Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 19039–19048.
DOI: 10.3934/math.2023971
Received: 23 March 2023
Revised: 30 May 2023
Accepted: 31 May 2023
Published: 06 June 2023

Research article

Liouville type theorem for weak solutions of nonlinear system for Grushin operator

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Abstract: In this paper, we prove Liouville type theorem of positive weak solution of nonlinear system for Grushin operator. We give some integral inequalities, which combine the method of moving plane with the integral inequality to get the result for nonlinear system.

Keywords: Liouville type theorem; Grushin operator; nonlinear system; method of moving planes
Mathematics Subject Classification: 35J60, 35J15

1. Introduction

In this paper, we study the Liouville type theorem of positive weak solution of the nonlinear system for Grushin operator

$$\begin{cases} -(\Delta_x u + (\alpha+1)^2|x|^{2\alpha}\Delta_y u) = f(v), & \text{in } \mathbb{R}^m \times \mathbb{R}^k, \\ -(\Delta_x v + (\beta+1)^2|x|^{2\beta}\Delta_y v) = g(u), & \text{in } \mathbb{R}^m \times \mathbb{R}^k, \end{cases} \tag{1.1}$$

where we denote the Grushin operators $L_\alpha u = \Delta_x u + (\alpha+1)^2|x|^{2\alpha}\Delta_y u$, $L_\beta u = \Delta_x u + (\beta+1)^2|x|^{2\beta}\Delta_y u$, $\alpha,\ \beta > 0$, and the right hand terms f and g are some continuous functions.

In the case of (1.1), $\alpha,\ \beta = 0$, it should be the Laplacian problem

$$\begin{cases} -\Delta u = v^p, & \text{in } \mathbb{R}^n, \\ -\Delta v = u^q, & \text{in } \mathbb{R}^n, \end{cases}$$

here, $n = m + k$. De Figueiredo and Felmer in [7] conjectured that the hyperbola curve

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{n} \ \ (p > 0,\ \ q > 0)$$

is the dividing curve between existence and nonexistence of the solution to the Laplacian problem. The conjecture is right for the radial solutions of the above problem, see Serrin and Zou [18, 19]. The authors in [7] proved that the above Laplacian problem has no positive solutions, when $0 < p, q \le \frac{n+2}{n-2}$ and $(p, q) \ne (\frac{n+2}{n-2}, \frac{n+2}{n-2})$. Guo and Liu [14] gave the Liouville type theorems for positive solutions of the following second order elliptic problem:

$$\begin{cases} -\Delta u = f(u, v), & \text{in } \mathbb{R}^n, \\ -\Delta v = g(u, v), & \text{in } \mathbb{R}^n, \end{cases}$$

when $n \ge 3$. The key tool in [14] is the method of moving planes which combined with integral inequalities.

For the Grushin operator, Yu [29] gave the nonexistence of positive solutions for the degenerate equation

$$-L_\alpha u = f(u) \ \text{ in } \mathbb{R}^m \times \mathbb{R}^k,$$

where $L_\alpha u = \Delta_x u + (\alpha + 1)^2 |x|^{2\alpha} \Delta_y u$, $\alpha > 0$, and f satisfies some assumptions.

We investigate the nonexistence result of the elliptic system involving of the Grushin operator and with general nonlinear terms. We call $(u,\ v)$ is a weak solution of system (1.1), if $(u,\ v)$ satisfies

$$\begin{cases} \displaystyle\int_{\mathbb{R}^m \times \mathbb{R}^k} \nabla_x u \nabla_x \varphi_1 + (\alpha + 1)^2 |x|^{2\alpha} \nabla_y u \nabla_y \varphi_1 dxdy = \int_{\mathbb{R}^m \times \mathbb{R}^k} f(v) \varphi_1 dxdy, \\ \displaystyle\int_{\mathbb{R}^m \times \mathbb{R}^k} \nabla_x v \nabla_x \varphi_2 + (\beta + 1)^2 |x|^{2\beta} \nabla_y v \nabla_y \varphi_2 dxdy = \int_{\mathbb{R}^m \times \mathbb{R}^k} g(u) \varphi_2 dxdy, \end{cases}$$

for any $\varphi_1, \varphi_2 \in C_0^1(\overline{\mathbb{R}^m \times \mathbb{R}^k})$.

