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Research article

A new approach to Jacobsthal, Jacobsthal-Lucas numbers and dual vectors

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Abstract: This paper gives a detailed study of a new generation of dual Jacobsthal and dual Jacobsthal-Lucas numbers using dual numbers. Also some formulas, facts and properties about these numbers are presented. In addition, a new dual vector called the dual Jacobsthal vector is presented. Some properties of this vector apply to various properties of geometry which are not generally known in the geometry of dual space. Finally, this study introduces the dual Jacobsthal and the dual Jacobsthal-Lucas numbers with coefficients of dual numbers. Some fundamental identities are demonstrated, such as the generating function, the Binet formulas, the Cassini's, Catalan's and d'Ocagne identities for these numbers.

Keywords: recurrences; integer sequences; dual number; dual angle; dual Jacobsthal numbers
Mathematics Subject Classification: 05A15, 11B37, 11Y55

1. Introduction

In 1873, W. K. Clifford initiated the study of dual numbers and these numbers were used at the beginning of the twentieth century by the German mathematician Eduard Study, who studied line geometry and kinematics to obtain the dual angle which measures the relative position of two skew lines in space. This study defines a dual angle as $\theta + \mathbf{d}\varepsilon$, where θ is the angle between the directions of two lines in three-dimensional space and $\mathbf{d}$ is a distance between them, [1–3].

In this paper, our objective is to conduct a detailed study of a new generation of the dual Jacobsthal number and dual Jacobsthal-Lucas numbers. We describe a new dual Jacobsthal vector to apply in the geometry of dual space.

A dual number is described as $\mathrm{U}=\mathfrak{u} + \varepsilon\mathfrak{u}^*$, where $\mathfrak{u}$ and $\mathfrak{u}^*$ are real numbers and ε is a symbol taken to satisfy $\varepsilon^2 = 0$ with $\varepsilon \neq 0$.

The addition and multiplication of two dual numbers $\mathrm{U}=\mathfrak{u} + \varepsilon\mathfrak{u}^*$ and $\mathrm{V}=\mathfrak{v} + \varepsilon\mathfrak{v}^*$ are defined by

$$\mathrm{U} + \mathrm{V}=(\mathfrak{u} + \mathfrak{v}) + \varepsilon(\mathfrak{u}^* + \mathfrak{v}^*)$$

and

$$\mathrm{UV}=\mathfrak{u}\mathfrak{v} + \varepsilon(\mathfrak{u}\mathfrak{v}^* + \mathfrak{u}^*\mathfrak{v}),$$

respectively. In mathematics and physics, *vector* is a term that refers to some quantities that cannot be expressed by a single number (a scalar) or to elements of some vector spaces as in [4].
The dual vector $\mathfrak{D}^3$ is defined by

$$\begin{aligned}\vec{\mathrm{U}} &= \vec{\mathfrak{u}} + \varepsilon\vec{\mathfrak{u}}^* \ ; \vec{\mathfrak{u}}, \vec{\mathfrak{u}}^* \epsilon \mathcal{R}^3 \\ &= (\mathfrak{u}_1 + \varepsilon\mathfrak{u}_1^*, \mathfrak{u}_2 + \varepsilon\mathfrak{u}_2^*, \mathfrak{u}_3 + \varepsilon\mathfrak{u}_3^*).\end{aligned}$$

The scalar product and cross product of dual vectors $\vec{\mathrm{U}} = \vec{\mathfrak{u}} + \varepsilon\vec{\mathfrak{u}}^*$ and $\vec{\mathrm{V}} = \vec{\mathfrak{v}} + \ \varepsilon\vec{\mathfrak{v}}^*$ are

$$\left\langle \vec{\mathrm{U}}, \vec{\mathrm{V}} \right\rangle = \left\langle \vec{\mathfrak{u}}, \vec{\mathfrak{v}} \right\rangle + \varepsilon\left(\left\langle \vec{\mathfrak{u}}, \vec{\mathfrak{v}}^* \right\rangle + \left\langle \vec{\mathfrak{u}}^*, \vec{\mathfrak{v}} \right\rangle\right), \tag{1.1}$$

and

$$\vec{\mathrm{U}} \times \vec{\mathrm{V}} = \left\langle \vec{\mathfrak{u}}, \vec{\mathfrak{v}} \right\rangle + \varepsilon\left(\vec{\mathfrak{u}} \times \vec{\mathfrak{v}}^* + \vec{\mathfrak{u}}^* \times \vec{\mathfrak{v}}\right), \tag{1.2}$$

respectively.

