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Research article

Subclasses of spiral-like functions associated with the modified Caputo's derivative operator

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Abstract: In this paper, the authors apply the modified Caputo's derivative operator, to introduce two new subclasses of spiral-like functions, namely the spiral-starlike functions and spiral-convex functions. In addition to this we, elaborate on the inclusion properties of these subclasses by considering the generalization of the Mittag-Leffler function and its integral transformation. Consequently, we obtain the subordination result for the functions in the class of spiral-like functions.

Keywords: spiral-like function; subordination; Mittag-Leffler function; fractional calculus; Caputo's derivative operator
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ denotes the class of functions whose members are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathcal{U}. \tag{1}$$

The set of all functions in (1) are analytic in the open unit disk $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$ and they are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose elements are univalent in $\mathcal{U}$. Also, let Ω be the class of all analytic functions, w in $\mathcal{U}$ that satisfy the conditions of $w(0) = 0$ and $|w(z)| < 1$. The two eminent subclasses of $\mathcal{S}$ are the class of starlike and convex functions (see Robertson [1]). A function $f \in \mathcal{A}$ given by (1) is said to be a starlike of order γ, $0 \leq \gamma < 1$, if and only if it meets with the following conditions,

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \gamma,\ z \in \mathcal{U}.$$

Note that the class of star-like functions belonging to $\mathcal{A}$ is denoted by $\mathcal{S}^*(\gamma)$. For example, we write $\mathcal{S}(0) =: \mathcal{S}^*$ (the standard class of starlike function), this implies that $f(\mathcal{U})$ is a starlike domain with respect to the origin. A function $f \in \mathcal{A}$ given by (1) is said to be the convex of order γ, $0 \leq \gamma < 1$, if and only if it meets the following criterion,

$$\Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma,\ z \in \mathcal{U}.$$

Note that the class of convex functions belonging to $\mathcal{A}$ is denoted by $K(\gamma)$. For $\gamma = 0$ the function $K(0) =: K$ represents the well-known standard class of convex functions. It is revealed from Alexander's duality relation (see [2]), that $f \in K \Leftrightarrow zf'(z) \in S^*$.

A function of $f \in \mathcal{A}$ is said to be spiral-like if and only if it meets the following conditions,

$$\Re e\left(e^{i\xi}\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathcal{U}, \xi \in \mathbb{C} \text{ but } |\xi| < \frac{\pi}{2}.$$

The class of spiral-like functions was introduced by [3]. Note that a function $f \in \mathcal{A}$ is said to be a convex spiral-like if $zf'(z)$ is spiral-like.

The spin of studies on spiral-like functions has emerged for about two decades. However, for the last few years, it has evoked the attention of many researchers. This is probably because of its interesting geometrical structures that emanate from a point and move farther away as it revolves around the point. The study of spiral-like functions includes many new subclasses, the characterization of spiral-like functions, and the relation between spiral-like and starlike functions (see [4]). Aharonov et al. [5] introduced a subclass of univalent functions on the unit disk of the complex plane whose image is spiral-shaped with respect to a boundary point and the uniformly Spiral-like functions as studied by Frasin et al. [6]. In a recent study by Srivastava et al. [7], the authors investigated several properties of spiral-like close-to-convex functions with respect to their conic domains. Further, studies on the Spiral-Like Harmonic univalent functions related to Quantum Calculus (q-difference operator) have been carried out by [8,9]. In another progression, a class of spiral-like functions of reciprocal order was unified with a well-known class of Janowski (see [10]).

The non-integer order integro-differential operators is referring to Fractional Calculus (FC), in which this study emerged in a letter written to Guillaume de L'Hospital by Gottfried Wilhelm Leibniz in 1695. The focus of their discussion was "what if the order of a derivative is half", and an answer to his question, W. Leibniz replied that "it will lead to a paradox, from which one day a useful consequence will be drawn" (for details see [11]). Literature review shows that the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral play a foremost role in the advancement of FC [12]. The Riemann-Liouville fractional derivative for the power function of

($x^{\beta}:\beta \in \mathcal{R}$) is given by [13,14],

$$D_{0+}^{\alpha}x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha},$$

where $n-1<\alpha<n, \beta \in R$ and $\beta > -1$.