The main result in this paper is the following Liouville type theorem of nonnegative weak solution for (1.1).

Theorem 1.1. *Let $(u, v) \in C(\mathbb{R}^m \times \mathbb{R}^k) \times C(\mathbb{R}^m \times \mathbb{R}^k)$ be a nonnegative weak solution of problem (1.1). Suppose that $f, g : [0, +\infty) \to \mathbb{R}$ are continuous functions satisfying*

(i) $f(t)$ and $g(t)$ are nondecreasing in $(0, +\infty)$,

(ii) $h(t) = \frac{f(t)}{t^{\frac{Q+2}{P-2}}}$ and $k(t) = \frac{g(t)}{t^{\frac{P+2}{Q-2}}}$ are nonincreasing in $(0, +\infty)$, here $Q = m+(\alpha+1)k > P = m+(\beta+1)k$. Then, $(u, v) = (c_1, c_2)$ for some constants c_1, c_2 satisfying $f(c_2) = g(c_1) = 0$.

In this paper, a key tool is some integral inequality, which was used in Terracini [21, 22]. And then there were some related results in the references [14, 23], etc. Quaas and Xia in [17] gave the Liouville theorem for fractional Lane-Emden system, which used the monotonicity argument for some suitable transformed functions with the method of moving planes, where some maximum principles which obtained by suitable barrier functions and a coupling argument for fractional Sobolev trace inequality in an infinity half cylinder be used. For this paper, since the extended Grushin operator is degenerated, and not conclude that $(u,\ v)$ belongs to C^2 class. To handle the weak solution and the degeneracy, we also borrow the idea that the integral inequality can play the role to the maximum principles when moving planes.

For the more information of the method of moving planes of elliptic equations, see [2, 4–6, 8, 12, 13], ect. There are also some results of the fractional operator and its related extension problems on Heisenberg group (see [3, 24–27], ect.) and Grushin manifold see [1, 11, 15, 20, 28, 29] and the references therein.

This paper is organized as follows. In Section 2, we collect some well known results, and give the proof of key integral inequalities. In Section 3, Theorem 1.1 will be proved with some conditions of f and g.

2. Preliminaries

We list some basic information about the Grushin operator. For the details can refer to [16].

Let a and b be two vectors in $\mathbb{R}^d$ for some $d \in \mathbb{N}^+$, define the inner product of them as $\langle a, b\rangle = \sum_{j=1}^{d} a_j b_j$, and $|a| = \langle a, a\rangle^{\frac{1}{2}}$ is the Euclidean norm. Let $\alpha > 0$ be a fixed constant (here we only list the case α, it also holds for β), for $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$, define the Grushin norm as

$$||z|| = (|x|^{2(\alpha+1)} + |y|^2)^{\frac{1}{2(\alpha+1)}}.$$

It is easy to check that this Grushin norm is 1-homogeneous, that for the Lie group dilations $\delta_\lambda = (\lambda x, \lambda^{\alpha+1} y), \lambda > 0$. Define the Grushin distance of two points z, z_0 in $\mathbb{R}^m \times \mathbb{R}^k$ by

$$d(z, z_0) = ||z - z_0||.$$

We define the open ball which radius R and centered at z_0 as

$$B(z_0, R) = \{z \in \mathbb{R}^m \times \mathbb{R}^k | \; d(z, z_0) < R\}.$$

A direct calculation gives that

$$|B(z_0, R)| = |B(0, R)| = R^Q |B(0, 1)|,$$

here $Q = m + (\alpha + 1)k$, and $|\cdot|$ denotes the Lebesgue measure. And the number Q is the homogeneous dimension of Grushin space $\mathbb{R}^m \times \mathbb{R}^k$. In order to define the Grushin gradient and operator in $\mathbb{R}^m \times \mathbb{R}^k$, we use Euclidean gradients ∇_x and ∇_y with respect to $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$. And we define

$$D_\alpha = (\nabla_x, (\alpha + 1)|x|^\alpha \nabla_y)$$

as the Grushin gradient, and set

$$\mathrm{div}_\alpha(f, g) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} + (\alpha + 1)|x|^\alpha \sum_{j=1}^{k} \frac{\partial g_j}{\partial y_j}, \quad (f, g) \in C^1(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}^k)$$

as the Grushin divergence. Then we also have $L_\alpha u = \mathrm{div}_\alpha D_\alpha u$. We list a Sobolev inequality in $\mathbb{R}^m \times \mathbb{R}^k$ (see [9, 10]).

