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Research article

SOR-based alternately linearized implicit iteration method for nonsymmetric algebraic Riccati equations

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Abstract: In this paper, we propose a class of successive over relaxation-based alternately linearized implicit iteration method for computing the minimal nonnegative solution of nonsymmetric algebraic Riccati equations. Under certain conditions, we prove the convergence of the iterative method. Finally, numerical examples are given to show the iterative method is efficient.

Keywords: nonsymmetric algebraic Riccati equations; minimal nonnegative solution; convergence; iterative; SORALI
Mathematics Subject Classification: 15A24, 15A57

1. Introduction

A nonsymmetric algebraic Riccati equation (NARE) is the matrix equation

$$\mathcal{R}(X) = XCX - XD - AX + B = 0, \tag{1.1}$$

for which A, B, C and D are known matrices whose sizes are $m \times m$, $m \times n$, $n \times m$ and $n \times n$ respectively, and $X \in \mathbb{R}^{m\times n}$ is an unknown matrix to be solved. Let

$$K = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}, \tag{1.2}$$

where K is a nonsingular or an irreducible singular M-matrix. This type of nonlinear matrix equation often appears in many fields of application. For instance, in Wiener-Hopf factorization of Markov chains [7, 20] and stochastic processes [2]. In addition, in transport theory, changes in the usual set of neutron transport equations [3, 5, 6, 14] are formulated as

$$\left\{ (\mu + \alpha)\tfrac{\partial}{\partial x} + 1 \right\}\varphi(x, \mu) = \frac{c}{2}\int_{-1}^{1} \varphi(x, \omega)d\omega, \tag{1.3}$$

$$\varphi(0,\mu) = f(\mu), \mu > -\alpha, |\mu| \leq 1, \tag{1.4}$$

$$\lim_{x\to\infty} \varphi(x,\mu) = 0, \tag{1.5}$$

where φ is the neutron flux, α ($0 \leq \alpha < 1$) is an angular shift, and c is the mean of the number of new particles produced by a collision, which is assumed to be conservative, that is $0 \leq c \leq 1$. Equation (1.3) is equivalent to Eq (1.1) by proper transformation. In [21], Peretz gave sufficient conditions for the existence of contractive solutions for the nonsymmetric algebraic Riccati equation and the sufficient conditions for the unique contractive solution. In [8], Guo introduced a new class of nonsymmetric algebraic Riccati equations, which was inspired by the study of linear quadratic differential games. In [4], Bai et al. established a class of alternately linearized implicit (ALI) iteration methods for computing the minimal nonnegative solution. In [11], Guan considered the numerical solution and proposed a modified alternately linearized implicit iteration method (MALI) for computing the minimal nonnegative solution of M-matrix algebraic Riccati equations. Because of the importance, the nonsymmetric algebraic Riccati equation has attracted considerable attention from many scholars, and many methods have been studied to solve them in recent years [9–13, 15–17, 19].

In general, Eq (1.1) may have solutions, there may be no solutions, and there may be infinite solutions. In practical applications, we are interested in the minimum nonnegative solution of Eq (1.1). Mainly inspired by [11], in this paper, we propose a class of successive over relaxation (SOR)-based alternately linearized implicit (SORALI) iteration method for computing the minimal nonnegative solution of nonsymmetric algebraic Riccati equations (NARE). It is proved that the matrix sequences generated by the developed iterative algorithm monotonically converge to a minimal nonnegative solution of NARE under appropriate conditions.

For convenience of description, we use the following notations and definitions throughout this paper. For any matrices $A, B \in \mathbb{R}^{m\times n}$ with $A = (a_{ij})$ and $B = (b_{ij})$, $A \leq B$ (or $A < B$) means $a_{ij} \leq b_{ij}$ (or $a_{ij} < b_{ij}$),$\forall i, j$. Thus we can define A is a positive matrix if $A > 0$, similarly we can define A is a nonnegative matrix if $A \geq 0$. A matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if $a_{ij} \leq 0$, $\forall i \neq j$. Obviously, for any Z-matrix A, it can be written as $A = sI - B$ with $B \geq 0$ and I is the identity matrix. A Z-matrix $A = sI - B$ is a nonsingular M-matrix if $s > \rho(B)$ and singular M-matrix if $s = \rho(B)$, where $\rho(B)$ is the spectral radius of B. $\|\cdot\|_\infty$ denotes the ∞-norm of a matrix.

The remainder of this paper is organized as follows: In Section 2, we give some preliminary knowledge and some previous results. In Section 3, we propose a class of SORALI iteration method for computing the minimal nonnegative solution of NARE, and convergence analysis of this method. In Section 4, we report some numerical results. In Section 5, we conclude the result in this paper and the possible work in the future of this paper is highlighted.

2. Preliminaries and previous results

In this section, we restate some preliminary knowledge and some previous results. The theory of nonsingular M-matrices will play an important role in our analysis, so several important properties of nonsingular M-matrices are described in the following lemmas.

Lemma 2.1. *([1]) Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix, then the following expressions are equivalent:*
(1) *A is a nonsingular M-matrix.*
(2) $A^{-1} \geq 0$.

