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*Research article*

# Analysis of fractional stochastic evolution equations by using Hilfer derivative of finite approximate controllability

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**Abstract:** The approximate controllability of a class of fractional stochastic evolution equations (FSEEs) are discussed in this study utilizes the Hilbert space by using Hilfer derivative. For different approaches, we remove the Lipschitz or compactness conditions and merely have to assume a weak growth requirement. The fixed point theorem, the diagonal argument, and approximation methods serve as the foundation for the study. The abstract theory is demonstratedusing an example. A conclusion is given at the end.

**Keywords:** stochastic calculus; Hilfer derivative; finite approximate controllability; semi-group; fixed point theory; stochastic evolution equations
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## 1. Introduction

Fractional calculus, fractional differential equations (FDEs) and the qualitative theory of these equations have all been included in the discipline of mathematical analysis over the past three decades, both on a theoretical level and in terms of its practical applications. Fundamentally, the theory of fractional calculus, the qualitative theory of fractional differential and fractional integro-differential equations, their numerical simulations, and symmetry analysis are mathematical analysis tools used to study arbitrary order integrals and derivatives, which unifies and generalizes the traditional notions of differentiation and integration. Nonlinear operators with fractional order are more practical than classical formulations. Numerous scientific disciplines, including fluid mechanics, physics, viscoelasticity, biology, chemistry, signal processing, dynamical systems, entropy theory, and others, can involve the qualitative theory of FDEs and fractional order operators. Because of this, the applications of the theory of fractional calculus and the qualitative theory of the aforementioned equations have drawn the attention of academics throughout the world, and many scholars have included them in their latest study.

The applicability of controllability in numerous disciplines of science and engineering for various types of linear and nonlinear dynamical systems has been taken into consideration in many publications using diverse methodologies, see [1]. It is to be stressed that there are numerous definitions of controllability for dynamical systems, including approximate controllability, exact controllability, null controllability, and others. See [2] for a comprehensive list of publications on the approximate controllability of semilinear evolution systems in abstract spaces. For further information about the exact controllability of differential control systems by various writers, check [3].

Due to their effective applications to issues in electricity, mechanics, physics, economics, and several fields of engineering, stochastic differential equations (SDEs) have gained a lot of attention in recent years. See [4] and its references for further information. Researchers specifically looked at the controllability of stochastic dynamical control systems in infinite dimensional spaces; for further information. The study of controllability issues for SEEs with nonlocal conditions has, however, received very little attention; for examples, see [5], where the authors assume that nonlocal item g is a fully continuous map. A coherent theory of integration for stochastic process integrals with respect to stochastic processes can be defined owing to the subject of mathematics that interacts with stochastic processes. It is employed to model systems with unexpected responses.

Nabulsi et al. [6] worked on the Vlasov equation, waves and dispersion relations in fractal dimensions. Montangero et al. [7] studied Loop-free tensor network, Nabulsi et al. [8, 9] a mapping from Schrodinger equation and [10] quantum effects in metal oxide semiconductor field effect transistor in fractal dimensions. Alfaro et al. [11] worked on the modelling of spatially heterogeneous nonlocal diffusion. Physics serves as a significant source of inspiration for researching fractional evolution equations. The fractional derivative in space and time is a component of fractional diffusion equations, which are abstract partial DEs. The fractional order diffusion-wave equation, for instance, was covered by EI-Sayed [12]. For the fractional diffusion equation, Eidelman and Kochubei [13] looked at the Cauchy problem. Fractal anomalous diffusion is described by fractional diffusion equations, as mentioned in [13]. A useful tool for describing the memory and inherited qualities of diverse materials and processes can be found in this class of equations. In order to approximate Brownian motion, Louis Bachelier and Albert Einstein developed the Wiener process, which bears Norbert Wiener's name. Since the 1970s, the Wiener process has been extensively employed in financial mathematics and economics to explain how stock values and bond interest rates have changed over time. Stochastic processes are a collection of random variables used to model the evolution of a system over time. Stochastic processes involve an element of randomness or uncertainty. In probability theory and similar topics, a stochastic or random process is typically described as a family of random variables. The issue of mild solutions for abstract DEs with fractional derivatives was examined in a few recent studies [14–16]. Since the description of mild solutions obtained from deviations of constant formulas in integer order abstract DEs cannot be effectively summarized to fractional order abstract DEs, Zhou and Jiao [17] give a suitable idea on mild solutions by using the laplace transform and probability density functions for the evolution equation with the Caputo fractional derivative. Zhou et al. [18] provided a suitable notion on mild solutions for the evolution problem with the RL fractional derivative utilizing the same methodology.

The Cauchy problems for linear and semilinear time fractional evolution equations with almost-sectorial operators was studied by Wang [19]. The generalized RL fractional derivative, or Hilfer fractional derivative (HFD), was proposed by Hilfer [20] and incorporates the RL fractional derivative as well as the Caputo fractional derivative. In the theoretical simulation of dielectric relaxation in glass-forming materials, this operator was present. In [21], Furati et al. discussed an initial value issue for a family of HFD based nonlinear FDEs. In [22], the Mittag-Leffler functions and Fox's H-function were used to derive the solution of a fractional diffusion equation with a HFD. To the best of our knowledge, the evolution equations using HFD have no results. Abuasbeh et al. [23–27] worked on Time-Fractional Initial Boundary Value Problems and fractional differential equation. Mouman et al. [28,29] studied simpson type inequalities and fractional integral pantograph differential equations. Boulares et al. [30] worked on fractional pantograph problems using fixed point theory and Ghafli et al. [31] studied topological structure of fractional control delay problem. Shafqat et al. [32] investigated the mild solution for the Navier-Stokes equation. Let $(\Omega_{pr}, \mathcal{G}, \{\mathcal{G}_r\}_{r\geq 0}, \mathbf{R^m})$ be a filtered complete probability space that meets the normal requirement which means that the filter is a right continuous increasing family and $\mathcal{G}_0$ contains all P-null sets. Let $\{e_k, k \in \mathbb{N}\}$ be a complete orthonormal basis of $\mathcal{K}$. $\{\mathcal{W}(r) : r \geq 0\}$ is a cylinderical K-valued Brownian motion or Weiner process specified on the probability space $(\Omega_{pr}, \mathcal{G}, \{\mathcal{G}_r\}_{r\geq 0}, \mathbf{R^m})$. We denote $\Im\mathrm{r}(Q) = \sum_{k=1}^{\infty} \alpha_k = \alpha < \infty$ with the finite trace nuclear covariance operator $Q \geq 0$ which satisfies that $Qe_k = \alpha_k e_k, k \in \mathbb{N}$.

Let $\{\mathcal{W}_k(r), k \in \mathbb{N}\}$ be a sequence of one-dimensional standard Weiner processes that are mutually independent on $\left(\Omega_{pr}, \mathcal{G}, \{\mathcal{G}_r\}_{r\geq 0}, \mathbf{R^m}\right)$ to the extent that

$$\mathcal{W}(r) = \sum_{k=1}^{\infty} \sqrt{\alpha_k}\mathcal{W}_k(r)\, e_k, r \geq 0.$$

Suppose that a finite trace nuclear covariance operator $\aleph \geq 0$ and a K-valued Brownian motion or Weiner process $\mathcal{W}(r) : r \geq 0$ distinct in filtered complete probability space $(\Omega_{pr}, \mathcal{G}, \{\mathcal{G}_r\}_{r\geq 0}, \mathbf{R^m})$.

In this paper, the FSEEs were used as a Hilfer derivative of finite approximate controllability as

$$\begin{cases} \mathcal{D}_{0^+}^{\upsilon,\mu}\chi(r) = \mathcal{A}\chi(r) + \mathcal{G}(r,\chi(r)) + \varpi(r,\chi(r))\frac{dw(r)}{dr} + \mathfrak{D}u(r), r \in (0, c], \\ I_{0^+}^{(1-\upsilon)(1-\mu)}\chi(0) = \chi_0, \end{cases} \tag{1.1}$$

where$\mathcal{D}_{0^+}^{\upsilon,\mu}$ is the HFD of order $0 < \upsilon \leq 1$ and $0 < \mu \leq 1$.

$\hbar$ and $\mathcal{K}$ are two separable Hilbert spaces (HS) and the state $\chi(.)$ takes its values in $\hbar$. Also, a closed linear operator $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \subset \hbar \to \hbar$ and infinitesimal generator -$\mathcal{A}$ of a $C_0 - semigroup$ $\Im(r)(r \geq 0)$ on $\hbar$.

