AIMS Mathematics

AIMS Mathematics, 8(8): 18881–18897.
DOI: 10.3934/math.2023961
Received: 18 March 2023
Revised: 20 May 2023
Accepted: 24 May 2023
Published: 05 June 2023

http://www.aimspress.com/journal/Math

Research article

On conjunctive complex fuzzification of Lagrange's theorem of ξ −CFSG

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Abstract: The application of a complex fuzzy logic system based on a linear conjunctive operator represents a significant advancement in the field of data analysis and modeling, particularly for studying physical scenarios with multiple options. This approach is highly effective in situations where the data involved is complex, imprecise and uncertain. The linear conjunctive operator is a key component of the fuzzy logic system used in this method. This operator allows for the combination of multiple input variables in a systematic way, generating a rule base that captures the behavior of the system being studied. The effectiveness of this method is particularly notable in the study of phenomena in the actual world that exhibit periodic behavior. The foremost aim of this paper is to contribute to the field of fuzzy algebra by introducing and exploring new concepts and their properties in the context of conjunctive complex fuzzy environment. In this paper, the conjunctive complex fuzzy order of an element belonging to a conjunctive complex fuzzy subgroup of a finite group is introduced. Several algebraic properties of this concept are established and a formula is developed to calculate the conjunctive complex fuzzy order of any of its powers in this study. Moreover, an important condition is investigated that determines the relationship between the membership values of any two elements and the membership value of the identity element in the conjunctive complex fuzzy subgroup of a group. In addition, the concepts of the conjunctive complex fuzzy order and index of a conjunctive complex fuzzy subgroup of a group are also presented in this article and their various fundamental algebraic attributes are explored structural. Finally, the conjunctive complex fuzzification of Lagrange's theorem for conjunctive complex fuzzy subgroups of a group is demonstrated.

Keywords: complex fuzzy set (CFS); complex fuzzy subgroup (CFSG); complex fuzzy normal subgroup (CFNSG); conjunctive complex fuzzy set (ξ −CFS); conjunctive complex fuzzy subgroup (ξ −CFSG); conjunctive complex fuzzy normal subgroup (ξ −CFNSG)
Mathematics Subject Classification: 08A72, 20N25, 03E72

1. Introduction

In the early 18th century, a star appeared on the horizon of group theory in the form of Lagrange's theorem, which played a key role in solving many complicated questions not only about algebra but also about geometry. This incredible result has a great impact on several branches of mathematics especially finite group theory, combinatorics and number theory. Many unraveling mathematical structures have been proved by this remarkable result. It is also a powerful tool to investigate the meaningful characterizations of finite groups as it provides a comprehensive structural view of subgroups. The notion of coset plays a central part to understand the ideology of this revolutionary result. One of the main advantages of this theorem is that it plays a key role in proving the famous Fermat's little theorem and its generalization, Euler's theorem in the field of number theory. One can easily visualize a probable connection between Lagrange's theorems to real life shows the relationship between group theory and real life. In addition, this result is considered a valuable source of abstract algebra, but it can be gradually integrated into physical situations. A good detail about the prosperous history of this noticeable theorem can be read in [14,16].

Presently, science and technology are characterized by complex processes and phenomena for which complete information is not always available. These situations lead us to design specific mathematical models to handle such types of systems having elements of uncertainty. The extension of classical set theory, namely, the fuzzy set theory helps to formulate the mechanism of large numbers of these models.

The theory of fuzzy sets plays a vital role to solve many physical situations in different branch of modern science, particularly, decision making, mathematical chemistry, biological classification and thermodynamics. Despite all these benefits, we still face major challenges to counter several physical situations based on a complex membership function. Complex fuzzy sets permit a natural extension to fuzzy set theory to counter such specific problems that cannot be addressed with one-dimensional grades of membership. The image processing and numerous periodic aspects of the forecast are the problems that can be handle through complex fuzzy sets in much effective way.

In modern days, the development of computer technologies, the accessibility of high-speed processors and numerous programming languages enable researchers to investigate and design many algorithms to solve physical phenomena on computers in various fields of science. However, to design high-precision models of a physical process, one must begin with its mathematical description and analysis in order to attain the specific outcomes of the problem under consideration. This helps to design highly efficient numerical methods that can effectively be applied to a computer. In this connection, the applications of the operator theory to engineering problems have grown considerably. This development is clearly connected to the wide variety of applications of both practical and theoretical interests. In many fields, various technical procedures contain similar mathematical structures that can be interpreted in the framework of general operator theory. Such a generalization allows to construct useful algorithms to solve a wide class of problems.

Zadeh [1] introduced the concept of a fuzzy set in 1965 as a means to address the challenges of managing vagueness in practical situations. The concept of fuzzy subgroups and its fundamental algebraic characteristics were established by Rosenfeld [2] in 1971. In 1981, Das [3] presented the idea of level subgroups of a fuzzy subgroup. One can study the comprehensive detail about the fundamental notions of fuzzy subgroups in [4–8]. Buckley [9] innovated the study of complex fuzzy numbers in 1989. The author [10] established the theory of complex fuzzy numbers to design a new formulation of differentiation by using this concept. Furthermore, some elementary properties of fuzzy contour integral in the complex plane were depicted by the same author in [11]. Many important properties of complex fuzzy numbers were formulated by Zhang [12]. An efficient fuzzy processor capable of dealing with complex fuzzy inference system was designed by Ascia et al. [13]. Ramote et al. [15] proponded the concept of CFS in 2002 and presented a comprehensive study of two novel operations, namely, reflection and rotation. In the recent times, the authors developed the theories of CFNSG [17], complex fuzzy hyper structure [18] and CFSG [19] over a complex fuzzy space. In 2017, the idea of CFS was used to initiate the notion of CFSG by Alsarahead and Ahmad [20]. In addition, the idea of complex intuitionistic fuzzy sets was innovated by the authors [21] in 2017. Moreover, the significant applications of this newly defined concept in the solution of decision making problems can be viewed in [22,23]. In 2018, an approximate parallelity preserving for complex fuzzy operators was established in [24] and several complex fuzzy geometric aggregation operators were investigated in [25]. Lvqing Bi et al. [26] and Songsong Dai et al. [27] defined two kinds of entropy measures for complex fuzzy sets and analyze their rotational invariance properties in 2019. Abd Ulzeez M. J. Alkouri et al. [28] gave the formal definition of bipolar complex fuzzy distance measure and some basic mathematical operations on bipolar CFS in 2020. Moreover, the authors [29] proposed the phenomenon of CFS based over linear conjunctive operator in 2020. Modernistic applications of complex fuzzy set can be seen in [30–39].

