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Research article

A Legendre-tau-Galerkin method in time for two-dimensional Sobolev equations

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Abstract: This work is devoted to present the Legendre space-time spectral method for two-dimensional (2D) Sobolev equations. Considering the asymmetry of the first-order differential operator, the Legendre-tau-Galerkin method is employed in time discretization and its multi-interval form is also investigated. In the theoretical analysis, rigorous proof of the stability and $L^2(\Sigma)$-error estimates is given for the fully discrete schemes in both single-interval and multi-interval forms. Being different from the general Legendre-Galerkin method, we specifically take the Fourier-like basis functions in space to save the computing time and memory in the algorithm of the proposed method. Numerical experiments were included to confirm that our method attains exponential convergence in both time and space and that the multi-interval form can achieve improved numerical results compared with the single interval form.

Keywords: 2D Sobolev equations; Fourier-like basis functions; Legendre space-time spectral method; exponential convergence
Mathematics Subject Classification: 65M12, 65M70

1. Introduction

The Sobolev equation plays an important role in partial differential equations (PDEs) because of its significant physical background, such as consolidation of clay [1], thermodynamics [2] and flow of fluids through fissured rock [3].

In this article, a Legendre-tau-Galerkin method in time and its multi-interval form will be considered for the following 2D Sobolev equations:

$$\begin{cases}\partial_t u(x,y,t) - \varepsilon\partial_t\Delta u(x,y,t) - \mu\Delta u(x,y,t) + \gamma u(x,y,t) = f(x,y,t), & (x,y,t) \in \Sigma := \Omega \times I,\\ u(x,y,-1) = u_0(x,y), & (x,y) \in \bar{\Omega},\\ u(x,y,t) = 0, & (x,y,t) \in \partial\Omega \times I,\end{cases} \tag{1.1}$$

where coefficients μ, ε, γ are known positive parameters. Though the the time interval is normally (0,T] and $T > 0$, we set the interval as $I = (-1, 1]$ and the spatial domain is $\Omega = (-1, 1) \times (-1, 1)$. We consider the time interval $I = (-1, 1]$ just to simplify the presentation of the theoretical analysis and the algorithm implementation process.

Because of the great difficulty in obtaining analytical solutions of PDEs, various numerical methods [4–7] have been proposed to approximate the exact solutions. For the Sobolev equations, there have been many studies investigating the numerical solutions. In [8–10], some finite volume element methods were presented in space to solve the two-dimensional Sobolev equations combined with the finite difference schemes in time. The continuous interior penalty finite element method, space-time continuous Galerkin method and finite difference streamline diffusion method were applied in [11–13] for solving Sobolev equations with convection-dominated term, respectively. In [14], discontinuous Galerkin scheme in space and Crank-Nicolson scheme in time were considered for approaching the exact solutions of generalized Sobolev equations. In [15], Shi and Sun studied an H^1-Galerkin mixed finite element method for solving Sobolev equations and presented the existence, uniqueness and superconvergence results of the discrete scheme. In [16, 17], a block-centered finite difference scheme and a time discontinuous Galerkin space-time finite element scheme for nonlinear Sobolev equations were established respectively, and stability and global convergence of the schemes were strictly proved. In [18], the Legendre spectral element method in space combined with the Crank-Nicolson finite difference technique in time were considered. In [19], the nonlinear periodic Sobolev equations were investigated by the Fourier spectral method.

As is well known, the spectral method is distinguished from other numerical methods by its exponential convergence, and when the spectral method is applied to time-dependent partial differential equations in both space and time (namely, space-time spectral method [20–25]), the mismatched accuracy caused by the spectral discretization in space and the finite difference method in time can be solved successfully. In [26], we constructed a space-time Legendre spectral scheme for the linear multi-dimensional Sobolev equations for the first time and the exponential convergence was obtained in both space and time. The main purpose of this paper is to study the multi-interval form of the Legendre space-time fully discrete scheme of two-dimensional Sobolev equations by dividing the time interval. It is worth noting that compared with the single interval method, the multi-interval spectral method [27–29] can adopt parallel computation, reduce the scale of the problem effectively and improve the flexibility of the algorithm. Considering the asymmetry of the first order differential operator, the fully discrete scheme is constructed by applying a Legendre-tau-Galerkin method in time based on the Legendre Galerkin method in space. In addition, we still apply the Fourier-like basis functions [30] in space to diagonalize the stiff matrix and the mass matrix simultaneously, which greatly saves the computing time and memory.

The organization of this article is as follows. In Section 2, we first provide some related notations, then establish the single interval Legendre space-time spectral fully discrete scheme of Eq (1.1) and give the stability analysis and $L^2(\Sigma)$-error estimates. In Section 3, we divide the time interval and develop the multi-interval Legendre space-time spectral fully discrete scheme of the equations, and then strictly prove the $L^2(\Sigma)$-error estimates. In Section 4, by using Fourier-like basis functions in space and selecting appropriate basis functions in time, we present the implementation of the multi-interval fully discrete scheme. In Section 5, numerical tests are included to access the efficiency and accuracy of the method. Finally, some conclusions are made in Section 6.

2. Theoretical analysis of single interval Legendre spectral method in time

Throughout the paper, the Sobolev spaces in spatial directions are the standard notations used, namely,

$$\|v(x,y)\|_{r,\Omega} = (\sum_{|\epsilon|\le r} \|D^{\epsilon} v(x,y)\|_{\Omega}^2)^{\frac{1}{2}}, \quad \forall v(x,y) \in H^r(\Omega), \tag{2.1}$$

where $\epsilon = (\epsilon_1, \epsilon_2)$ ($\epsilon_i \ge 0$ are integers and $|\epsilon| = \epsilon_1 + \epsilon_2$), $D^{\epsilon} v(x,y) = \frac{\partial^{|\epsilon|} v}{\partial x^{\epsilon_1} \partial y^{\epsilon_2}}$ and $\|\cdot\|_{r,\Omega}$ is denoted by $\|\cdot\|_{\Omega}$ when $r = 0$.

The temporal direction involves a weighted Sobolev space $L^2_{\omega_{\alpha,\beta}}(I)$ endowed with the norm and product

$$\|v(t)\|^2_{I,\omega_{\alpha,\beta}} = (v(t), v(t))_{I,\omega_{\alpha,\beta}} = \int_I v^2 \omega_{\alpha,\beta} \mathrm{d}t, \quad \forall v(t) \in L^2_{\omega_{\alpha,\beta}}(I), \tag{2.2}$$

where the weight function is $\omega_{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta}$. If $\alpha = \beta = 0$, the norm $\|\cdot\|_I$ and inner product $(\cdot,\cdot)_I$ are denoted in the space $L^2(I)$.

Furthermore, the weighted space-time Sobolev space $L^2_{\omega_{\alpha,\beta}}(I; H^r(\Omega))$ is endowed with the norm

$$\|v(x,y,t)\|_{L^2_{\omega_{\alpha,\beta}}(I;H^r(\Omega))} = (\int_I \|v(x,y,t)\|^2_{r,\Omega} \omega_{\alpha,\beta} \mathrm{d}t)^{\frac{1}{2}}, \quad \forall v(x,y,t) \in L^2_{\omega_{\alpha,\beta}}(I; H^r(\Omega)), \tag{2.3}$$

if $r = 0$, the norm $\|\cdot\|_{L^2_{\omega_{\alpha,\beta}}(I;H^r(\Omega))}$ is denoted by $\|\cdot\|_{L^2_{\omega_{\alpha,\beta}}(I;L^2(\Omega))}$; if $\alpha = \beta = 0$, the norm $\|\cdot\|_{L^2_{\omega_{\alpha,\beta}}(I;H^r(\Omega))}$ is denoted by $\|\cdot\|_{L^2(I;H^r(\Omega))}$; if $r = 0$ and $\alpha = \beta = 0$, the norm $\|\cdot\|_{L^2_{\omega_{\alpha,\beta}}(I;H^r(\Omega))}$ is denoted by $\|\cdot\|_{L^2(I;L^2(\Omega))} = \|\cdot\|_{\Sigma}$.

Let P_{ι} be a space of polynomials of degree $\le \iota$ on $[-1,1]$ and $L = (M,N)$, where M and N are a pair of given positive integers. In order to develop the single interval Legendre spectral method in time, we define

$$\begin{aligned} &V_N^0 = \{v \in P_N : v(\pm 1) = 0\}, \ \mathbf{V}_N^0 = V_N^0 \otimes V_N^0, \\ &V_M = \{v \in P_M : v(1) = 0\}. \end{aligned} \tag{2.4}$$

Then applying the Green's formula, we obtain the following single interval Legendre space-time fully discrete scheme of (1.1): Find $u_L \in \mathbf{V}_N^0 \otimes P_M(I)$ satisfying

$$\begin{cases} (\partial_t u_L, v)_{\Sigma} + \varepsilon(\partial_t \nabla u_L, \nabla v)_{\Sigma} + \mu(\nabla u_L, \nabla v)_{\Sigma} + \gamma(u_L, v)_{\Sigma} = (f, v)_{\Sigma}, \quad \forall v \in \mathbf{V}_N^0 \otimes V_M, \\ u_L(x,y,-1) = \mathbb{P}_N^1 u_0(x,y), \end{cases} \tag{2.5}$$

where $\mathbb{P}_N^1$ denote the spatial projection operator and its definition will be given below.

2.1. Stability analysis

Firstly, we introduce the definition of spatial projection operator and the corresponding lemma. Next, we present the existence, uniqueness and stability conclusion for the solution of (2.5).