It is noteworthy that for $\alpha = 1$ and $\beta = 1$, the above fractional differential equality reduced to the first order standard derivative of the power function $D_{0+}^{1}x^{1} = 1$. On the other hand, for $\alpha = \frac{1}{2}$ and $\beta = 1$, we obtain a transcendental function, $D^{\frac{1}{2}}x = 2\sqrt{\frac{x}{\pi}}$. One of the flaws in the first order integer derivative is that derivative of a constant number in classical calculus is zero, however, the fractional calculus determines the unique differentiation for different constants $D^{\alpha}k = \frac{kx^{-\alpha}}{\Gamma(1-\alpha)} \neq 0$. The disadvantage of Riemann-Liouville fractional derivative is that it is not consistent with the physical initial and boundary conditions problems. To overcome this difficulty, M. Caputo coined a variation in the Riemann-Liouville fractional derivative, now known as Caputo or Dzherbashyan-Caputo fractional derivative given by (see [15]).

$$D_{0}^{\alpha}f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{0}^{t}\frac{f^{n}(x)}{(x-\tau)^{\alpha-n+1}}d\tau, n-1<\alpha<n, n \in N \\ \dfrac{d^{n}f(x)}{dx^{n}}, \alpha = n \in N. \end{cases}$$

For a constant function (k), the Caputo fractional operator meets with the classical integer order derivative, which is comparatively more useful in a physical interpretation compared to Riemann-Liouville fractional operator.

For more than a decade, an increasing interest has been found in the advancement of fractional calculus and its physical application or analytical interpretation (see for examples [16,17]). In this study, the authors studied the application of spiral-like function in the FC and consequently proved the distortion theorems for fractional derivative and fractional integration with some geometric properties and coefficient inequalities. Utilizing the concepts of quantum calculus or fractional calculus results in introducing new subclasses of analytic functions. Hence, one can investigate some useful results such as coefficient estimates, subordination properties and Fekete-Szegö problem and consequently, open relevant problems for researchers such as distortion theorems, closure theorems, convolution properties and radii problems. Moreover, these results can be extended to multivalent functions and meromorphic functions. For that, we here follow the same pattern and we consider the modified Caputo's derivative operator that was introduced and studied by Salah and Darus [18], authors carried out a study on Caputo's differential operator hence determined the improved version of Caputo's derivative operator, known as the modified Caputo's derivative operator. The latter is defined for η a real number and λ $(\eta-1<\lambda\leq\eta<2)$,

$$J_{\eta,\lambda}f(z) := \frac{\Gamma(2+\eta-\lambda)}{\Gamma(\eta-\lambda)}z^{\lambda-\eta}\int_{0}^{z}\frac{\Omega^{\eta}f(t)}{(z-t)^{\lambda+1-\eta}}dt.$$

The summation form of the above operator can be expressed in the following manner.

$$J_{\eta,\lambda}f(z) := z + \sum_{n=2}^{\infty} \frac{\left(\Gamma(n+1)\right)^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} a_n z^n. \tag{2}$$

Remark 1.1. Noted that $J_{0,0}f(z) = f(z)$ and $J_{1,1}f(z) = zf'(z)$.

In this study, the authors considered the generalized form of Sălăgean derivative operator [19] and Libera integral operator [20] which was studied by Owa [21] as: $\Omega^{\eta} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z), \lambda \in \mathcal{R}$. For the applications of the Modified Caputo's derivative operator, one may refer to read the articles by [22–24]. Parametrically the Modified Caputo's derivative operator is expressed by $J_{\eta,\lambda}f(z)$ which is defined in Eq (2).

Next, we utilize the Modified Caputo's derivative operator to introduce the following two subclasses of spiral star-like functions and spiral convex functions.

Definition 1.1. For the real values of $\rho, \gamma, \lambda, \xi$ such that $0 \le \{\rho, \gamma\} < 1$ with $\lambda(\eta - 1 < \lambda \le \eta < 2)$ and $|\xi| < \frac{\pi}{2}$, a subclass $S_{\eta,\lambda}(\xi, \gamma, \rho)$, named as spiral-starlike function is defined by

$$\mathcal{S}_{\eta,\lambda}(\xi, \gamma, \rho) := \left\{ \mathcal{R}e\left(e^{i\xi} \frac{zJ'_{\eta.\lambda}f(z)}{(1-\rho)J_{\eta,\lambda}f(z) + \rho zJ'_{\eta.\lambda}f(z)} \right) > \gamma \cos \xi, z \in \mathcal{U}. \right\}. \tag{3}$$

Definition 1.2. For the real values of $\rho, \gamma, \lambda, \xi$ such that $0 \le \{\rho, \gamma\} < 1$ with $\lambda(\eta - 1 < \lambda \le \eta < 2)$ and $|\xi| < \frac{\pi}{2}$, a subclass $\mathcal{K}_{\eta,\lambda}(\xi, \gamma, \rho)$, named as spiral-convex function is defined by

$$\mathcal{K}_{\eta,\lambda}(\xi, \gamma, \rho) := \left\{ \mathcal{R}e\left(e^{i\xi} \frac{zJ''_{\eta.\lambda}f(z) + J'_{\eta.\lambda}f(z)}{J'_{\eta,\lambda}f(z) + \rho zJ''_{\eta.\lambda}f(z)} \right) > \gamma \cos \xi, z \in \mathcal{U}. \right\}. \tag{4}$$

Remark 1.2. For some initial values of ρ, η, λ and ξ, we found the following relations.