Complex fuzzy sets are a valuable tool for characterizing real-life problems that exhibit periodicity, as they provide a useful representation of the unpredictability and periodicity of the objects within these sets. However, their ability to handle real-world problems involving complex valued membership functions is inherently limited. To overcome this constraint, we propose a novel concept of complex fuzzy set based on a linear conjunctive operator, which offers increased flexibility and efficiency in modeling real world issues. Its application enables the systematic integration of multiple input variables that comprehensively represents the behavior of the studied system. This is accomplished by assigning a degree of membership to each input variable based on its relevance to the rule being defined. It is noteworthy that this method is highly efficacious in investigating phenomena that demonstrates periodicity in real-world situations. Despite all the significances and features of CFS, there is a lack of capacity to solve many real-life problems over a complex valued membership function. This encourages us to describe the concept of CFS over a linear conjunctive operator through which one can have the multiple options to investigate a specific real-world situation in much efficient way by choosing appropriate value of the parameter. In addition, it is important to note that the existing knowledge of FS and CFS lack the ability to dynamically adjust the parameter in accordance with the decision makers' risk aversion, which makes the multi attribute decision making solutions difficult to implement in real life. However, the approach presented in this article is very capable of addressing this weakness in this situation. Moreover, we also introduce conjunctive complex fuzzy subgroups (ξ −CFSG) using this method, which can effectively resolve key decision-making problems in areas such as image processing, crypto currencies and cyber-security. Our approach improves the effectiveness with which challenging real-world problems can be solved by expanding the capabilities of complex fuzzy sets. We anticipate that this method will have broad applications in various fields due to its ability to handle complex valued membership functions and the versatility offered by the linear conjunctive operator.

After a little discussion about the development of complex fuzzy subgroups, the remaining portion of this paper is shaped as: In Section 2, we study basic notions of ξ −CFS and ξ −CFSG, which are very important to understand the work presented in this paper. Section 3 contains the notion of conjunctive complex fuzzy order of an element of ξ −CFSG of a finite group along various fundamental algebraic characteristics of this phenomenon. In addition, we design a formula to calculate the conjunctive complex fuzzy order of an element for any of its power and determine a correlation between the membership values of any two elements and the membership grade of identity element of ξ −CFSG.The Section 4 deals with the notions of conjunctive complex fuzzy order and index of ξ −CFSG of a finite group and describes their numerous structural characteristics. Moreover, we utilize the study of these notions to establish the conjunctive complex fuzzification of Lagrange's theorem of ξ −CFSG.

2. Preliminaries

In this section, we provide some basic information of conjunctive complex fuzzy sets and conjunctive complex fuzzy subgroups to understand the topics covered in this paper.

Definition 2.1. [15] A CFS A defined on a universe of discourse U, is characterized by a membership function $\mu_A(m)$ that allocates each element of U to a unit circle C^* in complex plane and is written as $r_A(m)e^{i\omega_A(m)}$, where $r_A(m)$ denotes the real-valued function from U to the closed unit interval and $e^{i\omega_A(m)}$ is a periodic function whose periodic law and principal period are 2π and $0 < arg_A(m) \le 2\pi$, respectively.

Note that $\omega_A(m) = arg_A(m) + 2k\pi, k \in Z$ and $arg_A(m)$ is the principal argument.

Definition 2.2. [29] For any CFS A of U and an element $\xi \in C^*$ where $\xi = \alpha e^{\delta}$, $0 \le \alpha \le 1$ and $0 \le \delta \le 2\pi$. The conjunctive complex fuzzy set A^{ξ} relative to CFS A is an object of the form $\mu_{A^{\xi}}(m) = min\big(r_A(m)e^{i\omega_A(m)}, \alpha e^{i\beta}\big) = min\{r_A(m), \alpha\}e^{imin(\omega_A(m),\beta)} = r_{A^{\xi}}(m)e^{i\omega_{A^{\xi}}(m)}$ for any element m of U. Here, the real valued function is $r_{A^{\xi}}: U \to [0, 1]$ and $e^{i\omega_{A^{\xi}}}$ denotes a periodic function with periodicity 2π and $0 < arg_{A^{\xi}} \le 2\pi$.

For convenience, we write a conjunctive complex fuzzy set as ξ −CFS and $F^{\xi}(U)$ represents the family of ξ −CFS of U.

Definition 2.3. [17] Let A and B be any two CFS of a universe U, then

1) A is homogeneous CFS if $r_A(m) \le r_A(n)$ implies $\omega_A(m) \le \omega_A(n)$ and vice versa $\forall m, n \in U$.
2) A is homogeneous CFS with B if $r_A(m) \le r_B(n)$ implies $\omega_A(m) \le \omega_B(n)$ and vice versa $\forall m, n \in U$.

Definition 2.4. [17] Let A be a CFS of a group G. Then A is said to be a CFSG of G if the following conditions are satisfied for each element of G.

1) $\mu_A(mn) \ge \min\{\mu_A(m), \mu_A(n)\}$
2) $\mu_A(m^{-1}) \ge \mu_A(m)$.

Definition 2.5. [29] Let $A^{\xi}, B^{\xi} \in F^{\xi}(U)$. Then

1) A^{ξ} is said to be homogeneous ξ − CFS if $r_{A^{\xi}}(m) \le r_{A^{\xi}}(n)$ implies $\omega_{A^{\xi}}(m) \le \omega_{A^{\xi}}(n)$, $\forall\, m, n \in U$.
2) A^{ξ} is said to be homogeneous ξ − CFS with B^{ξ} if $r_{A^{\xi}}(m) \le r_{B^{\xi}}(n)$ implies $\omega_{A^{\xi}}(m) \le \omega_{B^{\xi}}(n)$, $\forall\, m, n \in U$.

Definition 2.6. [29] Let A^{ξ} be a ξ −CFS of a group G. Then A^{ξ} is said to be a conjunctive complex fuzzy subgroup (denoted by ξ −CFSG) if A^{ξ} satisfies the following conditions for all element m and n in G:

1) $\mu_{A^{\xi}}(mn) \ge min\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}(n)\}$.

2) $\mu_{A^\xi}(m^{-1}) \geq \mu_{A^\xi}(m)$.

We denote a conjunctive complex fuzzy subgroup by the ξ −CFSG whereas the family of ξ −CFSG of a group G is represented by $F^{\xi}(G)$ in this paper.

Definition 2.7. [17] A CFSG A of a group G is CFNSG(G) if $mA = Am, \forall m \in G$.

Definition 2.8. [29] Let $A^{\xi} \in F^{\xi}(G)$ and be a fixed element of the group G. The conjunctive complex fuzzy left coset of A^{ξ} in G is denoted by mA^{ξ} and is described as $\mu_{mA^\xi}(n) = \mu_{A^\xi}(m^{-1}n), \forall n \in G$ is called the conjunctive complex fuzzy left coset determined by m and A^{ξ}.

Definition 2.9. [29] Let $A^{\xi} \in F^{\xi}(G)$ then A^{ξ} is a conjunctive complex fuzzy normal subgroup (denoted by ξ −CFNSG) of G if $\mu_{A^\xi}(m) = \mu_{A^\xi}(n^{-1}mn), \forall\, m, n \in G$.

The above definition can also be visualized as: $\mu_{A^\xi}(mn) = \mu_{A^\xi}(nm)$.