The control function $u(.)$ is a Banach space (BS) of admissible control functions on a separable HS $\mathcal{U}$. The first innovative aspect of this article is that the methods employed in [33] are invalid for the current work since the nonlocal term g(x), defined by 1.1, depends on every value of x over the entire interval [0, b]. Under weaker conditions when $\chi_0$ lacks Lipschitz continuity or compactness, the approximate controllability conclusions are established using stochastic analysis, approximation techniques, a diagonal argument and Schauder fixed-point theorem. To put it more explicitly, the nonlocal term $\chi_0$ has just continuity and a few weak growth conditions, and it depends on every value of x throughout the whole interval [0, b]. The theorems discovered here build upon and complete those discovered in [34–37]. The second novel aspect of this article is that while stochastic dynamical systems are the focus of our investigation, the techniques we employ in this paper can be extended to investigate the approximate controllability of deterministic systems under initial conditions by appropriately utilizing abstract space and norm. The analogous findings for deterministic systems with local conditions are likewise novel. The third novelty, we use the HFD to find our results. As a result, only a few papers have been published on the study of controllability for equation 1.1. This study uses the Hilfer derivative to examine the fractional stochastic evolution equations' finite approximate controllability. Though the majority of them were first order differential equations, some researchers found FDE results in the literature. In our study, we obtained the results for Hilfer derivatives of order (0,1). In differential equation theory, stability plays a crucial role in both theory and application. As a result, controllability is a major area of research, and over the past 20 years, research articles on controllability for FDE have been published. We employ the fixed point theorem and the approximation technique with fractional stochastic evolution equations. Because of its wide range of applications in areas of science such engineering, robotics, mechanics, control, thermal systems, electrical, and signal processing, the theory of fractional stochastic evolution equation continues to attract researchers' attention. In Section 2, we offer some basic concepts and definitions that will be helpful in the whole work. We find a moderately mild solution to the system 1.1 in Section 3. In Section 4, we state and demonstrate the approximate finite controllability of the system 1.1. In Section 5, we use an example to explain our findings. In the last section, we illustrate the conclusion.

## 2. Preliminaries

We go over the notations, definitions, and introductions that will be utilised in this work in this portion.

**Lemma 2.1.** *[38]* If $\imath : [0,c] \times \hbar \to \mathcal{L}(\mathcal{K},\hbar)$ is continuous and $\chi \in \mathcal{B}\left([0,c], \mathcal{L}^2\left(\Omega_{pr},\hbar\right)\right)$ then

$$\mathrm{E}\left\|\int_{[0,c]} \imath(r,\chi(r))\, d\mathcal{W}(r)\right\|^2 \leq \Im\mathrm{r}(Q)\int_{[0,c]} \mathrm{E}\,\|k(r,\chi(r))\|^2 dr.$$

**Definition 2.1.** *[39]* For a mapping $z : (0,+\infty) \to \mathbf{R^m}$ of order $\mu > 0$, the RL fractional integral is given by

$$I_0^{\mu}\mathcal{Y}(r) = \frac{1}{\Gamma(\beta)}\int_0^r (r-\zeta)^{\mu-1}\mathcal{Y}(\zeta)\, d\zeta,$$

if the R.H.S is described pointwise on $0 < r < \infty$.

**Definition 2.2.** *[40]* For a function $\mathcal{Y} : [0, +\infty) \to \mathbf{R^m}$ of order $\mu > 0$, the RLF derivative is given by

$$\mathcal{D}_0^{\mu}\mathcal{Y}(r) = \frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dr}\right)^m \int_0^r \frac{\mathcal{Y}(\zeta)}{(r-\zeta)^{\mu-m+1}} d\zeta,$$

where $m = [\mu] + 1$, as long as the R.H.S is specified point-wise on $0 < r < \infty$.

**Definition 2.3.** *[40]* The CFD of a function of order $\mu > 0$ of a function z :$[0, +\infty) \to \mathbf{R^m}$ is given by

$${}^c\mathcal{D}_0^{\mu} z(r) = \mathcal{D}_0^{\mu}\left[z(\chi) - \sum_{k=0}^{m-1} \frac{r^k}{k!}\mathcal{Y}^{(k)}(0)\right],$$

where $m = [\mu] + 1$, as long as the R.H.S is described pointwise on $0 < r < \infty$.

**Definition 2.4.** *(HFD, see [20])* The generalized RL fractional derivative of order $0 \le \upsilon \le 1$ and $0 < \mu < 1$ with lower limit $a$ is defined as

$$D_{0^+}^{\upsilon,\mu} f(r) = I_{a^+}^{\upsilon(1-\mu)} \frac{d}{dr} I_{a^+}^{(1-\upsilon)(1-\mu)} f(r),$$

for operations that result in the existence of the phrase on the right.

**Remark 2.1.** (i) When $\upsilon = 0, 0 < \mu < 1$ and $a = 0$, the HFD corresponds to the classical RL fractional derivative:

$$D_{0^+}^{0,\mu} f(r) = \frac{d}{dr} I_{0^+}^{1-\mu} f(r) = D_{0^+}^{\mu} f(r).$$

(ii) When $\upsilon = 0, 0 < \mu < 1$ and $a = 0$, the HFD corresponds to the classical Caputo fractional derivative:

$$D_{0^+}^{1,\mu} f(r) = I_{0^+}^{1-\mu} \frac{d}{dr} f(r) =^c D_{0^+}^{\mu} f(r).$$

**Remark 2.2.** (i) If $f \in \mathcal{B}^n [0, +\infty)$, then

$${}^c\mathcal{D}_0^{\mu} f(r) = \frac{1}{\Gamma(m-\mu)} \int_0^r \frac{f^{(m)}(\zeta)}{(r-\zeta)^{\beta-m+1}} d\zeta = I_0^{m-\mu} f^{(m)}(r).$$

(ii) If z(r) is an abstract function with values in **E**, then the integrals that exist in Definitions 2.2–2.4 are defined in Bochner's interpretation.
(iii) A constant's Caputo derivative is 0.

**Lemma 2.2.** The Cauchy problem 1.1 is equivalent to the integral equation

$$\begin{aligned} \chi(r) &= \frac{\chi_0}{\Gamma(\upsilon(\mu)+\mu)} r^{(\upsilon-1)(1-\mu)} + \frac{1}{\Gamma(\mu)} \int_0^r (r-\zeta)^{\mu-1} [\mathcal{A}\chi(\zeta) + \mathcal{G}(\zeta, \chi(\zeta) + \mathfrak{D}u(\zeta))] d\zeta \\ &\quad + \int_0^r (r-\zeta)^{\mu-1} \varpi(\zeta, \chi(\zeta)) dW(\zeta), \quad r \in (0,1). \end{aligned}$$

The Wright function $M_\mu(\sigma)$ is defined by

$$M_\mu(\sigma) = \sum_{n=1}^{\infty} \frac{(-\sigma)^{m-1}}{(m-1)!\Gamma(1-\mu m)}, 0 < \mu < 1, \sigma \in C,$$

which satisfies the following equality:

$$\int_0^\infty \vartheta^\delta M_\mu(\vartheta) d\vartheta = \frac{\Gamma(1+\delta)}{\Gamma(1+\mu\delta)},\ for\ \vartheta \geq 0.$$

**Lemma 2.3.** *[41]* The operators $K_\mu(r)(r \geq 0)$ and $\mathfrak{S}_\mu(r)(r \geq 0)$ validate the below properties:

(i) For each fixed $r > 0$ , both operators are linear and bounded in $\hbar$ for each $\chi \in \hbar$

$$\left\|K_\mu(r)\chi\right\| \leq \mathcal{M}\|\chi\|, \left\|\mathfrak{S}_\mu(r)\chi\right\| \leq \frac{\mathcal{M}}{\Gamma(\mu)}\|\chi\|. \tag{2.1}$$

(ii) For all $\chi \in \hbar$, $r \to K_\mu(r)\chi$ and $r \to \mathfrak{S}_\mu(r)\chi$, are continuous functions from $[0,\infty)$ into $\mathcal{H}$, respectively.

(iii) These operators exhibit strong continuous behaviour.

(iv) These operators are norm-continuous if a semi-group $\mathfrak{I}(r)$ is compact as well as being likewise compact in $\hbar$, for $r > 0$.

**Definition 2.5.** Let $u \in \mathcal{L}^2_{\mathcal{G}}([0,c], \mathcal{U})$ then the mild solution on $[0,c]$ is claimed to be a stochastic process $\chi$ if $\chi \in \hbar\left([0,c], \mathcal{L}^2\left(\Omega_{pr}, \hbar\right)\right)$ and

(i) $\chi(r)$ is a measurable variable that has been adapted to $\mathcal{F}_r$.

(ii) The integral equation below is satisfied by the value $\chi(r)$:

$$\chi(r) = \mathfrak{S}_{\upsilon,\mu}(r)\chi_0 + \int_0^r K_\mu(r-\zeta)[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\zeta + \int_0^r K_\mu(r-\zeta)\varpi(\zeta,\chi(s))dW(\zeta).$$

**Lemma 2.4.** *[42] (Holder inequality)* Assume that $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(J, \mathbf{R^m})$, $g \in L^q(J, \mathbf{R^m})$, then $fg \in L^1(J, \mathbf{R^m})$ and

$$\|fg\|_{LJ} \leq \|f\|_{L^pJ}\|g\|_{L^q}.$$

**Definition 2.6.** *[43]*

(a) The system 1.1 is approximately controllable on the interval $[0,c]$ if $\overline{\mathbf{R^m}(c)} = \mathcal{L}^2(\Omega_{pr}, \hbar)$.