Lemma 2.1. *Let $D^1(\mathbb{R}^m \times \mathbb{R}^k)$ be the completion of $C_0^\infty(\mathbb{R}^m \times \mathbb{R}^k)$ under the seminorm of*

$$||u|| = (\int_{\mathbb{R}^m \times \mathbb{R}^k} (|\nabla_x u|^2 + (\alpha + 1)^2 |x|^{2\alpha} |\nabla_y u|^2) dxdy)^{\frac{1}{2}},$$

then the Sobolev inequality holds:

$$(\int_{\mathbb{R}^m \times \mathbb{R}^k} (|u|^{\frac{2Q}{Q-2}} dxdy)^{\frac{Q-2}{2Q}} \leq (\int_{\mathbb{R}^m \times \mathbb{R}^k} (|\nabla_x u|^2 + (\alpha + 1)^2 |x|^{2\alpha} |\nabla_y u|^2) dxdy)^{\frac{1}{2}}.$$

For the function $u \in D^1(\mathbb{R}^m \times \mathbb{R}^k)$ and $p = (x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^k$, let U be the Kelvin transformation of u with respect to point p, define as

$$U(z)=\frac{1}{||z-(x_0,y_0)||^{Q-2}}u\Big(\frac{z-(x_0,y_0)}{||z-(x_0,y_0)||^2}+(x_0,y_0)\Big),\ z\neq(x_0,y_0).$$

In this paper, let (u,v) be nonnegative in the Grushin manifolds $\mathbb{R}^m\times\mathbb{R}^k$, we follow the Kelvin transform in [16, 29] which centered at $p=0$. Then we obtain that U,V is continuous in $\mathbb{R}^m\times\mathbb{R}^k\setminus\{0\}$ of the Kelvin transformation of u,v. Obviously, U,V are continuous and nonnegative in $\mathbb{R}^m\times\mathbb{R}^k\setminus\{0\}$. And a direct computation shows that:

Lemma 2.2. *Let (u,v) be a nonnegative weak solution of system (1.1). Then (U,V) weakly satisfies the following system:*

$$\begin{cases}-L_\alpha U=\dfrac{1}{||z||^{Q+2}}f(||z||^{P-2}V), & \text{in } \mathbb{R}^m\times\mathbb{R}^k\setminus\{0\},\\ -L_\beta V=\dfrac{1}{||z||^{P+2}}g(||z||^{Q-2}U), & \text{in } \mathbb{R}^m\times\mathbb{R}^k\setminus\{0\}.\end{cases}\tag{2.1}$$

Moreover, (U,V) has decay at infinity as

$$\lim_{||z||\to\infty}||z||^{Q-2}U(z)=u(0),\quad \lim_{||z||\to\infty}||z||^{P-2}V(z)=v(0).\tag{2.2}$$

Let $\lambda\in\mathbb{R}$ and $z=(x,y)$. We define $T_\lambda=\{(x,y)\in\mathbb{R}^m\times\mathbb{R}^k|\, y_1=\lambda\}$, $\Sigma_\lambda=\{(x,y)\in\mathbb{R}^m\times\mathbb{R}^k|\, y_1>\lambda\}$, $p_\lambda=(0,\cdots,0,y_1=2\lambda,0,\cdots,0)$, $z_\lambda=(x_1,x_2,\cdots,x_m,2\lambda-y_1,y_2,\cdots,y_k)$, and denote $U_\lambda(z)=U(z_\lambda)$ and $V_\lambda(z)=V(z_\lambda)$. Then we can infer from problem (2.1) that U_λ and V_λ satisfy

$$\begin{cases}-L_\alpha U_\lambda=\dfrac{1}{||z_\lambda||^{Q+2}}f(||z_\lambda||^{P-2}V_\lambda), & \text{in } \mathbb{R}^m\times\mathbb{R}^k\setminus\{p_\lambda\},\\ -L_\beta V_\lambda=\dfrac{1}{||z_\lambda||^{P+2}}g(||z_\lambda||^{Q-2}U_\lambda), & \text{in } \mathbb{R}^m\times\mathbb{R}^k\setminus\{p_\lambda\}.\end{cases}\tag{2.3}$$

Let $U^\lambda(z)=U(z)-U(z_\lambda),\ V^\lambda(z)=V(X)-V(z_\lambda)$, we prove the following integral inequalities at firstly.