We denote a conjunctive complex fuzzy normal subgroup by ξ −CFNSG whereas the family of ξ −CFNSG of a group G is represented by $F^{\xi}N(G)$ in this paper.

3. Structural characterization of ξ −complex fuzzy order of an element of ξ −complex fuzzy subgroup

In the modern era, scientific and technological systems have become increasingly complex and multifaceted, often involving phenomena that are not fully understood or characterized by complete information. To tackle such complex systems and their uncertainties, specific mathematical models are required. In this context, the extension of classical set theory, known as fuzzy set theory, has emerged as a powerful tool to handle such situations. This theory has proven to be particularly useful in addressing a wide range of physical situations in various branches of modern science, including decision making, mathematical chemistry, biological classification and thermodynamics. Despite the benefits of fuzzy set theory, we still face significant challenges in dealing with physical situations that involve complex membership functions. Complex fuzzy sets provide a natural extension to fuzzy set theory, enabling the handling of specific problems that cannot be adequately addressed using one-dimensional grades of membership. Complex fuzzy sets are employed in control systems to regulate and optimize complex processes, such as manufacturing, robotics and traffic control. By adapting to changing conditions and uncertainties, complex fuzzy control systems can improve efficiency, safety and performance. By using the complex fuzzy sets, the modeling and analysis of these systems can be performed with higher accuracy and efficiency, resulting in superior outcomes and solutions to real-world problems. This highlights the novelty, versatility and practicality of conjunctive complex fuzzy sets in dealing with complex and uncertain systems, thereby advancing the application of complex fuzzy set theory to a wide range of fields. In addition, the elements of a group, along with their properties and relationships, provide crucial information about the structure of the group. By analyzing the significance of different elements, mathematicians can gain insights into the overall structure and properties of the group. The science of secure communication relies on the use of group theory to design and analyze encryption algorithms. The order of an element plays a crucial role in the design of such algorithms, as it determines the size of the group and the complexity of the computations required to break the encryption and also used to generate repeating patterns and animations in computer graphics. The above discussion motivates us to define the notion of a conjunctive complex fuzzy order of an element of a conjunctive complex fuzzy subgroup of a finite group G in this section. Many fundamental characteristics are highlighted that emerge as a result of this idea, which are vital to understand the underlying structural properties of such subgroups.

Definition 3.1. Consider a ξ −CFSG A^{ξ} and an element m of a finite group G. The least positive integer n is called conjunctive complex fuzzy order of m (denoted by $\xi - CFO_{A^{\xi}}(m)$) if $\mu_{A^{\xi}}(m^n) = \mu_{A^{\xi}}(e)$.

Example 3.2. Consider the group $G = \{1, -1, i, -i\}$ is given by:

$$\mu_A(m) = \begin{cases} 1e^{i1.9\pi} & if\ m \in \{1, -1\} \\ 0.5e^{i\pi} & otherwise \end{cases}.$$

The ξ −CFSG A^{ξ} of G with respect to the value of the parameter $\xi = 0.6e^{i1.1\pi}$ is obtained as:

$$\mu_{A^{\xi}}(m) = \begin{cases} 0.6e^{i1.1\pi} & if\ m \in \{1, -1\} \\ 0.5e^{i\pi} & otherwise \end{cases}.$$

In view of Definition 3.1, we have $\xi - CFO_{A^{\xi}}(1) = \xi - CFO_{A^{\xi}}(-1) = 1$
$\xi - CFO_{A^{\xi}}(i) = \xi - CFO_{A^{\xi}}(-i) = 4$.

Theorem 3.3. Let $A^{\xi} \in F^{\xi}(G)$ and $m \in G$ then $\xi - CFO_{A^{\xi}}(m) = \xi - CFO_{A^{\xi}}(m^{-1})$.

Proof. Given that A^{ξ} is a $\xi -$CFSG, we have $\mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(m^{-1})$. This implies that $\xi - CFO_{A^{\xi}}(m) = \xi - CFO_{A^{\xi}}(m^{-1})$.

Theorem 3.4. Let $A^{\xi} \in F^{\xi}N(G)$ and m be any fixed element of G then $\xi - CFO_{A^{\xi}}(m) = \xi - CFO_{A^{\xi}}(n^{-1}mn),\ \forall\, n \in G$.

Proof. By using Definition 2.6, we obtain $\mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(n^{-1}mn)$. Thus $\xi - CFO_{A^{\xi}}(m) = \xi - CFO_{A^{\xi}}(n^{-1}mn)$.

The subsequent example describes that the Theorem 3.4 is not true whenever A^{ξ} not conjunctive complex fuzzy normal subgroup in is G.

Example 3.5. The CFSG A of the dihedral group $\mathrm{D}_3 =< a, b \colon a^3 = b^2 = e, ba = a^2 b >$ is defined as:

$$\mu_A(m) = \begin{cases} 0.9e^{1.4i} & if\ m \in < b > \\ 0.5e^{0.9i} & otherwise \end{cases}.$$

The ξ −CFSG A^{ξ} of D_3 with respect to the value of the parameter $\xi = 0.8e^{1.2i}$ is obtained as:

$$\mu_{A^{\xi}}(m) = \begin{cases} 0.8e^{1.2i} & if\ m \in < b > \\ 0.5e^{0.9i} & otherwise \end{cases}.$$

Clearly A^{ξ} is not a $\xi -$ CFNSG. Moreover, $\xi - CFO_{A^{\xi}}(a) = 3 = \xi - CFO_{A^{\xi}}(a^{-1})$ and $\xi - CFO_{A^{\xi}}(b) = 1$. One can easily observe that $\xi - CFO_{A^{\xi}}(b) \neq \xi - CFO_{A^{\xi}}(a^{-1}ba)$.

Theorem 3.6. Let $A^{\xi} \in F^{\xi}N(G)$ then $\xi - CFO_{A^{\xi}}(mn) = \xi - CFO_{A^{\xi}}(nm)\ \ \forall\, m, n \in G$.

Proof. Consider $\xi - CFO_{A^{\xi}}(mn) = \xi - CFO_{A^{\xi}}((n^{-1}n)(mn)) = \xi - CFO_{A^{\xi}}(m^{-1}(nm)n)$.

The application of Theorem 3.4, the above relation yields that $\xi - CFO_{A^{\xi}}(m^{-1}(nm)n) = \xi - CFO_{A^{\xi}}(nm)$.

Theorem 3.7. Let $A^{\xi} \in F^{\xi}(G)$ and $m \in G$ then $\mu_{A^{\xi}}(m^k) \geq \mu_{A^{\xi}}(m), k \in Z$.