- A function $f \in \mathcal{A}$ is said to be starlike of order γ $(0 \le \gamma < 1)$, if and only if the $\mathcal{R}e\left(\frac{J_{1,1}f(z)}{J_{0,0}f(z)}\right) > \gamma$.
- A function $f \in \mathcal{A}$ is said to be convex of order γ $(0 \le \gamma < 1)$, if and only if the $\mathcal{R}e\left(1 + \frac{J_{1,1}f(z)}{J_{0,0}f(z)}\right) > \gamma$.
- A function $f \in \mathcal{A}$ is said to be spiral-like if and only if the $\mathcal{R}e\left(e^{-i\xi} \frac{J_{1,1}f(z)}{J_{0,0}f(z)}\right) > 0$.
- A function $f \in \mathcal{A}$ is said to be convex spiral-like if and only if $J_{1,1}f(z)$ is spiral-like.
- For $\mathcal{S}_{0,0}(0,0,0) = \mathcal{S}^*$(the standard star-like function).
- For $\mathcal{S}_{0,0}(0,\gamma,0) = \mathcal{S}^*(\gamma)$ known as the star-like of order γ.

2. The inclusion results

In this section, we obtain sufficient conditions for function $f(z)$ given by (1) to be a member of the classes $S_{\eta,\lambda}(\xi, \gamma, \rho)$ and $K_{\eta,\lambda}(\xi, \gamma, \rho)$ respectively. These conditions are presented in the following theorems.

Theorem 2.1. Let the function $f(z)$ be analytic in the open unit disk. Then $f(z)$ is in the class $\mathcal{S}_{\eta,\lambda}(\xi, \gamma, \rho)$ if $\left|\frac{zJ'_{\eta.\lambda}f(z)}{(1-\rho)J_{\eta,\lambda}f(z) + \rho zJ'_{\eta.\lambda}f(z)} - 1\right| < 1 - \sigma$ and $|\xi| \le \cos^{-1}\left(\frac{1-\sigma}{1-\xi}\right)$, where $0 \le \{\rho, \gamma\} < 1$, $(\eta - 1 < \lambda \le \eta < 2)$ and $|\xi| < \frac{\pi}{2}$.

Proof. Using the inequality that if $\left|\frac{zJ'_{\eta.\lambda}f(z)}{(1-\rho)J_{\eta,\lambda}f(z)+\rho zJ'_{\eta.\lambda}f(z)}-1\right|<1-\sigma$, then for $w(z)\in\Omega$, we can write

$$\frac{zJ'_{\eta.\lambda}f(z)}{(1-\rho)J_{\eta,\lambda}f(z)+\rho zJ'_{\eta.\lambda}f(z)}=1+(1-\sigma)\mathrm{w(z)}.$$

Thus, for spiral properties we have

$$\begin{aligned}\mathcal{R}e\left(e^{i\xi}\frac{zJ'_{\eta.\lambda}f(z)}{(1-\rho)J_{\eta,\lambda}f(z)+\rho zJ'_{\eta.\lambda}f(z)}\right)&=\mathcal{R}e\{e^{i\xi}[1+(1-\sigma)]\}\\&=\cos\xi+(1-\sigma)\,\mathcal{R}\{e^{i\xi}w(z)\}\\&\geq\cos\xi-(1-\sigma)\left|e^{i\xi}w(z)\right|\\&>\cos\xi-(1-\sigma)\\&\geq\lambda\cos\xi.\end{aligned}$$

As long $|\xi|\leq\cos^{-1}\left(\frac{1-\sigma}{1-\xi}\right)$, the function f belongs to the class $\mathcal{S}_{\eta,\lambda}(\xi,\gamma,\rho)$.

Observed that by putting $\sigma=1-(1-\lambda)cos\xi$ in Theorem 2.1, we deduce a corollary as under:

Corollary 2.1. For analytic function of the form (1), the $f\in\mathcal{S}_{\eta,\lambda}(\xi,\gamma,\rho)$ if the following inequality holds

$$\left|\frac{zJ'_{\eta.\lambda}f(z)}{(1-\rho)J_{\eta,\lambda}f(z)+\rho zJ'_{\eta.\lambda}f(z)}-1\right|<(1-\lambda)\cos\xi. \tag{5}$$

Theorem 2.2. A function f of the form (1) belongs to the class $S_{\eta,\lambda}(\xi,\gamma,\rho)$ if

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\,sec\,\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|\leq 1-\gamma$$

where $0\leq\{\rho,\gamma\}<1$, $(\eta-1<\lambda\leq\eta<2)$ and $|\xi|<\frac{\pi}{2}$.