Definition 2.1. *[31] Denote $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, then the orthogonal projection in space $\mathbb{P}_N^1 : H_0^1(\Omega) \to \mathbf{V}_N^0$ is given by*

$$(\nabla(\mathbb{P}_N^1 u - u), \nabla v)_\Omega = 0, \quad \forall v \in \mathbf{V}_N^0. \tag{2.6}$$

Lemma 2.1. *[31] If $v \in H_0^1(\Omega) \cap H^r(\Omega)$ and $r \geq 1$, we have*

$$N\|\mathbb{P}_N^1 v - v\|_\Omega + \|\nabla(\mathbb{P}_N^1 v - v)\|_\Omega \leq CN^{1-r}\|v\|_{r,\Omega}. \tag{2.7}$$

Theorem 2.1. *Assume that $u_0(x, y) \in H_0^1(\Omega) \cap H^r(\Omega)$ $(r \geq 1)$ and $f \in L^2(\Sigma)$, then the scheme (2.5) has a unique solution u_L satisfying*

$$\|\partial_t \nabla u_L\|_{\Sigma,\omega_{1,0}} + \|\nabla u_L\|_\Sigma + \|u_L\|_\Sigma \leq C(\|u_0\|_{1,\Omega} + \|f\|_\Sigma). \tag{2.8}$$

Proof. Taking $v = (1-t)\partial_t u_L (\in \mathbf{V}_N^0 \otimes V_M)$ and using the integration by parts, we can get for the left-hand side of (2.5)

$$(\partial_t u_L, (1-t)\partial_t u_L)_\Sigma = \|\partial_t u_L\|^2_{\Sigma,\omega_{1,0}}, \tag{2.9}$$

$$\varepsilon(\partial_t \nabla u_L, \nabla(1-t)\partial_t u_L)_\Sigma = \varepsilon(\partial_t \nabla u_L, (1-t)\partial_t \nabla u_L)_\Sigma = \varepsilon\|\partial_t \nabla u_L\|^2_{\Sigma,\omega_{1,0}}, \tag{2.10}$$

$$\gamma(u_L, (1-t)\partial_t u_L)_\Sigma = -2\gamma\|u_L(-1)\|^2_\Omega - \gamma(u_L, (1-t)\partial_t u_L)_\Sigma + \gamma(u_L, u_L)_\Sigma, \tag{2.11}$$

namely,

$$\gamma(u_L, (1-t)\partial_t u_L)_\Sigma = -\gamma\|u_L(-1)\|^2_\Omega + \frac{\gamma}{2}\|u_L\|^2_\Sigma, \tag{2.12}$$

similarly,

$$\mu(\nabla u_L, \nabla(1-t)\partial_t u_L)_\Sigma = -\mu\|\nabla u_L(-1)\|^2_\Omega + \frac{\mu}{2}\|\nabla u_L\|^2_\Sigma. \tag{2.13}$$

Additionally, by the Cauchy-Schwarz inequality and Young's inequality, the right-hand side of (2.5) can be estimated as

$$(f, (1-t)\partial_t u_L)_\Sigma \leq \|f\|_\Sigma \|(1-t)\partial_t u_L\|_\Sigma \leq \frac{1}{2}\|f\|^2_\Sigma + \|\partial_t u_L\|^2_{\Sigma,\omega_{1,0}}. \tag{2.14}$$

Collecting (2.9)–(2.14) leads to

$$\begin{aligned} &\|\partial_t u_L\|^2_{\Sigma,\omega_{1,0}} + \varepsilon\|\partial_t \nabla u_L\|^2_{\Sigma,\omega_{1,0}} + \frac{\mu}{2}\|\nabla u_L\|^2_\Sigma + \frac{\gamma}{2}\|u_L\|^2_\Sigma \\ \leq &\gamma\|u_L(-1)\|^2_\Omega + \mu\|\nabla u_L(-1)\|^2_\Omega + \frac{1}{2}\|f\|^2_\Sigma + \|\partial_t u_L\|^2_{\Sigma,\omega_{1,0}}, \end{aligned} \tag{2.15}$$

namely,

$$\|\partial_t \nabla u_L\|_{\Sigma,\omega_{1,0}} + \|\nabla u_L\|_\Sigma + \|u_L\|_\Sigma \le C(\|u_L(-1)\|_\Omega + \|\nabla u_L(-1)\|_\Omega + \|f\|_\Sigma). \tag{2.16}$$

For initial conditions $u_L(-1)$ and $\nabla u_L(-1)$ in (2.16), according to Lemma 2.1, we can easily get the following estimate:

$$\begin{aligned} \|u_L(-1)\|_\Omega + \|\nabla u_L(-1)\|_\Omega &= \|\mathbb{P}_N^1 u_0\|_\Omega + \|\nabla \mathbb{P}_N^1 u_0\|_\Omega \\ &\le \|\mathbb{P}_N^1 u_0 - u_0\|_\Omega + \|u_0\|_\Omega + \|\nabla(\mathbb{P}_N^1 u_0 - u_0)\|_\Omega + \|\nabla u_0\|_\Omega \\ &\le CN^{-r}\|u_0\|_{r,\Omega} + \|u_0\|_\Omega + CN^{1-r}\|u_0\|_{r,\Omega} + \|\nabla u_0\|_\Omega \\ &\le C\|u_0\|_{r,\Omega}. \end{aligned} \tag{2.17}$$

Thus, combining estimations (2.16) and (2.17), we immediately attain the stability conclusion.

Remark 2.1. *From stability conclusion, there exists a zero solution if $f = 0$ and $u_0 = 0$. In other words, we can easily obtain the existence and uniqueness of u_L.*

2.2. Error analysis

The purpose of this section is to show an $L^2(\Sigma)$-error estimate of the single interval Legendre space-time spectral method by applying the dual technique. Now, we first introduce following definition and lemma of the time projection operator which will be covered later..

Definition 2.2. *[28] The orthogonal projection in time $\mathbf{\Pi}_M : H^1(I) \to P_M(I)$ is given by*

$$(\mathbf{\Pi}_M u - u, v)_I = 0, \quad \forall v \in V_M, \tag{2.18}$$

and $\mathbf{\Pi}_M u(-1) = u(-1)$.

Lemma 2.2. *[28] (a) If $u \in H^\sigma(I)$ and $\sigma \ge 1$, then*

$$\|\mathbf{\Pi}_M u - u\|_{I,\omega_{l,-1}} \le CM^{\frac{1}{4}(1-l)-\sigma}\|\partial_t^\sigma u\|_{I,\omega_{\sigma-1,\sigma-1}}, \quad l = 0, 1. \tag{2.19}$$

(b) If $u \in H^\sigma(I)$ and $\sigma \ge 2$, then

$$\|\mathbf{\Pi}_M u - u\|_{I,\omega_{0,-1}} \le CM^{\frac{1}{8}-\sigma}\|\partial_t^\sigma u\|_{I,\omega_{\sigma-2,\sigma-2}}. \tag{2.20}$$

Let $U = \mathbb{P}_N^1 \mathbf{\Pi}_M u$. Now we decompose the error into: $u_L - u = (u_L - U) + (U - u)$ and denote $\tilde{u} = u_L - U$. So according to (2.5) we have

$$\begin{aligned} &(\partial_t \tilde{u}, v)_\Sigma + \varepsilon(\partial_t \nabla \tilde{u}, \nabla v)_\Sigma + \mu(\nabla \tilde{u}, \nabla v)_\Sigma + \gamma(\tilde{u}, v)_\Sigma \\ =&(\partial_t (u - U), v)_\Sigma + \varepsilon(\partial_t \nabla(u - U), \nabla v)_\Sigma + \mu(\nabla(u - U), \nabla v)_\Sigma + \gamma(u - U, v)_\Sigma, \quad \forall v \in \mathbf{V}_N^0 \otimes V_M. \end{aligned} \tag{2.21}$$

According to the Definitions 2.1 and 2.2, for the right-hand side terms of (2.21) we get

$$\begin{aligned} \varepsilon(\partial_t \nabla(u - U), \nabla v)_\Sigma &= \varepsilon(\partial_t \nabla(u - \mathbf{\Pi}_M u), \nabla v)_\Sigma + \varepsilon(\partial_t \nabla(\mathbf{\Pi}_M u - \mathbb{P}_N^1 \mathbf{\Pi}_M u), \nabla v)_\Sigma \\ &= \varepsilon(\partial_t \nabla(u - \mathbf{\Pi}_M u), \nabla v)_\Sigma, \end{aligned} \tag{2.22}$$

$$\begin{aligned}\mu(\nabla(u-U),\nabla v)_\Sigma &= \mu(\nabla(u-\mathbb{P}_N^1 u),\nabla v)_\Sigma + \mu(\nabla(\mathbb{P}_N^1 u-\mathbb{P}_N^1\mathbf{\Pi}_M u),\nabla v)_\Sigma \\ &= 0,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\gamma(u-U,v)_\Sigma &= \gamma(u-\mathbb{P}_N^1 u,v)_\Sigma + \gamma(\mathbb{P}_N^1 u-\mathbb{P}_N^1\mathbf{\Pi}_M u,v)_\Sigma \\ &= \gamma(u-\mathbb{P}_N^1 u,v)_\Sigma.\end{aligned} \tag{2.24}$$

Then (2.21) can be simplified as follows:

$$\begin{aligned}&(\partial_t\tilde{u},v)_\Sigma + \varepsilon(\partial_t\nabla\tilde{u},\nabla v)_\Sigma + \mu(\nabla\tilde{u},\nabla v)_\Sigma + \gamma(\tilde{u},v)_\Sigma \\ =&(\partial_t(u-U),v)_\Sigma + \varepsilon(\partial_t\nabla(I-\mathbf{\Pi}_M)u,\nabla v)_\Sigma + \gamma((I-\mathbb{P}_N^1)u,v)_\Sigma.\end{aligned} \tag{2.25}$$

According to the definition of $\tilde{u}$ and $\mathbf{\Pi}_M$, we note

$$\tilde{u}(x,y,-1) = u_L(x,y,-1) - \mathbb{P}_N^1\mathbf{\Pi}_M u(x,y,-1) = \mathbb{P}_N^1 u_0(x,y) - \mathbb{P}_N^1 u(x,y,-1) = 0. \tag{2.26}$$

Now, we give the $L^2(\Sigma)$-error estimates of the single interval Legendre space-time spectral scheme.

