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Research article

Global solutions to a nonlinear Fokker-Planck equation

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Abstract: In this paper, we construct global solutions to the Cauchy problem on a nonlinear Fokker-Planck equation near Maxwellian with small-amplitude initial data in Sobolev space $H_x^2L_v^2$ by a refined nonlinear energy method. Compared with the results of Liao et al. (Global existence and decay rates of the solutions near Maxwellian for non-linear Fokker-Planck equations, *J. Stat. Phys.*, **173** (2018), 222–241.), the regularity assumption on the initial data is much weaker.

Keywords: nonlinear Fokker-Planck equation; global existence; energy method; Cauchy problem; Priori estimates
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1. Introduction

In this paper, we are concerned on the Cauchy problem to a nonlinear Fokker-Planck equation as follows

$$\begin{cases}\partial_t F + v\cdot\nabla_x F = \rho\nabla_v\cdot(\nabla_v F + vF),\\ F(0,x,v) = F_0(x,v),\end{cases} \tag{1.1}$$

where the nonnegative unknown functions $F(t,x,v)$ is the distribution function of particles with position $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$ at time $t\geq 0$, and the density $\rho(t,x)$ is defined as $\rho=\int_{\mathbb{R}^3} F dv$.

In statistical mechanics, nonlinear Fokker-Planck equation is a partial differential equation which describes the Brownian motion of particles. This equation illustrates the evolution of particle probability density function with velocity, time and space position under the influence of resistance or random force. This equation is also widely used in various fields such as plasma physics, astrophysics, nonlinear hydrodynamics, theory of electronic circuitry and laser arrays, population dynamics, human movement sciences and marketing.

The global equilibrium for the nonlinear Fokker-Planck Eq (1.1) is the normalized global Maxwellian

$$\mu = \mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Therefore, we can define the perturbation $f = f(t, x, v)$ by

$$F(t, x, v) = \mu + \mu^{\frac{1}{2}} f(t, x, v),$$

then the Cauchy problem (1.1) of the nonlinear Fokker-Planck equation is reformulated as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \rho L f, \quad \rho = 1 + \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f dv, \\ f(0, x, v) = f_0(x, v) = \mu^{-\frac{1}{2}} (F_0(x, v) - \mu), \end{cases} \tag{1.2}$$

where the linear Fokker-Planck operator L is given by

$$Lf = \mu^{-\frac{1}{2}} \nabla_v \cdot (\mu \nabla_v (\mu^{-\frac{1}{2}} f)) = \Delta_v f + \frac{1}{4}(6 - |v|^2) f. \tag{1.3}$$

Let us define the velocity orthogonal projection

$$\mathbf{P} : L^2(\mathbb{R}^3_v) \to \text{Span}\left\{\mu^{\frac{1}{2}},\ v_i \mu^{\frac{1}{2}} (1 \leq i \leq 3)\right\}.$$

For any given function $f(t, x, v) \in L^2(\mathbb{R}^3_v)$, one has

$$\mathbf{P} f = a(t, x) \mu^{\frac{1}{2}} + b(t, x) \cdot v \mu^{\frac{1}{2}}, \tag{1.4}$$

with

$$a = \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f \, dv, \quad b = \int_{\mathbb{R}^3} v \cdot \mu^{\frac{1}{2}} f \, dv. \tag{1.5}$$

Then by the macro-micro decomposition introduced in [9], we get the decomposition of solutions $f(t, x, v)$ of the nonlinear Fokker-Planck Eq (1.1) as follows

$$f(t, x, v) = \mathbf{P} f(t, x, v) + \{\mathbf{I} - \mathbf{P}\} f(t, x, v), \tag{1.6}$$

where $\mathbf{I}$ denotes the identity operator, $\mathbf{P} f$ and $\{\mathbf{I} - \mathbf{P}\} f$ are called the macroscopic and the microscopic component of $f(t, x, v)$, respectively.

Before the statement of main result, we need list some notations used in this paper.