The norm of the dual vector $\vec{\mathrm{U}}$ is given by

$$\left\|\vec{\mathrm{U}}\right\| = \left\|\vec{\mathfrak{u}}\right\| + \frac{\left\langle \vec{\mathfrak{u}}, \vec{\mathfrak{u}}^* \right\rangle}{\left\|\vec{\mathfrak{u}}\right\|}.$$

Proposition 1. $\vec{\mathrm{U}}$ *is dual unit vector if and only if* $\left\|\vec{\mathfrak{u}}\right\| = 1$ *and* $\left\langle \vec{\mathfrak{u}}, \vec{\mathfrak{u}}^* \right\rangle = 0$ *[1].*

The Jacobsthal numbers and the Jacobsthal-Lucas numbers are given by the second-order recurrence relations as follows:

$$\mathbf{J}_n = \mathbf{J}_{n-1} + 2\mathbf{J}_{n-2}, \mathbf{J}_0 = 0, \mathbf{J}_1 = 1,$$

and

$$\mathbf{j}_n = \mathbf{j}_{n-1} + 2\mathbf{j}_{n-2}, \mathbf{j}_0 = 2, \ \mathbf{j}_1 = 1,$$

respectively, [5–12]. Also these numbers each have another expression of the respective forms

$$\mathbf{J}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{2^n - (-1)^n}{3}, \tag{1.3}$$

and

$$\mathbf{j}_n = \alpha^n + \beta^n = 2^n + (-1)^n. \tag{1.4}$$

The dual Jacobsthal numbers and the dual Jacobsthal-Lucas numbers, respectively, are as follows:

$$\mathcal{J}_n = \mathbf{J}_n + \varepsilon\mathbf{J}_{n+1}, \tag{1.5}$$

and

$$j_n = \mathbf{j}_n + \varepsilon\mathbf{j}_{n+1}, \tag{1.6}$$

where $\varepsilon^2 = 0$ with $\varepsilon \neq 0$.

2. Main identities on dual Jacobsthal and dual Jacobsthal-Lucas numbers

In this section, we obtain some fundamental identities such as the generating function, the Binet formulas, the Cassini's, Catalan's and d'Ocagne identities for the dual Jacobsthal numbers and the dual Jacobsthal-Lucas numbers.

Theorem 1. *The generating function of $\mathcal{J}_n$ is*

$$\mathfrak{h}(\mathrm{t}) = \frac{\mathcal{J}_0(1-\mathrm{t}) + \mathcal{J}_1\mathrm{t}}{1-\mathrm{t}-2\mathrm{t}^2}.$$

Proof. If $\mathfrak{h}(\mathrm{t})$ is a generating function for $\mathcal{J}_n$, then

$$\mathfrak{h}(\mathrm{t}) = \sum_{i=0}^{\infty} \mathcal{J}_i t^i.$$

Multiply the function $\mathfrak{h}(\mathrm{t})$ by t and t^2, respectively. We can write

$$\begin{aligned} \mathrm{t}\mathfrak{h}(\mathrm{t}) &= \mathcal{J}_0\mathrm{t} + \mathcal{J}_1^2\mathrm{t} + ... + \mathcal{J}_n^{n+1}\mathrm{t}, \\ \mathrm{t}^2\mathfrak{h}(t) &= \mathcal{J}_0\mathrm{t}^2 + \mathcal{J}_1\mathrm{t}^3 + ... + \mathcal{J}_n\mathrm{t}^{n+2}. \end{aligned}$$

Doing the needed arrangements, we will have

$$\begin{aligned} \mathfrak{h}(\mathrm{t})(1-\mathrm{t}-2\mathrm{t}^2) &= \mathcal{J}_0 + \mathrm{t}(\mathcal{J}_1 - \mathcal{J}_0), \\ \mathfrak{h}(\mathrm{t}) &= \sum_{i=0}^{\infty} \mathcal{J}_i\, \mathrm{t}^i \end{aligned}$$

which is the desired result. □

Theorem 2. *Let $\mathcal{J}_n$ and $\mathcal{J}_m$ be dual Jacobsthal numbers. For every $m, n \geq 1$,*

$$\mathcal{J}_n\mathcal{J}_m + \mathcal{J}_{n+1}\mathcal{J}_{m+1} = \frac{1}{9}(\,\mathbf{j}_{n+m} - 2^m(-1)^m\,\mathbf{j}_{n-m}) + \varepsilon\frac{1}{9}(2^{n+m+4} + 12\mathbf{J}_{n+m} + 2^m(-1)^m\,\mathbf{j}_{n-m}).$$