Proof. By Corollary 2.1, it is enough to show that the expressions in (5) is satisfied. Considering the left-hand side (L.H.S).

$$L.H.S=\left|\frac{zJ'_{\eta.\lambda}f(z)}{(1-\rho)J_{\eta,\lambda}f(z)+\rho zJ'_{\eta.\lambda}f(z)}-1\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}a_n z^n}{1+\sum_{n=2}^{\infty}(\rho n-\rho+1)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}a_n z^n}\right|$$

$$<\frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|}{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|}.$$

The last expression is bounded above by $(1-\gamma)\cos\xi$, if

$$\sum_{n=2}^{\infty}(1-\rho)(n-1)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|$$
$$<(1-\gamma)\cos\xi\left\{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|\right\}.$$

After some simple work the equivalent form we received from above expression is

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|\leq 1-\gamma.$$

The last expression proves the assertion of Theorem 2.2.

Theorem 2.3. Let the function $f(z)$ be analytic in the open unit disk. Then $f(z)$ is in the class $\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$ if $\left|\frac{zJ''_{\eta.\lambda}f(z)+J'_{\eta.\lambda}f(z)}{J'_{\eta,\lambda}f(z)+\rho zJ''_{\eta.\lambda}f(z)}-1\right|<1-\sigma$ and $|\xi|\leq\cos^{-1}\left(\frac{1-\sigma}{1-\xi}\right)$ hold, where $0\leq\{\rho,\gamma\}<1$, $(\eta-1<\lambda\leq\eta<2)$ and $|\xi|<\frac{\pi}{2}$.

Proof. Using the inequality that if $\left|\frac{zJ''_{\eta.\lambda}f(z)+J'_{\eta.\lambda}f(z)}{J'_{\eta,\lambda}f(z)+\rho zJ''_{\eta.\lambda}f(z)}-1\right|<1-\sigma$, then for $w(z)\in\Omega$, we can write

$$\frac{zJ''_{\eta.\lambda}f(z)+J'_{\eta.\lambda}f(z)}{J'_{\eta,\lambda}f(z)+\rho zJ''_{\eta.\lambda}f(z)}-1=1+(1-\sigma)\mathrm{w(z)}.$$

Thus, for spiral properties we have

$$\mathcal{R}e\left(e^{i\xi}\frac{zJ''_{\eta.\lambda}f(z)+J'_{\eta.\lambda}f(z)}{J'_{\eta,\lambda}f(z)+\rho zJ''_{\eta.\lambda}f(z)}-1\right)=\mathcal{R}e\{e^{i\xi}[1+(1-\sigma)]\}$$
$$=\cos\xi+(1-\sigma)\,\mathcal{R}\{e^{i\xi}w(z)\}$$
$$\geq\cos\xi-(1-\sigma)\left|e^{i\xi}w(z)\right|$$
$$>\cos\xi-(1-\sigma)$$
$$\geq\lambda\cos\xi.$$

As long $|\xi|\leq\cos^{-1}\left(\frac{1-\sigma}{1-\xi}\right)$, the function f belongs to the class $\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$.

Observed that by putting $\sigma=1-(1-\lambda)cos\xi$ in Theorem 2.3, we deduce a corollary as follow:

Corollary 2.2. For analytic function of the form (1), the $f\in\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$ if the following inequality holds

$$\left|\frac{zJ''_{\eta.\lambda}f(z)+J'_{\eta.\lambda}f(z)}{J'_{\eta,\lambda}f(z)+\rho zJ''_{\eta.\lambda}f(z)}-1\right|<(1-\lambda)\cos\xi. \tag{6}$$

Theorem 2.4. A function f of the form (1) belongs to the class $\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$ if

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]n\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|$$
$$\leq 1-\gamma,$$

where $0 \leq \{\rho, \gamma\} < 1$, $(\eta - 1 < \lambda \leq \eta < 2)$ and $|\xi| < \frac{\pi}{2}$.

Proof. By Corollary 2.2 it is enough to show that the expressions in (6) is satisfied, then by considering the left-hand side of (6) we obtain

$$\left|\frac{zJ''_{\eta.\lambda}f(z) + J'_{\eta.\lambda}f(z)}{J'_{\eta,\lambda}f(z) + \rho zJ''_{\eta.\lambda}f(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)n\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}a_n z^n}{1+\sum_{n=2}^{\infty}n(\rho n-\rho+1)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}a_n z^n}\right|$$

$$< \frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)n\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|}{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)n\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|}.$$

The last expression is bounded above by $(1-\gamma)\cos\xi$, if

$$\sum_{n=2}^{\infty}(1-\rho)(n-1)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|$$

$$< (1-\gamma)\cos\xi\left\{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n|\right\}.$$

After some simple calculation, the above inequality becomes

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}|a_n| \leq 1-\gamma.$$

The last expression proves the proclamation in Theorem 2.4.