- $A \lesssim B$ means that there is a constant $C > 0$ such that $A \leq CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.
- For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, the length of α is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We denote $\partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ and use ∂_i to denote ∂_{x_i} for each $i = 1, 2, 3$.

- For any function f and g, denote the inner product and norm as follows

$$\begin{aligned}
&\langle f, g\rangle := \int_{\mathbb{R}^3} fg dv, \quad |f|^2_{L^2_v} = \int_{\mathbb{R}^3} f^2 dv,\\
&|f|^2_\nu := |f|^2_{L^2_\nu} = \int_{\mathbb{R}^3} \left(|\nabla_v f|^2 + \nu(v)|f|^2\right) dv \text{ where } \nu(v) := 1 + |v|^2,\\
&\|f\|^2_\nu := \int_{\mathbb{R}^3} |f|^2_\nu dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(|\nabla_v f|^2 + \nu(v)|f|^2\right) dv dx,\\
&\|f\|^2 := \|f\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \text{ or } \|a\|^2 := \|a\|^2_{L^2(\mathbb{R}^3_x)}.
\end{aligned}$$

- Denoting the function spaces $H^N_x L^2_v$ and $H^N_x L^2_\nu$ with the norm as

$$\|f\|^2_{H^N_x L^2_v} = \sum_{|\alpha|\le N} \|\partial^\alpha_x f\|^2, \qquad \|f\|^2_{H^N_x L^2_\nu} = \sum_{|\alpha|\le N} \|\partial^\alpha_x f\|^2_\nu.$$

The basic properties of the linearized Fokker-Planck operator L in (1.3) can be referred in [3,6,7,10,15] as follows

$$\langle f, Lg\rangle = \langle Lf, g\rangle, \quad \mathrm{Ker}\, L = \mathrm{Span}\{\mu^{\frac{1}{2}}\}, \quad L(v\mu^{\frac{1}{2}}) = -v\mu^{\frac{1}{2}}, \tag{1.7}$$

and the Fokker-Planck operator L is coercive in the sense that there is a positive constant λ_0 such that

$$-\langle f, Lf\rangle = \int_{\mathbb{R}^3} |\nabla_v f + \frac{v}{2} f|^2 dv \ge \lambda_0 |\{\mathbf{I}-\mathbf{P}\}f|^2_\nu + |b|^2. \tag{1.8}$$

There are a lot of results about the global existence and large time behavior of solutions to the Fokker-Planck type equation. Such as for the Fokker-Planck-Boltzmann equation, DiPerna and Lions [4] first obtained the renormalized solution and established global existence for the Cauchy problem with large data. Li and Matsumura [12] proved that the strong solution for initial data near an absolute Maxwellian exist globally in time and tends asymptotically in the $L^\infty_y(L^1_\xi)$-norm to another time dependent self-similar Maxwellian in large time. The global existence and temporal decay estimates of classical solutions are established based on the nonlinear energy method developed in [9] under Grads angular cut-off in [17] and without cut-off in [16], respectively.

As for the Vlasov-Poisson-Fokker-Planck equation, Duan and liu [6] obtained the time-periodic small-amplitude solution in the three dimensional whole space by Serrins method. Hwang and Jang [10], Wang [18] obtained the global existence and the time decay of the solution. For the problem (1.1), the global existence is proved by combining uniform-in-time energy estimates and the decay rates of the solution is obtained by using the precise spectral analysis of the linearized Fokker-Planck operator as well as the energy method in [13]. Interested readers can refer to the references [2,7,8,12,14,19] for more related details.

For the nonlinear Fokker-Planck equation, Imbert and Mouhot [11] obtained the Hölder continuity by De Giorgi and Moser argument together with the averaging lemma. Liao et al. [13] deduced the global existence of the Cauchy problem to the equation based on the energy estimates and the decay rates of the solutions by using the precise spectral analysis of the linearized Fokker-Planck operator in Sobolev space $H^N_x, N \ge 4$. Also the new difficulty caused by the nonlinear term was resolved by additional tailored weighted-in-v energy estimates suitable for Fokker-Planck operators. However, in this paper, we find that we can deal with the difficult by using the definition of the linearized Fokker-Planck operator L in (1.3) which is not necessary to estimate the dissipation $\|Lg\|^2_{H^N_x L^2_v}$.