Proof. By using the equalities (1.3)–(1.5), and doing the necessary calculations, we will have

$$\begin{aligned} \mathcal{J}_n\mathcal{J}_m + \mathcal{J}_{n+1}\mathcal{J}_{m+1} &= \mathbf{J}_n\,\mathbf{J}_m + \mathbf{J}_{n+1}\mathbf{J}_{m+1} \\ &\quad +\varepsilon(\mathbf{J}_{n+1}\mathbf{J}_m + \mathbf{J}_n\mathbf{J}_{m+1} + \mathbf{J}_{n+1}\,\mathbf{J}_{m+2} + \mathbf{J}_{n+2}\mathbf{J}_{m+1}) \\ &= \frac{1}{9}(\alpha^{n+m} + \beta^{n+m}) - \frac{1}{9}(\alpha^n\beta^m + \alpha^n\beta^m)\frac{\alpha^{-m}\beta^{-m}}{\alpha^{-m}\beta^{-m}} \\ &\quad +\varepsilon\left(\frac{16\alpha^{n+m}}{9} + \frac{4(\alpha^{n+m} - \beta^{n+m})}{3} + \frac{1}{9}(\alpha^n\beta^m + \alpha^n\beta^m)\frac{\alpha^{-m}\beta^{-m}}{\alpha^{-m}\beta^{-m}}\right) \\ &= \frac{1}{9}(\alpha^{n+m} + \beta^{n+m}) - \frac{1}{9}(\alpha^n\beta^m + \alpha^n\,\beta^m)\frac{\alpha^{-m}\beta^{-m}}{\alpha^{-m}\beta^{-m}} \\ &\quad +\varepsilon\frac{1}{9}\left(2^{n+m+4} + 12\mathbf{J}_{n+m} + 2^m(-1)^m\mathbf{j}_{n-m}\right) \\ &= \frac{1}{9}(\mathbf{j}_{n+m} - \alpha^m\beta^m\mathbf{j}_{n-m}) + \varepsilon\frac{1}{9}(2^{n+m+4} + 12\mathbf{J}_{n+m} + 2^m(-1)^m\mathbf{j}_{n-m}). \end{aligned}$$

□

Theorem 3. ***(Binet formulas)*** *For every positive integer n, the following equalities hold*

$$\mathcal{J}_n = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{3}, \tag{2.1}$$

and

$$j_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n \tag{2.2}$$

where

$$\underline{\alpha} = 1 + \varepsilon\alpha \text{ and } \underline{\beta} = 1 + \varepsilon\beta. \tag{2.3}$$

Proof. By using the Binet formula for dual Jacobsthal number $\mathcal{J}_n$, we obtain

$$\begin{aligned} \mathcal{J}_n &= \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{3} \\ &= \frac{(1+\varepsilon\alpha)2^n - (1+\varepsilon\beta)(-1)^n}{3} \\ &= \frac{2^n - (-1)^n + \varepsilon(2^{n+1} - (-1)^{n+1})}{3} \\ &= \mathbf{J}_n + \mathbf{J}\varepsilon_{n+1}. \end{aligned}$$

If calculations are made similarly for the dual Jacobsthal-Lucas numbers, we obtain

$$\begin{aligned} j_n &= \underline{\alpha}\alpha^n + \underline{\beta}\beta^n \\ &= (1+\varepsilon\alpha)2^n + (1+\varepsilon\beta)(-1)^n \\ &= \mathbf{j}_n + \varepsilon\mathbf{j}_{n+1}. \end{aligned}$$

□

Theorem 4. ***(Catalan identity)*** *For the integers n, r such that $n \geq r$, then Catalan's identity is as follows:*

$$\mathcal{J}_{n+r}\mathcal{J}_{n-r} - \mathcal{J}_n^2 = \frac{(-1)^n 2^n}{9}((-1)^{r+1}2^r \ \mathbf{J}_r^2 + \ \varepsilon(2 - \mathbf{j}_{r+1})).$$

Proof. Using the relation in the equality (1.5), we can write

$$\mathcal{J}_{n+r}\mathcal{J}_{n-r} - \mathcal{J}_n^2 = \mathbf{J}_{n+r}\mathbf{J}_{n+r} - \mathbf{J}_n^2 + \varepsilon(\ \mathbf{J}_{n+r}\mathbf{J}_{n-r+1} + \mathbf{J}_{n+r+1} \ \mathbf{J}_{n-r} - 2\mathbf{J}_n \ \mathbf{J}_{n+1}).$$