3. Application on generalized Mittag-Leffler function

A Swedish mathematician discovered one of the functions characterizing the exponential behavior, known by his name "The Mittag-Leffler Function" (see [25]). The Mittag-Leffler function (M-L) has grown its importance through its vast application in various fields of science and engineering. In recent years, the application of Mittag-Leffler function has been observed in certain areas of physical and applied sciences such as probability and statistical distribution theory, fluid mechanics, biological problems, electric networks and others. This function arises naturally in the solution of integro-differential equations for example Leṽy flights, random walks and importantly, in the generalization of kinetic equations [26,27]. A considerable amount of literature has been investigated on the (M-L) function for its normalization and generalization, properties, applications, and extension. For more details, one may refer to [28–30].

In particular, the study of fractional generalization of kinetic equations, random walks, Levy flights, super-diffusive transport, complex systems and delayed fractional reaction-diffusion all involve fractional-order differential and integral equations, and the solutions inevitably include the Mittag-Leffler function see [31–34]. In addition, the one parameter Mittag-Leffler function was proposed as a solution for mathematical models in tourism and in biology (see [35,36]). The one-parameter Mittag-Leffler function $E_\alpha(z)$ for $\alpha \in \mathbb{C}$, with $\mathcal{Re}\ (\alpha > 0)$ (see [37,38]) is defined as:

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},\ z \in \mathbb{C}.$$

Further the extension of Mittag-Leffler function in two-parameters was studied by Wiman [39]. For all $\alpha, \beta \in \mathbb{C}$, with $\mathcal{Re}\ (\alpha, \beta > 0)$, the two parameters function $E_{\alpha,\beta}(z)$ is defined as:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ z \in \mathbb{C}.$$

In fact, many researchers have worked on the generalization of the Mittag-Leffler function (see [40]).

In this study, we confined our attention to the generalization given by Salah and Darus [41] as follows:

$$qF_{\alpha,\beta}^{\theta,k} = \sum_{n=0}^{\infty} \prod_{j=1}^{q} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} . \frac{z^n}{n!}.$$

Note that $(\theta)_v$ denotes the familiar Pochhammer symbol which is defined as:

$$(\theta)_v := \frac{\Gamma(\theta + v)}{\Gamma(\theta)} = \begin{cases} 1, & if\ v = 0, \theta \in \mathbb{C}\backslash\{0\} \\ \theta(\theta+1)\ldots(\theta+n-1), & if\ v = n \in \mathcal{N}, \theta \in \mathbb{C}, \end{cases}$$

$$(1)_n = n!, n \in \mathcal{N}_0, \mathcal{N}_0 = \mathcal{N} \cup \{0\}, \mathcal{N} = \{1,2,3,\ldots\},$$

and $\left(q \in \mathcal{N}, j = 1,2,3,\ldots q;\ \mathcal{Re}\{\theta_j,\ \beta_j\} > 0,\ and\ \mathcal{Re}\ \alpha_j > max\{0, \mathcal{Re}\ k_j - 1;\ \mathcal{Re}\ k_j\};\ \mathcal{Re}\ k_j > 0\right)$. Hence by using the convolution (or Hadamard product), we introduce the following operator:

$$q\mathbb{F}_{\alpha,\beta}^{\theta,k}(z) := \left[z.\, qF_{\alpha,\beta}^{\theta,k}(z)\right] * J_{\eta,\lambda} f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{\left(\Gamma(n+1)\right)^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} \prod_{j=1}^{q} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} . \frac{z^n}{n!}. \tag{7}$$

To obtain sufficient conditions for the operator in (7) to be a member of the classes $\mathcal{S}_{\eta,\lambda}\ (\xi, \gamma, \rho)$ and $\mathcal{K}_{\eta,\lambda}\ (\xi, \gamma, \rho)$ respectively, we compute the following:

$$q\mathbb{F}_{\alpha,\beta}^{\theta,k}(1) - 1 = \sum_{n=2}^{\infty} \frac{\left(\Gamma(n+1)\right)^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} \prod_{j=1}^{q} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \tag{8}$$

$$\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)'(1) - 1 = \sum_{n=2}^{\infty} \frac{\left(\Gamma(n+1)\right)^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} \prod_{j=1}^{q} \frac{1}{(n-1)!} . \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}, \tag{9}$$

$$\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)''(1) = \sum_{n=2}^{\infty} \frac{\left(\Gamma(n+1)\right)^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} \prod_{j=1}^{q} \frac{1}{(n-2)!} . \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}. \tag{10}$$