The rest of this paper is organized as follows. In Section 2, we give the main result of this paper. In Section 3, we deduce the microscopic and macroscopic dissipation by a refined energy method, respectively. Section 4 is devoted to close the a priori estimate, then the proof of main theorem is completed based on the continuation argument.

2. Main result

Now we define the energy norm and the corresponding dissipation rate norm, respectively, by

$$\mathcal{E}(t) \sim \sum_{|\alpha|\le 2} \|\partial^\alpha_x f\|^2, \tag{2.1}$$

$$\mathcal{D}(t) \sim \sum_{|\alpha|\le 2} \left(\|\partial^\alpha_x \{\mathbf{I}-\mathbf{P}\} f\|^2_v + \|\partial^\alpha_x b\|^2\right) + \sum_{|\alpha|\le 1} \|\partial^\alpha_x \nabla_x a\|^2. \tag{2.2}$$

With the above preparation in hand, our main result can be stated as follows.

Theorem 2.1. *Assume there exist a sufficiently small positive constant* ϵ_0 *such that* $F_0(x,v) = \mu + \mu^{\frac{1}{2}} f_0(x,v) \ge 0$ *satisfies* $\mathcal{E}(0) \le \epsilon_0$*, then the Cauchy problem* (1.2) *admits a unique global solution* $f(t,x,v)$ *satisfying* $F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v) \ge 0$*, and it holds that*

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s)ds \le \mathcal{E}(0), \tag{2.3}$$

for any $t > 0$*. In particular, we have the global energy estimate*

$$\sup_{t\ge 0} \|f(t)\|_{H^2_x L^2_v} \le \|f_0\|_{H^2_x L^2_v}.$$

Remark 2.1.
- *Compared with the integer Sobolev space* H^4_x *used in [13], the regularity assumption on the initial data in* H^2_x *is weaker by virtue of the Sobolev embedding in Lemma 3.1, especially the estimate of* $L^6(\mathbb{R}^3)$*.*
- *In order to overcome the difficulty from the nonlinear term, the authors in [13] need to estimate the dissipation* $\|Lg\|^2_{H^N_x L^2_v}$*. However, it seems to be not necessary for our estimates.*

3. Energy estimates

In this section, we will derive the energy estimates for the nonlinear Fokker-Planck equation. The first part is concerned on the estimates of the microscopic dissipation and the second part is about the estimates of macroscopic dissipation by the macroscopic equations similar as [13]. We need list the following lemma about Sobolev inequalities which are very important to obtain the corresponding energy estimates.

Lemma 3.1. *(See [1,5].) Let* $u \in H^2(\mathbb{R}^3)$*, then there is a constant* $C > 0$ *such that*

- $\|u\|_{L^\infty} \le C\|\nabla u\|^{\frac{1}{2}} \|\nabla^2 u\|^{\frac{1}{2}} \le C\|\nabla u\|_{H^1}$,
- $\|u\|_{L^6} \le C\|\nabla u\|$,
- $\|u\|_{L^q} \le C\|u\|_{H^1}$, $2 \le q \le 6$.

3.1. Estimates of microscopic dissipation

Firstly, we need the estimates of the microscopic dissipation for the solution f in (1.2).

Lemma 3.2. *It holds that*

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 2}\|\partial_x^\alpha f\|^2+\lambda_0\sum_{|\alpha|\le 2}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\sum_{|\alpha|\le 2}\|\partial_x^\alpha b\|^2\lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t), \tag{3.1}$$

for any $t>0$.