Theorem 2.2. *Suppose u_L and u are the solutions of the scheme* (2.5) *and problem* (1.1)*, respectively. If $u_0 \in H^r(\Omega)\cap H_0^1(\Omega)$ and $u \in H^\sigma(I;H^r(\Omega)\cap H_0^1(\Omega))$ for integers $r \geq 1$, then:*

(a) For $\sigma \geq 1$,

$$\begin{aligned}\|u-u_L\|_\Sigma \leq& C\Big\{N^{-r}(\|u\|_{L^2(I;H^r(\Omega))} + \|u-u_0\|_{L^2_{\omega_{0,-1}}(I;H^r(\Omega))}) \\ &+ M^{\frac14-\sigma}(\|\partial_t^\sigma u\|_{L^2_{\omega_{\sigma-1,\sigma-1}}(I;L^2(\Omega))} + N^{-r}\|\partial_t^\sigma u\|_{L^2_{\omega_{\sigma-1,\sigma-1}}(I;H^r(\Omega))} + \|\partial_t^\sigma u\|_{L^2_{\omega_{\sigma-1,\sigma-1}}(I;L^2(\Omega))})\Big\}.\end{aligned} \tag{2.27}$$

(b) For $\sigma \geq 2$,

$$\begin{aligned}\|u-u_L\|_\Sigma \leq& C\Big\{N^{-r}(\|u\|_{L^2(I;H^r(\Omega))} + \|u-u_0\|_{L^2_{\omega_{0,-1}}(I;H^r(\Omega))}) \\ &+ M^{\frac18-\sigma}(\|\partial_t^\sigma u\|_{L^2_{\omega_{\sigma-2,\sigma-2}}(I;L^2(\Omega))} + N^{-r}\|\partial_t^\sigma u\|_{L^2_{\omega_{\sigma-2,\sigma-2}}(I;H^r(\Omega))} + \|\partial_t^\sigma u\|_{L^2_{\omega_{\sigma-2,\sigma-2}}(I;L^2(\Omega))})\Big\}.\end{aligned} \tag{2.28}$$

Proof. In order to use the dual technique to attain the $L^2(\Sigma)$-error estimates, denote $H'(I) = \{v \in H^1(I) : v(-1) = 0\}$. Then for Eq (2.25) we can write its dual equation: For a given $g \in \mathbf{V}_N^0 \otimes P_M(I)$, obtain $u_g \in \mathbf{V}_N^0 \otimes V_M$ such that

$$(\partial_t e,u_g)_\Sigma + \varepsilon(\partial_t\nabla e,\nabla u_g)_\Sigma + \mu(\nabla e,\nabla u_g)_\Sigma + \gamma(e,u_g)_\Sigma = (g,e)_\Sigma,\ \ \forall e \in \mathbf{V}_N^0 \otimes (P_M(I)\cap H'(I)). \tag{2.29}$$

Firstly, we present the existence and uniqueness of u_g. Assuming $g = 0$ and taking $e = \partial_t^{-1}[\omega_{-1,0}u_g] \doteq \int_{-1}^t \frac{1}{1-s}u_g\mathrm{d}s$ in (2.29), then we have

$$(\partial_t e,u_g)_\Sigma = (\partial_t\partial_t^{-1}[\omega_{-1,0}u_g],u_g)_\Sigma = (\omega_{-1,0}u_g,u_g)_\Sigma = \|u_g\|^2_{\Sigma,\omega_{-1,0}}, \tag{2.30}$$

$$\begin{aligned}\varepsilon(\partial_t\nabla e,\nabla u_g)_\Sigma &= \varepsilon(\partial_t\nabla\partial_t^{-1}[\omega_{-1,0}u_g],\nabla u_g)_\Sigma = \varepsilon(\partial_t\partial_t^{-1}[\omega_{-1,0}\nabla u_g],\nabla u_g)_\Sigma \\ &= \varepsilon(\omega_{-1,0}\nabla u_g,\nabla u_g)_\Sigma = \varepsilon\|\nabla u_g\|^2_{\Sigma,\omega_{-1,0}},\end{aligned} \tag{2.31}$$

$$\begin{aligned}\mu(\nabla e,\nabla u_g)_\Sigma &= \mu(\partial_t^{-1}[\omega_{-1,0}\nabla u_g],\nabla u_g)_\Sigma = \mu(\varphi,(1-t)\varphi_t)_\Sigma\\ &= \mu\int_\Omega\Big((1-t)\varphi^2|_{-1}^1 - \int_I \varphi\partial_t((1-t)\varphi)\mathrm{d}t\Big)\mathrm{d}x\mathrm{d}y\\ &= -\mu\Big(\varphi,\partial_t((1-t)\varphi)\Big)_\Sigma = -\mu(\varphi,(1-t)\varphi_t))_\Sigma + \mu(\varphi,\varphi)_\Sigma,\end{aligned} \tag{2.32}$$

namely,

$$\mu(\nabla e,\nabla u_g)_\Sigma = \frac{\mu}{2}\|\partial_t^{-1}[\omega_{-1,0}\nabla u_g]\|_\Sigma^2, \tag{2.33}$$

where $\varphi = \partial_t^{-1}[\omega_{0,-1}\nabla u_g]$ and similarly

$$\gamma(\partial_t^{-1}[\omega_{-1,0}u_g],u_g)_\Sigma = \frac{\gamma}{2}\|\partial_t^{-1}[\omega_{-1,0}u_g]\|_\Sigma^2. \tag{2.34}$$

Collecting (2.30)–(2.34), we have

$$\|u_g\|_{\Sigma,\omega_{-1,0}}^2 + \varepsilon\|\nabla u_g\|_{\Sigma,\omega_{-1,0}}^2 + \frac{\mu}{2}\|\partial_t^{-1}[\omega_{-1,0}\nabla u_g]\|_\Sigma^2 + \frac{\gamma}{2}\|\partial_t^{-1}[\omega_{-1,0}u_g]\|_\Sigma^2 = 0. \tag{2.35}$$

Then $u_g = 0$.

Now, in order to derive the estimate of $\tilde{u}$, we first consider the estimates of u_g, $\partial_t u_g$, $\partial_t\nabla u_g$.

Taking $e = -(1+t)\partial_t u_g$ in (2.29), we can get

$$\begin{aligned}(\partial_t e,u_g)_\Sigma &= -(\partial_t((1+t)\partial_t u_g),u_g)_\Sigma\\ &= \int_\Omega\big(-(1+t)u_g\partial_t u_g)|_{-1}^1\mathrm{d}x\mathrm{d}y + ((1+t)\partial_t u_g,\partial_t u_g)_\Sigma = \|\partial_t u_g\|_{\Sigma,\omega_{0,1}}^2,\end{aligned} \tag{2.36}$$

$$\varepsilon(\partial_t\nabla e,\nabla u_g)_\Sigma = -\varepsilon\big(\partial_t((1+t)\partial_t\nabla u_g),\nabla u_g\big)_\Sigma = \varepsilon\|\partial_t\nabla u_g\|_{\Sigma,\omega_{0,1}}^2, \tag{2.37}$$

$$\gamma(e,u_g)_\Sigma = -\gamma\big((1+t)\partial_t u_g,u_g\big)_\Sigma = \gamma(u_g,u_g)_\Sigma + \gamma\big(u_g,(1+t)\partial_t u_g\big)_\Sigma = \frac{\gamma}{2}\|u_g\|_\Sigma^2, \tag{2.38}$$

$$\mu(\nabla e,\nabla u_g)_\Sigma = -\mu((1+t)\partial_t\nabla u_g,\nabla u_g)_\Sigma = \frac{\mu}{2}\|\nabla u_g\|_\Sigma^2. \tag{2.39}$$

Collecting (2.36)–(2.39), we can obtain

$$\begin{aligned}&\|\partial_t u_g\|_{\Sigma,\omega_{0,1}}^2 + \varepsilon\|\partial_t\nabla u_g\|_{\Sigma,\omega_{0,1}}^2 + \frac{\gamma}{2}\|u_g\|_\Sigma^2 + \frac{\mu}{2}\|\nabla u_g\|_\Sigma^2\\ =&(g,e)_\Sigma = -\big(g,(1+t)\partial_t u_g\big)_\Sigma \le \|g\|_\Sigma\|\partial_t u_g\|_{\Sigma,\omega_{0,2}} \le \sqrt{2}\|g\|_\Sigma\|\partial_t u_g\|_{\Sigma,\omega_{0,1}}.\end{aligned} \tag{2.40}$$

Then we get

$$\begin{aligned}&\|\partial_t u_g\|_{\Sigma,\omega_{0,1}} \le \sqrt{2}\|g\|_\Sigma,\\ &\|\partial_t\nabla u_g\|_{\Sigma,\omega_{0,1}} \le \sqrt{\frac{2}{\varepsilon}}\|g\|_\Sigma,\\ &\|u_g\|_\Sigma \le \frac{2}{\sqrt{\gamma}}\|g\|_\Sigma.\end{aligned} \tag{2.41}$$

Next, based on the above estimates we deduce the estimate of $\tilde{u}$. When $g = \tilde{u},\ e = \tilde{u}$ in (2.29) and utilizing integration by parts, we can see

$$\begin{aligned}\|\tilde{u}\|_\Sigma^2 =&(g, \tilde{u})_\Sigma = (\partial_t \tilde{u}, u_g)_\Sigma + \varepsilon(\partial_t \nabla \tilde{u}, \nabla u_g)_\Sigma + \mu(\nabla \tilde{u}, \nabla u_g)_\Sigma + \gamma(\tilde{u}, u_g)_\Sigma\\ =&(\partial_t (I - \mathbb{P}_N^1 \mathbf{\Pi}_M)u, u_g)_\Sigma + \varepsilon(\partial_t \nabla(I - \mathbf{\Pi}_M)u, \nabla u_g)_\Sigma + \gamma((I - \mathbb{P}_N^1)u, u_g)_\Sigma\\ =&-((I - \mathbb{P}_N^1 \mathbf{\Pi}_M)(u - u_0), \partial_t u_g)_\Sigma - \varepsilon((I - \mathbf{\Pi}_M)\nabla u, \partial_t \nabla u_g)_\Sigma + \gamma((I - \mathbb{P}_N^1)u, u_g)_\Sigma\\ \le&\|(I - \mathbb{P}_N^1 \mathbf{\Pi}_M)(u - u_0)\|_{\Sigma,\omega_{0,-1}} \|\partial_t u_g\|_{\Sigma,\omega_{0,1}} + \varepsilon\|(I - \mathbf{\Pi}_M)\nabla u\|_{\Sigma,\omega_{0,-1}} \|\partial_t \nabla u_g\|_{\Sigma,\omega_{0,1}}\\ &+ \gamma\|(I - \mathbb{P}_N^1)u\|_\Sigma \|u_g\|_\Sigma.\end{aligned} \tag{2.42}$$

Then we get

$$\begin{aligned}\|\tilde{u}\|_\Sigma \le& C(\|(I - \mathbb{P}_N^1 \mathbf{\Pi}_M)(u - u_0)\|_{\Sigma,\omega_{0,-1}} + \|(I - \mathbf{\Pi}_M)\nabla u\|_{\Sigma,\omega_{0,-1}} + \|(I - \mathbb{P}_N^1)u\|_\Sigma)\\ \le& C(\|(I - \mathbb{P}_N^1)(u - u_0)\|_{\Sigma,\omega_{0,-1}} + \|(I - \mathbf{\Pi}_M)u\|_{\Sigma,\omega_{0,-1}}\\ &+ \|(I - \mathbb{P}_N^1)(I - \mathbf{\Pi}_M)u\|_{\Sigma,\omega_{0,-1}} + \|(I - \mathbf{\Pi}_M)\nabla u\|_{\Sigma,\omega_{0,-1}} + \|(I - \mathbb{P}_N^1)u\|_\Sigma).\end{aligned} \tag{2.43}$$

Finally, according to the triangle inequality, we deduce

$$\begin{aligned}\|u - u_L\|_\Sigma \le& \|u - U\|_\Sigma + \|\tilde{u}\|_\Sigma \le \|u - \mathbb{P}_N^1 \mathbf{\Pi}_M u\|_\Sigma + \|\tilde{u}\|_\Sigma\\ \le& C(\|(I - \mathbb{P}_N^1)u\|_\Sigma + \|(I - \mathbf{\Pi}_M)u\|_{\Sigma,\omega_{0,-1}} + \|(I - \mathbb{P}_N^1)(I - \mathbf{\Pi}_M)u\|_{\Sigma,\omega_{0,-1}}\\ &+ \|(I - \mathbb{P}_N^1)(u - u_0)\|_{\Sigma,\omega_{0,-1}} + \|(I - \mathbf{\Pi}_M)\nabla u\|_{\Sigma,\omega_{0,-1}}).\end{aligned} \tag{2.44}$$

Then, by Lemmas 2.1 and 2.2, we directly derive the final $L^2(\Sigma)$-error estimates.