(b) For $\chi_c \in \mathcal{L}^2(\Omega_{pr}, \hbar)$ and $\epsilon > 0$ the system 1.1 is called finitely-approximately controllable on interval $[0,c]$ if $\exists$ a control $u_\epsilon \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{J}, \mathcal{U})$ such that the corresponding solution $\chi(c; u_\epsilon)$ of 1.1 satisfies the conditions:

$$\begin{aligned} \mathbf{E}\|\chi(c;u_\epsilon) - \chi_c\| &< \epsilon, \\ \pi_{\mathbf{E}}\chi(c;u_\epsilon) &= \pi_{\mathbf{E}}\chi_c. \end{aligned}$$

To demonstrate the main finding, we require the following limitations:

($\wp_1$) There is a Caratheodory continuous function $\mathcal{G} : [0, c] \times \hbar \to \hbar$, as well as a function $\xi_{\mathcal{G}} \in \mathcal{L}([0, c], \mathbf{R^{m+}})$ and nondecreasing continuous function $\psi_{\mathcal{G}} : \mathbf{R^{m+}} \to \mathbf{R^{m+}}$,

$$\mathbf{E}\|\mathcal{G}(r, \chi)\|^2 \leq \xi_{\mathcal{G}}(r)\psi_{\mathcal{G}}\left(\mathrm{E}\|\chi\|^2\right), a.e.t \in [0, c], \forall \chi \in \hbar.$$

($\wp_2$) There is a Caratheodory continuous function $\varpi : [0, c] \times \hbar \to \mathcal{L}_2^0$ as well as a function $\xi_{\varpi} \in \mathcal{L}^{\frac{1}{p}}([0, c], \mathbf{R^{m+}})$ for constant $p \in (0, 2\mu - 1)$ and a nondecreasing continuous function $\psi_{\varpi} : \mathbf{R^{m+}} \to \mathbf{R^{m+}}$ such that

$$\mathbf{E}\|\varpi(r, \chi)\|^2_{\mathcal{L}_2^0} \leq \xi_{\varpi}(r)\psi_{\varpi}\left(\mathbf{E}\|\chi\|^2\right), a.e.t \in [0, c], \forall \chi \in \hbar.$$

($\wp_3$) There is a Caratheodory continuous function $h : [0, c] \times \mathcal{H} \to \hbar$ as well as a function $\xi_h \in L([0, c], \mathbf{R^{m+}})$ and a nondecreasing continuous function $\psi_h : \mathbf{R^{m+}} \to \mathbf{R^{m+}}$ such that

$$\mathbf{E}\|h(r, \chi)\|^2 \leq \xi_h(r)\psi_h\left(\mathbf{E}\|\chi\|^2\right), a.e.t \in [0, c], \forall \chi \in \hbar.$$

($\wp_4$) The linear fractional differential system

$$\chi(r) = \mathfrak{S}_{\upsilon,\mu}(r)\chi_0 + \int_0^r K_\mu(r - \zeta)\mathfrak{D}u(\zeta)d\zeta \tag{2.2}$$

is approximately controllable in $[0, c]$.

It is understood that system approximately controllable on $[0, c]$ if the condition $\mathfrak{D}^* K_\mu^* (c - \zeta)\phi = 0, 0 \leq s \leq c$ implies that $\phi = 0$.

For any $\epsilon > 0$, we define an essential functional $\phi \in \hbar$,

$$\mathcal{J}_\epsilon(\phi, \chi) = \tfrac{1}{2}\int_0^c (c - \zeta)^{\mu-1}\mathrm{E}\left\|\mathfrak{D}^* K_\mu^*(c - \zeta)\phi\right\|^2 d\zeta + \epsilon\left(\mathbf{E}\|(I - \pi_\epsilon)\phi\|^2\right)^{\frac{1}{2}} - \mathbf{E}(\phi, \rho(r)), \tag{2.3}$$

where

$$\rho(\chi) = \chi_c - \mathfrak{S}_{\upsilon,\mu}(c)\chi_0 + \int_0^c K_\mu(c - \zeta)\mathcal{G}(\zeta, \chi(\zeta))d\zeta + \int_0^c K_\mu(c - \zeta)\varpi(\zeta, \chi(\zeta))dW(\zeta).$$

## 3. Definition of mild solution

In this section, we find the mild solution of system 1.1.

The system 1.1 is equivalent to the following integral equation:

$$\begin{aligned}\chi(r) = &\tfrac{\chi_0}{\Gamma(\upsilon(1-\mu)+\mu)} r^{(\upsilon-1)(1-\mu)} + \int_0^r (r - \zeta)^{\mu-1}[\mathcal{A}\chi(\zeta) + \mathcal{G}(\zeta, \chi(\zeta) + \mathfrak{D}u(\zeta))]d\zeta \\ &+ \int_0^r (r - \zeta)^{\mu-1}\varpi(\zeta, \chi(\zeta))dW(\zeta).\end{aligned} \tag{3.1}$$

**Theorem 3.1.** If Eq 3.1 holds, then

$$\chi(r) = \mathfrak{S}_{\upsilon,\mu}(r)\chi_0 + \int_0^r K_\mu(r - \zeta)[\mathcal{G}(\zeta, \chi(\zeta) + \mathfrak{D}u(\zeta))]d\zeta + \int_0^r K_\mu(r - \zeta)\varpi(\zeta, \chi(\zeta))\,dW(\zeta), \tag{3.2}$$

where

$K_\mu(r) = r^{\mu-1}P_\mu(r), P_\mu(r) = \int_0^\infty \mu\vartheta M_\mu(\vartheta)Q(r^\mu\vartheta)d\vartheta$ and $S_{\upsilon,\mu}(r) = I_{0^+}^{\upsilon(1-\mu)}K_\mu(r)$.

*Proof.* Let $\lambda > 0$. Taking laplace transform on both sides of Eq 3.1, we get

$$\begin{aligned} v(\lambda) &= \lambda^{(1-\upsilon)(1-\mu)}(\chi_0) + \frac{1}{\lambda^{\mu}}\mathcal{A}\chi(\lambda) + \frac{1}{\lambda^{\mu}}\omega(\lambda) + \frac{1}{\lambda}\varsigma(\lambda) \\ &= \lambda^{\upsilon(\mu-1)}\int_0^{\infty} e^{-\lambda\mu_\zeta}Q(\zeta)\chi_0 d\zeta + \int_0^{\infty} e^{-\lambda\mu_\zeta}Q(\zeta)\omega(\lambda)d\zeta + \int_0^{\infty} e^{-\lambda\mu_\zeta}Q(\zeta\varsigma(\lambda))d\zeta \qquad (3.3) \end{aligned}$$

provided that the integrals in (3.3) exist, where I is the identity operator defined on X. Let

$$\psi_\mu(\theta) = \frac{\mu}{\theta^{\mu+1}}M_\mu(\theta)^{-\mu},$$

whose Laplace transform is given by

$$\int_0^{\infty} e^{-\lambda\theta}\psi_\mu(\theta)d\theta = e^{-\lambda^{\mu}},\ where\ \mu \in (0,1). \qquad (3.4)$$