Proof. Step 1. $\alpha=0$. Multiply $(1.2)_1$ by f and integrate over $\mathbb{R}_v^3\times\mathbb{R}_x^3$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^2dvdx-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} fLfdvdx=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} afLfdvdx. \tag{3.2}$$

By using (1.5) and (1.7), we have

$$\begin{aligned}\langle L\mathbf{P}f,f\rangle&=\langle L(a\mu^{\frac{1}{2}}),f\rangle+\langle L(b\cdot v\mu^{\frac{1}{2}}),f\rangle\\&=a\langle L(\mu^{\frac{1}{2}}),f\rangle+b\langle L(v\mu^{\frac{1}{2}}),f\rangle\\&=-b\langle v\mu^{\frac{1}{2}},f\rangle=-|b|^2.\end{aligned} \tag{3.3}$$

Similarly, we can get

$$\begin{aligned}\langle L\{\mathbf{I}-\mathbf{P}\}f,f\rangle&=\langle L\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle+\langle L\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f\rangle\\&=\langle L\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle+\langle L\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f\rangle\\&=\langle L\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle.\end{aligned} \tag{3.4}$$

Therefore, by (3.3) and (3.4) and the definition of L (1.3), we can obtain

$$\begin{aligned}\langle Lf,f\rangle=\langle L\mathbf{P}f,f\rangle+\langle L\{\mathbf{I}-\mathbf{P}\}f,f\rangle&=\langle L\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle-|b|^2\\&=\langle\Delta_v\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle+\frac{3}{2}\langle\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle\\&\quad-\langle|v|^2\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle-|b|^2\\&=-|\nabla_v\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}^2+\frac{3}{2}|\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}^2-|v\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}^2-|b|^2,\end{aligned}$$

where we have used the integration by parts of v, i.e.,

$$\langle\Delta_v\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle=-\langle\nabla_v\{\mathbf{I}-\mathbf{P}\}f,\nabla_v\{\mathbf{I}-\mathbf{P}\}f\rangle=-|\nabla_v\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}^2.$$

Consequently,

$$\begin{aligned}|\langle Lf,f\rangle|&=|\nabla_v\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}^2+\frac{3}{2}|\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}^2+|v\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}^2+|b|^2\\&\le C(|\{\mathbf{I}-\mathbf{P}\}f|_\nu^2+|b|^2).\end{aligned}$$

Furthermore, Sobolev embedding in Lemma 3.1 yields

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} afLfdvdx\Big|&\le\int_{\mathbb{R}^3}|a||\langle f,Lf\rangle|dx\lesssim\int_{\mathbb{R}^3}|a|(|\{\mathbf{I}-\mathbf{P}\}f|_\nu^2+|b|^2)dx\\&\le\|a\|_{L_x^\infty}^2(\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|b\|^2)\\&\lesssim\|\nabla_x a\|_{H_x^1}\mathcal{D}(t)\lesssim\mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned}$$

Therefore, from (3.2) we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2+\lambda_0\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|b\|^2\lesssim\mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Step 2. $1\le|\alpha|\le 2$. Taking ∂_x^α of $(1.2)_1$ yields

$$\partial_t\partial_x^\alpha f+v\cdot\nabla_x\partial_x^\alpha f=L\partial_x^\alpha f+\partial_x^\alpha(aLf), \tag{3.5}$$

Multiply above equation by $\partial_x^\alpha f$ and integrate over $\mathbb{R}_v^3\times\mathbb{R}_x^3$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\partial_x^\alpha f|^2dvdx-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x^\alpha fL(\partial_x^\alpha f)dvdx\\ =\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{|\beta|\le|\alpha|}C_\alpha^\beta\partial_x^{\alpha-\beta}aL(\partial_x^\beta f)(\partial_x^\alpha f)dvdx.\end{aligned} \tag{3.6}$$

Case 1. $\beta=0$. The estimates of the last term in above equation is as follows by the definition of L (1.3):