(3) $Av > 0$ *for some positive vector* $v \in \mathbb{R}^n$.
(4) *All eigenvalues of A have positive real part.*

Lemma 2.2. *([18]) Let* $A \in \mathbb{R}^{n\times n}$ *be a nonsingular M-matrix. If a Z-matrix* $B \in \mathbb{R}^{n\times n}$ *satisfies* $B \geq A$*, then B is also a nonsingular M-matrix.*

Remark 2.1. *From Lemma* 2.2*, we can easily conclude that for any real number* $\gamma \geq 0$*,* $B = \gamma I + A$ *is a nonsingular M-matrix.*

Lemma 2.3. *([1]) Let* $A \in \mathbb{R}^{n\times n}$ *be a nonsingular M-matrix, then any principal sub-matrix of A is also a nonsingular M-matrix.*

Remark 2.2. *([11]) From Lemma* 2.3*, we can know that if A is a nonsingular M-matrix and is partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} *and* A_{22} *are square matrices, then* A_{11} *and* A_{22} *are also nonsingular M-matrices.*

Lemma 2.4. *([4, 9, 11, 15]) If K in* (1.2) *is a nonsingular M-matrix or an irreducible singular M-matrix, then Eq* (1.1) *has a unique minimal nonnegative solution S. If K is nonsingular, then A-SC and D-CS are also nonsingular M-matrices. If K is irreducible, then* $S > 0$ *and A-SC and D-CS are also irreducible M-matrices.*

Many methods for finding the minimal nonnegative solution of NARE have been studied, such as Newton iteration, fixed-point iteration, structure-preserving doubling algorithm, nonlinear block Gauss-Seidel iterative, ALI iteration, MALI iterative and so on [4, 7, 11–13, 15].

In [11], the modified alternately linearized implicit iteration (MALI) can be written as follows:

$$\begin{cases} X_{k+\frac{1}{2}}(\alpha I + L_D) = (\alpha I - A + X_k C)X_k + X_k U_D + B, \\ (\beta I + L_A)X_{k+1} = X_{k+\frac{1}{2}}(\beta I - D + CX_{k+\frac{1}{2}}) + U_A X_{k+\frac{1}{2}} + B. \end{cases} \tag{2.1}$$

In the above formula, $D = L_D - U_D$, where L_D is the lower triangular part of D and U_D is the strictly upper triangular part of D. Similarly, $A = L_A - U_A$, where L_A is the lower triangular part of A and U_A is the strictly upper triangular part of A. $\alpha > 0$ and $\beta > 0$ are two known parameters, set $X_0 = 0 \in \mathbb{R}^{m\times n}$, computing X_{k+1} from X_k by solving Eq (2.1), $k = 0, 1, \cdots$, until $\{X_k\}$ converges.

Because of the special structure of coefficient matrices in Eq (2.1), the method of MALI costs less CPU time than ALI for solving NARE.

3. SORALI iteration and convergence analysis of SORALI iteration

Considering the special structure of the coefficient matrices and motivated by [11], we use the idea of the SOR iteration to split the matrices A and D respectively. Let

$$A = (\frac{1}{\omega}D_A - L_A) - (\frac{1-\omega}{\omega}D_A + U_A),$$

where D_A is the main diagonal part of A, $-L_A$ is the strictly lower triangular part of A, $-U_A$ is the strictly upper triangular part of A, and ω is the relaxation factor. Similarly, let

$$D = (\frac{1}{\omega}D_D - L_D) - (\frac{1-\omega}{\omega}D_D + U_D),$$

where D_D is the main diagonal part of D, $-L_D$ is the strictly lower triangular part of D, $-U_D$ is the strictly upper triangular part of D and ω is the relaxation factor. ω makes the diagonal entries of the matrices D_A and D_D inverted more dominant and thus the related matrices to be inverted have a smaller condition number, leading to more accurate inverses and eventually to a more accurate solution and to a better convergence rate.

Then, we propose the method of SORALI iteration as follows:

$$\begin{cases} X_{k+\frac{1}{2}}(\alpha I + \frac{1}{\omega}D_D - L_D) = (\alpha I - A + X_k C)X_k + X_k(\frac{1-\omega}{\omega}D_D + U_D) + B, \\ (\beta I + \frac{1}{\omega}D_A - L_A)X_{k+1} = X_{k+\frac{1}{2}}(\beta I - D + CX_{k+\frac{1}{2}}) + (\frac{1-\omega}{\omega}D_A + U_A)X_{k+\frac{1}{2}} + B, \end{cases} \tag{3.1}$$

where α, β are two given positive parameters. Obviously, if $\omega = 1$, (3.1) will be reduced to (2.1).

Next, we discuss the convergence of the SORALI iteration. As preparation, we first show several lemmas about the SORALI iteration method.

Lemma 3.1. *Let S be the minimal nonnegative solution of the NARE* (1.1) *and sequence $\{X_k\}$ be generated by the method of the SORALI iteration. Then the following equalities hold for any integer $k \geq 0$:*

(1) $(X_{k+\frac{1}{2}} - S)(\alpha I + \frac{1}{\omega}D_D - L_D) = (\alpha I - A + SC)(X_k - S) + (X_k - S)(CX_k + \frac{1-\omega}{\omega}D_D + U_D)$.