Lemma 2.3. *For any fixed $\lambda>0$, $U^\lambda\in L^{\frac{2Q}{Q-2}}(\Sigma_\lambda)\bigcap L^\infty(\Sigma_\lambda)$, $V^\lambda\in L^{\frac{2P}{P-2}}(\Sigma_\lambda)\bigcap L^\infty(\Sigma_\lambda)$, such that*

$$\int_{\Sigma_\lambda}\left|D_\alpha U^{\lambda+}\right|^2dxdy\le C_\lambda\Big(\int_{A_\lambda^1}\frac{1}{||z||^{\frac{PQ(P+Q-4)}{P+Q}}}dxdy\Big)^{\frac{2(P+Q)}{PQ}}\int_{\Sigma_\lambda}\left|D_\beta V^{\lambda+}\right|^2dxdy,\tag{2.4}$$

$$\int_{\Sigma_\lambda}\left|D_\beta V^{\lambda+}\right|^2dxdy\le C_\lambda\Big(\int_{A_\lambda^2}\frac{1}{||z||^{\frac{PQ(P+Q-4)}{P+Q}}}dxdy\Big)^{\frac{2(P+Q)}{PQ}}\int_{\Sigma_\lambda}\left|D_\alpha U^{\lambda+}\right|^2dxdy,\tag{2.5}$$

where $D_\alpha=(\nabla_x,(\alpha+1)|x|^\alpha\nabla_y)$, $D_\beta=(\nabla_x,(\beta+1)|x|^\beta\nabla_y)$, $A_\lambda^1=\{z\in\Sigma_\lambda|\ V^\lambda\ge0\}$, $A_\lambda^2=\{z\in\Sigma_\lambda|\ U^\lambda\ge0\}$, $U^{\lambda+}=\max\{U^\lambda,0\}$, $V^{\lambda+}=\max\{V^\lambda,0\}$, $C_\lambda>0$ is a constant which is nonincreasing in λ.

Proof. We just prove the inequality (2.4), the proof of (2.5) is omitted. For any fixed $\lambda>0$, we know that U and $U^{\lambda+}\le U\in L^{\frac{2Q}{Q-2}}(\Sigma_\lambda)\bigcap L^\infty(\Sigma_\lambda)$.

For $\varepsilon>0$ small, choose a smooth cut-off function $\eta_\varepsilon\in C_0^\infty(\mathbb{R}^m\times\mathbb{R}^k)$ such that $0\le\eta_\varepsilon\le1$, $\eta_\varepsilon(z)=1$ for $2\varepsilon\le||z-p_\lambda||\le\varepsilon^{-1}$, $\eta_\varepsilon=0$ for $||z-p_\lambda||\le\varepsilon$ or $||z-p_\lambda||\ge2\varepsilon^{-1}$, with that $|D_\alpha\eta_\varepsilon|\le C\varepsilon^{-1}$ for

$\varepsilon \leq \|z - p_\lambda\| \leq 2\varepsilon$ and $|D_\alpha \eta_\varepsilon| \leq C\varepsilon$ for $\varepsilon^{-1} \leq \|z - p_\lambda\| \leq 2\varepsilon^{-1}$, where the positive constant C is independent of ε. By the assumption $h(t) = \frac{f(t)}{t^{\frac{Q+2}{P-2}}}$, for $z \in \Sigma_\lambda$, we notice that U and U_λ satisfy

$$-L_\alpha U = h(\|z\|^{P-2} V(z)) V(z)^{\frac{Q+2}{P-2}}, \quad \text{in } \mathbb{R}^m \times \mathbb{R}^k \setminus \{0\}, \tag{2.6}$$

and

$$-L_\alpha U_\lambda = h(\|z_\lambda\|^{P-2} V_\lambda(z)) V_\lambda(z)^{\frac{Q+2}{P-2}}, \quad \text{in } \mathbb{R}^m \times \mathbb{R}^k \setminus \{p_\lambda\}. \tag{2.7}$$