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x^\alpha aLf\partial_x^\alpha fdvdx&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x^\alpha a(\Delta_v f+\frac{1}{4}(6-|v|^2)f)\partial_x^\alpha fdvdx\\ &=\underbrace{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x^\alpha a\Delta_v f\partial_x^\alpha fdvdx}_{J_1}+\underbrace{\frac{3}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x^\alpha af\partial_x^\alpha fdvdx}_{J_2}.\\ &\quad\underbrace{-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x^\alpha a|v|^2f\partial_x^\alpha fdvdx}_{J_3}\end{aligned} \tag{3.7}$$

Using the integration by parts of v, Hölder inequality and Sobolev embedding in Lemma 3.1 to get

$$\begin{aligned}|J_1|&=\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_x^\alpha a\nabla_v f\partial_x^\alpha\nabla_v fdvdx\Big|\le\int_{\mathbb{R}^3}|\partial_x^\alpha a||\nabla_v f|_{L_v^2}|\partial_x^\alpha\nabla_v f|_{L_v^2}dx\\ &\le\|\partial_x^\alpha a\|_{L_x^2}\|\nabla_v f\|_{L_x^\infty L_v^2}\|\partial_x^\alpha\nabla_v f\|_{L_x^2L_v^2}\lesssim\|\partial_x^\alpha a\|_{L_x^2}\|\nabla_x\nabla_v f\|_{H_x^1L_v^2}\|\partial_x^\alpha\nabla_v f\|_{L_x^2L_v^2}\\ &\lesssim\mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t),\end{aligned} \tag{3.8}$$

where we have used

$$\begin{aligned}\|\nabla_x\nabla_v f\|_{H_x^1L_v^2}&\lesssim\|\nabla_x\nabla_v\mathbf{P}f\|_{H_x^1L_v^2}+\|\nabla_x\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|_{H_x^1L_v^2}\\ &\lesssim\|\nabla_x a\|_{H_x^1}+\|\nabla_x b\|_{H_x^1}+\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{H_x^1L_v^2}\lesssim\mathcal{D}^{\frac{1}{2}}(t),\end{aligned}$$

$$\begin{aligned}\|\partial_x^\alpha\nabla_v f\|_{L_x^2L_v^2}&\lesssim\|\partial_x^\alpha\nabla_v\mathbf{P}f\|_{L_x^2L_v^2}+\|\partial_x^\alpha\nabla_v\{\mathbf{I}-\mathbf{P}\}f\|_{L_x^2L_v^2}\\ &\lesssim\|\partial_x^\alpha a\|_{L_x^2}+\|\partial_x^\alpha b\|_{L_x^2}+\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_{L_x^2L_v^2}\lesssim\mathcal{D}^{\frac{1}{2}}(t).\end{aligned}$$

Similarly, we can easily get

$$|J_2|\lesssim\mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t),\quad |J_3|\lesssim\mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Case 2. $\beta = 1$. The estimates of the last term in (3.6):

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_x^{\alpha-\beta} a L(\partial_x^\beta f)(\partial_x^\alpha f) dvdx = \underbrace{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_x^{\alpha-\beta} a \Delta_v \partial_x^\beta f \partial_x^\alpha f dvdx}_{J_4} + \underbrace{\frac{3}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_x^{\alpha-\beta} a \partial_x^\beta f \partial_x^\alpha f dvdx}_{J_5} \underbrace{- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_x^{\alpha-\beta} a |v|^2 \partial_x^\beta f \partial_x^\alpha f dvdx}_{J_6} . \quad (3.9)$$

Using the similar techniques to estimate J_1, we have

$$\begin{aligned} |J_4| &= \Big| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_x^{\alpha-\beta} a \partial_x^\beta \nabla_v f \partial_x^\alpha \nabla_v f dvdx \Big| \le \int_{\mathbb{R}^3} |\partial_x^{\alpha-\beta} a| |\partial_x^\beta \nabla_v f|_{L_v^2} |\partial_x^\alpha \nabla_v f|_{L_v^2} dx \\ &\le \|\partial_x^{\alpha-\beta} a\|_{L_x^3} \|\partial_x^\beta \nabla_v f\|_{L_x^6 L_v^2} \|\partial_x^\alpha \nabla_v f\|_{L_x^2 L_v^2} \lesssim \|\partial_x^{\alpha-\beta} a\|_{H_x^1} \|\nabla_x \partial_x^\beta \nabla_v f\|_{L_x^2 L_v^2} \|\partial_x^\alpha \nabla_v f\|_{L_x^2 L_v^2} \\ &\lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t). \end{aligned} \quad (3.10)$$