(2) $(X_{k+\frac{1}{2}} - X_k)(\alpha I + \frac{1}{\omega}D_D - L_D) = \mathcal{R}(X_k)$.

(3) $\mathcal{R}(X_{k+\frac{1}{2}}) = (\alpha I - A + X_{k+\frac{1}{2}}C)(X_{k+\frac{1}{2}} - X_k) + (X_{k+\frac{1}{2}} - X_k)(CX_k + \frac{1-\omega}{\omega}D_D + U_D)$.

(4) $(\beta I + \frac{1}{\omega}D_A - L_A)(X_{k+1} - S) = (X_{k+\frac{1}{2}} - S)(\beta I - D + CS) + (\frac{1-\omega}{\omega}D_A + U_A + X_{k+\frac{1}{2}}C)(X_{k+\frac{1}{2}} - S)$.

(5) $(\beta I + \frac{1}{\omega}D_A - L_A)(X_{k+1} - X_{k+\frac{1}{2}}) = \mathcal{R}(X_{k+\frac{1}{2}})$.

(6) $\mathcal{R}(X_{k+1}) = (X_{k+1} - X_{k+\frac{1}{2}})(\beta I - D + CX_{k+\frac{1}{2}}) + (\frac{1-\omega}{\omega}D_A + U_A + X_{k+1}C)(X_{k+1} - X_{k+\frac{1}{2}})$.

Proof. We only need to prove (1)–(3), because (4)–(6) can be derived similarly.

We first verify (1). As a matter of fact,

$$\mathcal{R}(S) = SCS - SD - AS + B = 0.$$

Similar to the first equation in (3.1), we have

$$S(\alpha I + \frac{1}{\omega}D_D - L_D) = (\alpha I - A + SC)S + S(\frac{1-\omega}{\omega}D_D + U_D) + B.$$

Thus

$$\begin{aligned} & (X_{k+\frac{1}{2}} - S)(\alpha I + \frac{1}{\omega}D_D - L_D) \\ = \ & X_{k+\frac{1}{2}}(\alpha I + \frac{1}{\omega}D_D - L_D) - S(\alpha I + \frac{1}{\omega}D_D - L_D) \\ = \ & (\alpha I - A + X_k C)X_k + X_k(\frac{1-\omega}{\omega}D_D + U_D) + B - ((\alpha I - A + SC)S + S(\frac{1-\omega}{\omega}D_D + U_D) + B) \end{aligned}$$

$$
\begin{aligned}
&= (\alpha I - A)X_k + X_kCX_k + X_k(\frac{1-\omega}{\omega}D_D + U_D) - (\alpha I - A)S - SCS - S(\frac{1-\omega}{\omega}D_D + U_D) \\
&= (\alpha I - A)(X_k - S) + X_k(CX_k + \frac{1-\omega}{\omega}D_D + U_D) - SCS - S(\frac{1-\omega}{\omega}D_D + U_D) \\
&= (\alpha I - A + SC)(X_k - S) - SCX_k + SCS + X_k(CX_k + \frac{1-\omega}{\omega}D_D + U_D) - SCS - S(\frac{1-\omega}{\omega}D_D + U_D) \\
&= (\alpha I - A + SC)(X_k - S) + X_k(CX_k + \frac{1-\omega}{\omega}D_D + U_D) - S(CX_k + \frac{1-\omega}{\omega}D_D + U_D) \\
&= (\alpha I - A + SC)(X_k - S) + (X_k - S)(CX_k + \frac{1-\omega}{\omega}D_D + U_D).
\end{aligned}
$$

(2) Directly calculates the equation to the left

$$
(X_{k+\frac{1}{2}} - X_k)(\alpha I + \frac{1}{\omega}D_D - L_D) = X_{k+\frac{1}{2}}(\alpha I + \frac{1}{\omega}D_D - L_D) - X_k(\alpha I + \frac{1}{\omega}D_D - L_D).
$$

Substituting the first equation of (3.1) into the above equation, we get

$$
\begin{aligned}
&(X_{k+\frac{1}{2}} - X_k)(\alpha I + \frac{1}{\omega}D_D - L_D) \\
&= (\alpha I - A + X_kC)X_k + X_k(\frac{1-\omega}{\omega}D_D + U_D) + B - X_k(\alpha I + \frac{1}{\omega}D_D - L_D) \\
&= \alpha X_k - AX_k + X_kCX_k + X_k(\frac{1-\omega}{\omega}D_D + U_D) + B - \alpha X_k - X_k(\frac{1}{\omega}D_D - L_D) \\
&= X_kCX_k - X_k(\frac{1}{\omega}D_D - L_D - (\frac{1-\omega}{\omega}D_D + U_D)) - AX_k + B \\
&= X_kCX_k - X_kD - AX_k + B \\
&= \mathcal{R}(X_k).
\end{aligned}
$$