Multiply the above equations by the test function $\psi = U^{\lambda+}\eta_\varepsilon$ and denote $\phi_\varepsilon = U^{\lambda+}\eta_\varepsilon^2$, we deduce that

$$\begin{aligned}
&\int_{\Sigma_\lambda \cap \{2\varepsilon \leq |z-p_\lambda| \leq \varepsilon^{-1}\}} \left|D_\alpha U^{\lambda+}\right|^2 dz \\
&\leq \int_{\Sigma_\lambda} |D_\alpha \psi|^2 dz = \int_{\Sigma_\lambda} D_\alpha U^{\lambda+} \cdot D_\alpha \phi_\varepsilon dz + \int_{\Sigma_\lambda} (U^{\lambda+})^2 |D_\alpha \eta_\varepsilon|^2 dz = I + I_\varepsilon.
\end{aligned} \tag{2.8}$$

We estimate I_ε at first.Write $B_\varepsilon^+ = \{z \in \Sigma_\lambda | \varepsilon \leq \|z - p_\lambda\| \leq 2\varepsilon \text{ or } \varepsilon^{-1} \leq \|z - p_\lambda\| \leq 2\varepsilon^{-1}\}$, then we have

$$\int_{B_\varepsilon^+} |D_\alpha \eta_\varepsilon|^Q dz \leq C.$$

Hence, following from the Hölder inequality, we obtain

$$I_\varepsilon \leq (\int_{B_\varepsilon^+} (u^{\lambda+})^{\frac{2Q}{Q-2}} dz)^{\frac{Q-2}{Q}} (\int_{B_\varepsilon^+} |D_\alpha \eta_\varepsilon|^Q dz)^{\frac{2}{Q}} \leq C (\int_{B_\varepsilon^+} (u^{\lambda+})^{\frac{2Q}{Q-2}} dz)^{\frac{Q-2}{Q}} \to 0$$

as $\varepsilon \to 0$.

We estimate I at first. Since $\|z\| > \|z_\lambda\|$, and h is nonincreasing. If $v(z) \geq v(z_\lambda) \geq 0$, then $-h(\|z\|^{Q-2} V) \geq -h(\|z_\lambda\|^{Q-2} V_\lambda)$. By (2.6) and (2.7), we have

$$\begin{aligned}
I &= \int_{\Sigma_\lambda} D_\alpha U^{\lambda+} \cdot D_\alpha \phi_\varepsilon dz \\
&= -\int_{A_\lambda^1} L_\alpha U^\lambda \cdot \phi_\varepsilon dxdy \\
&= \int_{A_\lambda^1} [h(\|z\|^{P-2} V(z)) V(z)^{\frac{Q+2}{P-2}} - h(\|z_\lambda\|^{P-2} (V_\lambda)(z)) (V_\lambda(z))^{\frac{Q+2}{P-2}}] \phi_\varepsilon dxdy \\
&\leq \int_{A_\lambda^1} h(\|z\|^{P-2} V(z)) [V(z)^{\frac{Q+2}{P-2}} - (V_\lambda(z))^{\frac{Q+2}{P-2}}] \phi_\varepsilon dxdy \\
&\leq C'_\lambda \int_{A_\lambda^1} V^{\frac{Q-P+4}{P-2}} V^{\lambda+} \phi_\varepsilon dxdy \leq C'_\lambda \int_{A_\lambda^1} \frac{1}{\|z\|^{Q-P+4}} V^{\lambda+} \phi_\varepsilon dxdy \\
&= C'_\lambda \int_{A_\lambda^1} \frac{1}{\|z\|^{Q-P+4}} U^{\lambda+} V^{\lambda+} (\eta_\varepsilon)^2 dxdy \\
&\leq C_\lambda (\int_{A_\lambda^1} \frac{1}{\|z\|^{\frac{PQ(Q-P+4)}{P+Q}}})^{\frac{P+Q}{PQ}} (\int_{\partial\Sigma_\lambda} (U^{\lambda+})^{\frac{2Q}{Q-2}} dxdy)^{\frac{Q-2}{2Q}} (\int_{\partial\Sigma_\lambda} (V^{\lambda+})^{\frac{2P}{P-2}} dxdy)^{\frac{P-2}{2P}}.
\end{aligned}$$