3. Theoretical analysis of multi-interval Legendre spectral method in time

In order to construct the multi-interval form of the Legendre space-time spectral fully discrete scheme, we take $a_1 = -1$, $a_{K+1} = 1$, $a_k < a_{k+1}$ and denote $I_k = (a_k, a_{k+1}]$, $c_k = a_{k+1} - a_k$, $d_k = c_k/2$, namely, $I = \bigcup_{k=1}^{K} I_k$, where K is a known positive integer.

Denote

$$\|v(t)\|_{I,\omega_{\alpha,\beta}}^2 = \sum_{I_k} \|v^k(t)\|_{I_k,\tilde{\omega}_{\alpha,\beta}}^2, \quad \forall v(t) \in L^2_{\omega_{\alpha,\beta}}(I), \tag{3.1}$$

where $v^k(t) = v(t)|_{I_k}$, $\tilde{\omega}_{\alpha,\beta} = \frac{(1-t)^\alpha}{d_k^\alpha}(2 - \frac{1-t}{d_k})^\beta$ $(t \in I_k)$ and the definition of $\|v^k(t)\|_{I_k,\tilde{\omega}_{\alpha,\beta}}^2$ is presented in Section 2.

Moreover, let $\Sigma_k = \Omega \times I_k$ and denote

$$\|v(x, y, t)\|_{L^2_{\omega_{\alpha,\beta}}(I;H^r(\Omega))}^2 = \sum_{I_k} \|v^k(x, y, t)\|_{L^2_{\tilde{\omega}_{\alpha,\beta}}(I_k;H^r(\Omega))}^2, \quad \forall v(x, y, t) \in L^2_{\omega_{\alpha,\beta}}(I; H^r(\Omega)), \tag{3.2}$$

where $v^k(t) = v(t)|_{\Sigma_k}$ and the definition of $\|v^k(x, y, t)\|_{L^2_{\tilde{\omega}_{\alpha,\beta}}(I_k;H^r(\Omega))}$ is presented in Section 2.

Let $M = (M_1, \cdots, M_K)$ and $L = (N, M)$. Define the space of trial and test functions in time

$$\begin{aligned}&X_K^M = Y_K^M \cap H^1(I),\ \ Y_K^M = \{v : v|_{I_k} \in P_{M_k}(I_k),\ 1 \le k \le K\},\\ &\widetilde{X}_K^M = \{v : v = (1 - t)q(t),\ q(t) \in Y_K^{M-1}\},\end{aligned} \tag{3.3}$$

where $P_{M_k}(I_k)$ is a space of polynomials of degree $\leq M_k$ on time span I_k and $M-1 = (M_1-1, \cdots, M_K-1)$.

Then applying the Green's formula, we can write the multi-interval Legendre space-time spectral fully discrete scheme of (1.1) as: Find $u_L^K \in \mathbf{V}_N^0 \otimes X_K^M$ satisfying

$$\begin{cases}(\partial_t u_L^K, v)_\Sigma + \varepsilon(\partial_t \nabla u_L^K, \nabla v)_\Sigma + \mu(\nabla u_L^K, \nabla v)_\Sigma + \gamma(u_L^K, v)_\Sigma = (f, v)_\Sigma, \quad \forall v \in \mathbf{V}_N^0 \otimes \widetilde{X}_K^M, \\ u_L^K(x, y, -1) = \mathbb{P}_N^1 u_0(x, y). \end{cases} \tag{3.4}$$

The stability analysis of (3.4) is similar to the single interval method, so we only provide the error analysis of the scheme. Firstly, we introduce the following definition and lemma of the multi-interval projection operator in time.

Definition 3.1. *[28] The orthogonal projection in time $\mathbf{\Pi}_M : H^1(I) \to X_K^M$ is given by*

$$(\partial_t(\mathbf{\Pi}_M u - u), v)_I = 0, \quad \forall v \in \widetilde{X}_K^M, \tag{3.5}$$

with $\mathbf{\Pi}_M u(-1) = u(-1)$.

Lemma 3.1. *[28] If $u \in H^\sigma(I)$ and $\sigma \geq 1$, $\bar{M} = \min_{1\leq k\leq K} M_k$, we have*

$$\|\partial_t^l(\mathbf{\Pi}_M u - u)\|_{I,\omega_{l,l-1}} \leq C\bar{M}^{l-1} \sum_{k=1}^{K} (d_k^{-1} M_k)^{1-\sigma} \|\partial_t^\sigma u^k\|_{I_k}, \quad l = 0, 1, \tag{3.6}$$

where $u^k = u|_{I_k}$.

Denote $U_K = \mathbb{P}_N^1 \mathbf{\Pi}_M u$. Now, we decompose the error into: $u - u_L^K = (u - U_K) + (U_K - u_L^K)$ and denote $\tilde{u}_K = U_K - u_L^K$. We note $\tilde{u}_K(x, y, -1) = 0$. Then according to the scheme (3.4), $\forall v \in \mathbf{V}_N^0 \otimes \widetilde{X}_K^M$, we have

$$\begin{aligned}&(\partial_t \tilde{u}_K, v)_\Sigma + \varepsilon(\partial_t \nabla \tilde{u}_K, \nabla v)_\Sigma + \mu(\nabla \tilde{u}_K, \nabla v)_\Sigma + \gamma(\tilde{u}_K, v)_\Sigma \\ &= (\partial_t(U_K - u), v)_\Sigma + \varepsilon(\partial_t \nabla(U_K - u), \nabla v)_\Sigma + \mu(\nabla(U_K - u), \nabla v)_\Sigma + \gamma(U_K - u, v)_\Sigma.\end{aligned} \tag{3.7}$$

By using the Definitions 2.1 and 3.1, we can see for some terms of the formula (3.7)

$$(\partial_t(\mathbb{P}_N^1 \mathbf{\Pi}_M u - u), v)_\Sigma = (\partial_t(\mathbb{P}_N^1 \mathbf{\Pi}_M u - \mathbb{P}_N^1 u), v)_\Sigma + (\partial_t(\mathbb{P}_N^1 - I)u, v)_\Sigma = (\partial_t(\mathbb{P}_N^1 - I)u, v)_\Sigma, \tag{3.8}$$

$$\varepsilon(\partial_t \nabla(\mathbb{P}_N^1 \mathbf{\Pi}_M u - u), \nabla v)_\Sigma = \varepsilon(\partial_t(\mathbf{\Pi}_M \nabla \mathbb{P}_N^1 u - \nabla \mathbb{P}_N^1 u), \nabla v)_\Sigma + \varepsilon(\nabla(\mathbb{P}_N^1 \partial_t u - \partial_t u, \nabla v)_\Sigma = 0, \tag{3.9}$$

$$\begin{aligned}\mu(\nabla(\mathbb{P}_N^1 \mathbf{\Pi}_M u - u), \nabla v)_\Sigma &= \mu(\nabla(\mathbf{\Pi}_M \mathbb{P}_N^1 u - \mathbf{\Pi}_M u), \nabla v)_\Sigma + \mu(\nabla(\mathbf{\Pi}_M - I)u, \nabla v)_\Sigma \\ &= \mu(\nabla(\mathbf{\Pi}_M - I)u, \nabla v)_\Sigma.\end{aligned} \tag{3.10}$$

Then (3.7) can be simplified as follows:

$$\begin{aligned}&(\partial_t \tilde{u}_K, v)_\Sigma + \varepsilon(\partial_t \nabla \tilde{u}_K, \nabla v)_\Sigma + \mu(\nabla \tilde{u}_K, \nabla v)_\Sigma + \gamma(\tilde{u}_K, v)_\Sigma \\ =&(\partial_t(\mathbb{P}_N^1 - I)u, v)_\Sigma + \mu(\nabla(\mathbf{\Pi}_M - I)u, \nabla v)_\Sigma + \gamma((\mathbb{P}_N^1 \mathbf{\Pi}_M - I)u, v)_\Sigma, \quad \forall v \in V_N^0 \otimes \widetilde{X}_K^M.\end{aligned} \tag{3.11}$$

Now, we give the $L^2(\Sigma)$-error estimate of the multi-interval Legendre space-time spectral scheme.

Theorem 3.1. *Suppose u and u_L^K are the solutions of the problem* (1.1) *and scheme* (3.4), *respectively. If $u \in H^\sigma(I; H^r(\Omega) \cap H_0^1(\Omega))$ for integers $r, \sigma \geq 1$, then*

$$\begin{aligned}\|u - u_L^K\|_\Sigma \leq C\Big\{ & N^{-r}\Big(\|u_t\|_{L^2(I;H^r(\Omega))} + \|u\|_{L^2(I;H^r(\Omega))}\Big) \\ & + \bar{M}^{-1}\sum_{k=1}^{K}(d_k^{-1}M_k)^{1-\sigma}\Big(\|\partial_t^\sigma \nabla u^k\|_{L^2(I_k;L^2(\Omega))} \\ & + \|\partial_t^\sigma u^k\|_{L^2(I_k;L^2(\Omega))} + N^{-r}\|\partial_t^\sigma u^k\|_{L^2(I_k;H^r(\Omega))}\Big)\Big\},\end{aligned} \tag{3.12}$$

where $u^k = u|_{\Sigma_k}$.