(3) Subtracts $X_{k+\frac{1}{2}}(\frac{1-\omega}{\omega}D_D + U_D)$ from both sides of the first equation of (3.1), we obtain

$$
\begin{aligned}
&X_{k+\frac{1}{2}}(\alpha I + \frac{1}{\omega}D_D - L_D - (\frac{1-\omega}{\omega}D_D + U_D)) \\
&= (\alpha I - A + X_kC)X_k + X_k(\frac{1-\omega}{\omega}D_D + U_D) + B - X_{k+\frac{1}{2}}(\frac{1-\omega}{\omega}D_D + U_D),
\end{aligned}
$$

that is

$$
X_{k+\frac{1}{2}}D = (\alpha I - A + X_kC)X_k + X_k(\frac{1-\omega}{\omega}D_D + U_D) + B - X_{k+\frac{1}{2}}(\alpha I + \frac{1-\omega}{\omega}D_D + U_D).
$$

Thus,

$$
\begin{aligned}
\mathcal{R}(X_{k+\frac{1}{2}}) &= X_{k+\frac{1}{2}}CX_{k+\frac{1}{2}} - X_{k+\frac{1}{2}}D - AX_{k+\frac{1}{2}} + B \\
&= X_{k+\frac{1}{2}}CX_{k+\frac{1}{2}} - (\alpha I - A + X_kC)X_k - X_k(\frac{1-\omega}{\omega}D_D + U_D) - B + X_{k+\frac{1}{2}}(\alpha I + \frac{1-\omega}{\omega}D_D + U_D) \\
&\quad -AX_{k+\frac{1}{2}} + B \\
&= (\alpha I - A + X_{k+\frac{1}{2}}C)X_{k+\frac{1}{2}} - (\alpha I - A + X_{k+\frac{1}{2}}C)X_k + X_{k+\frac{1}{2}}CX_k - X_kCX_k
\end{aligned}
$$

$$-X_k(\frac{1-\omega}{\omega}D_D+U_D)+X_{k+\frac{1}{2}}(\frac{1-\omega}{\omega}D_D+U_D)$$
$$=(\alpha I-A+X_{k+\frac{1}{2}}C)(X_{k+\frac{1}{2}}-X_k)+(X_{k+\frac{1}{2}}-X_k)CX_k+(X_{k+\frac{1}{2}}-X_k)(\frac{1-\omega}{\omega}D_D+U_D)$$
$$=(\alpha I-A+X_{k+\frac{1}{2}}C)(X_{k+\frac{1}{2}}-X_k)+(X_{k+\frac{1}{2}}-X_k)(CX_k+\frac{1-\omega}{\omega}D_D+U_D).$$

Similarly, we can get (4)–(6). □

Lemma 3.2. *Assume that S is the minimal nonnegative solution of the NARE* (1.1)*, K in* (1.2) *is a nonsingular M-matrix and the matrix sequence $\{X_k\}$ is generated by the method of SORALI iteration. Let α,β and ω be the prescribed parameters of* (3.1) *such that*

$$\alpha\geq max\{a_{ii}\},\qquad \beta\geq max\{d_{ii}\},\qquad 0<\omega\leq 1,$$

where a_{ii} and d_{ii} are the ith diagonal elements of the matrices A and D, respectively. Then the following inequalities hold for any integer $k\geq 0$:

$$0\leq X_{k+\frac{1}{2}}\leq S,\qquad 0\leq X_{k+1}\leq S. \tag{3.2}$$

Proof. Because K is a nonsingular M-matrix, A and D are also nonsingular M-matrices by Remark 2.2 and B and C are nonnegative matrices by the definition of M-matrix. Therefore, it is known from the splitting of A and D that $\frac{1}{\omega}D_D-L_D$ and $\frac{1}{\omega}D_A-L_A$ are both nonsingular M-matrices. We have $\alpha I+\frac{1}{\omega}D_D-L_D$ and $\beta I+\frac{1}{\omega}D_A-L_A$ are both nonsingular M-matrices by Remark 2.1, and

$$(\alpha I+\frac{1}{\omega}D_D-L_D)^{-1}\geq 0,\qquad (\beta I+\frac{1}{\omega}D_A-L_A)^{-1}\geq 0$$

by Lemma 2.1.

Next, we prove the conclusion (3.2) according to the mathematical induction principle. When $k=0$, we get

$$X_{\frac{1}{2}}(\alpha I+\frac{1}{\omega}D_D-L_D)=B$$

by Eq (3.1), so

$$X_{\frac{1}{2}}=B(\alpha I+\frac{1}{\omega}D_D-L_D)^{-1}\geq 0. \tag{3.3}$$

On the other hand, from Lemma 3.1 (1), we obtain

$$(X_{\frac{1}{2}}-S)(\alpha I+\frac{1}{\omega}D_D-L_D)=-(\alpha I-A+SC)S-S(\frac{1-\omega}{\omega}D_D+U_D)\leq 0,$$

thus

$$X_{\frac{1}{2}}-S=-((\alpha I-A+SC)S+S(\frac{1-\omega}{\omega}D_D+U_D))(\alpha I+\frac{1}{\omega}D_D-L_D)^{-1}\leq 0,$$