Proof. The result is obvious in the frame work of mathematical induction for the values. $k = 0$ and 1. Moreover, for $k = 2$, we have

$$\mu_{A^{\xi}}(m^2) = \mu_{A^{\xi}}(m.m) \geq \min\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}(m)\} = \mu_{A^{\xi}}(m).$$

Let the statement be true for $k = n$. Consider

$$\mu_{A^{\xi}}(m^{n+1}) = \mu_{A^{\xi}}(m^n.m) \geq \min\{\mu_{A^{\xi}}(m^n), \mu_{A^{\xi}}(m)\} = \mu_{A^{\xi}}(m).$$

This completes the induction. Moreover, if $k < 0$ then $\mu_{A^{\xi}}(m^k) = \mu_{A^{\xi}}(m^{-1})^{-k} \geq \mu_{A^{\xi}}(m)$.

Remark 3.8. If $(o(m),k)=1$ then $\mu_{A^\xi}(m^k)=\mu_{A^\xi}(m)$ for any integer k.
Theorem 3.9. Let $A^\xi\in F^\xi(G)$. For $m\in G$, if $\mu_{A^\xi}(m^{k_1})=\mu_{A^\xi}(e)$ for some integers k_1, then $\xi-CFO_{A^\xi}(m)|k_1$.
Proof. Let $\xi-CFO_{A^\xi}(m)=n$. Then by Euclidean Algorithm, there exist integers k_2 and k_3 such that $k_1=nk_2+k_3$, where $0\le k_3<n$. Consider

$$\mu_{A^\xi}(m^{k_3})=\mu_{A^\xi}(m^{k_1-nk_2})=\mu_{A^\xi}(m^{k_1}(m^n)^{-k_2})\ge \min\{\mu_{A^\xi}(m^{k_1}),\mu_{A^\xi}(m^n)^{-k_2}\}$$
$$\ge \min\{\mu_{A^\xi}(e),\mu_{A^\xi}(m^n)\}=\min\{\mu_{A^\xi}(e),\mu_{A^\xi}(e)\}.$$

It follows that $\mu_{A^\xi}(m^{k_3})=\mu_{A^\xi}(e)$.
Hence $k_3=0$ by the minimality of n.
The subsequent theorem establishes a condition in which the membership values of any two elements of ξ −CFSG coincides with the membership value of identity element of ξ −CFSG.
Theorem 3.10. Let $\forall\, m,n\in G$, $(\xi-CFO_{A^\xi}(m),\xi-CFO_{A^\xi}(n))=1$, $mn=nm$ and $\mu_{A^\xi}(mn)=\mu_{A^\xi}(e)$. Then $\mu_{A^\xi}(m)=\mu_{A^\xi}(n)=\mu_{A^\xi}(e)$.
Proof. Let $\xi-CFO_{A^\xi}(m)=k_1$ and $\xi-CFO_{A^\xi}(n)=k_2$. In view of Theorem 3.7 and Remark 3.8, we have $\mu_{A^\xi}(e)=\mu_{A^\xi}(mn)\le\mu_{A^\xi}((mn)^{k_2}=\mu_{A^\xi}(m^{k_2}n^{k_2})$. It follows that $\mu_{A^\xi}(m^{k_2}n^{k_2})=\mu_{A^\xi}(e)$.
Now $\mu_{A^\xi}(m^{k_2})=\mu_{A^\xi}(m^{k_2}n^{k_2}n^{-k_2})\ge\min\{\mu_{A^\xi}(m^{k_2}n^{k_2}),\mu_{A^\xi}(n^{-k_2})\}=\min\{\mu_{A^\xi}(e),\mu_{A^\xi}(e)\}$.
Thus $\mu_{A^\xi}(m^{k_2})=\mu_{A^\xi}(e)$.
By applying the Theorem 3.9, we have $k_1|k_2$. However, $(k_1,k_2)=1$. Thus $k_1=1$. Hence $\mu_{A^\xi}(m)=\mu_{A^\xi}(e)$. Similarly, $\mu_{A^\xi}(n)=\mu_{A^\xi}(e)$.
The following formula facilitates us to calculate the ξ-complex fuzzy order of an element for any of its power.
Theorem 3.11. If $\xi-CFO_{A^\xi}(m)=k_1$ then $\xi-CFO_{A^\xi}(m^{k_2})=\frac{k_1}{(k_1,k_2)}$, for some integer k_2 and $m\in G$.

Proof. Assume that $\xi-CFO_{A^\xi}(m^{k_2})=r$ and $(k_1,k_2)=d$. Consider

$$\mu_{A^\xi}\left((m^{k_2})^{\frac{k_1}{d}}\right)=\mu_{A^\xi}\left((m^{k_1})^{\frac{k_2}{d}}\right)$$
$$\ge\mu_{A^\xi}(e^{\frac{k_2}{d}})$$
$$=\mu_{A^\xi}(e).$$

By using Theorem 3.9, we have r divides $\frac{k_1}{d}$.
Moreover, $(k_1,k_2)=d$, therefore $k_1p+k_2q=d$, for some $p,q\in Z$. Now

$$\mu_{A^\xi}(m^{rd})=\mu_{A^\xi}\left(m^{r(k_1p+k_2q)}\right)\quad=\mu_{A^\xi}(m^{rk_1p}m^{rk_2q})\ge\min\{\mu_{A^\xi}(m^{k_1})^{rp}),\mu_{A^\xi}(m^{k_2})^{rq})\}\ge$$
$$\min\{\mu_{A^\xi}(m^{k_1}),\mu_{A^\xi}(m^{k_2r})\}\ge\min\{\mu_{A^\xi}(m^{k_1}),\mu_{A^\xi}(m^{k_2})^r=\min\{\mu_{A^\xi}(e),\mu_{A^\xi}(e)\}\}.$$

Thus $\mu_{A^\xi}(m^{rd})\ge\mu_{A^\xi}(e)$.
The application of Theorem 3.9 in the above relation, we have $k_1|rd$ and hence $r=\frac{k_1}{d}$.
Theorem 3.12. Let $\xi-CFO_{A^\xi}(m)=k_1$ $(k_1,k_2)=1, k_1,k_2\in Z$ and $m\in G$. Then $\mu_{A^\xi}(m^{k_2})=\mu_{A^\xi}(m)$.
Proof. By using the fact k_1 and k_2 are relatively prime, we have $k_1r+k_2s=1$, for some $r,s\in Z$.

Then

$$\mu_{A^\xi}(m) = \mu_{A^\xi}(k_1 r + k_2 s)$$
$$= \mu_{A^\xi}((m^{k_1})^r (m^{k_2})^s)$$
$$\geq \min\{\mu_{A^\xi}(m^{k_1})^r, \mu_{A^\xi}(m^{k_2})^s\}$$
$$= \min\{\mu_{A^\xi}(e), \mu_{A^\xi}(m^{k_2})\}.$$

Thus

$$\mu_{A^\xi}(m) \geq \mu_{A^\xi}(m^{k_2}). \tag{3.1}$$

Moreover,

$$\mu_{A^\xi}(m^{k_2}) \geq \mu_{A^\xi}(m). \tag{3.2}$$

By comparing (3.1) and (3.2), we get $\mu_{A^\xi}(m^{k_2}) = \mu_{A^\xi}(m)$.