If $d_k = d, M_k = M$, we have

$$\begin{aligned}\|u - u_L^K\|_\Sigma \leq C\Big\{ & N^{-r}\Big(\|u_t\|_{L^2(I;H^r(\Omega))} + \|u\|_{L^2(I;H^r(\Omega))}\Big) \\ & + dM^{-\sigma}\Big(\|\partial_t^\sigma \nabla u\|_{L^2(I;L^2(\Omega))} + \|\partial_t^\sigma u\|_{L^2(I;L^2(\Omega))} + N^{-r}\|\partial_t^\sigma u\|_{L^2(I;H^r(\Omega))}\Big)\Big\}.\end{aligned} \tag{3.13}$$

Proof. Similar to the analysis of the $L^2(\Sigma)$-error estimate of the single interval scheme, we also take the dual technique to give the proof. Considering the dual equation of (3.11), for a given $g \in \mathbf{V}_N^0 \otimes X_K^M$, we obtain $u_g \in \mathbf{V}_N^0 \otimes \widetilde{X}_K^M$ such that

$$(\partial_t w, u_g)_\Sigma + \varepsilon(\partial_t \nabla w, \nabla u_g)_\Sigma + \mu(\nabla w, \nabla u_g)_\Sigma + \gamma(w, u_g)_\Sigma = (g, w)_\Sigma, \quad \forall w \in \mathbf{V}_N^0 \otimes (Y_K^M \cap H'(I)). \tag{3.14}$$

The existence and uniqueness of u_g can be easily attained, so we focus on using the dual equation (3.14) to present the $L^2(\Sigma)$-error estimate.

In order to derive the estimate of $\tilde{u}$, we take $g = \tilde{u}_K$ and $w = \tilde{u}_K$ in Eq (3.14),

$$\begin{aligned}\|\tilde{u}_K\|_\Sigma^2 = & (g, \tilde{u}_K)_\Sigma = (\partial_t \tilde{u}_K, u_g)_\Sigma + \varepsilon(\partial_t \nabla \tilde{u}_K, \nabla u_g)_\Sigma + \mu(\nabla \tilde{u}_K, \nabla u_g)_\Sigma + \gamma(\tilde{u}_K, u_g)_\Sigma \\ = & (\partial_t(\mathbb{P}_N^1 - I)u, u_g)_\Sigma + \mu(\nabla(\mathbf{\Pi}_M - I)u, \nabla u_g)_\Sigma + \gamma((\mathbb{P}_N^1\mathbf{\Pi}_M - I)u, u_g)_\Sigma \\ \leq & \|(\mathbb{P}_N^1 - I)u_t\|_{\Sigma,\omega_{1,0}}\|u_g\|_{\Sigma,\omega_{-1,0}} + \mu\|(\mathbf{\Pi}_M - I)\nabla u\|_{\Sigma,\omega_{1,0}}\|\nabla u_g\|_{\Sigma,\omega_{-1,0}} \\ & + \gamma\|(\mathbb{P}_N^1\mathbf{\Pi}_M - I)u\|_{\Sigma,\omega_{1,0}}\|u_g\|_{\Sigma,\omega_{-1,0}}.\end{aligned} \tag{3.15}$$

Now, to deduce the estimates of of u_g and ∇u_g, taking $w = \int_{-1}^t \frac{1}{1-s}u_g \mathrm{d}s \doteq \partial_t^{-1}[\omega_{-1,0}u_g]$ in Eq (3.14),

$$\begin{aligned}& \|u_g\|_{\Sigma,\omega_{-1,0}}^2 + \varepsilon\|\nabla u_g\|_{\Sigma,\omega_{-1,0}}^2 + \frac{\mu}{2}\|\partial_t^{-1}[\omega_{-1,0}\nabla u_g]\|_\Sigma^2 + \frac{\gamma}{2}\|\partial_t^{-1}[\omega_{-1,0}u_g]\|_\Sigma^2 \\ = & (g, \partial_t^{-1}[\omega_{-1,0}u_g])_\Sigma \leq \|g\|_\Sigma\|\partial_t^{-1}[\omega_{-1,0}u_g]\|_\Sigma,\end{aligned} \tag{3.16}$$

then we get

$$\begin{aligned}& \|\partial_t^{-1}[\omega_{-1,0}u_g]\|_\Sigma \leq \frac{2}{\gamma}\|g\|_\Sigma, \\ & \|u_g\|_{\Sigma,\omega_{-1,0}} \leq \sqrt{\frac{2}{\gamma}}\|g\|_\Sigma, \\ & \|\nabla u_g\|_{\Sigma,\omega_{-1,0}} \leq \sqrt{\frac{2}{\gamma\varepsilon}}\|g\|_\Sigma.\end{aligned} \tag{3.17}$$

Taking estimates (3.17) into inequality (3.15) have

$$\|\tilde{u}_K\|_\Sigma^2 \le C(\|(I-\mathbb{P}_N^1)u_t\|_{\Sigma,\omega_{1,0}}\|g\|_\Sigma + \|(\mathbf{\Pi}_M - I)\nabla u\|_{\Sigma,\omega_{1,0}}\|g\|_\Sigma + \|(\mathbf{\Pi}_M\mathbb{P}_N^1 - I)u\|_{\Sigma,\omega_{1,0}}\|g\|_\Sigma), \tag{3.18}$$

namely, error estimate of $\tilde{u}_K$ is that

$$\begin{aligned}\|\tilde{u}_K\|_\Sigma &\le C(\|(I-\mathbb{P}_N^1)u_t\|_{\Sigma,\omega_{1,0}} + \|(\mathbf{\Pi}_M - I)\nabla u\|_{\Sigma,\omega_{1,0}} + \|(\mathbf{\Pi}_M\mathbb{P}_N^1 - I)u\|_{\Sigma,\omega_{1,0}})\\ &\le C(\|(I-\mathbb{P}_N^1)u_t\|_\Sigma + \|(\mathbf{\Pi}_M - I)\nabla u\|_\Sigma + \|(\mathbf{\Pi}_M\mathbb{P}_N^1 - I)u\|_\Sigma).\end{aligned} \tag{3.19}$$

Finally, according to the triangle inequality, we deduce

$$\begin{aligned}\|u-u_L^K\|_\Sigma \le& \|u-U_K\|_\Sigma + \|\tilde{u}_K\|_\Sigma\\ \le& C(\|(\mathbb{P}_N^1 - I)u_t\|_\Sigma + \|(\mathbf{\Pi}_M - I)\nabla u\|_{\Sigma,\omega_{0,-1}} + \|(\mathbf{\Pi}_M\mathbb{P}_N^1 - I)u\|_\Sigma)\\ \le& C(\|(\mathbb{P}_N^1 - I)u_t\|_\Sigma + \|(\mathbf{\Pi}_M - I)\nabla u\|_{\Sigma,\omega_{0,-1}}\\ &+ \|(\mathbb{P}_N^1 - I)u\|_\Sigma + \|(\mathbf{\Pi}_M - I)\nabla u\|_{\Sigma,\omega_{0,-1}} + \|(\mathbb{P}_N^1 - I)(\mathbf{\Pi}_M - I)\nabla u\|_{\Sigma,\omega_{0,-1}}).\end{aligned} \tag{3.20}$$

4. Implementation

In this section, we present the detailed implementation of the multi-interval case by taking Fourier-like basis functions in space and the appropriate basis functions in time. Regarding the single interval case, please see [26] for more information.

Let

$$\begin{aligned}&X_k^{M_k} = \{v : v = (1-t)q(t), q(t) \in P_{M_k-1}(I_k), t \in I_k\}\\ &u^{(k)}(x,y,t) := u_L^K(x,y,t)|_{\Sigma_k}, \quad f^{(k)}(x,y,t) := f(x,y,t)|_{\Sigma_k}, \quad 1 \le k \le K.\end{aligned} \tag{4.1}$$

Then, according to the scheme (3.4), we can see for each k $(1 \le k \le K)$, we find $u^{(k)} \in \mathbf{V}_N^0 \otimes P_{M_k}(I_k)$ such that

$$\begin{cases}(\partial_t u^{(k)}, v^{(k)})_{\Sigma_k} + \varepsilon(\partial_t \nabla u^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \mu(\nabla u^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \gamma(u^{(k)}, v^{(k)})_{\Sigma_k} = (f^{(k)}, v^{(k)})_{\Sigma_k},\\ u^{(k)}(x,y,a_k) = u^{(k-1)}(x,y,a_k), \qquad \forall v^{(k)} \in \mathbf{V}_N^0 \otimes X_k^{M_k},\end{cases} \tag{4.2}$$

where $u^{(0)}(x,y,a_1) = \mathbb{P}_N^1 u_0(x,y)$.

Furthermore, let

$$u^{(k)}(x,y,t) = w^{(k)}(x,y,t) + u^{(k-1)}(x,y,a_k),$$

then the scheme (4.2) can be converted into: Find $w^{(k)} \in \mathbf{V}_N^0 \otimes P_{M_k}(I_k)$ such that

$$\begin{cases}\ (\partial_t w^{(k)}, v^{(k)})_{\Sigma_k} + \varepsilon(\partial_t \nabla w^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \mu(\nabla w^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \gamma(w^{(k)}, v^{(k)})_{\Sigma_k}\\ =(f^{(k)}, v^{(k)})_{\Sigma_k} - \mu(\nabla u^{(k-1)}(a_k), \nabla v^{(k)})_{\Sigma_k} - \gamma(u^{(k-1)}(a_k), v^{(k)})_{\Sigma_k}, \quad \forall v^{(k)} \in \mathbf{V}_N^0 \otimes X_k^{M_k},\\ \ w^{(k)}(x,y,a_k) = 0.\end{cases} \tag{4.3}$$

In order to the operability of the scheme (4.3), we try to separate it into two parts: Find $w_q^{(k)} \in \mathbf{V}_N^0 \otimes P_{M_k}(I_k)$ $(q = 1,2)$ such that

$$\begin{cases}\ (\partial_t w_1^{(k)}, v^{(k)})_{\Sigma_k} + \varepsilon(\partial_t \nabla w_1^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \mu(\nabla w_1^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \gamma(w_1^{(k)}, v^{(k)})_{\Sigma_k}\\ =(f^{(k)}, v^{(k)})_{\Sigma_k} - \mu(\nabla \mathbb{P}_N^1 u_0, \nabla v^{(k)})_{\Sigma_k} - \gamma(\mathbb{P}_N^1 u_0, v^{(k)})_{\Sigma_k}, \quad \forall v^{(k)} \in \mathbf{V}_N^0 \otimes X_k^{M_k},\\ \ w_1^{(k)}(x,y,a_k) = 0,\end{cases} \tag{4.4}$$

$$\begin{cases} (\partial_t w_2^{(k)}, v^{(k)})_{\Sigma_k} + \varepsilon(\partial_t \nabla w_2^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \mu(\nabla w_2^{(k)}, \nabla v^{(k)})_{\Sigma_k} + \gamma(w_2^{(k)}, v^{(k)})_{\Sigma_k} \\ =\mu(\nabla u^{(k-1)}(a_k), \nabla v^{(k)})_{\Sigma_k} + \gamma(u^{(k-1)}(a_k), v^{(k)})_{\Sigma_k} - \mu(\nabla \mathbb{P}_N^1 u_0, \nabla v^{(k)})_{\Sigma_k} - \gamma(\mathbb{P}_N^1 u_0, v^{(k)})_{\Sigma_k}, \\ w_2^{(k)}(x, y, a_k) = 0, \qquad \forall v^{(k)} \in \mathbf{V}_N^0 \otimes X_k^{M_k}, \end{cases} \tag{4.5}$$