that is

$$X_{\frac{1}{2}}\leq S. \tag{3.4}$$

Analogously, from Eq (3.1), we have

$$(\beta I+\frac{1}{\omega}D_A-L_A)X_1=X_{\frac{1}{2}}(\beta I-D+CX_{\frac{1}{2}})+(\frac{1-\omega}{\omega}D_A+U_A)X_{\frac{1}{2}}+B\geq 0,$$

thus

$$X_1 = (\beta I + \frac{1}{\omega}D_A - L_A)^{-1}(X_{\frac{1}{2}}(\beta I - D + CX_{\frac{1}{2}}) + (\frac{1-\omega}{\omega}D_A + U_A)X_{\frac{1}{2}} + B) \geq 0. \tag{3.5}$$

From Lemma 3.1 (4), we obtain

$$(\beta I + \frac{1}{\omega}D_A - L_A)(X_1 - S) = (X_{\frac{1}{2}} - S)(\beta I - D + CS) + (\frac{1-\omega}{\omega}D_A + U_A + X_{\frac{1}{2}}C)(X_{\frac{1}{2}} - S).$$

From (3.4), we get

$$X_1 - S = (\beta I + \frac{1}{\omega}D_A - L_A)^{-1}((X_{\frac{1}{2}} - S)(\beta I - D + CS) + (\frac{1-\omega}{\omega}D_A + U_A + X_{\frac{1}{2}}C)(X_{\frac{1}{2}} - S)) \leq 0,$$

that is

$$X_1 \leq S. \tag{3.6}$$

So we have shown that

$$0 \leq X_{\frac{1}{2}} \leq S, \qquad 0 \leq X_1 \leq S. \tag{3.7}$$

Assume that (3.2) holds for $k = l - 1$, that is to say

$$0 \leq X_{l-\frac{1}{2}} \leq S, \qquad 0 \leq X_l \leq S. \tag{3.8}$$

From Eq (3.1), when $k = l$, we obtain

$$X_{l+\frac{1}{2}}(\alpha I + \frac{1}{\omega}D_D - L_D) = (\alpha I - A + X_lC)X_l + X_l(\frac{1-\omega}{\omega}D_D + U_D) + B \geq 0,$$

then

$$X_{l+\frac{1}{2}} = ((\alpha I - A + X_lC)X_l + X_l(\frac{1-\omega}{\omega}D_D + U_D) + B)(\alpha I + \frac{1}{\omega}D_D - L_D)^{-1} \geq 0. \tag{3.9}$$

In addition, from Lemma 3.1 (1), we get

$$(X_{l+\frac{1}{2}} - S)(\alpha I + \frac{1}{\omega}D_D - L_D) = (\alpha I - A + SC)(X_l - S) + (X_l - S)(CX_l + \frac{1-\omega}{\omega}D_D + U_D),$$

then by (3.8)

$$X_{l+\frac{1}{2}} - S = ((\alpha I - A + SC)(X_l - S) + (X_l - S)(CX_l + \frac{1-\omega}{\omega}D_D + U_D))(\alpha I + \frac{1}{\omega}D_D - L_D)^{-1} \leq 0,$$

that is

$$X_{l+\frac{1}{2}} \leq S. \tag{3.10}$$

Analogously, from Eq (3.1), we have

$$(\beta I + \frac{1}{\omega}D_A - L_A)X_{l+1} = X_{l+\frac{1}{2}}(\beta I - D + CX_{l+\frac{1}{2}}) + (\frac{1-\omega}{\omega}D_A + U_A)X_{l+\frac{1}{2}} + B \geq 0,$$

then

$$X_{l+1} = (\beta I + \frac{1}{\omega}D_A - L_A)^{-1}(X_{l+\frac{1}{2}}(\beta I - D + CX_{l+\frac{1}{2}}) + (\frac{1-\omega}{\omega}D_A + U_A)X_{l+\frac{1}{2}} + B) \geq 0. \tag{3.11}$$

In addition, from Lemma 3.1 (4), we obtain

$$(\beta I + \frac{1}{\omega}D_A - L_A)(X_{l+1} - S) = (X_{l+\frac{1}{2}} - S)(\beta I - D + CS) + (\frac{1-\omega}{\omega}D_A + U_A + X_{l+\frac{1}{2}}C)(X_{l+\frac{1}{2}} - S) \leq 0,$$

thus

$$X_{l+1} - S = (\beta I + \frac{1}{\omega}D_A - L_A)^{-1}((X_{l+\frac{1}{2}} - S)(\beta I - D + CS) + (\frac{1-\omega}{\omega}D_A + U_A + X_{l+\frac{1}{2}}C)(X_{l+\frac{1}{2}} - S)) \leq 0,$$

that is

$$X_{l+1} \leq S. \tag{3.12}$$

In summary, (3.2) holds for $k = l$. Thus, the proof is completed. □

Lemma 3.3. *Suppose that the assumption is as in Lemma* 3.2*, and* $X_0 = 0$ *is the initial matrix such that* $\mathcal{R}(X_0) \geq 0$*. Then for any integer* $k \geq 0$*, it holds that*

$$X_k \leq X_{k+\frac{1}{2}} \leq X_{k+1}, \qquad \mathcal{R}(X_{k+\frac{1}{2}}) \geq 0, \qquad \mathcal{R}(X_{k+1}) \geq 0. \tag{3.13}$$