Similarly, we can easily get

$$|J_5| \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t), \quad |J_6| \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Case 3. $\beta = \alpha$. It holds that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a L(\partial_x^\alpha f)(\partial_x^\alpha f) dvdx = \underbrace{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a \Delta_v \partial_x^\alpha f \partial_x^\alpha f dvdx}_{J_7} + \underbrace{\frac{3}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a \partial_x^\alpha f \partial_x^\alpha f dvdx}_{J_8} \underbrace{- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a |v|^2 \partial_x^\alpha f \partial_x^\alpha f dvdx}_{J_9} \quad (3.11)$$

Using the similar techniques to estimate J_1, we have

$$\begin{aligned} |J_7| &= \Big| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a \partial_x^\alpha \nabla_v f \partial_x^\alpha \nabla_v f dvdx \Big| \le \int_{\mathbb{R}^3} |a| |\partial_x^\alpha \nabla_v f|_{L_v^2} |\partial_x^\alpha \nabla_v f|_{L_v^2} dx \\ &\lesssim \|a\|_{L_x^\infty} \|\partial_x^\alpha \nabla_v f\|_{L_x^2 L_v^2} \|\partial_x^\alpha \nabla_v f\|_{L_x^2 L_v^2} \lesssim \|\nabla_x a\|_{H_x^1} \|\partial_x^\alpha \nabla_v f\|_{L_x^2 L_v^2}^2 \\ &\lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t). \end{aligned} \quad (3.12)$$

Similarly, we can easily get

$$|J_8| \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t), \quad |J_9| \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Take the summation over $1 \le |\alpha| \le 2$ to get

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2 + \lambda_0 \sum_{|\alpha|\le 2} \|\partial_x^\alpha \{\mathbf{I}-\mathbf{P}\} f\|_v^2 + \sum_{|\alpha|\le 2} \|\partial_x^\alpha b\|^2 \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Thus we complete the proof of Lemma 3.2.

3.2. Estimates of macroscopic dissipation

Now we give the estimate of the macroscopic component a by the macroscopic equations.

Lemma 3.3. *It holds that*

$$\begin{aligned}\frac{d}{dt}\sum_{|\alpha|\le 1}\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx+\sum_{|\alpha|\le 1}\|\nabla_x\partial_x^\alpha a\|^2 \lesssim &\sum_{|\alpha|\le 1}\|\nabla_x\partial_x^\alpha b\|^2+\sum_{|\alpha|\le 1}\|\nabla_x\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2\\ &+\sum_{|\alpha|\le 1}\|\partial_x^\alpha b\|^2+\mathcal{E}^{\frac12}(t)\mathcal{D}(t).\end{aligned} \tag{3.13}$$

Proof. Firstly, multiplying $(1.2)_1$ by $\mu^{\frac12}$ and $v\mu^{\frac12}$ respectively, then integrating with respect to v over $\mathbb{R}^3$ to obtain

$$\partial_t a+\nabla_x\cdot b=0, \tag{3.14}$$

and

$$\partial_t b+\nabla_x a+\nabla_x\cdot\langle v\otimes v\mu^{\frac12},\{\mathbf{I}-\mathbf{P}\}f\rangle+(a+1)b=0. \tag{3.15}$$

Secondly, taking ∂_x^α of (3.15) for $|\alpha|\le 1$ to get

$$\partial_x^\alpha\partial_t b+\nabla_x\partial_x^\alpha a+\nabla_x\cdot\langle v\otimes v\mu^{\frac12},\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\rangle+\partial_x^\alpha(ab)+\partial_x^\alpha b=0.$$