From the equalities (1.3) and (1.4), we will have

$$\begin{aligned} \mathbf{J}_{n+r}\mathbf{J}_{n+r} - \mathbf{J}_n^2 &= -\alpha^{n+r}\beta^{n+}(\frac{\alpha^r - \ \beta^r}{3})^2 \\ &= \frac{1}{9}(-1)^{n+r+1}2^{n+r} \ \mathbf{J}_r^2, \end{aligned}$$

and

$$\begin{aligned} \varepsilon(\mathbf{J}_{n+r}\mathbf{J}_{n-r+1} + \mathbf{J}_{n+r+1} \ \mathbf{J}_{n-r} - 2\mathbf{J}_n \ \mathbf{J}_{n+1}) &= \frac{1}{9}\alpha^n\beta^n(2 - \alpha^{r+1} - \ \beta^{r+1}) \\ &= \frac{1}{9}(-1)^n 2^n(2 - \mathbf{j}_{r+1}); \end{aligned}$$

we obtain that

$$\mathcal{J}_{n+r}\mathcal{J}_{n-r}-\mathcal{J}_n^2=\frac{(-1)^n2^n}{9}((-1)^{r+1}2^r\,\mathbf{J}_r^2+\ \varepsilon(2-\mathbf{j}_{r+1})).$$

□

Theorem 5. ***(Cassini identity)*** *For $n\geq 1$, we have*

$$\mathcal{J}_{n+r}\mathcal{J}_{n-r}-\mathcal{J}_n^2=\frac{(-1)^n2^n}{9}(2-3\varepsilon).$$

Proof. If we write $r=1$ in Theorem 4, the proof is completed. □

Theorem 6. ***(d'Ocagne's identity)*** *For any integer n and m, we have*

$$\mathcal{J}_{m+1}\mathcal{J}_n-\mathcal{J}_m\mathcal{J}_{n+1}=(-1)^m2^m\mathbf{J}_{n-m}(1+\varepsilon).$$

Proof. From the equality (1.5)

$$\mathcal{J}_{m+1}\mathcal{J}_n-\mathcal{J}_m\mathcal{J}_{n+1}=\mathbf{J}_n\mathbf{J}_{m+1}-\mathbf{J}_{n+1}\mathbf{J}_m+\varepsilon(\mathbf{J}_n\mathbf{J}_{m+2}-\mathbf{J}_{n+2}\mathbf{J}_m). \tag{2.4}$$

From the equalities (1.3) and (1.4), and a straightforward computation, we have

$$\begin{aligned}&\mathbf{J}_n\mathbf{J}_{m+1}-\mathbf{J}_{n+1}\mathbf{J}_m\\ =\ &\left(\frac{\alpha^n-\beta^n}{3}\frac{\alpha^{m+1}-\beta^{m+1}}{3}-\frac{\alpha^{n+1}-\beta^{n+1}}{3}\frac{\alpha^m-\beta^m}{3}\right)\frac{\alpha^{-m}\beta^{-m}}{^{-m}\,\beta^{-m}}\\ =\ &\frac{1}{9}\alpha^m\beta^m(\alpha^{n-m}-\beta^{n-m})=\alpha^m\beta^m\mathbf{J}_{n-m}\end{aligned}$$

where from the last equality, we will have

$$\mathbf{J}_n\mathbf{J}_{m+1}-\mathbf{J}_{n+1}\,\mathbf{J}_m=2^m(-1)^m\,\mathbf{J}_{n-m} \tag{2.5}$$

and

$$\varepsilon(\mathbf{J}_n\mathbf{J}_{m+2}-\mathbf{J}_{n+2}\mathbf{J}_m)=\varepsilon 2^m(-1)^m\,\mathbf{J}_{n-m}. \tag{2.6}$$

Substituting the equalities (2.5) and (2.6) in (2.4), we obtain

$$\mathcal{J}_{m+1}\mathcal{J}_n-\mathcal{J}_m\mathcal{J}_{n+1}=(-1)^m2^m\mathbf{J}_{n-m}(1+\varepsilon).$$

□

3. Dual Jacobsthal vectors

In this part, we will give dual Jacobsthal vectors and geometric properties of them.
A dual Jacobsthal vector in $\mathfrak{D}^3$ is given by

$$\overrightarrow{\mathcal{J}}_n=\overrightarrow{\mathbf{J}}_n+\varepsilon\overrightarrow{\mathbf{J}}_{n+1},$$

where

$$\overrightarrow{\mathbf{J}}_n=(\mathbf{J}_n,\mathbf{J}_{n+1},\mathbf{J}_{n+2})\text{ and }\overrightarrow{\mathbf{J}}_{n+1}=(\mathbf{J}_{n+1},\mathbf{J}_{n+2},\mathbf{J}_{n+3}) \tag{3.1}$$

are real vectors in $\mathcal{R}^3$.