Theorem 3.1. Let the function $f(z)$ be analytic in the open unit disk, then $q\mathbb{F}_{\alpha,\beta}^{\theta,k}(z) \in \mathcal{S}_{\eta,\lambda}(\xi,\gamma,\rho)$, if the following postulation holds:

$$[(1-\rho)se\,c\,\xi+\rho(1-\gamma)]\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)'(1)+(1-\rho)(1-\gamma-se\,c\,\xi)q\mathbb{F}_{\alpha,\beta}^{\theta,k}(1)\le 2(1-\gamma). \tag{11}$$

Proof. By Theorem 2.2, it is sufficient to show that

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)se\,c\,\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)} \times\frac{1}{n!}\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_jn}}{\left(\beta_j\right)_{\alpha_jn}}\le 1-\gamma. \tag{12}$$

The left-hand side of (12) can be written as:

$$\begin{aligned}J_1 &:=\sum_{n=2}^{\infty}[(1-\rho)sec\xi(n-1)\\ &\quad+(1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}.\frac{1}{n!}\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_jn}}{\left(\beta_j\right)_{\alpha_jn}}\\ &=[(1-\rho)se\,c\,\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}.\frac{1}{(n-1)!}\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_jn}}{\left(\beta_j\right)_{\alpha_jn}}\\ &\quad+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}.\frac{1}{n!}\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_jn}}{\left(\beta_j\right)_{\alpha_jn}}.\end{aligned}$$

Therefore, by using the expressions (8) and (9), we get

$$\begin{aligned}j_1 =\ &[(1-\rho)\sec\,\xi+\rho(1-\gamma)]\left[\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)'(1)-1\right]\\ &+(1-\rho)(1-\gamma-\sec\,\xi)\left[q\mathbb{F}_{\alpha,\beta}^{\theta,k}(1)-1\right]\\ =\ &[(1-\rho)\sec\,\xi+\rho(1-\gamma)]\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)'(1)+(1-\rho)(1-\gamma-\sec\,\xi)q\mathbb{F}_{\alpha,\beta}^{\theta,k}(1)\\ &-(1-\gamma).\end{aligned}$$

Thus, from assumption (11), it follows that $j_1\le 1-\gamma$. This implies that the assumption stated in Theorem 2.2 holds and hence, $q\mathbb{F}_{\alpha,\beta}^{\theta,k}(\mathrm{z}) \in \mathcal{S}_{\eta,\lambda}(\xi,\gamma,\rho)$.

Theorem 3.2. Let the function $f(z)$ be analytic in the open unit disk, then $q\mathbb{F}_{\alpha,\beta}^{\theta,k}(\mathrm{z}) \in \mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$, if the following hypothesis holds:

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)''(1)+(1-\gamma)\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)'(1)\le 2(1-\gamma). \tag{13}$$

Proof. In view of Theorem 2.4 and by convolution defined in (7), it is sufficient to verify that

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)se\,c\,\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}$$
$$\times\prod_{j=1}^{q}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\leq 1-\gamma. \tag{14}$$

The left-hand side of (14) can be expressed as:

$$J_2 := \sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi$$
$$+(1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\prod_{j=1}^{q}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}$$

$$=[(1-\rho)se\,c\,\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{1}{(n-2)!}.\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\prod_{j=1}^{q}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}$$
$$+(1-\gamma)\sum_{n=2}^{\infty}\frac{1}{(n-1)!}.\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\prod_{j=1}^{q}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}.$$

Therefore, by using the expressions (9) and (10), we get

$$j_2=[(1-\rho)\sec\xi+\rho(1-\gamma)]\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)''(1)+(1-\gamma)\left[\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)'(1)-1\right].$$

Thus, by assumption (13), it follows that $j_2\leq 1-\gamma$. This implies that the assumption stated in Theorem 2.4 holds. Consequently, $q\mathbb{F}_{\alpha,\beta}^{\theta,k}(\mathrm{z})\in\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$.