Multiply the above equation by $\nabla_x\partial_x^\alpha a$ and integrate with respect to x to obtain

$$\begin{aligned}\|\nabla_x\partial_x^\alpha a\|^2=&-\int_{\mathbb{R}^3}\partial_x^\alpha\partial_t b\nabla_x\partial_x^\alpha a dx-\int_{\mathbb{R}^3}\nabla_x\cdot\langle v\otimes v\mu^{\frac12},\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\rangle\nabla_x\partial_x^\alpha a dx\\ &-\int_{\mathbb{R}^3}\partial_x^\alpha(ab)\nabla_x\partial_x^\alpha a dx-\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx.\end{aligned} \tag{3.16}$$

Using (3.14) to get

$$\begin{aligned}\int_{\mathbb{R}^3}\partial_x^\alpha\partial_t b\nabla_x\partial_x^\alpha a dx&=\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx-\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha\partial_t a dx\\ &=\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx-\|\nabla_x\partial_x^\alpha b\|^2.\end{aligned} \tag{3.17}$$

By Young's inequality, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\nabla_x\cdot\langle v\otimes v\mu^{\frac12},\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\rangle\nabla_x\partial_x^\alpha a dx\Big|&\lesssim\int_{\mathbb{R}^3}|\nabla_x\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f|_{L_v^2}|\nabla_x\partial_x^\alpha a|dx\\ &\lesssim\eta\|\nabla_x\partial_x^\alpha a\|^2+C_\eta\|\nabla_x\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2,\end{aligned} \tag{3.18}$$

and

$$\Big|\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx\Big|\lesssim\eta\|\nabla_x\partial_x^\alpha a\|^2+C_\eta\|\partial_x^\alpha b\|^2, \tag{3.19}$$

where $\eta>0$ is a sufficiently small universal constant and $C_\eta>0$. Using Sobolev embedding in Lemma 3.1 to derive

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\partial_x^\alpha ab\nabla_x\partial_x^\alpha a dx\Big|&\lesssim\|\partial_x^\alpha a\|_{L_x^2}\|b\|_{L_x^\infty}\|\nabla_x\partial_x^\alpha a\|_{L_x^2}\\ &\lesssim\|\partial_x^\alpha a\|_{L_x^2}\|\nabla_x b\|_{H_x^1}\|\nabla_x\partial_x^\alpha a\|_{L_x^2}\lesssim\mathcal{E}^{\frac12}(t)\mathcal{D}(t),\end{aligned}$$

and

$$\Big|\int_{\mathbb{R}^3} a\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx\Big| \lesssim \|a\|_{L_x^\infty}\|\partial_x^\alpha b\|_{L_x^2}\|\nabla_x\partial_x^\alpha a\|_{L_x^2}$$
$$\lesssim \|\nabla_x a\|_{H_x^1}\|\partial_x^\alpha b\|_{L_x^2}\|\nabla_x\partial_x^\alpha a\|_{L_x^2} \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Thus we can obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \partial_x^\alpha(ab)\nabla_x\partial_x^\alpha a dx\Big| &= \Big|\int_{\mathbb{R}^3}(\partial_x^\alpha ab + a\partial_x^\alpha b)\nabla_x\partial_x^\alpha a dx\Big| \\ &\le \int_{\mathbb{R}^3}|\partial_x^\alpha a||b||\nabla_x\partial_x^\alpha a|dx + \int_{\mathbb{R}^3}|a||\partial_x^\alpha b||\nabla_x\partial_x^\alpha a|dx \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned} \tag{3.20}$$

Combining (3.17)–(3.20) with (3.16) to derive, for $|\alpha| \le 1$

$$\begin{aligned}\frac{d}{dt}\sum_{|\alpha|\le1}\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx + \sum_{|\alpha|\le1}\|\nabla_x\partial_x^\alpha a\|^2 &\lesssim \sum_{|\alpha|\le1}\|\nabla_x\partial_x^\alpha b\|^2 + \sum_{|\alpha|\le1}\|\nabla_x\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2 \\ &\quad + \sum_{|\alpha|\le1}\|\partial_x^\alpha b\|^2 + \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t),\end{aligned}$$

where we take $\eta > 0$ sufficiently small enough. Thus the proof of Lemma 3.3 is completed.