Theorem 7. *If $\overrightarrow{\mathcal{J}}_n$ is a dual unit vector, then*

$$7(-1)^n 4^n + (-1)^n 2^n - 2 = 0 \text{ or } \mathbf{j}_n = 7(-1)^n 2^{2n+1}.$$

Proof. By using the equalities (1.1), (1.3), (1.5) and (3.1), we get the following statements:

$$\begin{aligned} \left\|\overrightarrow{\mathcal{J}}_n\right\| &= \sqrt{\mathbf{J}_n^2 + \mathbf{J}_{n+1}^2 + \mathbf{J}_{n+2}^2} \\ &= \sqrt{(\frac{\alpha^n - \beta^n}{3})^2 + (\frac{\alpha^{n+1} - \beta^{n+1}}{3})^2 + (\frac{\alpha^{n+2} - \beta^{n+2}}{3})^2} \\ &= \sqrt{\frac{21\alpha^{2n} + 3\beta^{2n} + 6\,\alpha^n\beta^n}{9}} \\ &= \sqrt{\frac{7.4^n + 2^n(-1)^n + 1}{3}} \end{aligned}$$

and

$$\left\langle \overrightarrow{\mathbf{J}}_n, \overrightarrow{\mathbf{J}}_{n+1} \right\rangle = \mathbf{J}_n\mathbf{J}_{n+1} + \mathbf{J}_{n+1}\mathbf{J}_{n+2} + \mathbf{J}_{n+2}\mathbf{J}_{n+3}. \tag{3.2}$$

Doing the necessary calculations, we will have

$$\left\langle \overrightarrow{\mathbf{J}}_n, \overrightarrow{\mathbf{J}}_{n+1} \right\rangle = \frac{7.2^{2n+1} - (-1)^n\mathbf{j}_n}{3}.$$

Using proposition 1 and above calculations, we easily reach the result. □

Theorem 8. *Let $\overrightarrow{\mathcal{J}}_n$ and $\overrightarrow{\mathcal{J}}_m$ be dual Jacobsthal vectors. The scalar and cross products of $\overrightarrow{\mathcal{J}}_n$ and $\overrightarrow{\mathcal{J}}_m$ are*

$$\left\langle \overrightarrow{\mathcal{J}}_n, \overrightarrow{\mathcal{J}}_m \right\rangle = \frac{7}{3}\,\mathbf{j}_{n+m}(1 + 4\varepsilon) + \frac{2^m(-1)^m}{3}\,\mathbf{j}_{n-m}(1 + \varepsilon), \tag{3.3}$$

and

$$\overrightarrow{\mathcal{J}}_n \times \overrightarrow{\mathcal{J}}_m = 2^n(-1)^n\mathbf{j}_{n-m}(1 + \varepsilon)(2\overrightarrow{e}_1 + \overrightarrow{e}_2 - \overrightarrow{e}_3).$$

Proof. From the equalities (1.1), (1.4), (1.6) and (3.2),

$$\begin{aligned} &\left\langle \overrightarrow{\mathcal{J}}_n, \overrightarrow{\mathcal{J}}_m \right\rangle \\ = \;& \mathbf{J}_n\mathbf{J}_m + \mathbf{J}_{n+1}\mathbf{J}_{m+1} + \mathbf{J}_{n+2}\mathbf{J}_{m+2} \\ &+\varepsilon(\mathbf{J}_n\mathbf{J}_{m+1} + \mathbf{J}_m\mathbf{J}_{n+1} + \mathbf{J}_{n+1}\mathbf{J}_{m+2} + \mathbf{J}_{n+2}\mathbf{J}_{m+1} + \mathbf{J}_{m+3}\mathbf{J}_{n+2} + \mathbf{J}_{n+3}\mathbf{J}_{m+2}). \end{aligned}$$

By doing the necessary calculations, we obtain

$$\begin{aligned} &= \mathbf{J}_n\mathbf{J}_m + \mathbf{J}_{n+1}\mathbf{J}_{m+1} + \mathbf{J}_{n+2}\mathbf{J}_{m+2} \\ &= \frac{7}{3}(\alpha^{n+m} + \beta^{n+m}) + \frac{\alpha^m\beta^m}{9}(1 + \alpha\beta + \alpha^2\beta^2)(\alpha^{n-m} + \beta^{n-m}) \\ &= \frac{7}{3}\mathbf{j}_{n+m} + \frac{2^m(-1)^m}{3}\mathbf{j}_{n-m} \end{aligned}$$

and

$$\mathbf{J}_n\mathbf{J}_{m+1} + \mathbf{J}_m\mathbf{J}_{n+1} + \mathbf{J}_{n+1}\mathbf{J}_{m+2} + \mathbf{J}_{n+2}\mathbf{J}_{m+1} + \mathbf{J}_{m+3}\mathbf{J}_{n+2} + \mathbf{J}_{n+3}\mathbf{J}_{m+2}$$

$$= \frac{28}{3}(\alpha^{n+m} + \beta^{n+m}) - \frac{1}{9}(\alpha + \beta)(1 + \alpha\beta + \quad \alpha^2\beta^2)(\alpha^n\beta^m + \alpha^m\beta^n)$$
$$= \frac{28}{3}(\alpha^{n+m} + \beta^{n+m}) - \frac{1}{9}(\alpha + \beta)(1 + \alpha\beta + \quad \alpha^2\beta^2)(\alpha^n\beta^m + \alpha^m\beta^n)\frac{\alpha^{-m}\beta^{-m}}{\alpha^{-m}\beta^{-m}}.$$