Using (3.4), we have

$$\begin{aligned} \int_0^{\infty} e^{-\lambda\mu_\zeta}Q(\zeta)\chi_0 d\zeta &= \int_0^{\infty} \mu r^{\mu-1}e^{-(\lambda r)^{\mu}}Q(r^{\mu})\chi_0 dr \\ &= \int_0^{\infty}\int_0^{\infty} \mu\psi_\mu(\theta)e^{-\lambda r\theta}Q(r^{\mu})r^{\mu-1}\chi_0 d\theta dr \\ &= \int_0^{\infty}\int_0^{\infty} \mu\psi_\mu(\theta)e^{-\lambda r}Q\Big(\frac{r^{\mu}}{\theta^{\mu}}\Big)\frac{r^{\mu-1}}{\theta^{\mu}}\chi_0 d\theta dr \\ &= \int_0^{\infty} e^{-\lambda r}\Big[\mu\int_0^{\infty}\psi_\mu(\theta)Q\Big(\frac{r^{\mu}}{\theta^{\mu}}\Big)\frac{r^{\mu-1}}{\theta^{\mu}}\chi_0 d\theta\Big]dr \\ &= \int_0^{\infty} e^{-\lambda r}r^{\mu-1}P_\mu(r)\chi_0 dr, \qquad (3.5) \\ \int_0^{\infty} e^{-\lambda_\zeta^{\mu}}Q(\zeta)\omega(\lambda)d\zeta &= \int_0^{\infty} \mu r^{\mu-1}e^{-(\lambda r)^{\mu}}Q(r^{\mu})e^{-\lambda\zeta}[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\zeta dr \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mu r\psi_\mu(\theta)e^{-(\lambda r\theta)}Q(r^{\mu})e^{-\lambda\zeta}r^{\mu-1}[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\theta d\zeta dr \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mu r\psi_\mu(\theta)e^{-(\lambda r\theta)}Q\Big(\frac{r^{\mu}}{\theta^{\mu}}\Big)\frac{r^{\mu-1}}{\theta^{\mu}}[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\theta d\zeta dr \\ &= \int_0^{\infty} e^{-\lambda r}\Big[\mu\int_0^{r}\int_0^{\infty}\psi_\mu(\theta)Q\Big(\frac{(r-\zeta)^{\mu}}{\theta^{\mu}}\Big)\frac{(r-\zeta)^{\mu-1}}{\theta^{\mu}}[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\theta d\zeta\Big]dr \\ &= \int_0^{\infty} e^{-\lambda r}\Big[\int_0^{r}(r-\zeta)^{\mu-1}P_\mu(r-\zeta)\frac{(r-\zeta)^{\mu-1}}{\theta^{\mu}}[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\zeta\Big]dr, \quad (3.6) \\ \lambda^{\mu-1}\int_0^{\infty} e^{-\lambda^{\mu}\zeta}Q(\zeta)\varsigma(\lambda)d\zeta &= \lambda^{\mu-1}\int_0^{\infty} e^{-\lambda^{\mu}\zeta}Q(\zeta)e^{-\lambda\zeta}\mathfrak{G}(\zeta,\chi(\zeta))dW(\zeta) \\ &= \lambda^{\mu-1}\int_0^{\infty}\int_0^{\infty} \mu r^{\mu-1}e^{-(\lambda r)^{\mu}}Q(r^{\mu})e^{-\lambda\zeta}\mathfrak{G}(\zeta,\chi(\zeta))d\zeta dW(r) \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \omega\psi_\beta(\omega)e^{-\lambda r\omega}Q(r^{\mu})e^{-\lambda\zeta}\mathfrak{G}(\zeta,\chi(\zeta))d\omega d\zeta dW(r). \qquad (3.7) \end{aligned}$$

Since the Laplace inverse transform of $\lambda^{\upsilon(\mu-1)}$ is

$$\mathcal{L}^{-1}(\lambda^{\upsilon(\mu-1)}) = \begin{cases} {}^{c}_{0}\frac{r^{\upsilon(1-\mu)-1}}{\Gamma(\upsilon(1-\mu))}, 0 < \upsilon \leq 1, \\ \delta(r), \upsilon = 0, \end{cases}$$

where $\delta(r)$ is the Delta function. Thus, by 3.3, 3.6 and 3.7 in Eq 3.2, we get

$$\begin{aligned} \chi(r) &= (L^{-1}(\lambda^{\upsilon(\mu-1)} * K_\mu(r))\chi_0 + \int_0^r K_\mu(r-\zeta)[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\zeta + \int_0^r K_\mu(r-\zeta)\varpi(\zeta,\chi(\zeta))\,dW(\zeta) \\ &= (I_{0^+}^{\upsilon(1-\mu)}K_\mu(r))\chi_0 + \int_0^r K_\mu(r-\zeta)[\mathcal{G}(\zeta,\chi(\zeta) + \mathfrak{D}u(\zeta))]d\zeta + \int_0^r K_\mu(r-\zeta)\varpi(\zeta,\chi(\zeta))\,dW(\zeta) \\ &= \mathfrak{S}_{\upsilon,\mu}(r)\chi_0 + \int_0^r K_\mu(r-\zeta)[\mathcal{G}(\zeta,\chi(s) + \mathfrak{D}u(\zeta))]d\zeta + \int_0^r K_\mu(r-\zeta)\varpi(\zeta,\chi(\zeta))\,dW(\zeta). \end{aligned} \tag{3.8}$$

This completes the proof.

## 4. Results on finite approximate controllability

**Lemma 4.1.** Assume that the assumptions $(\wp_1) \to (\wp_3)$ are satisfied. Then the following conclusions hold:

(1) In $\mathcal{B}_g$, $\rho$ is continuous.
(2) $\left\{\rho(\chi) : \chi \in \mathcal{B}_g\right\}$ is relatively compact in $\hbar$.

*Proof.* Let $\chi_m \to \chi$ in $\mathcal{B}_g$, then we have
$\mathcal{G}(r,\chi_m(r)) \to \mathcal{G}(r,\chi(r)), \varpi(r,\chi_m(r)) \to \varpi(r,\chi(r)), h(r,\chi_m(r)) \to h(r,\chi(r))\,(n \to \infty)$.
Furthermore, for any $r \in [0,c]$, using Holder Inequality 2.4 and Lebesgue dominated convergence Theorem 2.1 on Eq 3.8, we get

$$\begin{aligned} &\mathbf{E}\left\|\int_0^c K_\mu(c-\zeta)[\mathcal{G}(r,\chi_n(r)) - \mathcal{G}(r,\chi(r))]d\zeta\right\|^2 \leq \mathcal{M}^2 \int_0^c d\zeta \int_0^c \|[\mathcal{G}(\zeta,\chi_m(\zeta)) - \mathcal{G}(\zeta,\chi(\zeta))]\|^2 d\zeta \\ &\leq c^2\mathcal{M}^2 \int_0^c \left\|[\mathcal{G}(\zeta,\chi_n(\zeta)) - \mathcal{G}(\zeta,\chi(\zeta))]\right\|^2 d\zeta \to 0\,(m \to \infty). \end{aligned} \tag{4.1}$$

Similarly,

$$\begin{aligned} &\left\|\int_0^c K_\mu(c-\zeta)\left[\varpi(\zeta,\chi_m(\zeta)) - \varpi(\zeta,\chi(\zeta))\right]d\zeta\right\|^2 \leq \mathfrak{I}r(Q)\,\mathcal{M}^2 \int_0^c \|\varpi(\zeta,\chi_m(\zeta)) - \varpi(\zeta,\chi(\zeta))\|^2 dW(\zeta) \\ &\to 0\,(m \to \infty). \end{aligned} \tag{4.2}$$

And

$$\left\|\mathfrak{S}_\beta(c)\,(h(\chi_m) - h(\chi))\right\|^2 \leq \left(\frac{\mathcal{M}}{\Gamma(\beta)}\right)^2 \|h(\chi_m) - h(\chi)\|^2 \to 0\,(m \to \infty). \tag{4.3}$$

Using the inequality found above, we arrive at

$$\begin{aligned} &\|(\rho(\chi_m) - \rho(\chi))\|^2 \leq 3\mathrm{E}\left\|\mathfrak{S}_\beta(c)\,(h(\chi_m) - h(\chi))\right\|^2 + 3\mathbf{E}\left\|\int_0^c K_\mu(c-\zeta)\left[\mathcal{G}(\zeta,\chi_m(\zeta)) - \mathcal{G}(\zeta,\chi(\zeta))\right]d\zeta\right\|^2 \\ &+3\mathbf{E}\left\|\int_0^c K_\mu(c-\zeta)\left[\varpi(\zeta,\chi_m(\zeta)) - \mathfrak{G}(\zeta,\chi(\zeta))\right]ds\right\| \to 0\,(m \to \infty). \end{aligned} \tag{4.4}$$

Therefore, $\rho$ is continuous in $\mathcal{B}_g$.

Secondly, define an operator $\mathcal{Y}^{\epsilon,v}$ for all $\epsilon \in (0, c)$ and all $v > 0$ on $\mathcal{B}_g$ by the formula

$$\begin{aligned}(\mathcal{Y}^{\epsilon,v}\chi)(c) &= \int_0^{c-\epsilon}\int_v^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\mathcal{G}(\zeta,\chi(\zeta))\,d\omega d\zeta + \int_0^{c-\epsilon}\int_v^{\infty}\varphi_\beta(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\varpi(\zeta,\chi(\zeta))\,d\omega dW\zeta \\ &= \Im(\epsilon^\mu v)\int_0^{c-\epsilon}\int_v^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega-\epsilon^\mu v\right)\mathcal{G}(\zeta,\chi(\zeta))\,d\omega d\zeta + \Im(\epsilon^\mu v)\int_0^{c-\epsilon}\int_v^{\infty}\varphi_\mu(\omega) \\ &\quad \Im\left((c-\zeta)^\mu\omega-\epsilon^\mu v\right)\varpi(\zeta,\chi(\zeta))\,d\omega dW\zeta.\end{aligned}$$