By Lemma 2.1, Sobolev trace inequality on extended Grushin manifolds, letting $\varepsilon \to 0$ in (2.8), we get

$$\begin{aligned}
&\int_{\Sigma_\lambda} \left|D_\alpha U^{\lambda+}\right|^2 dz \\
&\leq C_\lambda \Big(\int_{A_\lambda^1} \frac{1}{||z||^{\frac{PQ(Q-P+4)}{P+Q}}}\Big)^{\frac{P+Q}{PQ}} \Big(\int_{\partial\Sigma_\lambda} (U^{\lambda+})^{\frac{2Q}{Q-2}} dxdy\Big)^{\frac{Q-2}{2Q}} \Big(\int_{\partial\Sigma_\lambda} (V^{\lambda+})^{\frac{2P}{P-2}} dxdy\Big)^{\frac{P-2}{2P}} \\
&\leq C_\lambda \Big(\int_{A_\lambda^1} \frac{1}{||z||^{\frac{PQ(Q-P+4)}{P+Q}}}\Big)^{\frac{P+Q}{PQ}} \Big(\int_{\Sigma_\lambda} \left|D_\alpha U^{\lambda+}\right|^2 dz\Big)^{\frac{1}{2}} \Big(\int_{\Sigma_\lambda} \left|D_\beta V^{\lambda+}\right|^2 dz\Big)^{\frac{1}{2}}.
\end{aligned}$$

Hence, (2.4) is holds. □

3. Proof of Theorem 1.1

We start moving planes from some place.

Lemma 3.1. *There exist $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $U^\lambda(z) \leq 0$ and $V^\lambda(z) \leq 0$ for any $z \in \Sigma_\lambda$.*

Proof. If $\lambda > 0$ large enough, then

$$\int_{A_\lambda^1} \frac{1}{||z||^{\frac{PQ(Q-P+4)}{P+Q}}} \leq \int_{\Sigma_\lambda} \frac{1}{||z||^{\frac{PQ(Q-P+4)}{P+Q}}} \to 0, \quad \text{as } \lambda \to 0,$$

we have

$$C_\lambda \Big(\int_{A_\lambda^1} \frac{1}{||z||^{\frac{PQ(Q-P+4)}{P+Q}}}\Big)^{\frac{2(P+Q)}{PQ}} < 1, \quad \text{for all } \lambda \geq \lambda_0,$$

and

$$C_\lambda \Big(\int_{A_\lambda^2} \frac{1}{||z||^{\frac{PQ(Q-P+4)}{P+Q}}}\Big)^{\frac{2(P+Q)}{PQ}} < 1, \quad \text{for all } \lambda \geq \lambda_0.$$

By Lemma 2.3, for $U^\lambda \in L^{\frac{2Q}{Q-2}}(\Sigma_\lambda) \bigcap L^\infty(\Sigma_\lambda)$, $V^\lambda \in L^{\frac{2P}{P-2}}(\Sigma_\lambda) \bigcap L^\infty(\Sigma_\lambda)$, we deduce

$$\int_{\Sigma_\lambda} \left|D_\alpha U^{\lambda+}\right|^2 dz = 0$$

and

$$\int_{\Sigma_\lambda} \left|D_\beta V^{\lambda+}\right|^2 dz = 0$$

for all $\lambda \geq \lambda_0$. Thus, for $\lambda > 0$ large enough, we obtain that $U^\lambda(X) \leq 0$ and $V^\lambda(X) \leq 0$, for all $z \in \Sigma_\lambda$. □

The above process provides a starting point of moving planes, now we can go on moving the planes. Let

$$\Lambda = \inf\{\lambda > 0 |\ U_\mu(z) \leq 0, V_\mu(z) \leq 0, \forall z \in \Sigma_\mu, \mu > \lambda\}.$$

Lemma 3.2. *If $\Lambda > 0$, then $U^\Lambda(z) \equiv 0$ and $V^\Lambda(z) \equiv 0$ for any $z \in \Sigma_\Lambda$.*

Proof. By the continuity of u and v, we have $U^{\Lambda}(z) \leq 0$ and $V^{\Lambda}(z) \leq 0$ for any $z \in \Sigma_{\Lambda}$.

