional stochastic differential inclusions, *Fractal Fract.*, **6** (2022), 721. https://doi.org/10.3390/fractalfract6120721
23. T. Sathiyaraj, J. Wang, P. Balasubramaniam, Controllability and optimal control for a class of time-delayed fractional stochastic integro-differential systems, *Appl. Math. Optimi.*, **84** (2021), 2527–2554. https://doi.org/10.1007/s00245-020-09716-w
24. X. Ma, X. B. Shu, J. Mao, Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay, *Stoch. Dynam.*, **20** (2020), 2050003. https://doi.org/10.1142/S0219493720500033
25. M. Liu, L. Chen, X. B. Shu, The existence of positive solutions for Φ-Hilfer fractional differential equation with random impulses and boundary value conditions, *Wave. Random Complex Media*, 2022, 1–19. https://doi.org/10.1080/17455030.2023.2176695
26. L. Shu, X. B.Shu, J. Mao, Approximate controllability and existence of mild solutions for Riemann-Liouville fractional stochastic evolution equations with nonlocal conditions of order $1 < \alpha < 2$, *Fract. Calc. Appl. Anal.*, **22** (2019), 1086–1112. https://doi.org/10.1515/fca-2019-0057
27. Y. Guo, X. B. Shu, F. Xu, C. Yang, HJB equation for optimal control system with random impulses, *Optimization*, 2022, 1–25. https://doi.org/10.1080/02331934.2022.2154607
28. L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.*, **40** (1991), 11–19. https://doi.org/10.1080/00036819008839989
29. J. Alzabut, A. G. M. Selvam, R. A. El-Nabulsi, V. Dhakshinamoorthy, M. E. Samei, Asymptotic stability of nonlinear discrete fractional pantograph equations with non-local initial conditions, *Symmetry*, **13** (2021), 473. https://doi.org/10.3390/sym13030473
30. K. Abuasbeh, R. Shafqat, Fractional Brownian motion for a system of fuzzy fractional stochastic differential equation, *J. Math.*, **2022** (2022), 3559035. https://doi.org/10.1155/2022/3559035

31. K. Abuasbeh, R. Shafqat, A. Alsinai, M. Awadalla, Analysis of the mathematical modelling of COVID-19 by using mild solution with delay caputo operator, *Symmetry*, **15** (2023), 286. https://doi.org/10.3390/sym15020286
32. K. Abuasbeh, R. Shafqat, A. Alsinai, M. Awadalla, Analysis of controllability of fractional functional random integroevolution equations with delay, *Symmetry*, **15** (2023), 290. https://doi.org/10.3390/sym15020290
33. K. Abuasbeh, R. Shafqat, A. U. K. Niazi, M. Awadalla, Oscillatory behavior of solution for fractional order fuzzy neutral predator-prey system, *AIMS Math.*, **7** (2022), 20383–20400. https://doi.org/10.3934/math.20221117
34. A. Moumen, R. Shafqat, A. Alsinai, H. Boulares, M. Cancan, M. B. Jeelani, Analysis of fractional stochastic evolution equations by using Hilfer derivative of finite approximate controllability, *AIMS Math.*, **7** (2023), 16094–16114. https://doi.org/10.3934/math.2023821
35. A. Moumen, R. Shafqat, Z. Hammouch, A. U. K. Niazi, M. B. Jeelani, Stability results for fractional integral pantograph differential equations involving two Caputo operators, *AIMS Math.*, **8** (2023), 6009–6025. https://doi.org/10.3934/math.2023303
36. A. A. A. Ghafli, R. Shafqat, A. U. K. Niazi, K. Abuasbeh, M. Awadalla, Topological structure of solution sets of fractional control delay problem, *Fractal Fract.*, **7** (2023), 59. https://doi.org/10.3390/fractalfract7010059
37. R. Sakthivel, R. Ganesh, S. M. Anthoni, Approximate controllability of fractional nonlinear differential inclusions, *Appl. Math. Comput.*, **225** (2013), 708–717. https://doi.org/10.1016/j.amc.2013.09.068
38. R. Sakthivel, S. Suganya, S. M. Anthoni, Approximate controllability of fractional stochastic evolution equations, *Comput. Math. Appl.*, **63** (2012), 660–668. https://doi.org/10.1016/j.camwa.2011.11.024
39. Y. Ren, L. Hu, R. Sakthivel, Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay, *J. Comput. Appl. Math.*, **235** (2011), 2603–2614. https://doi.org/10.1016/j.cam.2010.10.051
40. I. Podlubny, Fractional differential equations, *Math. Sci. Eng.*, 1999, 340.
41. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Cont Dyn. S*, **13** (2020), 709–722. https://doi.org/10.3934/dcdss.2020039
42. B. C. Dhage, Multi-valued mappings and fixed points II, *Tamkang J. Math.*, **37** (2006), 27–46. https://doi.org/10.5556/j.tkjm.37.2006.177

AIMS Press