The solution of the scheme (4.3) is obtained by

$$w^{(k)}(x, y, t) = w_1^{(k)}(x, y, t) - w_2^{(k)}(x, y, t). \tag{4.6}$$

The choice of basis functions $\phi_n(x)$ and $\phi_s(y)$ $(0 \le n, s \le N-2)$ can refer [26]. Regarding the temporal local trail functions $\psi_m^k(t)$ and test functions $\tilde{\psi}_m^k(t)$ $(0 \le m \le M_k - 1)$, please see [28]. Let

$$\begin{aligned} w^{(k)}(x, y, t) &= \sum_{m=0}^{M_k-1} \sum_{n,s=0}^{N-2} w_{m,n,s}^{(k)} \phi_n(x)\phi_s(y)\psi_m^k(t), \ W^{(k)} = (w_{m,n,s}^{(k)})_{M_k \times (N-1)^2}, \\ w_q^{(k)}(x, y, t) &= \sum_{m=0}^{M_k-1} \sum_{n,s=0}^{N-2} w_{q,m,n,s}^{(k)} \phi_n(x)\phi_s(y)\psi_m^k(t), \ W^{(k)} = (w_{q,m,n,s}^{(k)})_{M_k \times (N-1)^2}, \quad q = 1, 2, \end{aligned} \tag{4.7}$$

and $v^{(k)}(x, y, t) = \phi_{n'}(x)\phi_{s'}(y)\tilde{\psi}_{m'}^{(k)}(t)$, where $n', s' = 0, 1, \cdots, N-2$ and $m' = 0, 1, \cdots, M_k - 1$, we can see that $W^{(k)} = W_1^{(k)} - W_2^{(k)}$.

Denote the sets of LGL points and weights in spatial directions by $\{x_{\tilde{n}}, \varpi_{\tilde{n}}\}_{\tilde{n}=0}^{N+1}$ and $\{y_{\tilde{s}}, \varpi_{\tilde{s}}\}_{\tilde{s}=0}^{N+1}$, and denote the set of LGL points in time span I_k by $\{t_{\tilde{m}}, \dot{\varpi}_{\tilde{m}}\}_{\tilde{m}=0}^{M_k+1}$. Define

$$\begin{aligned} &\tilde{\Psi}^{(k)} = (\tilde{\psi}_m^{(k)}(t_{\tilde{m}}))_{M_k \times (M_k+2)}, \quad \mathcal{F}^{(k)} = (\mathfrak{f}_{\tilde{m},\tilde{n},\tilde{s}})_{(M_k+2)\times(N+2)^2}, \\ &\mathfrak{f}_{\tilde{m},\tilde{n},\tilde{s}} = \frac{b_k}{2}(f^{(k)}(x_{\tilde{n}}, y_{\tilde{s}}, t_{\tilde{m}}) + \mu\Delta\mathbb{P}_N^1 u_0(x_{\tilde{n}}, y_{\tilde{s}}) - \gamma\mathbb{P}_N^1 u_0(x_{\tilde{n}}, y_{\tilde{s}})). \end{aligned} \tag{4.8}$$

Then, we can get the matrix form of the scheme (4.4) as follows:

$$\begin{aligned} &(C^{(k)} + \gamma D^{(k)})W_1^{(k)}[\mathrm{Diag}(\lambda_n) \otimes \mathrm{Diag}(\lambda_s)] + (\varepsilon C^{(k)} + \mu D^{(k)})W_1^{(k)}[I_{N-1} \otimes \mathrm{Diag}(\lambda_n) + \mathrm{Diag}(\lambda_s) \otimes I_{N-1}] \\ =&\tilde{\Psi}^{(k)}\mathrm{Diag}(\dot{\varpi}_{\tilde{m}})\mathcal{F}^{(k)}[(\Phi_x\mathrm{Diag}(\varpi_{\tilde{n}}))^T \otimes (\Phi_y\mathrm{Diag}(\varpi_{\tilde{s}}))^T], \end{aligned} \tag{4.9}$$

where matrices $\mathrm{Diag}(\lambda_n)$, $\mathrm{Diag}(\lambda_s)$, I_{N-1}, Φ_x and Φ_y in spatial directions are given in [26], and matrices $C^{(k)}$ and $D^{(k)}$ $(1 \le k \le K)$ in temporal direction are given in [28]. Then, by the properties of matrix multiplication in [32], Eq (4.9) can be formulated as

$$\mathcal{A}^{(k)}\mathrm{vec}(W_1^{(k)}) = \mathrm{vec}(\tilde{\Psi}^{(k)}\mathrm{Diag}(\dot{\varpi}_{\tilde{m}})\mathcal{F}^{(k)}[(\Phi_x\mathrm{Diag}(\varpi_{\tilde{n}}))^T \otimes (\Phi_y\mathrm{Diag}(\varpi_{\tilde{s}}))^T]), \tag{4.10}$$

where

$$\begin{aligned} \mathcal{A}^{(k)} =&\mathrm{Diag}(\lambda_n) \otimes \mathrm{Diag}(\lambda_s) \otimes (C^{(k)} + \gamma D^{(k)}) \\ &+ [I_{N-1} \otimes \mathrm{Diag}(\lambda_s) + \mathrm{Diag}(\lambda_n) \otimes I_{N-1}] \otimes (\varepsilon C^{(k)} + \mu D^{(k)}). \end{aligned} \tag{4.11}$$

Now, we try to get the matrix form of scheme (4.5). According to the processing means in [28], similarly, we have

$$u^{(k-1)}(x, y, a_k) = \sum_{n,s=0}^{N-2} \rho_{n,s}^{(k)} \phi_n(x)\phi_s(y) + u^{(0)}(x, y, a_1), \quad \rho_{n,s}^{(k)} = 2\sum_{l=1}^{k-1} \sum_{m=0}^{M_l-1} w_{m,n,s}^{(l)}. \tag{4.12}$$

Then

$$\begin{aligned}&\mu(\nabla u^{(k-1)}(a_k),\nabla v^{(k)})_{\Sigma_k}+\gamma(u^{(k-1)}(a_k),v^{(k)})_{\Sigma_k}-\mu(\nabla\mathbb{P}_N^1u_0,\nabla v^{(k)})_{\Sigma_k}-\gamma(\mathbb{P}_N^1u_0,v^{(k)})_{\Sigma_k}\\&=\rho_{n,s}^{(k)}(\mu(\lambda_n+\lambda_s)+\gamma\lambda_n\lambda_s)\sigma_m^{(k)},\end{aligned}\tag{4.13}$$

where $\{\lambda_n\}_{n=0}^{N-2}$ are corresponding eigenvalues of mass matrices and $\{\sigma_m^{(k)}\}_{m=0}^{M_k-1}$ can refer to (4.33) in [28]. Denote $\lambda_{n,s}=\mu(\lambda_n+\lambda_s)+\gamma\lambda_n\lambda_s$, then according to the values of $\{\sigma_m^{(k)}\}_{m=0}^{M_k-1}$ we can get the matrix form of scheme (4.5) as follows:

$$\begin{aligned}&(C^{(k)}+\gamma D^{(k)})W_2^{(k)}[\mathrm{Diag}(\lambda_n)\otimes\mathrm{Diag}(\lambda_s)]+(\varepsilon C^{(k)}+\mu D^{(k)})W_2^{(k)}[I_{N-1}\otimes\mathrm{Diag}(\lambda_s)+\mathrm{Diag}(\lambda_n)\otimes I_{N-1}]\\&=F^{(k)},\end{aligned}\tag{4.14}$$

where

$$F^{(k)}=(\eta_0^{(k)},\eta_1^{(k)},\mathbf{0},\cdots,\mathbf{0})^T,\quad \eta_i^{(k)}=(\sigma_i^{(k)}\rho_{n,s}^{(k)}\lambda_{n,s})_{1\times(N-1)^2},\quad i=0,1.\tag{4.15}$$

The Eq (4.14) above can also be converted to a form similar to Eq (4.10).

In summary, the algorithm can be implemented as follows:

(1) For each $k\geq 1$, compute $W_1^{(k)}$ by (4.9).

(2) Obviously, $W^{(1)}=W_1^{(1)}$. For each $k\geq 2$, assume that $W^{(k-1)}$ have been obtained, then $W_2^{(k)}$ is obtained by (4.14) easily.

(3) $W^{(k)}=W_1^{(k)}-W_2^{(k)}$ for each $k\geq 2$.

5. Numerical experiments

We mainly devote this section to demonstrate the accuracy and efficiency of the multi-interval Legendre space-time spectral method by utilizing numerical examples for the 2D Sobolev equations. Regarding the numerical results of the single interval Legendre space-time spectral methods for the multi-dimensional Sobolev equations, one can refer [26].

Example 5.1. We consider the 2D Sobolev equations (1.1) on the time interval $I=(0,2]$ with the following exact solution:

$$u(x,y,t)=e^{\frac{1}{t+0.2}}(\sin\pi x\sin\pi y+(1-x^2)(1-y^2)).\tag{5.1}$$

In this example, the two-interval Legendre space-time spectral method is applied to attain the numerical solution u_L. We divide the time interval $I=(0,2]$ into $I_1=(0,0.3]$ and $I_2=(0.3,2]$, namely, $a_1=0,a_2=0.3$ and $a_3=2$. Under the premise of setting constants $\varepsilon=\mu=\gamma=1$, we compare the images of numerical solutions u_L and exact solutions u at different times in Figures 1 and 2. From these figures we can deduce that the image of the numerical solutions very well simulate the image of the exact solutions.

To show the spectral accuracy of the proposed method, we plot the maximum point-wise errors and $L^2(\Sigma)$-errors using semi-log coordinates in Figure 3. The numerical results indicate that the proposed method obtained the exponential convergence in both time and space.