Then the set $\left\{(\mathcal{Y}^{\epsilon,v}\chi)(c) : \chi \in \mathcal{B}_g\right\}$ is relatively compact in $\mathcal{B}_g$ because $\Im(\epsilon^\mu v)$ is compact. We denote

$$(\mathcal{Y}_1\chi)(c) = \int_0^c\int_0^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\mathcal{G}(\zeta,\chi(\zeta))\,d\omega d\zeta + \int_0^c\int_0^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\varpi(\zeta,\chi(\zeta))\,d\omega dW\zeta. \tag{4.5}$$

Furthermore,

$$\begin{aligned}\|(\mathcal{Y}_1\chi)(c)-(\mathcal{Y}^{\epsilon,v}\chi)(c)\|^2 &\le 4\mathbf{E}\left\|\int_0^c\int_0^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\mathcal{G}(\zeta,\chi(\zeta))\,d\omega d\zeta\right\|^2 \\ &\quad +4\mathbf{E}\left\|\int_0^c\int_0^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\varpi(\zeta,\chi(\zeta))\,d\omega dW(\zeta)\right\|^2 \\ &\quad -4\mathbf{E}\left\|\int_0^{c-\epsilon}\int_v^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\mathcal{G}(\zeta,\chi(\zeta))\,d\omega d\zeta\right\|^2 \\ &\quad -4\mathbf{E}\left\|\int_0^{c-\epsilon}\int_v^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\varpi(\zeta,\chi(\zeta))\,d\omega dW(\zeta)\right\|^2 \\ &\le 4\mathbf{E}\left\|\int_0^c\int_0^{v}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\mathcal{G}(\zeta,\chi(\zeta))\,d\omega d\zeta\right\|^2 \\ &\quad +4\mathbf{E}\left\|\int_0^c\int_0^{v}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\varpi(\zeta,\chi(\zeta))\,d\omega dW(\zeta)\right\|^2 \\ &\quad +4\mathbf{E}\left\|\int_{c-\epsilon}^c\int_v^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\mathcal{G}(\zeta,\chi(\zeta))\,d\omega d\zeta\right\|^2 \\ &\quad +4\mathbf{E}\left\|\int_{c-\epsilon}^c\int_v^{\infty}\varphi_\mu(\omega)\,\Im\left((c-\zeta)^\mu\omega\right)\varpi(\zeta,\chi(\zeta))\,d\omega dW(\zeta)\right\|^2 \\ &\le 4\mathcal{M}^2\int_0^c d\zeta\int_0^c\mathbf{E}\|\mathcal{G}(\zeta,\chi(\zeta))\|^2 ds\left(\int_0^v\varphi_\mu(\omega)\,d\omega\right)^2\end{aligned}$$

$$+4\mathcal{M}^2\int_{c-\epsilon}^{c}ds\int_{c-\epsilon}^{c}\mathbf{E}\|\mathcal{G}(\zeta,\chi(\zeta))\|^2d\zeta+4\Im r(Q)\mathcal{M}^2\int_0^c d\zeta\mathbf{E}\|\varpi(\zeta,\chi(\zeta))\|_{\mathcal{L}_2^0}{}^2d\zeta\left(\int_0^v\varphi_\mu(\omega)\,d\omega\right)^2$$

$$+4\Im r(Q)\mathcal{M}^2\int_{c-\epsilon}^{c}d\zeta\mathbf{E}\|\varpi(\zeta,\chi(\zeta))\|_{\mathcal{L}_2^0}{}^2d\zeta$$

$$\leq \frac{4\mathcal{M}^2c^2\psi_{\mathcal{G}}(\mathbf{R^m})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]}}{2}\left(\int_0^v\varphi_\mu(\omega)\,d\omega\right)^2+4\mathcal{M}^2\psi_{\mathcal{G}}(\mathbf{R})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]}\frac{\epsilon^{2\mu-1}}{2\mu-1}$$

$$+4\Im r(Q)\mathcal{M}^2\psi_{\varpi}(\mathbf{R})\left(\frac{1-p}{2\mu-1-p}\right)^{1-p}\|\xi_{\varpi}\|_{\mathcal{L}^{\frac{1}{p}}}\left(\int_0^v\varphi_\mu(\omega)\,d\omega\right)^2$$

$$+4\Im r(Q)\mathcal{M}^2\beta^2\psi_{\varpi}(\mathbf{R})\left(\frac{1-p}{2\mu-1-p}\right)^{1-p}\|\xi_{\varpi}\|_{\mathcal{L}^{\frac{1}{p}}}\epsilon^{2\mu-1-p}\to 0\,(\epsilon,v\to 0)\,(n\to\infty). \tag{4.6}$$

Therefore, there are relatively compact set $\left\{(\mathcal{Y}^{\varepsilon,v}\chi)(c):\chi\in\mathcal{B}_g\right\}$ arbitrary close to the set $\left\{(\mathcal{Y}^{\varepsilon,v}\chi)(c):\chi\in\mathcal{B}_g\right\}$ in $\hbar$. Hence $\left\{\rho(\chi):\chi\in\mathcal{B}_g\right\}$ is relatively compact in $\hbar$.

**Lemma 4.2.** If assumptions $(\wp_1)$–$(\wp_4)$ are satisfied, then the following conclusions hold for any $\chi\in\mathcal{B}_g$,

(i) $\mathrm{E}\|u_\epsilon(r,\chi)\|^2\leq\mathcal{L}_u$;
(ii) $u_\epsilon(r,\chi)$ is continuous in $\mathcal{B}_g$, where $\mathcal{L}_u=\|\mathfrak{D}\|^2\left(\frac{\mathcal{M}}{\Gamma(\mu)}\right)^2\mathcal{L}_\epsilon$.

**Lemma 4.3.** Assume that in a HS,-$\mathcal{A}$ generates a compact of uniformly bounded operators. Let $(\wp_1)$–$(\wp_4)$ hold, the following condition is satisfied:

$(\wp_5)$ There is a constant $\sigma\in$ (0,c) such that for any $r\in[0,c]$ $\mathcal{G}(r,\chi_1(r))=\mathcal{G}(r,\chi_2(r))$, $\varpi(r,\chi_1(r))=\varpi(r,\chi_2(r))$, $h(r,\chi_1(r))=h(r,\chi_2(r))$, where $\chi_1,\chi_2\in\hbar\left([0,c],\mathcal{L}^2\left(\Omega_{pr},\hbar\right)\right)\chi_1(r)=\chi_2(r)\,(r\in[\sigma,c])$.

The nonlocal problem will have at least one mild solution in $\mathcal{B}_g$ if there is a +ve constant $\mathbf{R^m}$ such that

$$3\mathcal{M}^2c\psi_k(\mathbf{R})\|\xi_k\|_{\mathcal{L}[0,c]}+3c_0\left(2\psi_{\mathcal{G}}(\mathbf{R})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]}+2c\|\mathfrak{D}\|^2\mathcal{L}_u\right)+3c_1\Im r(Q)\psi_{\varpi}(\mathbf{R})\|\xi_{\varpi}\|_{\mathcal{L}^{\frac{1}{p}}}\leq\mathbf{R}, \tag{4.7}$$

where

$$c_0=\frac{\mathcal{M}^2c^2}{2},c_1=\mathcal{M}^2\left(\frac{1-p}{2-p}\right)^{1-p}c^{2-p}.$$

*Proof.* For any $r>0$, define

$$\mathcal{B}_g(\sigma)=\left\{\chi\in\hbar\left([\sigma,c],\mathcal{L}^2\left(\Omega_{pr},\hbar\right)\right):\mathbf{E}\|\chi(r)\|^2\leq r,\forall r\in[\sigma,c]\right\}.$$

It is obvious that, a function $\Upsilon\in\mathcal{B}_g$ exists that satisfies $\chi(r)=\Upsilon(r)$, $r\in[\sigma,c]$. Define the following mappings on $\mathcal{B}_g(\sigma)$ by

$$(\mathcal{G}^*\chi)(r)=\mathcal{G}(r,\Upsilon(r)),r\in[0,c]\quad(\varpi^*\chi)(r)=\varpi(r,\Upsilon(r)),r\in[0,c]\quad h^*(\chi)=h(\Upsilon).$$

Then by conditions $(\wp_1)$–$(\wp_3)$ and $(\wp_5)$, it is easy to see that $\mathcal{G}^*, \varpi^*, h^*$ is well defined on $\mathcal{B}_g(\sigma)$ and continuous. In addition,

$$\begin{aligned} \|(\mathcal{G}^*\chi)(r)\|^2 &\leq \xi_{\mathcal{G}}(r)\psi_{\mathcal{G}}\left(\mathbf{E}\|\chi\|^2\right), a.e.t,, r \in [0,c], \forall \chi \in \mathcal{B}_g(\sigma) \\ \|(\varpi^*\chi)(r)\|^2_{\mathcal{L}_2^0} &\leq \xi_{\varpi}(r)\psi_{\varpi}\left(\mathbf{E}\|\chi\|^2\right), a.e.t, r \in [0,c], \forall \chi \in B_g(\sigma) \\ \mathbf{E}\|h^*(\chi)\|^2 &\leq c\psi_h(h)\|\xi_h\|_{\mathcal{L}[0,c]}, \forall \chi \in B_g(\sigma). \end{aligned}$$