Proof. We prove the Lemma 3.3 by the mathematical induction principle. In fact, for $k = 0$, $0 = X_0 \leq X_{\frac{1}{2}}$ by (3.2). According to Lemma 3.1 (3), we get

$$\mathcal{R}(X_{\frac{1}{2}}) = (\alpha I - A + X_{\frac{1}{2}}C)X_{\frac{1}{2}} + X_{\frac{1}{2}}(\frac{1-\omega}{\omega}D_D + U_D) \geq 0. \tag{3.14}$$

By making use of Lemma 3.1 (5), we get

$$(\beta I + \frac{1}{\omega}D_A - L_A)(X_1 - X_{\frac{1}{2}}) = \mathcal{R}(X_{\frac{1}{2}}),$$

then

$$X_1 - X_{\frac{1}{2}} = (\beta I + \frac{1}{\omega}D_A - L_A)^{-1}\mathcal{R}(X_{\frac{1}{2}}) \geq 0,$$

that is

$$X_{\frac{1}{2}} \leq X_1. \tag{3.15}$$

From Lemma 3.1 (6), we obtain

$$\mathcal{R}(X_1) = (X_1 - X_{\frac{1}{2}})(\beta I - D + CX_{\frac{1}{2}}) + (\frac{1-\omega}{\omega}D_A + U_A + X_1C)(X_1 - X_{\frac{1}{2}}) \geq 0 \tag{3.16}$$

by (3.15). Therefore, we we have shown that

$$X_0 \leq X_{\frac{1}{2}} \leq X_1, \qquad \mathcal{R}(X_{\frac{1}{2}}) \geq 0, \qquad \mathcal{R}(X_1) \geq 0. \tag{3.17}$$

Assume that (3.13) holds for $k = l - 1$, that is to say

$$X_{l-1} \leq X_{l-\frac{1}{2}} \leq X_l, \qquad \mathcal{R}(X_{l-\frac{1}{2}}) \geq 0, \qquad \mathcal{R}(X_l) \geq 0. \tag{3.18}$$

By making use of Lemma 3.1 (2), we get

$$(X_{l+\frac{1}{2}} - X_l)(\alpha I + \frac{1}{\omega}D_D - L_D) = \mathcal{R}(X_l),$$

thus

$$X_{l+\frac{1}{2}} - X_l = \mathcal{R}(X_l)(\alpha I + \frac{1}{\omega}D_D - L_D)^{-1} \geq 0,$$

that is

$$X_l \leq X_{l+\frac{1}{2}}. \tag{3.19}$$

From Lemma 3.1 (3), we obtain

$$\mathcal{R}(X_{l+\frac{1}{2}}) = (\alpha I - A + X_{l+\frac{1}{2}}C)(X_{l+\frac{1}{2}} - X_l) + (X_{l+\frac{1}{2}} - X_l)(CX_l + \frac{1-\omega}{\omega}D_D + U_D) \geq 0 \tag{3.20}$$

by (3.19). By Lemma 3.1 (5), we get

$$(\beta I + \frac{1}{\omega}D_A - L_A)(X_{l+1} - X_{l+\frac{1}{2}}) = \mathcal{R}(X_{l+\frac{1}{2}}),$$

thus

$$X_{l+1} - X_{l+\frac{1}{2}} = (\beta I + \frac{1}{\omega}D_A - L_A)^{-1}\mathcal{R}(X_{l+\frac{1}{2}}) \geq 0,$$

that is

$$X_{l+\frac{1}{2}} \leq X_{l+1}. \tag{3.21}$$

From Lemma 3.1 (6), we obtain

$$\mathcal{R}(X_{l+1}) = (X_{l+1} - X_{l+\frac{1}{2}})(\beta I - D + CX_{l+\frac{1}{2}}) + (\frac{1-\omega}{\omega}D_A + U_A + X_{l+1}C)(X_{l+1} - X_{l+\frac{1}{2}}) \geq 0. \tag{3.22}$$

In summary, (3.13) holds for $k = l$. Thus, the proof is completed. □

The convergence property of the SORALI can be given by Lemmas 3.2 and 3.3. We show this result by the following theorem.

Theorem 3.1. *Let K defined in* (1.2) *be an M-matrix. For the SORALI* (3.1)*, if the parameters are chosen as*

$$\alpha \geq max\{a_{ii}\}, \qquad \beta \geq max\{d_{ii}\}, \qquad 0 < \omega \leq 1,$$

then the sequence $\{X_k\}$ generated by the SORALI (3.1) *is well defined, monotonically increasing and converges to the unique minimum nonnegative solution S of NARE* (1.1).