Theorem 3.3. Let the integral operator $I(z)$ be given by $I(z)=\int_0^z\frac{q\mathbb{F}_{\alpha,\beta}^{\theta,k}(\mathrm{t})}{t}dt,\ z\in\mathbb{U}$, then $I(z)\in S_{\eta,\lambda}\,(\xi,\gamma,\rho)$ if the following condition holds:

$$[(1-\rho)\sec\,\xi+\rho(1-\gamma)]\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}\right)'(1)+(1-\rho)(1-\gamma-\sec\,\xi)q\mathbb{F}_{\alpha,\beta}^{\theta,k}(1)\leq 2(1-\gamma).$$

Proof. The summation form of the integral operator $I(z)$ can be expressed as:

$$I(z)=z+\sum_{n=2}^{\infty}\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\prod_{j=1}^{q}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}.\frac{z^n}{n!}.\frac{1}{n}. \tag{15}$$

In view of Theorem 2.2, it is sufficient to show that

$$\sum_{n=2}^{\infty}n[(1-\rho)(n-1)sec\xi+(1-\gamma)(1+n\rho-\rho)].\frac{1}{n}\frac{\left(\Gamma(n+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{n!\,\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}$$
$$\times\prod_{j=1}^{q}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\leq 1-\gamma.$$

After some simple calculation the equivalent expression, we get is as under.

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)].\frac{\left(\Gamma(n+1)\right)^{2}\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{n!\,\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}$$
$$\times\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_j n}}{\left(\beta_j\right)_{\alpha_j n}}\leq 1-\gamma.$$

Thus, the proof of Theorem 3.3 is noticeably parallel to that of Theorem 2.2.

Theorem 3.4. Let the integral operator $I(z)$ be given by $I(z)=\int_0^z\frac{q\mathbb{F}_{\alpha,\beta}^{\theta,k}(\mathrm{t})}{t}dt,\ z\in\mathbb{U}$, then $I(z)\in\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$ if the following condition holds:

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\left(q\mathbb{F}_{\alpha,\beta}^{\theta,k}(\mathrm{t})(1)-1\right)+(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{q\mathbb{F}_{\alpha,\beta}^{\theta,k}(\mathrm{t})(t)}{t}-1\right)dt$$
$$\leq 1-\gamma.$$

Proof. In the light of Theorem 2.4 and the power series of the integral operator $I(z)$ in (15), it is sufficient to confirm the expression that

$$\sum_{n=2}^{\infty}\frac{1}{n}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\left(\Gamma(n+1)\right)^{2}\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{n!\,\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}$$
$$\times\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_j n}}{\left(\beta_j\right)_{\alpha_j n}}\leq 1-\gamma. \tag{16}$$

The left-hand side of the (16) can be written as:

$$j_3:=\sum_{n=2}^{\infty}\frac{1}{n}[(1-\rho)(n-1)se\,c\,\xi+(1-\gamma)(1+n\rho-\rho)]$$
$$\times\frac{\left(\Gamma(n+1)\right)^{2}\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{n!\,\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_j n}}{\left(\beta_j\right)_{\alpha_j n}}$$
$$=\sum_{n=2}^{\infty}\ [(1-\rho)se\,c\,\xi+\rho(1-\gamma)]\frac{\left(\Gamma(n+1)\right)^{2}\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{n!\,\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_j n}}{\left(\beta_j\right)_{\alpha_j n}}$$
$$+(1-\rho)(1-\gamma-se\,c\,\xi)\sum_{n=2}^{\infty}\frac{1}{n}\cdot\frac{\left(\Gamma(n+1)\right)^{2}\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{n!\,\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\prod_{j=1}^{q}\frac{\left(\theta_j\right)_{k_j n}}{\left(\beta_j\right)_{\alpha_j n}}.$$

By taking (10) into account, the above expression the final outcomes we get is as under.

$$J_3\leq\ \ [(1-\rho)\sec\ \xi+\rho(1-\gamma)]\left[q\mathbb{F}_{\alpha,\beta}^{\theta,k}(1)-1\right]$$
$$+(1-\rho)(1-\gamma-\sec\ \xi)\int_0^1\left(\frac{q\mathbb{F}_{\alpha,\beta}^{\theta,k}(t)}{t}-1\right)dt.$$

Therefore, if the assumption in Theorem 3.4 holds, this means that $J_3 \leq 1-\gamma$ and hence proved that $I(z) \in K_{\eta,\lambda}(\xi,\gamma,\rho)$.

4. Subordination

The concept of subordination between the analytic functions was initiated by Lindelöf 1909 [42] and this has motivated many researchers to solve differential subordination equations. There is some literature that involves subordination while studying the estimation of the coefficient bounds in both the study of univalent and bi-univalent functions (see [43] for example). The author introduced different subclasses of star-like and convex functions by using the concept of subordination. Nevertheless, it is a powerful tool for the solution of many questions in the theory of functions, especially in the geometric function theory of a complex variable. The subordination principle is stated as under:

Let $f(z), g(z)$ and $w(z)$ be analytic functions in the open unit disc $\mathcal{U}$. The function $f(z)$ is said to be subordinate to $g(z)$, expressed as $f(z), g(z)$ if there exists a Schwartz function w, that is $w(z) = 0, |w(z)| < 1$ and $f(z) = g(w(z))$. Particularly, if the function $g(z)$ is univalent in the unit disc $\mathcal{U}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(D) \subset g(D). \tag{17}$$

The purpose of this section is to obtain subordination results for the class $S_{\eta,\lambda}(\xi,\gamma,\rho)$. Let us recall those definitions and lemmas which assist in achieving our results.