Define an operator $\mathcal{G}_\sigma$ on $\mathcal{B}_g(\sigma)$ as follows:

$$(\mathcal{G}_\sigma\chi)(r) = \mathfrak{S}_{\upsilon,\mu}(r)\chi_0 + \int_0^r K_\mu(r-\zeta)\left[(\mathcal{G}^*\chi)(\zeta) + \mathfrak{D}u^*(\zeta,\chi)\right]d\zeta + \int_0^r K_\mu(r-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta), r \in [\sigma, c],$$

here $u_\epsilon^*(\zeta,\chi)$ is defined by $u_\epsilon^*(r,\chi) = \mathfrak{D}^*K_\mu^*(c-r)\Phi_\epsilon^*(\chi)$ and $\Phi_\epsilon^*(\chi)$ is the critical point of $\mathcal{J}_\epsilon(\phi,\chi)$ of $\rho^*(\chi)$ where

$$\rho^*(\chi) = \chi_c - \left[\mathfrak{S}_{\upsilon,\mu}(r)\chi_0 + \int_0^r K_\mu(r-\zeta)(\mathcal{G}^*\chi)(\zeta)\,d\zeta + \int_0^c K_\mu(r-\zeta)K_\mu(c-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right].$$

Evidently, the results in Lemma 4.3 hold for $u_\epsilon^*(\zeta,\chi)$. Then, using schauder's fixed point theorem, we show that $\mathcal{G}_\sigma$ has a fixed point. To do so, we first verify that there is a positive number $\mathbf{R^m}$ such that $\mathcal{G}_\sigma$ maps $\mathcal{B}_g(\sigma)$ into itself. For any $\chi \in \mathcal{B}_g(\sigma)$ and $r \in [\sigma, c]$ , it follows

$$\begin{aligned} \mathbf{E}\|(\mathcal{G}_\sigma\chi)(r)\|^2 &\leq 3\mathbf{E}\left\|\mathfrak{S}_{\upsilon,\mu}(r)\chi_0\right\|^2 + 3\mathbf{E}\left\|\int_0^r K_\mu(r-\zeta)\left[(\mathcal{G}^*\chi)(\zeta) + \mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2 \\ &\quad +3\mathbf{E}\left\|\int_0^r K_\mu(r-\zeta)(\varpi^*\chi)(s)\,dW(\zeta)\right\|^2 \\ &\leq \frac{3\mathcal{M}^2 b\psi_h(r)\|\xi_h\|_{\mathcal{L}[0,c]}}{\Gamma^2(\mu)} + \frac{3\mathcal{M}^2 b^2}{2}\int_0^r \mathbf{E}\|(\mathcal{G}^*\chi)(\mu) + \mathfrak{D}u^*(\zeta,\chi)\|^2 d\zeta \\ &\quad +3\Im r(Q)\mathcal{M}^2\int_0^r \mathbf{E}\,\|(\varpi^*\chi)(\zeta)\|^2_{\mathcal{L}_2^0}\,dW(\zeta) \\ &\leq 3\mathcal{M}^2 c\psi_h(\mathbf{R})\|\xi_h\|_{\mathcal{L}[0,c]} + 3c_0\left(2\psi_{\mathcal{G}}(\mathbf{R})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]} + 2c\|\mathfrak{D}\|^2\mathcal{L}_u\right) \\ &\quad +3c_1\Im r(Q)\psi_\varpi(\mathbf{R})\|\xi_\varpi\|_{\mathcal{L}^{\frac{1}{p}}} \\ &\leq \mathbf{R}. \end{aligned}$$

As a result, $\mathcal{G}_\sigma$ maps $\mathcal{B}_g(\sigma)$ to $\mathcal{B}_g(\sigma)$. Second, we can show that $\mathcal{G}_\sigma : \mathcal{B}_g(\sigma) \to \mathcal{B}_g(\sigma)$ is a continuous operators and the set $\left\{(\mathcal{G}_\sigma\chi)(r) : \chi \in \mathcal{B}_g(\sigma)\right\}$ is relatively compact in $\hbar$ for $r \in [\sigma, c]$. We'll demonstrate that $\mathcal{G}_\sigma\left(\mathcal{B}_g(\sigma)\right)$ is an equicontinuous family of functions on $[\sigma, c]$. For any $\chi \in \mathcal{B}_g(\sigma)$ and $\sigma \leq r_1 < r_2 \leq c$, we get that

$$\mathbf{E}\|(\mathcal{G}_\sigma\chi)(r_2) - (\mathcal{G}_\sigma\chi)(r_1)\|^2$$

$$
\begin{aligned}
&= 5\mathbf{E}\left\|\left(\mathfrak{S}_\mu(r_2)-\mathfrak{S}_\mu(r_1)\right)(\chi_0)\right\|^2+5\mathbf{E}\left\|\int_0^{r_2}K_\mu(r_2-\zeta)\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\quad+5\mathbf{E}\left\|\int_0^{r_2}K_\mu(r_2-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2-5\mathbf{E}\left\|\int_0^{r_1}K_\mu(r_1-\zeta)\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\quad-5\mathbf{E}\left\|\int_0^{r_1}K_\mu(r_1-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2\\
&= 7\mathbf{E}\left\|\left(\mathfrak{S}_\beta(r_2)-\mathfrak{S}_\beta(r_1)\right)(\chi_0)\right\|^2+7\mathbf{E}\left\|\int_{r_1}^{r_2}K_\mu(r_2-\zeta)\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|\\
&\quad+7\mathbf{E}\left\|\int_0^{r_1}K_\mu(r_2-\zeta)\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2+7\mathbf{E}\left\|\int_0^{r_1}K_\mu(r_1-\zeta)\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\quad+7\mathbf{E}\left\|\int_{r_1}^{r_2}K_\mu(r_2-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2+7\mathbf{E}\left\|\int_0^{r_1}K_\mu(r_2-\zeta)K_\mu(r_2-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2\\
&\quad+7\mathbf{E}\left\|\int_0^{r_1}K_\mu(r_1-\zeta)-K_\mu(r_1-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2\\
&= I_0+I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

In order to prove that $\mathbf{E}\|(\mathcal{G}_\sigma\chi)(r_2)-(\mathcal{G}_\sigma\chi)(r_1)\|^2\to 0\,(r_2-r_1\to 0)$, we only need to check $I_i\to 0$ independently of $\chi\in\mathcal{B}_g(\sigma)$ when $(r_2-r_1\to 0)$ for $i=0,1,2,...,6$. Clearly, $I_0\to 0$ as $(r_2-r_1\to 0)$

$$
\begin{aligned}
I_1 &= 7\mathbf{E}\left\|\int_{r_1}^{r_2}K_\mu(r_2-\zeta)\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\le 7\mathcal{M}^2\int_{r_1}^{r_2}d\zeta\int_{r_1}^{r_2}\mathbf{E}\|(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\|^2d\zeta\\
&\le 7\mathcal{M}^2\left(2\psi_{\mathcal{G}}(\mathbf{R})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]}+2\|\mathfrak{D}\|^2\mathcal{L}_u c\right).\frac{(r_2-r_1)^2}{2}\to 0\,(r_2-r_1\to 0),\\
I_4 &= 7\mathbf{E}\left\|\int_{r_1}^{r_2}K_\mu(r_2-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2\\
&\le 7\Im r(\mathcal{Q})\,\mathcal{M}^2\int_{r_1}^{r_2}\|(\varpi^*\chi)(\zeta)\|^2d\zeta+7\Im r(\mathcal{Q})\,\mathcal{M}^2\psi_\varpi\left(\mathcal{R}^{\Updownarrow}\right)\|\xi_\varpi\|_{\mathcal{L}^{\frac{1}{P}}}(r_2-r_1)^{1-P}\to 0\,(r_2-r_1\to 0).
\end{aligned}
$$