4. Global existence

This section is devoted to proving our main result based on the continuation argument. First, we need to close the a priori estimate.

Proposition 4.1. *There is a small positive constant $M > 0$ such that if*

$$\sup_{0\le t\le T}\mathcal{E}(f(t)) \le M$$

for any $0 < T < \infty$, then it holds that

$$\frac{d}{dt}\mathcal{E}(t) + \mathcal{D}(t) \le 0. \tag{4.1}$$

Proof. Taking the linear combination (3.1) + $\kappa \times$ (3.13) with $\kappa > 0$ sufficiently small to get

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{|\alpha|\le2}\|\partial_x^\alpha f\|^2 + \kappa\sum_{|\alpha|\le1}\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx\Big) + \kappa\sum_{|\alpha|\le1}\|\nabla_x\partial_x^\alpha a\|^2 \\ &+ \lambda_0\sum_{|\alpha|\le2}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + \sum_{|\alpha|\le2}\|\partial_x^\alpha b\|^2 \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).\end{aligned} \tag{4.2}$$

Noticing that

$$\sum_{|\alpha|\le1}\Big|\int_{\mathbb{R}^3}\partial_x^\alpha b\nabla_x\partial_x^\alpha a dx\Big| \le \frac{1}{2}\sum_{|\alpha|\le1}[\|\partial_x^\alpha b\|^2 + \|\nabla_x\partial_x^\alpha a\|^2] \le \sum_{|\alpha|\le2}\|\partial_x^\alpha f\|^2,$$

then we have

$$-\kappa \sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2 \le \kappa \sum_{|\alpha|\le 1} \int_{\mathbb{R}^3} \partial_x^\alpha b \nabla_x \partial_x^\alpha a dx \le \kappa \sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2,$$

i.e.,

$$(1-\kappa) \sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2 \le \sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2 + \kappa \sum_{|\alpha|\le 1} \int_{\mathbb{R}^3} \partial_x^\alpha b \nabla_x \partial_x^\alpha a dx \le (1+\kappa) \sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2.$$

Consequently, let $\kappa > 0$ be small enough, it holds

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2 + \kappa \sum_{|\alpha|\le 1} \int_{\mathbb{R}^3} \partial_x^\alpha b \nabla_x \partial_x^\alpha a dx \sim \sum_{|\alpha|\le 2} \|\partial_x^\alpha f\|^2 \sim \mathcal{E}(t).$$

By (4.2) and the definition of $\mathcal{D}(t)$ (2.2), it derives to

$$\frac{d}{dt}\mathcal{E}(t) + \mathcal{D}(t) \lesssim \mathcal{E}^{\frac{1}{2}}(t)\mathcal{D}(t).$$

Finally, choosing $M > 0$ to be small enough, then the desired estimate (4.1) is obtained.

Proof of Theorem 2.1. Firstly, the local-in-time existence and uniqueness of the solutions to the Cauchy problem (1.2) can be established by performing the standard arguments as in [13]. To extend the local solution into the global one, we can deduce that

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s) ds \le \mathcal{E}(0),$$

from (4.1) in Proposition 4.1 by virtue of the smallness assumption on $\mathcal{E}(0)$. Combining this with the local existence, the global existence of solution and uniqueness follows immediately from the standard continuity argument. This completes the proof of the global existence and the uniform estimate of Theorem 2.1.

5. Conclusions

This paper proves the global existence to the Cauchy problem on a nonlinear Fokker- Planck equation near Maxwellian with small-amplitude initial data by a refined nonlinear energy method. And the regularity assumption on the initial data is much weaker by virtue of the Sobolev embedding inequalities.

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Conflict of interest

The authors declare that they have no competing interests.

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