In Tables 1 and 2, we show $L^2(\Sigma)$-errors in space and time, respectively, mainly to compare the numerical effects of the proposed method applying the Fourier-like basis functions and the traditional basis functions. We can then observe that multi-interval method taking Fourier-like basis functions attain better efficiency.

In Table 3, we compare the temporal $L^2(\Sigma)$-errors obtained by using the single-interval Legendre spectral method and the two-interval Legendre spectral method for the same N and $\varepsilon = \mu = \gamma = 1$. One can find that the two-interval method show a great improvement in accuracy compared with the single-interval method.

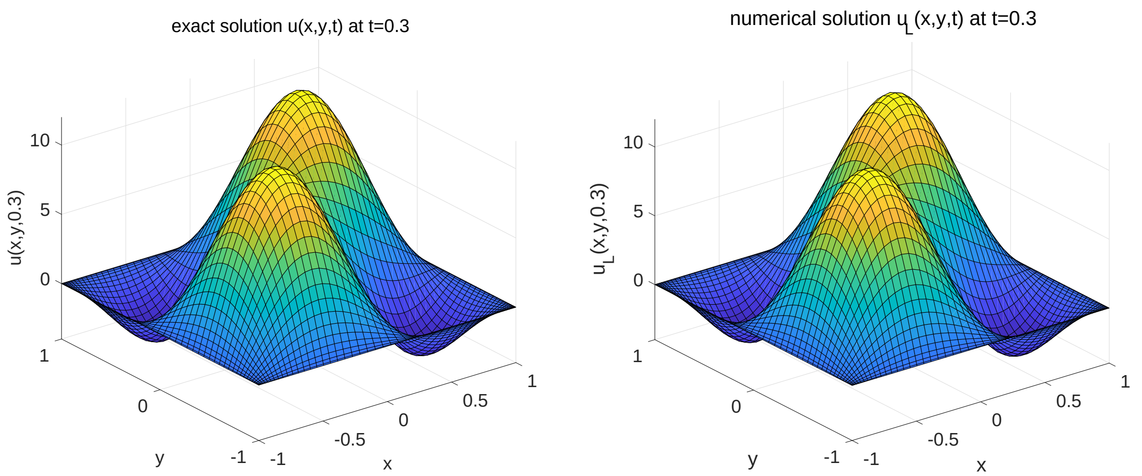


Figure 1. (Left) The exact solution u at time $t = 0.3$. (Right) The numerical solution u_L by two-interval Legendre spectral method at time $t = 0.3$ for $(M_1, M_2) = (40, 40)$ and $N = 50$. Take $\varepsilon = \mu = \gamma = 1$.

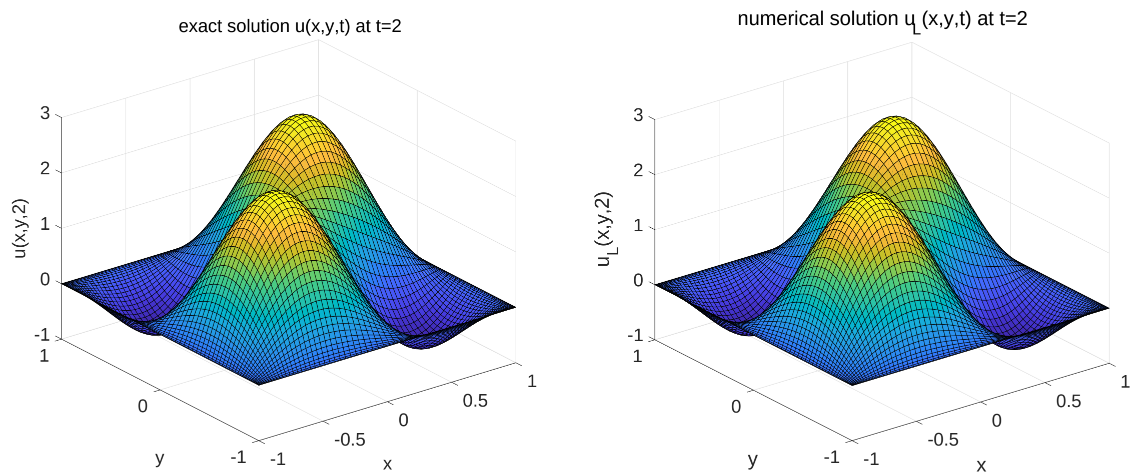


Figure 2. (Left) The exact solution u at time $t = 2$. (Right) The numerical solution u_L by two-interval Legendre spectral method at time $t = 2$ for $(M_1, M_2) = (60, 60)$ and $N = 70$. Take $\varepsilon = \mu = \gamma = 1$.

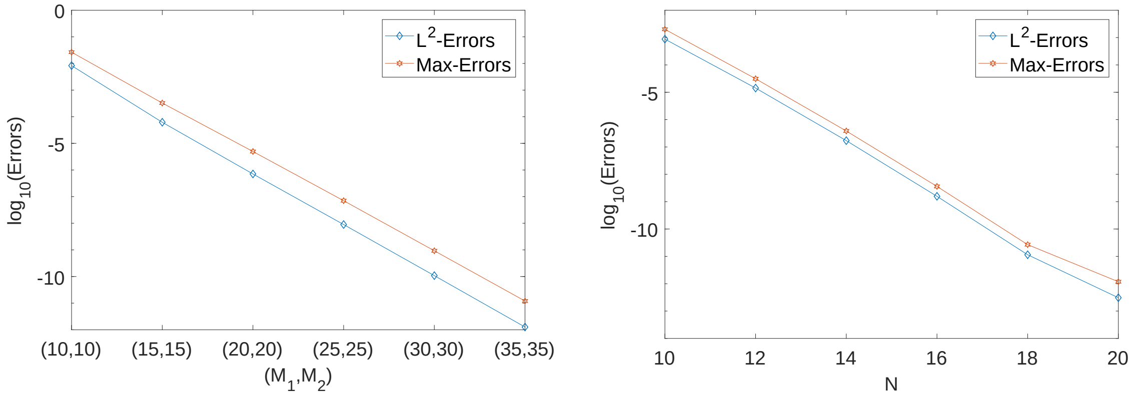


Figure 3. Spectral accuracy. (Left) Temporal errors by taking $N = 15$. (Right) Spatial errors by taking $(M_1, M_2) = (35, 35)$. Take $\varepsilon = \mu = \gamma = 1$.

Table 1. Spatial errors. Take $(M_1, M_2) = (40, 40)$ and $\varepsilon = \mu = \gamma = 1$.

N	Fourier-like basis		Standard basis		Order
	$\|\|u_L - u\|\|$	CPU(s)	$\|\|u_L - u\|\|$	CPU(s)	
12	1.4243E-05	0.3309	1.4243E-05	9.7683	
14	1.7056E-07	0.3463	1.7056E-07	26.4171	$N^{-28.71}$
16	1.5700E-09	0.3552	1.5700E-09	55.0611	$N^{-35.11}$
18	1.1462E-11	0.3734	1.1462E-11	122.2270	$N^{-41.77}$
20	3.0801E-13	0.3892	3.0801E-13	237.8722	$N^{-34.32}$

Table 2. Temporal errors. Take $N = 20$ and $\varepsilon = \mu = \gamma = 1$.

$M = (M_1, M_2)$	Fourier-like basis		Standard basis		Order
	$\|\|u_L - u\|\|$	CPU(s)	$\|\|u_L - u\|\|$	CPU(s)	
(10,10)	8.3000E-03	0.3431	8.3000E-03	4.6588	
(15,15)	6.2018E-05	0.3495	6.2018E-05	13.9264	$M^{-12.08}$
(20,20)	7.0981E-07	0.3579	7.0981E-07	30.4591	$M^{-15.54}$
(25,25)	8.9560E-09	0.3621	8.9560E-09	61.2795	$M^{-19.59}$
(30,30)	1.0813E-10	0.3785	1.0813E-10	94.7840	$M^{-24.22}$
(35,35)	1.2687E-12	0.3862	1.2687E-12	150.8296	$M^{-28.84}$

Table 3. Temporal errors. Take $\varepsilon = \mu = \gamma = 1$.

N	Single interval method			Two-interval method		
	M	$\|\|u_L - u\|\|$	Order	$M = (M_1, M_2)$	$\|\|u_L - u\|\|$	Order
20	20	2.3710E-01		(10,10)	8.3000E-03	
20	30	3.4000E-03	$M^{-10.47}$	(15,15)	6.2018E-05	$M^{-12.08}$
20	40	3.9093E-05	$M^{-15.52}$	(20,20)	7.0981E-07	$M^{-15.54}$
20	50	3.8171E-07	$M^{-20.74}$	(25,25)	8.9560E-09	$M^{-19.59}$
20	60	3.2869E-09	$M^{-26.08}$	(30,30)	1.0813E-10	$M^{-24.23}$
20	70	2.5627E-11	$M^{-31.49}$	(35,35)	1.2687E-12	$M^{-28.84}$

Example 5.2. In this example we consider the 2D Sobolev equations (1.1) on time interval $I = (0, 2]$ with the unknown exact solution. The source term is $f = 0$ and the initial condition is taken as

$$u_0(x, y) = \begin{cases} (1 - x^2)\sin(\pi y), & (x, y) \in [-1, 0] \times [-1, 1], \\ (1 - y^2)\sin(\pi x), & (x, y) \in (0, 1] \times [-1, 1]. \end{cases} \tag{5.2}$$

We also consider the two-interval Legendre space-time spectral method in time. We divide the time interval $I = (0, 2]$ into $I_1 = (0, 1.3]$ and $I_2 = (1.3, 2]$, namely, $a_1 = 0, a_2 = 1.3$ and $a_3 = 2$. In Figure 4, we depict the numerical solutions $u_L(x, y, t)$ at $t = 1.3$ and $t = 2$ with $(M_1, M_2) = (30, 30)$, $N = 60$ and $\varepsilon = \mu = \gamma = 1$.

For the unknown of the exact solution, there is no uniform standard to compare the efficiency of the single interval Legendre space-time spectral method with the two-interval method. Thus in Figure 5 by taking the numerical solutions obtained under $(M_1, M_2) = (30, 30)$ and $N = 60$ as a reference, we plot the temporal and spatial errors of the proposed method with $\varepsilon = \mu = \gamma = 1$. One can clearly observe that two-interval Legendre space-time spectral method possess the spectral accuracy both in time and space.