Similarly, for $I_2$ and $I_5$, we get

$$
\begin{aligned}
I_2 &= 7\mathbf{E}\left\|\int_0^{r_1}\left[(r_2-\zeta)^{\mu-1}-(r_1-\zeta)^{\mu-1}\right]K_\mu(r_2-\zeta)\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\leq 7\mathcal{M}^2\int_0^{r_1}\left[(r_2-\zeta)^{\mu-1}-(r_1-\zeta)^{\mu-1}\right]^2d\zeta\int_0^{r_1}\mathbf{E}\|(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\|^2ds\\
&\leq 7\mathcal{M}^2\left(2\psi_{\mathcal{G}}(\mathbf{R})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]}+2\|\mathfrak{D}\|^2\mathcal{L}_uc\right)\int_0^{r_1}\left[(r_2-\zeta)^{\mu-1}-(r_1-\zeta)^{\mu-1}\right]^2d\zeta\to 0\,(r_2-r_1\to 0),\\
I_5 &= 7\mathbf{E}\left\|\int_0^{r_1}\left[(r_2-\zeta)^{\mu-1}-(r_1-\zeta)^{\beta-1}\right]K_\mu(r_2-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\,d\zeta\right\|^2\\
&\leq 7\mathfrak{I}r(Q)\,\mathcal{M}^2\int_0^{r_1}\left[(r_2-\zeta)^{\mu-1}-(r_1-s)^{\mu-1}\right]^2\mathbf{E}\|(\varpi^*\chi)(\zeta)\|^2d\zeta\\
&\leq 7\mathcal{M}^2\left(2\psi_{\varpi}(\mathbf{R})\|\xi_{\varpi}\|_{\mathcal{L}^{\frac{1}{P}}}\right)\left(\int_0^{r_1}\left[(r_2-\zeta)^{\mu-1}-(r_1-\zeta)^{\mu-1}\right]^{\frac{2}{1-P}}d\zeta\right)^{1-P}\to 0\,(r_2-r_1\to 0).
\end{aligned}
$$

Additionally, if $0<\epsilon<r_1$ is sufficiently small for $I_3$ and $I_6$, we derive the following inequalities:

$$
\begin{aligned}
I_3 &= 7\mathbf{E}\left\|\int_0^{r_1}[K_\mu(r_2-\zeta)-K_\mu(r_1-\zeta)]\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\leq 14\mathbf{E}\left\|\int_0^{r_1-\epsilon}[K_\mu(r_2-\zeta)-K_\mu(r_1-\zeta)]\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\quad+14\mathbf{E}\left\|\int_{r_1-\epsilon}^{r_1}[K_\mu(r_2-\zeta)-K_\mu(r_1-\zeta)]\left[(\mathcal{G}^*\chi)(\zeta)+\mathfrak{D}u_\epsilon{}^*(\zeta,\chi)\right]d\zeta\right\|^2\\
&\leq 14\sup_{\zeta\in[0,r_1-]}\left\|K_\mu(r_2-\zeta)-K_\mu(r_1-\zeta)\right\|^2\left(2\psi_{\mathcal{G}}(\mathbf{R})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]}+2\|\mathfrak{D}\|^2\mathcal{L}_uc\right)\times\frac{r_1^2-\epsilon^2}{2}\\
&\quad+56\mathcal{M}^2\left(2\psi_{\mathcal{G}}(\mathbf{R})\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}[0,c]}+2\|\mathfrak{D}\|^2\mathcal{L}_uc\right)\frac{\epsilon^2}{2}\to 0\,(r_2-r_1\to 0\ \ and\ \ \epsilon\to 0),\\
I_6 &= 7\mathbf{E}\left\|\int_0^{r_1}K_\mu(r_2-\zeta)-K_\mu(r_1-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2\\
&\leq 14\mathbf{E}\left\|\int_0^{r_1-\epsilon}K_\mu(r_2-\zeta)-K_\mu(r_1-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2
\end{aligned}
$$

$$+14\mathbf{E}\left\|\int_{r_1-\epsilon}^{r_1} K_\mu(r_2-\zeta)-K_\mu(r_1-\zeta)(\varpi^*\chi)(\zeta)\,dW(\zeta)\right\|^2$$

$$\leq \frac{14\Im r(Q)(1-p)\psi_\varpi(\mathbf{R})}{2-p}\sup_{\zeta\in[0,r_1-\zeta]}(K_\mu(r_2-\zeta)-K_\mu(r_1-s))\left\|\xi_{\mathcal{G}}\right\|_{\mathcal{L}^{\frac{1}{p}}}\times\left(r_1^{\frac{2-p_2}{1-p}}-\epsilon\right)$$

$$+56\Im r(Q)\mathcal{M}^2\psi_\varpi(\mathbf{R})\|\xi_\varpi\|_{\mathcal{L}^{\frac{1}{p}}}\to 0\,(r_2-r_1\to 0\ \ and\ \ \epsilon\to 0).$$

Overall, $I_i\to 0$ is equal to $r_2-r_1\to 0$ and $\epsilon\to 0$, implying that ${}_1(\mathcal{B}_g(\sigma))$ is equicontinuous. According to the Schauder fixed point theorem, $\mathcal{G}_\sigma$ has atleast one fixed point in $\bar{\chi}\in\mathcal{B}_g(\sigma)$,

$$\overline{\chi}(r) = \mathfrak{S}_\mu(\overline{\chi}_0)+\int_0^r K_\mu(r-\zeta)\left[(\mathcal{G}^*\overline{\chi})(\zeta)+\mathfrak{D}u_{\epsilon}{}^*(\zeta,\overline{\chi})\right]d\zeta+\int_0^r K_\mu(r-\zeta)(\varpi^*\overline{\chi})(\zeta)\,dW(\zeta),\ r\in[\sigma,c],$$

$$\overline{z}(r) = \mathfrak{S}_\mu(r)(\overline{\chi}_0+h^*(\overline{\chi}))+\int_0^r(r-\zeta)^{\mu-1}K_\mu(r-\zeta)\left[(\mathcal{G}^*\overline{\chi})(\zeta)+\mathfrak{D}u_{\epsilon}{}^*(\zeta,\overline{\chi})\right]d\zeta$$

$$+\int_0^r(r-\zeta)^{\mu-1}K_\mu(r-\zeta)(\varpi^*\overline{\chi})(\zeta)\,dW(\zeta),\quad r\in[0,c].$$

Clearly, $\overline{\chi}(r)=\overline{z}(r)$ for $r\in[\sigma,c]$. From the definitions of $\mathcal{G}^*,h^*,\varpi^*$, it follows immediately that

$$\overline{z}(r)=\mathfrak{S}_\mu(r)(\overline{\jmath}_0+h(\overline{z}))+\int_0^r(r-\zeta)^{\mu-1}K_\mu(r-\zeta)\left[\mathcal{G}(\zeta,\overline{z}(\zeta))+\mathfrak{D}u_{\epsilon}{}^*(\zeta,\overline{\jmath})\right]ds$$
$$+\int_0^r(r-\zeta)^{\beta-1}K_\mu(r-\zeta)\,\varpi(\zeta,\overline{\jmath}(\zeta))\,dW(\zeta),\quad r\in[0,c],$$

that is, $\overline{\jmath}$ is mild solution of given system in $B_g$.

For each $\sigma\in(0,c)$ and arbitrary $\chi\in\hbar([0,c],\mathcal{L}^2(\Omega_{pr},\hbar)$ write

$$(\mathcal{L}_\sigma\chi)(r)=\begin{cases}\chi(\sigma), & r\in[0,\sigma],\\ \chi(r), & r\in[\sigma,c],\end{cases}\tag{4.8}$$

and

$$\begin{aligned}\mathcal{G}_\sigma(r,\chi(r)) &= \mathcal{G}_\sigma(r,\chi(r)),\quad r\in[0,c],\\ \varpi_\sigma(r,\chi(r)) &= \varpi_\sigma(r,\chi(r)),\quad r\in[0,c],\\ h_\sigma(r,\chi(r)) &= h_\sigma(r,\chi(r)),\quad r\in[0,c].\end{aligned}$$

It is easy to see that $\mathcal{G}_\sigma$, $\varpi_\sigma$ and $h_\sigma$ defined above satisfy condition $(\wp_5)$, thus we obtain.

**Lemma 4.4.** Assume that in a HS $\mathcal{H}$, $\mathcal{A}$ generates a compact $C_0$ semigroup $\Im(r)(r\geq 0)$ of uniformly bounded operators. Let Assumptions $(\wp_1)$–$(\wp_4)$ hold. Then the following nonlocal problem:

$$\begin{cases}\mathcal{D}_{0^+}^{\upsilon,\mu}\chi(r)=\mathcal{A}\chi(r)+\mathcal{G}_\sigma(r,\chi(r))+\varpi_\sigma(r,\chi(r))\frac{dw(r)}{dr}+\mathfrak{D}u_\epsilon(r,\mathcal{L}_\sigma\chi),\ r\in[0,c],\\ I_{0^+}^{(1-\upsilon)(1-\mu)}\chi(0)=\chi_0.\end{cases}\tag{4.9}$$

**Lemma 4.5.** Assume that in a HS $\hbar$, $\mathscr{A}$ generates a compact $C_0$ semigroup $\Im(r)(r \geq 0$ of uniformly bounded operators. Let Assumptions $(\wp_1) - (\wp_4)$ satisfied, then fractional stochastic control system with 1.1 has atleast one mild solution $\chi \in \hbar\,[\sigma, c]\,, \mathcal{L}^2\left(\Omega_{pr}, \hbar\right)$ in provided that there exists a positive constant $\mathbf{R^m}$.