Proof. According to Lemmas 3.2 and 3.3, we know that $X_k \geq 0$ and is monotonically increasing with upper bound. Suppose there is an S_* that satisfies $\lim_{k\to\infty} = S_*$. By making use of Lemma 3.2, we obtain $S_* \leq S$. In addition, taking the limit in the SORALI (3.1), we have S_* is a solution of NARE (1.1). Because S is the minimum nonnegative solution of NARE (1.1), we get $S \leq S_*$. Thus $S = S_*$, the proof is completed. □

Remark 3.1. *Theorem* 3.1 *is only proved theoretically so that the SORALI* (3.1) *is convergent under* $0 < \omega \leq 1$. *In fact, it might still be convergent even if ω is out of* $(0, 1]$. *Therefore, further research is needed in the selection of the parameter ω. Unfortunately, we did not give an answer in this paper. This will be our future work.*

4. Numerical experiments

In this section, two examples are given to show the effectiveness of the SORALI (3.1). All the experiments are implemented under MATLAB R2016a running on a personal computer with an Intel Core i5 CPU (3.30 GHz) and 4 GB RAM. The numerical behaviours of the SORALI will be compared with the MALI in [11] with respect to the number of iterations (IT), CPU time and the relative residue (RES, see [11]) which is defined to be

$$RES := \frac{\|\mathcal{R}(X)\|_\infty}{\|XCX\|_\infty + \|XD\|_\infty + \|AX\|_\infty + \|B\|_\infty}.$$

The iteration termination condition is $RES < 10^{-12}$ or the number of iterations exceeds 2000. In both experiments, we all choose $\alpha = max\{a_{ii}\}$, $\beta = max\{d_{ii}\}$ and $\omega = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2$, respectively to observe the experimental results.

Example 4.1. ([4]) Considering the SORALI (3.1), for which

$$A = D = Tridiag(-I, T, -I) \in \mathcal{R}^{n\times n}$$

are block tridiagonal matrices,

$$C = \frac{1}{50} tridiag(1, 2, 1) \in \mathcal{R}^{n\times n}$$

is a tridiagonal matrix, and

$$B = AS + SD - SCS,$$

such that

$$S = \frac{1}{50} ee^T \in \mathcal{R}^{n\times n}$$

is the minimal nonnegative solution of the SORALI (3.1). Here,

$$T = tridiag(-1, 4 + \frac{200}{(m+1)^2}, -1) \in \mathcal{R}^{m\times m}, \quad e = (1, 1, \cdots, 1)^T \in \mathcal{R}^n,$$

and $n = m^2$. For $m = 8$, $m = 10$, $m = 15$ and $m = 30$, the numerical results are listed in Tables 1–9 .

Table 1. Numerical results of Example 4.1 ($\omega = 0.25$).

m	Methods	IT	CPU	RES
8	SORALI	71	0.0363	9.2220e-13
10	SORALI	98	0.2050	7.6969e-13
15	SORALI	247	3.7553	9.0748e-13
30	SORALI	280	235.7742	9.4447e-13

Table 2. Numerical results of Example 4.1 (ω = 0.5).

m	Methods	IT	CPU	RES
8	SORALI	38	0.0243	7.4415e-13
10	SORALI	53	0.1158	7.1117e-13
15	SORALI	136	2.2079	9.9288e-13
30	SORALI	161	135.0248	9.5215e-13

Table 3. Numerical results of Example 4.1 (ω = 0.75).

m	Methods	IT	CPU	RES
8	SORALI	27	0.0166	5.4078e-13
10	SORALI	38	0.0781	6.2481e-13
15	SORALI	100	1.5827	8.2157e-13
30	SORALI	121	100.4319	8.9631e-13

Table 4. Numerical results of Example 4.1 (ω = 1).

m	Methods	IT	CPU	RES
8	SORALI	21	0.0136	6.9648e-13
10	SORALI	30	0.0294	8.3184e-13
15	SORALI	81	0.4496	9.5754e-13
30	SORALI	100	39.3327	9.6586e-13

Table 5. Numerical results of Example 4.1 (ω = 1.25).

m	Methods	IT	CPU	RES
8	SORALI	18	0.0148	3.1938e-13
10	SORALI	26	0.0509	4.4744e-13
15	SORALI	70	1.1274	9.4045e-13
30	SORALI	88	73.1086	8.8938e-13

Table 6. Numerical results of Example 4.1 (ω = 1.5).

m	Methods	IT	CPU	RES
8	SORALI	18	0.0130	6.3672e-13
10	SORALI	23	0.0254	3.7593e-13
15	SORALI	63	0.3004	8.0621e-13
30	SORALI	80	32.2483	8.1755e-13

Table 7. Numerical results of Example 4.1 ($\omega = 1.75$).

m	Methods	IT	CPU	RES
8	SORALI	24	0.0145	8.8852e-13
10	SORALI	30	0.0535	8.3134e-13
15	SORALI	71	1.0175	8.0621e-13
30	SORALI	82	62.6056	8.7195e-13

Table 8. Numerical results of Example 4.1 ($\omega = 2$).

m	Methods	IT	CPU	RES
8	SORALI	32	0.0256	9.0222e-13
10	SORALI	42	0.0801	8.5909e-13
15	SORALI	69	1.0933	8.0621e-13
30	SORALI	150	129.5271	7.7757e-13