In the last equality, we take into account that $\alpha = 2$ and $\beta = -1$; thus, we obtain

$$= \frac{28}{3}\mathbf{j}_{n+m} - \frac{2^m(-1)^m}{3}\mathbf{j}_{n-m}.$$

From the equalities (1.2), (1.3), (1.5) and (1.6), we obtain

$$\overrightarrow{\mathcal{J}}_n \times \overrightarrow{\mathcal{J}}_m = \overrightarrow{\mathbf{J}}_n \times \overrightarrow{\mathbf{J}}_m + \varepsilon(\overrightarrow{\mathbf{J}}_n \times \overrightarrow{\mathbf{J}}_{m+1} + \overrightarrow{\mathbf{J}}_{n+1} \times \overrightarrow{\mathbf{J}}_m). \tag{3.4}$$

We compute the following expressions:

$$\overrightarrow{\mathbf{J}}_n \times \overrightarrow{\mathbf{J}}_m = \begin{bmatrix} \overrightarrow{e}_1 & \overrightarrow{e}_2 & \overrightarrow{e}_3 \\ \mathbf{J}_n & \mathbf{J}_{n+1} & \mathbf{J}_{n+2} \\ \mathbf{J}_m & \mathbf{J}_{m+1} & \mathbf{J}_{m+2} \end{bmatrix} = (-1)^m 2^m \mathbf{J}_{n-m}(-2\overrightarrow{e}_1 - \overrightarrow{e}_2 + \overrightarrow{e}_3). \tag{3.5}$$

In the equality (3.4), if the dual part is calculated similarly, then

$$\overrightarrow{\mathbf{J}}_n \times \overrightarrow{\mathbf{J}}_{m+1} + \overrightarrow{\mathbf{J}}_{n+1} \times \overrightarrow{\mathbf{J}}_m = (-1)^m 2^m \mathbf{J}_{n-m}(-2\overrightarrow{e}_1 - \overrightarrow{e}_2 + \overrightarrow{e}_3). \tag{3.6}$$

Substituting the equalities (3.5) and (3.6) in (3.4), we obtain

$$\overrightarrow{\mathcal{J}}_n \times \overrightarrow{\mathcal{J}}_m = (-1)^m 2^m \mathbf{J}_{n-m}(-2\overrightarrow{e}_1 - \overrightarrow{e}_2 + \overrightarrow{e}_3)(1 + \varepsilon).$$

□

Proposition 2. *(**E. Study mapping**) Let* $\overrightarrow{\mathrm{U}} = \vec{u} + \varepsilon\vec{u}^*$ *be the unit dual vector corresponding to the directed line l in $\mathcal{R}^3$. The unit real vector $\vec{u}$ is the direction vector of the line l, and the real vector $\vec{u}^*$ determines the position of l [2] (see Figure 1).*

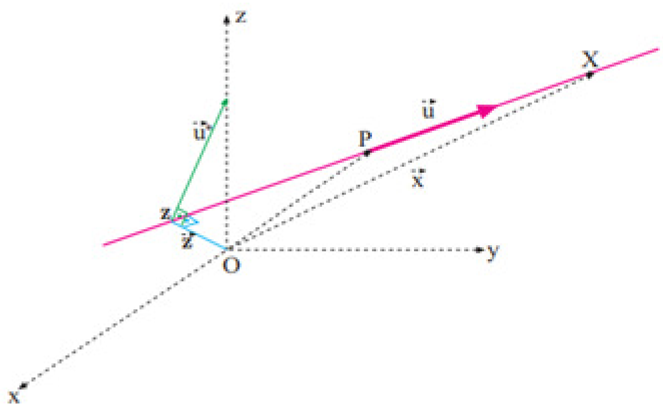

Figure 1. E. Study mapping.