Corollary 3.13. Let A^ξ be a fuzzy subgroup of a group G then $\xi - CFO_{A^\xi}(m) | o(m), \forall m \in G$.

Corollary 3.14. Let A^ξ be a fuzzy subgroup of a group G then $\xi - CFO_{A^\xi}(m) | o(G)$.

Theorem 3.15. Let $\xi - CFO_{A^\xi}(m) = k_1$, $\forall m \in G$. If $x \equiv y(mod n)$, where $x, y \in Z$ then $\xi - CFO_{A^\xi}(m^x) = \xi - CFO_{A^\xi}(m^y)$.

Proof. Let $\xi - CFO_{A^\xi}(m^x) = t$ and $\xi - CFO_{A^\xi}(m^y) = s$. Now let $x = y + k_1 n$ for some $n \in Z$, we have

$$\mu_{A^\xi}((m^x)^s) = \mu_{A^\xi}((m^{y+k_1 n})^s)$$
$$= \mu_{A^\xi}((m^y)^s (m^{k_1})^{ns})$$
$$\geq \min\{\mu_{A^\xi}(m^y)^s, \mu_{A^\xi}(m^{k_1})^n\}$$
$$\geq \min\{\mu_{A^\xi}(e), \mu_{A^\xi}(m^{k_1})\}.$$

Thus $\mu_{A^\xi}((m^x)^s) = \mu_{A^\xi}(e)$

This means that,

$$t | s. \tag{3.3}$$

Similarly,

$$s | t. \tag{3.4}$$

By comparing (3.3) and (3.4), we have $t = s$.

In the following result, we prove a formula to obtain ξ-complex fuzzy order of the product of any two elements of ξ −CFSG.

Theorem 3.16. If $\left(\xi - CFO_{A^\xi}(m),\ \xi - CFO_{A^\xi}(n)\right) = 1$ and $mn = nm, \forall m, n \in G$, then $\xi - CFO_{A^\xi}(mn) = \left[\xi - CFO_{A^\xi}(m)\right] \times \left[\xi - CFO_{A^\xi}(n)\right]$.

Proof. Suppose that $\xi - CFO_{A^\xi}(mn) = k_1$, $\xi - CFO_{A^\xi}(m) = k_2$ and $\xi - CFO_{A^\xi}(n) = k_3$. Consider

$$\mu_{A^\xi}((mn)^{k_2 k_3}) = \mu_{A^\xi}((m)^{k_2 k_3}(n)^{k_2 k_3}) \geq \min\{\mu_{A^\xi}((m^{k_3})^{k_2}), \mu_{A^\xi}((n^{k_3})^{k_2})\}.$$

Thus $\mu_{A^\xi}((mn)^{k_2 k_3}) \geq \mu_{A^\xi}(e)$.

By using the Theorem 3.9 in the above theorem, we have

$$k_2k_3|k_1. \tag{3.5}$$

Since $(k_2, k_3) = 1$, therefore either $k_2|k_1$ or $k_3|k_1$. Assume that $k_2|k_1$, then by applying the Theorem 3.11, we have

$$\xi - CFO_{A^\xi}(m^{k_1}) = \frac{k_2}{(k_1,k_2)}. \tag{3.6}$$

In view of Theorem 3.9, in the above relation for $\xi - CFO_{A^\xi}(n^{k_1})$ establishes the following relation:

$$\xi - CFO_{A^\xi}(n^{k_1}) = \frac{k_3}{(k_1,k_3)}. \tag{3.7}$$

Again from Theorem 3.9 Eqs (3.6) and (3.7) we obtain $\left(\xi - CFO_{A^\xi}(m^{k_2}),\ \xi - CFO_{A^\xi}(n^{k_3})\right) = 1$. By applying Theorem 3.9 Eqs (3.6) and (3.7) and using the above stated fact leads us to note that $\mu_{A^\xi}(e) = \mu_{A^\xi}(m^{k_1}) = \mu_{A^\xi}(n^{k_1})$. This means that

$$k_1|k_2k_3. \tag{3.8}$$

By comparing (3.5) and (3.8), we have the required result.
It is important to note that the condition $mn = nm$ in the Theorem 3.16 cannot be ignored for the validity of the result. The consequent example illustrates the otherwise situation.
Example 3.17. The CFSG A of the symmetric group $S_3 = \{1, (12), (13), (23), (132), (123)\}$ is defined as:

$$\mu_A(m) = \begin{cases} 1e^{1.9i} & if\ x = 1, \\ 0.4e^{0.8i} & otherwise \end{cases}.$$

The ξ −CFSG A^ξ of S_3 with respect to the value of the parameter $\xi = 0.7e^{1.1i}$ is obtained as:

$$\mu_{A^\xi}(m) = \begin{cases} 0.7e^{1.1i} & if\ x = e \\ 0.4e^{0.8i} & otherwise \end{cases}.$$

Consider $m = (1\ 2)$ and $n = (1\ 3)$. Note that $\xi - CFO_{A^\xi}(m) = 2$, $\xi - CFO_{A^\xi}(n) = 2$, $\xi - CFO_{A^\xi}(mn) = \xi - CFO_{A^\xi}(nm) = 3$ and $mn \neq nm$.–
Theorem 3.18. Let m, n be any two generators of a finite group G and let $A^\xi \in F^\xi(G)$ then $\xi - CFO_{A^\xi}(m) = \xi - CFO_{A^\xi}(n)$.
Proof. **Case-I.** Suppose that $O(G) = \alpha$. In view of given condition, we have $m^\alpha = n^\alpha = e$.
Let for some $\beta \in Z$, we have $n = m^\beta$, then $(\alpha, \beta) = 1$. Furthermore, by using Theorem 3.12 in the above equation, we obtain $\xi - CFO_{A^\xi}(m) = \xi - CFO_{A^\xi}(m^\alpha) = \xi - CFO_{A^\xi}(n)$.
Case-II. If G is infinite cyclic group then by using Theorem 3.11, we have $n = m^{-1}$.
Theorem 3.19. Every ξ −CFSG A^ξ of a finite cyclic group G admits the following properties:
1) If $O(m) = O(n)$ then $\xi - CFO_{A^\xi}(m) = \xi - CFO_{A^\xi}(n)$.
2) If $O(m)$ divides $O(n)$ then $\xi - CFO_{A^\xi}(n)$ divides $\xi - CFO_{A^\xi}(m)$.
3) If $O(m) > O(n)$, then $\xi - CFO_{A^\xi}(m) \geq \xi - CFO_{A^\xi}(n)$.
Proof. Let $O(G) = k_2$ and a be a generator of G. Then $m = a^r, n = a^s$ and $\xi - CFO_{A^\xi}(a) = k_1$, where r, s and $k_1 \in Z$. In view of Theorem 3.18, k_1 is regardless of the specific choice of generator a of G. We know that $O(m) = {k_2}/{(k_2, r)}$ and $O(n) = {k_2}/{(k_2, s)}$.The application of the Theorem 3.11, we have $\xi - CFO_{A^\xi}(m) = {k_1}/{(k_1, r)}$ and $\xi - CFO_{A^\xi}(n) = {k_1}/{(k_1, s)}$ Moreover, by applying the

application of Corollary 3.14, yields that $k_1|k_2$.