Definition 4.1. A complex valued sequence $\{b_n\}_{n=1}^{\infty}$ is said to be subordinating sequence if for every analytic, univalent and convex function, we receive

$$\sum_{n=1}^{\infty} a_n b_n \prec f(z). \tag{18}$$

Lemma 4.1. [44] A complex valued sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\mathcal{R}e\left(1+2\sum_{n=1}^{\infty} b_n z^n\right) > 0, z \in \mathcal{U}. \tag{19}$$

Theorem 4.1. Let $g(z)$ be a convex function and $f(z)$ be a member of the class $S_{\eta,\lambda}(\xi,\gamma,\rho)$, then the following relations hold.

$$\frac{[(1-\rho)\sec\xi + (1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{2\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}(f*g)(z) \prec g(z), \tag{20}$$

$$\phi(2,\eta,\lambda) = \frac{\left(\Gamma(2+1)\right)^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(2+\eta-\lambda+1)\Gamma(2-\eta+1)} = \frac{4}{(2+\eta-\lambda)(2-\eta)}, \tag{21}$$

$$\mathcal{R}e\{f(z)\} > -\frac{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}, \tag{22}$$

and, the constant factor

$$\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{2\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}$$

involved in (20) cannot be replaced by further larger number.

Proof. Let f belongs to $S_{\eta,\lambda}(\xi,\gamma,\rho)$, then for a convex function $g(z)=z+\sum_{n=1}^{\infty}c_n z^n$ we obtain the left hand side of (20) by using the Hadamard product.

$$\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{2\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}(f*g)(z)$$

$$=\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{2\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}\left(z+\sum_{n=2}^{\infty}c_n\,a_n z^n\right).$$

Using the Definition 4.1, the consequence of subordination holds if

$$\left\{\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{2\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}a_n\right\}_{n=1}^{\infty}$$

is a sequence of subordinating factor. By the virtue of Lemma 4.1 equivalently we get

$$\mathcal{R}e\left\{1+\sum_{n=1}^{\infty}\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}a_n z^n\right\}>0. \tag{23}$$

Clearly, the expression

$$\frac{[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\phi(n,\eta,\lambda)}{1-\lambda}$$

is an increasing function for $n\geq 2$, this means the expression

$$\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{1-\lambda}$$
$$\leq\frac{[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\phi(n,\eta,\lambda)}{1-\lambda}.$$

Thus, by letting $|z|=r$, we deduce

$$\mathcal{R}e\left\{1+\sum_{n=1}^{\infty}\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}a_n z^n\right\}$$

$$=\mathcal{R}e\left\{\begin{array}{c}1+\dfrac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}z\\ +\displaystyle\sum_{n=2}^{\infty}\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}a_n z^n\end{array}\right\}$$

$$\geq 1-\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}r$$
$$-\sum_{n=2}^{\infty}\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}|a_n|r^n$$

$$\geq 1-\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}r$$
$$-\frac{1-\xi}{\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}r^2>0.$$

This confirms the validity of assertion (23). As a result (20) holds.

In order to prove (21), we simply consider the convex function

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n,$$

and, let us assume that if

$$\mathcal{F}(z):=z-\frac{1-\xi}{\{1-\lambda+[(1-\rho)\,sec\,\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}z^2,$$

then we obtain the following subordination

$$\frac{[(1-\rho)\,sec\,\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{2\{1-\lambda+[(1-\rho)\,sec\,\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}\mathcal{F}(z)\prec g(z).$$

This shows that the constant

$$\frac{[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)}{2\{1-\lambda+[(1-\rho)\sec\xi+(1-\gamma)(1+\rho)]\phi(2,\eta,\lambda)\}}$$

involved in (20) cannot be replaced by any larger one. This completes the proof of our theorem.

5. Conclusions

In this paper, we considered the well-known concept of spiral-like functions to introduce two new subclasses: the spiral-starlike and the spiral-convex by the means of the modified Caputo's derivative operator. For these subclasses, we studied the inclusion results. In addition, we investigated further inclusion properties of these subclasses by application on the generalization of the Mittag-Leffler function and its integral transform. Finally, we obtain the subordination result for the functions in the class of spiral-starlike functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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