Proof. Let us consider a dual unit vector $\overrightarrow{U}=\vec{u}+\varepsilon\vec{u}^*$. The directed line corresponding with $\overrightarrow{U}$ is in the direction of $\vec{u}$ according to the E. Study mapping; let P, X be points on the line and the origin be considered as the point O. Then, the line equation is given by

$$\overrightarrow{OX}=\overrightarrow{OP}+r\overrightarrow{u},$$

where $r\in I\subset\mathcal{R}$ is a parameter. A point P is on the line of vectors $\overrightarrow{u},\vec{u}^*$ if and only if

$$\vec{u}\times\vec{u}^*=\ \vec{u}\times(OP\times\overrightarrow{u}),$$

and then

$$\overrightarrow{OP}=\overrightarrow{u}\times\vec{u}^*+\left\langle\overrightarrow{u},\overrightarrow{OP}\right\rangle\overrightarrow{u},$$

and

$$\overrightarrow{OX}=\overrightarrow{u}\times\vec{u}^*+(\ \left\langle\vec{u},\overrightarrow{OP}\right\rangle+r\ )\overrightarrow{u}.$$

By taking a new parameter $t=\left\langle\overrightarrow{u},\overrightarrow{OP}\right\rangle+r$ and $\overrightarrow{OX}=\overrightarrow{x}$, we will have the result as

$$\overrightarrow{x}=\overrightarrow{u}\times\vec{u}^*+t\vec{u}. \tag{3.7}$$

□

Theorem 9. *If $\overrightarrow{\mathcal{J}}_{0_n}$ is a dual Jacobsthal vector and its unitized vector of $\overrightarrow{\mathcal{J}}_{0_n}$ is $\overrightarrow{\mathcal{J}}_n=\overrightarrow{\mathbf{J}}_n+\ \varepsilon\overrightarrow{\mathbf{J}}_{n+1}$, whose terminal point is on the dual unit sphere, then the equation of the oriented line that corresponds to $\overrightarrow{\mathcal{J}}_n$ in $\mathcal{R}^3$ is given by*

$$\overrightarrow{x}_n=(-1)^n2^n(\frac{2}{9}+t\mathbf{J}_n)\overrightarrow{e}_1+(t\mathbf{J}_{n+1})\overrightarrow{e}_2+8\frac{1}{9}+t\mathbf{J}_{n+2})\overrightarrow{e}_3),$$

where $t\in I\subset R$ is a parameter.

Proof. By using the equalities (1.2), (1.3), (1.5) and (3.6), we obtain the equation of the oriented line which corresponds with $\overrightarrow{\mathcal{J}}_n$ as follows:

$$\begin{aligned}\overrightarrow{x_n}&=\overrightarrow{\mathbf{J}}_n\times\overrightarrow{\mathbf{J}}_{n+1}+t\overrightarrow{\mathbf{J}}_n,\\&=\begin{bmatrix}\overrightarrow{e}_1&\overrightarrow{e}_2&\overrightarrow{e}_3\\\mathbf{J}_n&\mathbf{J}_{n+1}&\mathbf{J}_{n+2}\\\mathbf{J}_m&\mathbf{J}_{m+1}&\mathbf{J}_{m+2}\end{bmatrix}+t(\mathbf{J}_n\overrightarrow{e}_1+\mathbf{J}_{n+1}\overrightarrow{e}_2+\mathbf{J}_{n+2}\overrightarrow{e}_3)\\&=(-1)^n2^n(\frac{2}{9}+t\mathbf{J}_n)\overrightarrow{e}_1+(t\mathbf{J}_{n+1})\overrightarrow{e}_2+(\frac{1}{9}+t\mathbf{J}_{n+2})\overrightarrow{e}_3).\end{aligned}$$

□

Definition 1. ***(Dual angle)*** *The dual angle represents the relative displacement and orientation between two lines $\mathbf{d}_1$ and $\mathbf{d}_2$ in $\mathfrak{D}^3$. The dual angle is defined as*

$$\varphi=\theta+\varepsilon\theta^*.$$

This concept was given by Study in 1903. θ the primary component of θ^* is the projected angle between $\mathbf{d}_1$ and $\mathbf{d}_2$, and θ^* is the shortest distance between lines $\mathbf{d}_1$ and $\mathbf{d}_2$ (see Figure 2).

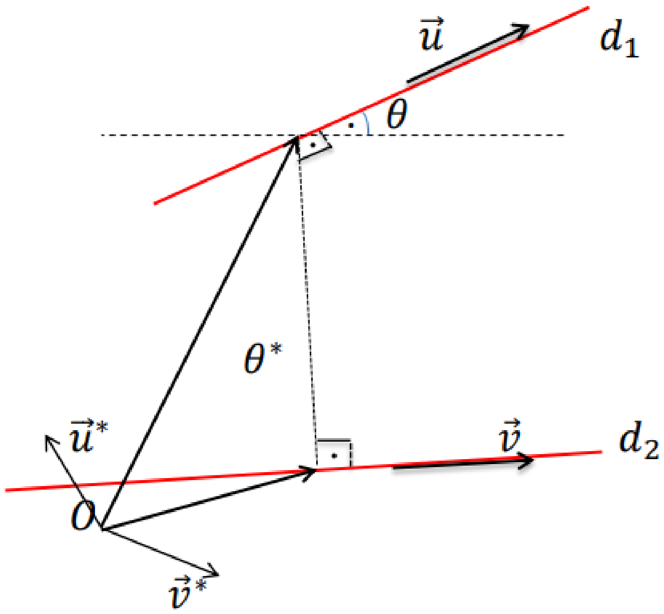

Figure 2. Dual angle.