1) Given that $O(m) = O(n)$. This implies that $O(a^r) = O(a^s)$. This further implies that $(r, k_2) = (s, k_2)$. This shows that $(r, k_1) = (s, k_1)$.

Hence

$$\xi - CFO_{A^\xi}(m) = \xi - CFO_{A^\xi}(n).$$

2) Given that $O(m)|O(n)$, then $(r, k_2)|(s, k_2)$. This implies that $(r, k_1)|(s, k_1)$. Thus $\xi - CFO_{A^\xi}(m)|\xi - CFO_{A^\xi}(n)$.
3) Given that $O(m) > O(n)$, then $(r, k_2) < (s, k_2)$. This implies that $(r, k_1) < (s, k_1)$. Also $k_1|k_2$, thus $\xi - CFO_{A^\xi}(m) \geq \xi - CFO_{A^\xi}(n)$.

Theorem 3.20. Every ξ −CFSG A^ξ of a finite cyclic group G admits the following properties:

1) If $\xi - CFO_{A^\xi}(m)|\xi - CFO_{A^\xi}(n)$ then $\mu_{A^\xi}(m) \geq \mu_{A^\xi}(n)$.
2) *If* $\xi - CFO_{A^\xi}(m) = \xi - CFO_{A^\xi}(n)$ then $\mu_{A^\xi}(m) = \mu_{A^\xi}(n)$.

Proof.

1) Suppose a is a generator of G. Let $m = a^r$, $n = a^s$ and $\xi - CFO_{A^\xi}(a) = k_1$. By Lemma 3.18 k_1 is regardless of the specific choice of a generator a of G. Then $\xi - CFO_{A^\xi}(m) = k_1|(r, k_1)$ and $\xi - CFO_{A^\xi}(n) = k_1|(s, k_1)$ by Theorem 3.11 let $r = h(r, k_1)$ and $s = i(s, k_1)$ and $p = j(s, k_1) = k(r, k_1)$ for some $h, i, j, k \in \mathbb{Z}$. Given that $\xi - CFO_{A^\xi}(m)|\xi - CFO_{A^\xi}(n)$ then $(s, k_1)|(r, k_1)$. Thus s divides $ri = h(r, k_1)i$ and k_1 divides $rj = h(r, k_1)j$, we have

$$\mu_{A^\xi}(m) = \mu_{A^\xi}(a^r)$$

$$= \mu_{A^\xi}(a^{r(iv+jw)}) \text{ for some } v, w \in \mathbb{Z} \text{ since } (i, j) = 1$$

$$= \mu_{A^\xi}\left(a^{riv}a^{rjw}\right) \geq \min\{\mu_{A^\xi}\left(a^{riv}\right), \mu_{A^\xi}(a^{rjw})$$

$$\geq \min\{\mu_{A^\xi}(a^s), \mu_{A^\xi}(a^p)\} = \min\{\mu_{A^\xi}(n), \mu_{A^\xi}(e)\}.$$

Hence $\mu_{A^\xi}(m) = \mu_{A^\xi}(n)$.

2) Same as first

In the following example, we show that if $\xi - CFO_{A^\xi}(m) \geq \xi - CFO_{A^\xi}(n)$ then $\mu_{A^\xi}(n) \geq \mu_{A^\xi}(m)$ is not true.

Example 3.21. The CFSG A of the group $G = \{1, a, a^2, a^3, a^4, a^5\}$ is defined as:

$$\mu_A(m) = \begin{cases} 1e^{1.9i} & if\ \ m = 1 \\ 0.5e^{i} & if\ m \in \{a^2, a^4\} \\ 0.3e^{0.7i} & otherwise \end{cases}.$$

The ξ −CFSG A^ξ of G with respect to the value of the parameter $\xi = 0.7e^{1.2i}$ is obtained as:

$$\mu_{A^\xi}(m) = \begin{cases} 0.7e^{1.2i} & if\ \ m = 1 \\ 0.5e^{i} & if\ m \in \{a^2, a^4\} \\ 0.3e^{0.7i} & otherwise \end{cases}.$$

Note that $\xi - CFO_{A^\xi}(a^2) = 3$ and $\xi - CFO_{A^\xi}(a^3) = 2$. However, $\mu_{A^\xi}(a^2) = 0.5e^{i}$ and $\mu_{A^\xi}(a^3) = 0.3e^{0.7i}$.

Theorem 3.22. Let a be any generator of a finite cyclic subgroup H of unit group G. Let $A^\xi \in F^\xi(G)$. If $O(m)|O(n)$ then $\mu_{A^\xi}(m) \geq \mu_{A^\xi}(n)$, for all $m, n \in H$.

Proof. Suppose $O(m) = p$ and $O(n) = q$ for some $p, q \in N$. Let $m = a^r$ and $n = a^s$ for any $r, s \in N$. We have $a^{rp} = e = a^{sqp}$. Thus, $m = n^q$. So $\mu_{A^\xi}(m) = \mu_{A^\xi}(n^q) \geq \mu_{A^\xi}(n)$.

The subsequent example illustrates that the above result is not true for any choice of elements of G.
Example 3.23. The CFSG A of U_{24} is defined as follows:

$$\mu_A(m)=\begin{cases} 1e^{1.9i} & if\ m=1 \\ 0.6e^{1.1i} & if\ m\in\{3,7,13,17\}. \\ 0.3e^{0.8i} & if\ m\in\{9,11,19\} \end{cases}$$

The ξ −CFSG A^{ξ} of U_{20} with respect to the value of the parameter $\xi=0.8e^{1.2i}$ is obtained as:

$$\mu_{A^{\xi}}(m)=\begin{cases} 0.8e^{1.5i} & if\ m=1 \\ 0.6e^{1.1i} & if\ m\in\{3,7,13,17\}. \\ 0.3e^{0.8i} & if\ m\in\{9,11,19\} \end{cases}$$

Note that $O(11)=2$ and $O(17)=4$ in U_{24}.
Clearly, $O(11)$ divides $O(17)$ but $\mu_{A^{\xi}}(17)<\mu_{A^{\xi}}(11)$.