*Proof.* To begin with, let $\{\sigma_m : m \in \mathbf{N}\}$ be a decreasing sequence in $(0, c)$ with $\lim\limits_{m\to\infty} \sigma_m = 0$. For every n,

$$\begin{cases} \mathcal{D}_{0^+}^{\upsilon,\mu}\chi\,(r) = \mathscr{A}\chi\,(r) + \mathcal{G}_{\sigma}\,(r,\chi\,(r)) + \varpi_{\sigma}\,(r,\chi\,(r))\,\frac{dw(r)}{dr} + \mathfrak{D}u_{\epsilon}\,(r, \mathcal{L}_{\sigma}\chi)\,, \quad r \in [0, c], \\ I_{0^+}^{(1-\upsilon)(1-\mu)}\chi(0) = \chi_0, \end{cases} \tag{4.10}$$

has mild solution $\chi_m \in \mathcal{B}_g$ if constant $\mathbf{R^m}$ satisfies , which is expressed by

$$\begin{aligned} \chi_n\,(r) \quad &= \quad \mathfrak{S}_{\upsilon,\mu}\,(r)\chi_0 + \int_0^r K_{\mu}\,(r-\zeta)\,[\mathcal{G}_{\sigma_m}\,(\zeta,\chi_m\,(\zeta)) + \mathfrak{D}u_{\epsilon}\,(\zeta, \mathcal{L}_{\sigma_n}\chi_m)]\,d\zeta \\ &\quad + \int_0^r K_{\mu}\,(r-\zeta)\,\varpi_{\sigma_m}\,(\zeta,\chi_m\,(\zeta))\,dW\,(\zeta)\,, \quad r \in [0, c]\,, \end{aligned}$$

$$v_m\,(r) = \begin{cases} \chi_n\,(\sigma_m)\,, & r \in [0, \sigma_m], \\ \chi_n\,(r)\,, & r \in [\sigma_m, c], \end{cases} \tag{4.11}$$

then $v_m \in \mathcal{B}_g$. In view of definitions $\mathcal{G}_{\sigma_m}, h_{\sigma_m}, \varpi_{\sigma_m}$, we conclude that

$$\begin{aligned} &\chi_m\,(r) = \mathfrak{S}_{\upsilon,\mu}\,(r)\,v_0 + \int_0^r K_{\mu}\,(r-\zeta)\,[\mathcal{G}\,(\zeta, v_m\,(\zeta)) + \mathfrak{D}u_{\epsilon}\,(\zeta, v_m)]\,d\zeta \\ &+ \int_0^r K_{\mu}\,(r-\zeta)\,\varpi\,(\zeta, v_m\,(\zeta))\,dW\,(\zeta)\,, \quad r \in [0, c]. \end{aligned} \tag{4.12}$$

Furthermore, we will show that the $\{\chi_m : m \in \mathbb{N})$ is precompact. For this purpose, we introduce the following definition:

$$\begin{aligned} \eta_n\,(r) \quad &= \quad \mathfrak{S}_{\upsilon,\mu}v_0, \\ \varphi_m\,(r) \quad &= \quad \int_0^r K_{\mu}\,(r-\zeta)\,[\mathcal{G}\,(\zeta, v_m\,(\zeta)) + \mathfrak{D}u_{\epsilon}\,(\zeta, v_m)]\,d\zeta + \int_0^r K_{\mu}\,(r-\zeta)\,\varpi\,(\zeta, v_m\,(\zeta))\,dW\,(\zeta)\,, \quad r \in [0, c]. \end{aligned}$$

Therefore, we only need to show that the set $\{\eta_m : m \in \mathbb{N}\}$ and $\{\varphi_m : m \in \mathbb{N}\}$ are percompact in $\hbar([0, c]), \mathcal{L}^2(\Omega_{pr}, \hbar)$. From the expression of $v_m(r)$ we know that $v_m(r) \in \mathcal{B}_g$. This implies that $(\wp_1)$–$(\wp_3)$ hold for $\mathcal{G}(\zeta, v_m(\zeta)), \varpi(\zeta, v_m(\zeta))$. Furthermore, $u_{\epsilon}(\zeta, v_m)$ satisfies the estimates $(i)$ and $(ii)$ of Lemma 4.2. As a result, using arguments close to those used in the proof of Lemma 4.3, it is easy to show that the set $\{\eta_m : n \in \mathbb{N}\}$ is precompact in $\hbar([0, c]), \mathcal{L}^2(\Omega_{pr}, \hbar$. We'll also demonstrate that the range $\{\varphi_m : m \in \mathbb{N}\}$ is precompact in $\hbar([0, c]), \mathcal{L}^2(\Omega_{pr}, \hbar)$.

## 5. Application

The following fractional stochastic control scheme exemplifies the key result:

$$\begin{cases} \frac{\partial^{\frac{2}{3}}}{\partial r^{\frac{2}{3}}}\chi(\jmath,r) - \frac{\partial^2\chi(\jmath,r)}{\partial\chi^2} = \left(\frac{r\chi(\jmath,r)}{2(1+|\chi(\jmath,r)|)} + \chi(\jmath,r)\right) + \left(e^r\chi(\jmath,r) + \cos\chi(\jmath,r)\right)\frac{dW(r)}{dr} + u(\jmath,r), \\ r\in[0,c], \quad z\in(0,1], \\ \chi(0,r) = \chi(1,r), \quad r\in[0,c], \\ \chi(\jmath,0) = \chi_0, \quad \jmath\in(0,1], \end{cases} \tag{5.1}$$

where $\mathcal{W}(r)$ is a standard one dimensional Brownian motion defined on the filtered probability space $\left(\Omega_{pr}, \mathcal{G}, \{\mathcal{G}_r\}_{r\geq 0}, \mathbf{R^m}\right)$. To write the above system into the abstract form of 1.1. Let $\hbar = \mathbf{E} = \mathcal{U}=\mathcal{L}^2(0,1]$ with norm $\|.\|$. Define the operator $\mathcal{A}:\mathfrak{D}(\mathcal{A})\subset\mathcal{X}\to\hbar$ by

$$\mathcal{A}v = -v'', v\in\mathfrak{D}(\mathcal{A}),$$

$\mathfrak{D}(\mathcal{A}) = \{v\in\hbar, v, v'$ are absolutely continuous, $v''\in\mathcal{H}, v(0) = v(1) = 0\}$. We know that -$\mathcal{A}$ generates a compact, analytic semigroup $\mathfrak{I}(r)$ $(r\geq 0)$ in $\hbar$ and

$$\mathfrak{I}(r)v = \sum_{m=1}^{\infty} e^{-m^2 r}(v, v_m)v_m, \left\|\mathfrak{I}(r)\right\| \leq e^{-r},$$

where $v_m = \sqrt{2}cos(ns)$, m = 1,2,3,... is the orthogonal set of eigenvectors in $\mathcal{A}$. For any r $\in[0,c]$, let $\chi(r)(\jmath) = \chi(\jmath,r)$, $\mathfrak{D}u(r)(\jmath) = u(\jmath,r)$, $\mathcal{G}(r,\chi(r))(z) = \frac{r\chi(\jmath,r)}{2(1+|\chi(\jmath,r)|)} + \chi(\jmath,r)$, $\varpi(r,\chi(r))(\jmath) = e^r\chi(\jmath,r) + \cos\chi(\jmath,r)$. Then above problem can be rewritten in the abstract form of (1.1). In addition,

$$\begin{aligned} \|\sigma(r,\chi)\|^2 &= \int_0^1 \left|\frac{1}{1+e^e}\frac{\chi(\jmath,r)}{(1+\chi^2(\jmath,r))}\right|^2 d\jmath \\ &\leq \frac{1}{2}\int_0^1 |\chi(\jmath,r)|^2 d\jmath \\ &= \frac{1}{2}\|\chi(r)\|^2, \\ \|h(r,\chi\|^2 &= \int_0^1 \left|r^2 cos\left(\frac{\chi(\jmath,r)}{r}\right)\right|^2 d\jmath \\ &\leq r^2\int_0^1 |\chi(z,r)|^2 d\jmath \\ &= r^2\|\chi(r)\|^2. \end{aligned}$$

So, the assumption $(\wp_1)$ and $(\wp_2)$ hold $\xi_f(r) = \frac{r^2}{2}, \xi_\sigma(r) = \frac{1}{2}, \xi_h(r) = r^2$, and $\psi_f(\zeta) = \psi_\sigma(\zeta) = \psi_h(s) = s$. As a result, all of the hypotheses $(\wp_1)$ through $(\wp_3)$ are true.

## 6. Conclusions

This work investigates the HS by using Hilferderivative-based approximate controllability for a category of FSEEs. We eliminate the Lipschitz condition or compactness requirement found in several literatures, leaving only a weak growth condition on the nonlocal term. Additionally, our future study will focus on the regularity of mild solutions for FSEEs with nonlocal beginning conditions.

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## Conflict of interest

The authors declare no conflict of interest.

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