The scalar product of any unit dual vectors $\vec{\mathrm{U}} = \vec{\mathrm{u}} + \varepsilon\vec{\mathrm{u}}^*$ and $\vec{\mathrm{V}} = \vec{\mathrm{v}} + \varepsilon\vec{\mathrm{v}}^*$ is obtained as follows:

$$\left\langle \vec{\mathrm{U}}, \vec{\mathrm{V}} \right\rangle = \cos\varphi = \cos\theta - \varepsilon\theta^* \sin\theta, \tag{3.8}$$

where $\varphi = \theta + \varepsilon\theta^*$ is a dual angle [2].

Theorem 10. *From equalities* (3.3) *and* (3.8), *the scalar product of* $\overrightarrow{\mathcal{J}}_n$ *and* $\overrightarrow{\mathcal{J}}_m$ *is*

$$\begin{aligned} \left\langle \overrightarrow{\mathcal{J}}_n, \overrightarrow{\mathcal{J}}_m \right\rangle &= \frac{7\mathbf{j}_{n+m} + 2^m(-1)^m\mathbf{j}_{n-m}}{3} + \varepsilon\frac{28\mathbf{j}_{n+m} + 2^m(-1)^m\mathbf{j}_{n-m}}{3}, \\ &= \cos\varphi = \cos\theta - \varepsilon\theta^* \sin\theta. \end{aligned}$$

In the last the equality, φ is a dual angle if and only if the following cases are true.

Case 1. *If* $\cos\theta = 0$ *and* $\theta^* \neq 0$, *then*

$$\left\langle \overrightarrow{\mathcal{J}}_n, \overrightarrow{\mathcal{J}}_m \right\rangle = \cos\varphi = -\varepsilon\theta^*,$$

where

$$-\frac{28\,\mathbf{j}_{n+m} + 2^m(-1)^m\mathbf{j}_{n-m}}{3} = \theta^*.$$

Therefore, lines $\overrightarrow{\mathcal{J}}_n$ *and* $\overrightarrow{\mathcal{J}}_m$ *are perpendicular but not intersecting.*

Case 2. *If* $\theta^* = 0$, *then*

$$\left\langle \overrightarrow{\mathcal{J}}_n, \overrightarrow{\mathcal{J}}_m \right\rangle = \cos\varphi = \cos\theta = \frac{7\mathbf{j}_{n+m} + 2^m(-1)^m\mathbf{j}_{n-m}}{3}$$

$$\theta = cos^{-1}(\frac{7\,\mathbf{j}_{n+m} + 2^m(-1)^m \mathbf{j}_{n-m}}{3}),$$

lines $\overrightarrow{\mathcal{J}}_n$ *and* $\overrightarrow{\mathcal{J}}_m$ *are intersecting.*

Case 3. *If* $\theta = \frac{\pi}{2}$ *and* $\theta^* = 0$*, then*

$$\left\langle \overrightarrow{\mathcal{J}}_n, \overrightarrow{\mathcal{J}}_m \right\rangle = cos\varphi = cos\frac{\pi}{2} = \frac{7\mathbf{j}_{n+m} + 2^m(-1)^m \mathbf{j}_{n-m}}{3},$$

the last equality is

$$\frac{\mathbf{j}_{n+m}}{\mathbf{j}_{n-m}} = \frac{2^m(-1)^{m+1}}{7},$$

and lines $\overrightarrow{\mathcal{J}}_n$ *and* $\overrightarrow{\mathcal{J}}_m$ *are perpendicular and intersecting.*

Case 4. *If* $\theta = 0$*, then*

$$\begin{aligned} cos\varphi &= 1, \\ \varepsilon &= \frac{3 - 7\ \ 2^{2n+1} - (-1)^n \mathbf{j}_n}{28\,\mathbf{j}_{2n+1} - 2^n(-1)^n}. \end{aligned}$$

Thus, lines $\overrightarrow{\mathcal{J}}_n$ *and* $\overrightarrow{\mathcal{J}}_m$ *are parallel.*

4. Conclusions

In this study, the dual Jacobsthal numbers and the dual Jacobsthal-Lucas numbers with coefficients of dual numbers have been introduced. We obtained some fundamental identities such as the generating function, the Binet formulas, the Cassini's, Catalan's and d'Ocagne identities for these numbers. In addition, some properties of dual Jacobsthal vectors to exert in geometry of dual space are obtain.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no competing interests.

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