4. ξ −complex fuzzification of Lagrange's theorem of ξ −complex fuzzy subgroup

The significance of fuzzy logic lies in its ability to handle uncertainty, imprecision and linguistic variables, which are inherent in many real-world problems. By providing a flexible and intuitive framework for modeling and reasoning, fuzzy logic enhances decision-making processes, improves control strategies and enables the development of intelligent systems across diverse fields, ultimately contributing to the advancement of science, technology and problem-solving capabilities. However, when the number of variables and fuzzy rules increase, the computational complexity of processing and analyzing fuzzy sets may become prohibitively high. To overcome the limitations of traditional fuzzy sets, complex fuzzy logic offers enhanced computational efficiency. By extending fuzzy sets to include complex-valued membership functions, complex fuzzy sets provide a more flexible and powerful framework for modeling and analyzing complex systems. One of the key advantages of complex fuzzy sets is their ability to handle situations that cannot be adequately addressed with one-dimensional grades of membership. This logic allows the representation of intricate relationships, uncertainties and interdependencies among variables in a more nuanced manner. This capability is particularly beneficial in various fields such as image processing, signal processing and forecasting, where complex patterns and dynamic interactions are prevalent. However, the efficacy of complex fuzzy sets in tackling real-world problems that involve complex-valued membership functions is intrinsically constrained. These limitations arise from the inherent complexity and computational burden associated with managing and manipulating complex-valued membership functions. The intricate nature of such functions poses challenges in terms of their representations, interpretations and practical applicability. Moreover, Complex-valued membership functions require sophisticated mathematical techniques and algorithms for their definition and manipulation. To tackle the aforementioned constraint, we present a novel concept of complex fuzzy sets utilizing a linear conjunctive operator. This innovative approach provides enhanced flexibility and efficiency in the modeling of real-world problems within this domain. This advancement enables more accurate and effective modeling and analysis of real-world phenomenon. Additionally, our novel concept extends beyond complex fuzzy sets to include conjunctive complex fuzzy subgroups (ξ −CFSG). This extension leverages the power of complex fuzzy sets in resolving significant decision-making challenges in the diverse domain. By harnessing the flexibility and versatility afforded by complex-valued membership functions and the linear conjunctive operator, this approach offers a promising framework for tackling real-world challenges and driving advancements in multiple domains. The primary aim of this section is to present the concepts of index and order of a conjunctive complex fuzzy subgroup of a finite group G. Additionally, the conjunctive complex fuzzification of Lagrange's theorem of conjunctive complex fuzzy subgroup (ξ −CFSG) is presented in this study.

Definition 4.1. Let $A^{\xi} \in F^{\xi}(G)$. The conjunctive complex fuzzy order of A^{ξ} (denoted by ξ −CFO(A^{ξ})) is the least common multiple of conjunctive complex fuzzy order of all elements of G.

Example 4.2. The CFSG A of the group $G = \{e, a, b, ab\}$ is defined as:

$$\mu_A(m) = \begin{cases} 0.9e^{1.9i} & if\ m \in \{e, ab\} \\ 0.6e^{0.7i} & if\ m \in \{a, b\} \end{cases}.$$

The ξ-CFSG A^{ξ} of G with respect to the value of the parameter $\xi = 0.7e^{1.1i}$ is obtained as:

$$\mu_{A^{\xi}}(m) = \begin{cases} 0.7e^{1.1i} & if\ m \in \{e, ab\} \\ 0.6e^{0.7i} & if\ m \in \{a, b\} \end{cases}.$$

Note that $\xi - CFO_{A^{\xi}}(e) = \xi - CFO_{A^{\xi}}(ab) = 1$ and $\xi - CFO_{A^{\xi}}(a) = \xi - CFO_{A^{\xi}}(b) = 2$. In view of above definition, ξ −CFO$(A^{\xi}) = 2$.

Theorem 4.3. For any $A^{\xi} \in F^{\xi}N(G)$, where G is a finite group. Then a map $\overline{A^{\xi}}: G/A^{\xi} \to [0,1]$ by $\mu_{\overline{A^{\xi}}}(mA^{\xi}) = \mu_{A^{\xi}}(m), \forall m \in G$ is a ξ −CFSG of G/A^{ξ}.

Proof. Let $\mu_{\overline{A^{\xi}}}(mA^{\xi}), \mu_{\overline{A^{\xi}}}(nA^{\xi}) \in \overline{A^{\xi}}$ where $mA^{\xi}, nA^{\xi} \in G/A^{\xi}$. Consider

$$\mu_{\overline{A^{\xi}}}(mA^{\xi} \circ nA^{\xi}) = \mu_{\overline{A^{\xi}}}(mnA^{\xi}) = \mu_{A^{\xi}}(mn) \geq \min\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}(n)\}.$$

It follows that

$$\mu_{\overline{A^{\xi}}}(mA^{\xi} \circ nA^{\xi}) \geq \min\{\mu_{\overline{A^{\xi}}}(mA^{\xi}), \mu_{\overline{A^{\xi}}}(nA^{\xi})\}.$$

Moreover,

$$\mu_{\overline{A^{\xi}}}(m^{-1}A^{\xi}) = \mu_{A^{\xi}}(m^{-1}) = \mu_{A^{\xi}}(m) = \mu_{\overline{A^{\xi}}}(mA^{\xi}).$$

This shows that A^{ξ} is a ξ −CFSG of G/A^{ξ}.

Definition 4.4. Let $A^{\xi} \in F^{\xi}N(G)$ and G be a finite group G. Then $\overline{A^{\xi}}$ defined in Theorem 4.3 is called the conjunctive complex fuzzy quotient group of G determined by A^{ξ}.

Theorem 4.5. Let $A^{\xi} \in F^{\xi}(G)$ and $m \in G$. Then $\mu_{A^{\xi}}(mn) = \mu_{A^{\xi}}(n)$ for all $n \in G$, if and only if $\mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(e)$.

Proof. Suppose that $\mu_{A^{\xi}}(mn) = \mu_{A^{\xi}}(n), \forall n \in G$. We obtain the required result by taking $n = e$ in the above relation.

Conversely: Let $\mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(e)$. Since A^{ξ} is ξ −CFSG, therefore, $\mu_{A^{\xi}}(n) \leq \mu_{A^{\xi}}(e), \forall n \in G$. This means that $\mu_{A^{\xi}}(n) \leq \mu_{A^{\xi}}(m), \forall n \in G$.

Moreover, $\mu_{A^{\xi}}(mn) \geq \min\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}(n)\}$. Therefore, we have

$$\mu_{A^{\xi}}(mn) \geq \mu_{A^{\xi}}(n), \forall n \in G. \tag{4.1}$$

But $\mu_{A^{\xi}}(n) = \mu_{A^{\xi}}(m^{-1}mn) \geq \min\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}(mn)\}$.

This shows that

$$\mu_{A^\xi}(n) \geq \mu_{A^\xi}(mn), \forall n \in G. \tag{4.2}$$

By comparing (4.1) and (4.2), we get

$$\mu_{A^\xi}(mn) = \mu_{A^\xi}(n).$$

Remark 4.6. It is important to note that if $\mu_{A^\xi}(m) = \mu_{A^\xi}(e)$ Then $\mu_{A^\xi}(mn) = \mu_{A^\xi}(nm), \forall n \in G.$

Theorem 4.7. For any $A^\xi \in F^\xi N(G)$ and G/A^ξ is a quotient group of G by A^ξ. Then there is a natural homomorphism f between G and G/A^ξ as follows: $f(m) = mA^\xi, \forall m \in G$ with Kernel$f = \{m \in G: \mu_{A^\xi}(m) = \mu_{A^\xi}(e)\}$.