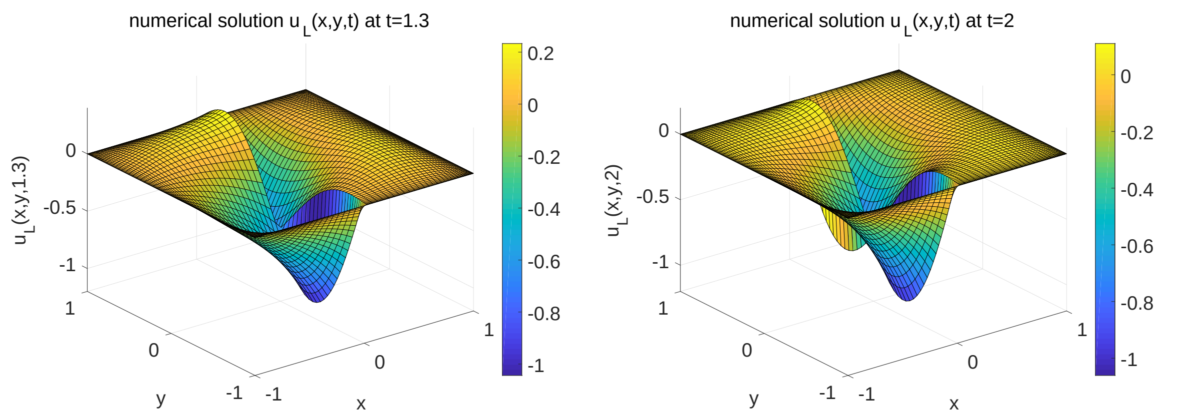
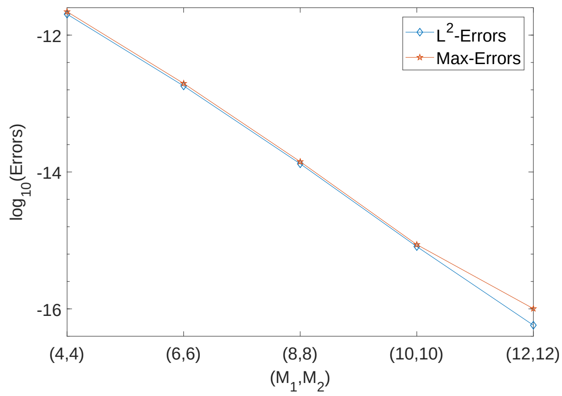


Figure 4. The numerical solutions u_L by two-interval Legendre space-time spectral method at (Left) $t = 1.3$ and (Right) $t = 2$ for $(M_1, M_2) = (30, 30)$ and $N = 60$. Take $\varepsilon = \mu = \gamma = 1$.



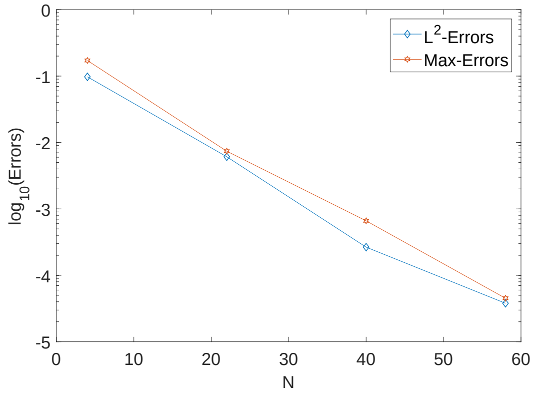


Figure 5. Spectral accuracy. (Left) Temporal errors by taking $N = 60$. (Right) Spatial errors by taking $(M_1, M_2) = (30, 30)$. Take $\varepsilon = \mu = \gamma = 1$.

Example 5.3. This example is devoted to exploring the 2D Sobolev equations (1.1) on time interval $I = (0, 1]$ with the exact solution, which is not regular enough and unknown in advance. The source term is $f = 0$ and the initial condition is taken as, see Figure 6,

$$u_0(x,y) = \begin{cases} \max[0, 0.3 - 0.5(|x+0.1| + |y+0.1|)] + 0.5, & \sqrt{(x-0.5)^2 + (y-0.5)^2} < 0.2, \\ \max[0, 0.3 - 0.5(|x+0.1| + |y+0.1|)], & \text{otherwise}. \end{cases} \tag{5.3}$$

We divide the time interval $I = (0, 1]$ into $I_1 = (0, 0.7]$ and $I_2 = (0.7, 1]$ to consider the two-interval Legendre space-time spectral method. In Figure 7, we depict the numerical solutions $u_L(x, y, t)$ derived by the proposed method at $t = 0.7$ and $t = 1$ with $(M_1, M_2) = (30, 30)$, $N = 60$ and $\varepsilon = \mu = \gamma = 1$.

In Figure 8, by taking the numerical solutions obtained under $(M_1, M_2) = (30, 30)$ and $N = 60$ as a reference, we present the temporal and spatial errors with $\varepsilon = \mu = \gamma = 1$. One can clearly observe that our method possess spectral accuracy both in time and space.

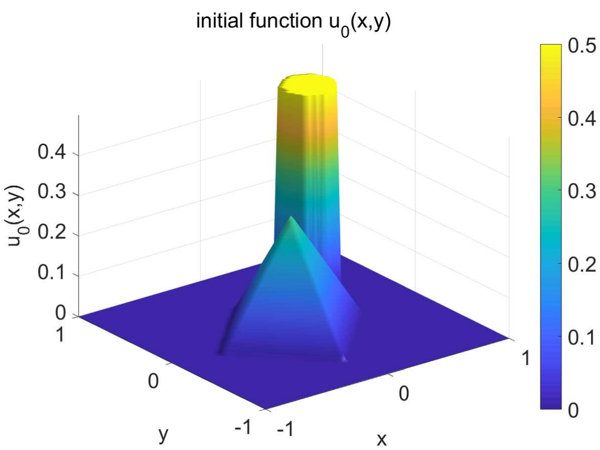


Figure 6. Initial function $u_0(x, y)$.

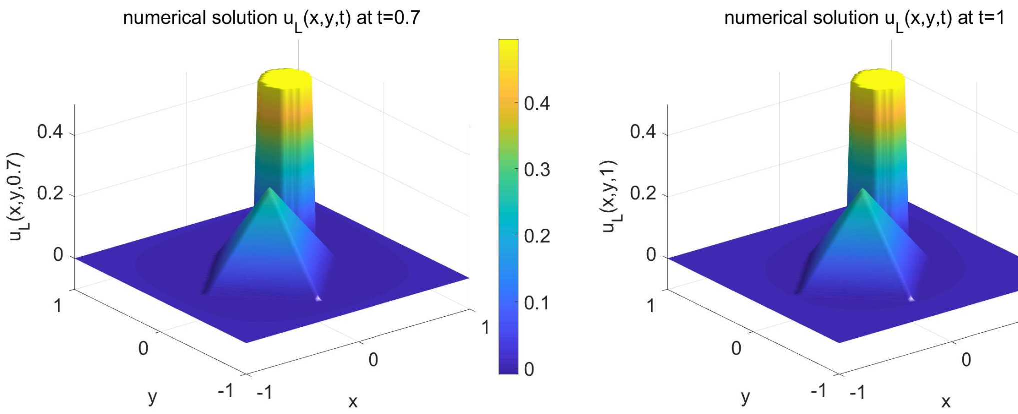


Figure 7. The numerical solutions u_L by two-interval Legendre space-time spectral method at (Left) $t = 0.7$ and (Right) $t = 1$ for $(M_1, M_2) = (30, 30)$ and $N = 60$. Take $\varepsilon = \mu = \gamma = 1$.

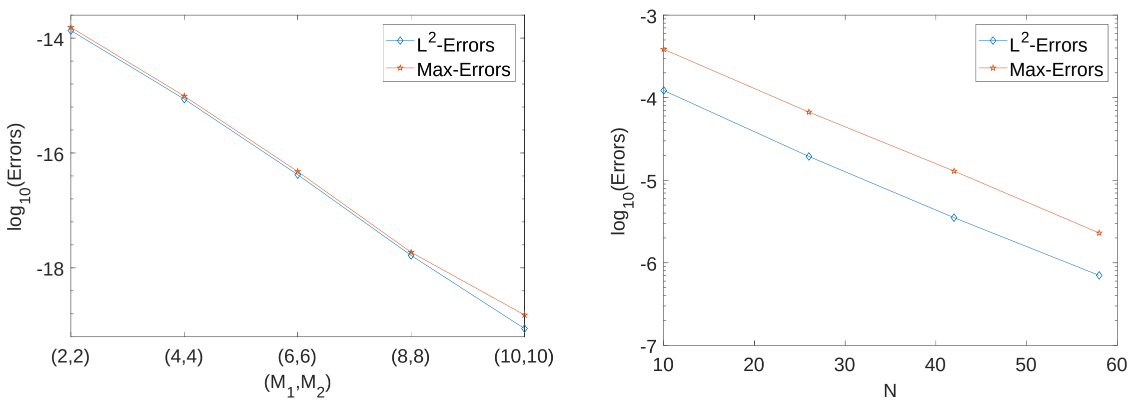


Figure 8. Spectral accuracy. (Left) Temporal errors by taking $N = 60$. (Right) Spatial errors by taking $(M_1, M_2) = (30, 30)$. Take $\varepsilon = \mu = \gamma = 1$.

6. Discussion and conclusions

As previously seen, the spectral method is commonly used to formulate the numerical scheme in space combined with the finite difference method in time, but in most cases, the infinite accuracy in space and finite accuracy in time leads to a unbalanced scheme. In order to avoid this problem, in this paper we use the Legendre-Galerkin method in space and the Legendre-tau-Galerkin method in time to study two-dimensional Sobolev equations. We have succeeded in obtaining spectral convergence in both time and space. In particular, the multi-interval form of the proposed method is also considered. In the theoretical analysis, we not only prove the stability of the single and multi-interval numerical scheme, but also give strict proof of the $L^2(\Sigma)$-error estimates by using the dual technique, where a better error estimate is obtained for the single interval form and the optimal error estimate is obtained for multi-interval form. Compared with the single interval method, the multi-interval spectral method succeeds in reducing the scale of problems, adopting the parallel computers and improving the flexibility of algorithm. Another highlight of this paper is that the Fourier-like basis functions, different from the traditional basis functions, are adopted for the Legendre spectral method in space. Since the mass matrix obtained by Fourier-like basis functions is a diagonal matrix, the computing time and memory can be effectively reduced in the implementation of the algorithm. Numerical experiments show that our method can attain the spectral accuracy both in time and space, and the multi-interval method in time is more efficient than the single one.

In a future study, we will extend our method to the numerical solutions of the nonlinear Sobolev equation by using appropriate technique to deal with the nonlinear terms effectively.

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Conflict of interest

We declare no conflicts of interest in this paper.

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