Suppose on the contrary that $V^{\Lambda}(z) \not\equiv 0$ in Σ_{Λ}, then we have

$$\begin{aligned} h(\|z\|^{P-2}V(z))V(z)^{\frac{Q+2}{P-2}} &= \frac{f(\|z\|^{P-2}V(z))}{\|z\|^{Q+2}} \leq \frac{f(\|z\|^{P-2}V_{\Lambda}(z))}{\|z\|^{Q+2}} \leq \frac{f(\|z_{\Lambda}\|^{P-2}V_{\Lambda}(z))}{\|z_{\Lambda}\|^{Q+2}} \\ &= h(\|z_{\Lambda}\|^{P-2}(V_{\Lambda})(z))(V_{\Lambda}(z))^{\frac{Q+2}{P-2}}. \end{aligned}$$

Using the Lemma 2.1 and the strong maximum principle to $V^{\Lambda}(z)$, we obtain that $V^{\Lambda}(z) \leq 0$, and then $V^{\Lambda}(z) < 0$ in Σ_{Λ}. This strict inequality shows that the characteristic function $\mathcal{X}_{A_{\lambda}^{2}} \to 0$ a.e. in $\mathbb{R}^m \times \mathbb{R}^k$, when $\lambda \to \Lambda$. By the dominated convergence theorem, we know

$$\lim_{\lambda \to \Lambda} C_{\lambda}\Big(\int_{A_{\lambda}^{2}} \frac{1}{\|z\|^{\frac{PQ(Q-P+4)}{P+Q}}}\Big)^{\frac{2(P+Q)}{PQ}} = 0,$$

therefore, for $\lambda \in (\Lambda - \delta, \Lambda)$, we see

$$C_{\lambda}\Big(\int_{A_{\lambda}^{1}} \frac{1}{\|z\|^{\frac{PQ(Q-P+4)}{P+Q}}}\Big)^{\frac{2(P+Q)}{PQ}} \cdot C_{\lambda}\Big(\int_{A_{\lambda}^{2}} \frac{1}{\|z\|^{\frac{PQ(Q-P+4)}{P+Q}}}\Big)^{\frac{2(P+Q)}{PQ}} < 1,$$

where $\delta > 0$ is a sufficiently small constant. Using the previous argument, we obtain that $V^{\lambda}(z) \leq 0$ and $U^{\lambda}(z) \leq 0$ for any $z \in \Sigma_{\lambda}$, which contradicts with the definition of Λ. □

If $\Lambda = 0$, for any $(x, y_1, y_2, \cdots, y_n) \in \Sigma_0$, we get $U(x, y_1, y_2, \cdots, y_n) \leq U(x, -y_1, y_2, \cdots, y_n)$ and $V(x, y_1, y_2, \cdots, y_n) \leq V(x, -y_1, y_2, \cdots, y_n)$. One can move the planes in the contrary direction, which will show that $U(x, y_1, y_2, \cdots, y_n) \geq U(x, -y_1, y_2, \cdots, y_n)$ and $V(x, y_1, y_2, \cdots, y_n) \geq V(x, -y_1, y_2, \cdots, y_n)$. Hence, we have

$$U(x, y_1, y_2, \cdots, y_n) = U(t, x, -y_1, y_2, \cdots, y_n)$$

and

$$V(x, y_1, y_2, \cdots, y_n) = V(t, x, -y_1, y_2, \cdots, y_n).$$

For any point can be chosen as the center of Kelvin transform, then U and V must be independent of the variable y_1. Similarly, we can process the above procedure of the directions $y_2, \cdots, y_n$. Then we obtain that U, V are independent of y. However, this claim implies that U and V satisfy the system

$$\begin{cases} -\Delta_x u = f(v), & \text{in } \mathbb{R}^m, \\ -\Delta_x v = g(u), & \text{in } \mathbb{R}^m, \end{cases}$$

By [7, 14], we have that $(u, v) = (c_1, c_2)$ for some constants c_1, c_2 with $f(c_2) = g(c_1) = 0$. This completes the proof of Theorem 1.1.

4. Conclusions

In this paper, the Liouville type theorem of positive weak solution of nonlinear system for Grushin operator be given by the method of moving plane. In stead of the maximum principles, we introduce some integral inequality, which be used in the processes of moving planes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Key Scientific Research Project of the Colleges and Universities in Henan Province (No. 22A110013), the Key Specialized Research and Development Breakthrough Program in Henan Province (No. 222102310265), and the Natural Science Foundation of Henan Province of China (No. 222300420499).

Conflict of interest

The author declares that he has no conflict of interest.

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