Table 9. Comparison of the numerical results of MALI and SORALI ($\omega = 1.5$) in Example 4.1.

m	Methods	IT	CPU	RES
8	MALI	21	0.0136	6.9648e-13
	SORALI	18	0.0130	6.3672e-13
10	MALI	30	0.0294	8.3184e-13
	SORALI	23	0.0254	3.7593e-13
15	MALI	81	0.4496	9.5754e-13
	SORALI	63	0.3004	8.0621e-13
30	MALI	100	39.3327	9.6586e-13
	SORALI	80	32.2483	8.1755e-13

Example 4.2. ([11]) Considering the SORALI (3.1), for which A, B, C and D are generated by the following rules. Set $R = rand(2n, 2n)$, then set $W = diag(Re) - R$, where $e = (1, 1, \cdots, 1)^T \in \mathcal{R}^{2n}$; and finally, define

$$D = W(1:n, 1:n) + I, \qquad C = -W(1:n, n+1:2n),$$
$$B = -W(n+1:2n, 1:n), \qquad A = W(n+1:2n, n+1:2n) + I,$$

where I is the identity matrix of size n. The matrices A, B, C and D generated by the above rules guarantee that K in (1.2) is a nonsingular M-matrix. For $n = 50$, $n = 100$, $n = 500$ and $n = 1000$, the numerical results are listed in Tables 10–18.

Table 10. Numerical results of Example 4.2 (ω = 0.25).

n	Methods	IT	CPU	RES
50	SORALI	203	0.0719	8.9290e-13
100	SORALI	314	0.2249	7.6291e-13
500	SORALI	521	38.2340	9.5280e-13
1000	SORALI	836	412.2531	8.3215e-13

Table 11. Numerical results of Example 4.2 (ω = 0.5).

n	Methods	IT	CPU	RES
50	SORALI	118	0.0513	7.6230e-13
100	SORALI	227	0.1034	8.5367e-13
500	SORALI	401	26.5261	7.9658e-13
1000	SORALI	621	307.3852	6.3689e-13

Table 12. Numerical results of Example 4.2 (ω = 0.75).

n	Methods	IT	CPU	RES
50	SORALI	98	0.0413	6.5895e-13
100	SORALI	152	0.0963	8.3210e-13
500	SORALI	341	19.3680	6.1283e-13
1000	SORALI	428	215.0362	8.0698e-13

Table 13. Numerical results of Example 4.2 (ω = 1).

n	Methods	IT	CPU	RES
50	SORALI	76	0.0239	7.9896e-13
100	SORALI	107	0.0781	8.9683e-13
500	SORALI	225	14.3587	9.4359e-13
1000	SORALI	309	177.4588	9.3316e-13

Table 14. Numerical results of Example 4.2 (ω = 1.25).

n	Methods	IT	CPU	RES
50	SORALI	87	0.0402	8.5630e-13
100	SORALI	138	0.0885	9.7528e-13
500	SORALI	236	13.2681	8.5374e-13
1000	SORALI	287	160.3321	6.0289e-13

Table 15. Numerical results of Example 4.2 (ω = 1.5).

n	Methods	IT	CPU	RES
50	SORALI	62	0.0203	6.7452e-13
100	SORALI	86	0.0685	8.4461e-13
500	SORALI	183	12.3760	8.9754e-13
1000	SORALI	251	137.2604	9.5769e-13

Table 16. Numerical results of Example 4.2 (ω = 1.75).

n	Methods	IT	CPU	RES
50	SORALI	92	0.0510	5.6480e-13
100	SORALI	146	0.9230	9.6531e-13
500	SORALI	271	15.0635	7.8793e-13
1000	SORALI	298	171.5628	6.8261e-13

Table 17. Numerical results of Example 4.2 (ω = 2).

n	Methods	IT	CPU	RES
50	SORALI	121	0.1021	6.8480e-13
100	SORALI	167	2.012	8.3532e-13
500	SORALI	323	21.8512	9.3653e-13
1000	SORALI	532	325.5620	7.2980e-13

Table 18. Comparison of the numerical results of MALI and SORALI (ω = 1.5) in Example 4.2.

n	Methods	IT	CPU	RES
50	MALI	76	0.0239	7.9896e-13
	SORALI	62	0.0203	6.7452e-13
100	MALI	107	0.0781	8.9683e-13
	SORALI	86	0.0685	8.4461e-13
500	MALI	225	14.3587	9.4359e-13
	SORALI	183	12.3760	8.9754e-13
1000	MALI	309	177.4588	9.3316e-13
	SORALI	251	137.2604	9.5769e-13

From the above two numerical examples, we can see that when the size of the matrices become larger, the SORALI has a slight advantage over the MALI in terms of iteration number and CPU time.

5. Conclusions

In this paper, we propose a class of SORALI iteration method for computing the minimal nonnegative solution of NARE, and have proved the convergence of the method. Finally, two numerical examples illustrate the effectiveness of the proposed method. We hope to continue to study the better range of the parameter ω in theory in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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