Proof. Consider $f(mn) = (mn)A^\xi = (mA^\xi) \circ (nA^\xi) = f(m)f(n),\ m, n \in G.$

Therefore, we obtain f as a homomorphism between the groups G and G/A^ξ.

$$Kerf = \{m \in G: f(m) = A^\xi\}$$

$$= \{m \in G: mA^\xi = A^\xi\}$$

$$= \{m \in G: (mA^\xi)y = (A^\xi)y, \forall y \in G\}.$$

In view of Definition 2.5, we have $Kerf = \{m \in G: \mu_{A^\xi}(ym^{-1}) = \mu_{A^\xi}(y)\}$.

By using the Theorem 4.5 in the above equation, we have $\mu_{A^\xi}(m) = \mu_{A^\xi}(e)$. Consequently,

$$Kerf = \{m \in G: \mu_{A^\xi}(m) = \mu_{A^\xi}(e)\}.$$

Remark 4.8. Note that $|Kerf| = \xi - CFO(A^\xi)$.

Definition 4.9. The cardinality of the quotient group $\boldsymbol{G/A^\xi}$ is called the index of $\boldsymbol{\xi}$ −CFSG $\boldsymbol{A^\xi}$ and is symbolized by $[\boldsymbol{G}: \boldsymbol{A^\xi}]$.

Example 4.10. The CFSG A of a symmetric group $S_3 = \{1, (12), (23), (13), (123), (132)\}$ is defined as:

$$\mu_A(m) = \begin{cases} 0.92e^{1.8i} & if\ \ m \in \{1, (123), (132)\} \\ 0.6e^i & otherwise \end{cases}.$$

The ξ −CFSG A^ξ of G with respect to the value of the parameter $\xi = 0.7e^{1.1i}$ is obtained as:

$$\mu_{A^\xi}(m) = \begin{cases} 0.7e^{1.1i} & ifm \in \{1, (123), (132)\} \\ 0.6e^i & \text{otherwise} \end{cases}.$$

The set of all $0.7e^{1.1i}$-complex fuzzy distinct left cosets of S_3 by $A^{0.7e^{1.1i}}$ is given by:

$$G/A^\xi = \{A^{0.7e^{1.1i}}, mA^{0.7e^{1.1i}}\}.$$

This means that $[G: A^{.7e^{1.1i}}] = card(G/A^\xi) = 2.$

Theorem 4.11. (Conjunctive Complex Fuzzy version of Lagrange's Theorem of ξ −CFSG) Let $A^\xi \in F^\xi(G)$, where G is finite group. Then $[G: A^\xi]$ divides $O(G)$.

Proof. In the frame work of Theorem 4.7, f is a homomorphism between G and G/A^ξ, where $G/A^\xi = \{mA^\xi: m \in G\}$, where mA^ξ is described in Definition 2.7. One can easily observe that G/A^ξ is finite as G is finite.

Consider, the subgroup H of G as:

$$H = \{m \in G: mA^\xi = eA^\xi\}. \tag{4.3}$$

By using Theorem 4.7 in the above relation, we get $H = \{m \in G: \mu_{A^\xi}(m) = \mu_{A^\xi}(e)\}$.

The left decomposition of G as a disjoint union of cosets of G modulo H is given by:

$$G = x_1H \cup x_2H \cup \ldots\ldots \cup x_kH \tag{4.4}$$

where $\boldsymbol{x_iH = H}$. Now we illustrate that there is a conjunctive complex fuzzy left coset in $\boldsymbol{G/A^{\xi}}$ corresponding to each coset $\boldsymbol{x_iH}$ given in (4.4) and also establish the injectivity of this correspondence. Consider any coset $\boldsymbol{x_iH}$. Let $\boldsymbol{h} \in \boldsymbol{H}$, then

$$f(x_ih) = x_ihA^{\xi} = x_iA^{\xi}hA^{\xi} = x_iA^{\xi}eA^{\xi} = x_iA^{\xi}.$$

It shows that each element of x_iA^{ξ} is mapped to conjunctive complex fuzzy coset under the function f.

Moreover, we formulate a natural correspondence $\tilde{f}: \{x_iH: 1 \leq i \leq m\} \to G/A^{\xi}$ defined as: $\tilde{f}(x_iH) = x_iA^{\xi},\ 1 \leq i \leq m$. The correspondence $\tilde{f}$ is injective as for if $x_iA^{\xi} = x_jA^{\xi}$, then $x_i^{-1}x_jA^{\xi} = eA^{\xi}$. By using (4.3), we have $x_i^{-1}x_j \in H$. This means that $x_iH = x_jH$ and hence $\tilde{f}$ is one-to-one.

The above discussion clearly indicates that $[G: H]$ and $[G: A^{\xi}]$ are equal. As $[G: H]|O(G)$, so $[G: A^{\xi}]$ also divides $O(G)$.

Corollary 4.12. Let $A^{\xi} \in F^{\xi}(G)$ and G be a finite group then $\xi - CFO(A^{\xi})$ divides $O(G)$.

Example 4.13. In view of example 4.10, we have, $[G: A^{\xi}] = 2$ and $O(G) = 6$. This means that $[G: A^{\xi}]$ divides $O(G)$.

5. Conclusions

The notions of conjunctive complex fuzzy order of an element and conjunctive complex fuzzy order of ξ −CFSG have been innovated in this work. Many key algebraic postulates of these concepts have been established. A useful mechanism has been designed to calculate the conjunctive complex fuzzy order of an element for any of its power and a correlation between the membership values of any two elements and the membership value of identity element of ξ −CFSG have been determined in this article. In addition, the notions of conjunctive complex fuzzy order and index of a ξ −CFSG have been proposed and various elementary structural properties of these concepts have been proved to highlight the significance of these newly defined ideologies. Furthermore, the conjunctive complex fuzzification of Lagrange's theorem for conjunctive complex fuzzy subgroups of a group has been presented in this article. The main limitation of this study is its computational complexity. Modeling complex systems using complex fuzzy sets can require a significant amount of computation and memory resources. Additionally, designing the membership functions for complex fuzzy sets can be challenging, and the results may be highly sensitive to the choice of parameters. Another limitation is the difficulty in interpreting the results obtained from complex fuzzy logic models, which may require expert knowledge and can be less intuitive compared to traditional mathematical models. Finally, the lack of standardized methods for designing and evaluating complex fuzzy logic models can make it difficult to compare and reproduce results across different applications and domains. The main emphasis of future endeavors will be directed towards the development of a comprehensive decision analysis tool that incorporates the linear conjunctive operator. The ultimate objective will be enhancing the practicality and applicability of this tool in real-world contexts. In addition, the approach presented in this article can be intend to address multi-attribute decision-making problems in medicine, image processing and cyber security.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare no conflict of interest.

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