AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 14111–14131.
DOI: 10.3934/math.2023721
Received: 20 December 2022
Revised: 18 March 2023
Accepted: 27 March 2023
Published: 14 April 2023

Research article

Flat modules and coherent endomorphism rings relative to some matrices

Yuedi Zeng*

Department of Mathematics and Finance, Fujian Key Laboratory of Financial Information Processing, Putian University, Putian 351100, China

* **Correspondence:** Email: yuedizeng@163.com.

Abstract: Let N be a left R-module with the endomorphism ring $S = \mathrm{End}(_RN)$. Given two cardinal numbers α and β and a matrix $A \in S^{\beta\times\alpha}$, N is called flat relative to A in case, for each $x \in l_{N^{(\beta)}}(A) = \{u \in N^{(\beta)} \mid uA = 0\}$, there are a positive integer k, $y \in N^k$ and a $k \times \beta$ row-finite matrix C over S such that $CA = 0$ and $x = yC$. It is shown that N_S is flat relative to a matrix A if and only if $l_{N^{(\beta)}}(A)$ is generated by N. S is called left coherent relative to A if $\mathrm{Ker}(_SS^{(\beta)} \to_S S^{(\beta)}A)$ is finitely generated. It is shown that S is left coherent relative to A if and only if $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated left S-module if and only if $l_{N^n}(A)$ has an add(N)-precover (add(N) denotes the category of all direct summands of finite direct sums of copies of $_RN$). Regarding applications, new necessary and sufficient conditions for epic (monic, having the unique mapping property) add(N)-precovers of $l_{N^{(\beta)}}(A)$ are investigated. Also, some new characterizations of left n-semihereditary rings and von Neumann regular rings are given.

Keywords: coherent; flat; endomorphism ring; precover; quasi-injective
Mathematics Subject Classification: 16D40, 16D50, 16P70

1. Introduction

Let N be a left R-module with the endomorphism ring $S = \mathrm{End}(_RN)$. Obviously, $_RN_{\mathrm{End}(_RN)}$ is right balanced. In [1], Mao investigated the (m, n)-flatness and (m, n)-coherence of S. Motivated by [2–6], m and n are superseded by two (possibly infinite) cardinal numbers α and β when investigating some homological properties of modules and rings so that some known results can be extended.

Recall that a left R-module N with $S = \mathrm{End}(_RN)$ is flat over S [7, Lemma 19.19] if and only if for each $A \in S^{n\times m}$ and $x \in l_{N^n}(A)$, there are a positive integer k, $y \in N^k$ and a $k \times n$ matrix C over S such that $CA = 0$ and $x = yC$. A left R-module N with $S = \mathrm{End}(_RN)$ is called flat relative to a matrix $A \in S^{\beta\times\alpha}$ in case, for each $x \in l_{N^{(\beta)}}(A)$, there are a positive integer k, $y \in N^k$ and $C \in\mathrm{RFM}_{k\times\beta}(S)$ such that $CA = 0$ and $x = yC$. In Section 2, it is shown that N_S is flat relative to a matrix $A \in S^{\beta\times\alpha}$ if and only if $l_{N^{(\beta)}}(A)$ is generated by N. It is shown that N_S is flat relative to a matrix $A \in\mathrm{RFM}_{\beta\times\alpha}(S)$ if and only if $\mathrm{Tor}_1^S(N, S^{(\alpha)}/S^{(\beta)}A) = 0$. Next, N is called (α,β)-flat over its endomorphism ring if N_S is flat relative to all $A \in S^{\beta\times\alpha}$. A united frame is provided to investigate n-projective [8], finite-projective [3], (m,n)-flat [1,9] and flat modules over their endomorphism rings. The notion of quasi-injectivity is also extended to quasi-injectivity relative to A. It is proven that the following conditions are equivalent for a left R-module N which is flat relative to a matrix $A \in \mathrm{RFM}_{\beta\times\alpha}(S)$:

(1) S is a left injective ring relative to A (S is called left injective relative to A [6] in case, for every $h \in \mathrm{Hom}_S(S^{(\beta)}A, S)$, there exists $g \in \mathrm{Hom}_S(S^{(\alpha)}, S)$ such that $h = \eta g$, where η: $S^{(\beta)}A \to S^{(\alpha)}$ is the inclusion).

(2) If σ: $N^{(\beta)} \to N$ is a left R-homomorphism with $l_{N^{(\beta)}}(A) \subseteq Ker(\sigma)$, then $\sigma \in AS^{\alpha}$.

(3) N is quasi-injective relative to A (N is called quasi-injective relative to A in case, for every $h \in \mathrm{Hom}_R(N^{(\beta)}A, N)$, there exists $g \in \mathrm{Hom}_R(N^{(\alpha)}, N)$ such that $h = \eta g$ where $\eta : N^{(\beta)}A \to N^{(\alpha)}$ is the inclusion).

Thus the condition "$\beta < \infty$ or ${}_RN$ is finitely generated" in [6, Proposition 3.11(2)] is superfluous. At the same time, a new necessary and sufficient condition of left (m,n)-injective endomorphism rings is given.

Recall that a ring S is said to be left coherent (see [10]) in case each finitely generated left ideal of S is finitely presented, or equivalently, any finitely generated submodule of any finitely generated free left S-module is finitely presented. Let N be a left R-module with $S = \mathrm{End}({}_RN)$ and $A \in S^{\beta\times\alpha}$. The ring S is called left coherent relative to $A \in S^{\beta\times\alpha}$ if $\mathrm{Ker}({}_SS^{(\beta)} \to_S S^{(\beta)}A)$ is finitely generated. S is said to be (α, n)-coherent provided that it is left coherent relative to all matrices $A \in S^{n\times\alpha}$. For example:

(1) S is left (m,n)-coherent [1,9] if it is left coherent relative to all matrices $A \in S^{n\times m}$.

(2) S is left coherent [10] if and only if it is left $(1,n)$-coherent for all positive integers n if and only if it is left (m,n)-coherent for all positive integers m and n.

(3) S is left π-coherent [11] if and only if it is left (α,β)-coherent for all cardinal numbers $\beta \in \mathbb{N}$ and all cardinal numbers α.

In Section 3, it is shown that the following conditions are equivalent for a left R-module N with $S = \mathrm{End}({}_RN)$ and a matrix $A \in S^{n\times\alpha}$:

(1) S is left coherent relative to A.

(2) $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated left S-module.

(3) $l_{N^n}(A)$ has an $\mathrm{add}(N)$-precover.

Next, the new necessary and sufficient conditions are investigated for the epic (monic, having the unique mapping property) $\mathrm{add}(N)$-precovers of $l_{N^n}(A)$. For example, it is shown that $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated projective left S-module if and only if $l_{N^n}(A)$ has an $\mathrm{add}(N)$-cover with the unique mapping property (Theorem 3.11). Moreover, the specific form of the cover is given. We get a new necessary and sufficient condition of [1, Proposition 3.4]. Regarding applications, some new characterizations of left n-semihereditary rings and von Neumann regular rings are obtained.

Throughout this article, R is an associative ring with identity, and all modules are unitary. Let N be a left R-module with $S = \mathrm{End}({}_RN)$. Note that $\mathrm{add}(N)$ denotes the category of all direct summands of finite direct sums of copies of ${}_RN$. Let α and β be two fixed cardinal numbers. $S^{\beta\times\alpha}$ ($\mathrm{RFM}_{\beta\times\alpha}(S)$) stands for the set of all $\beta\times\alpha$ full (row-finite) matrices over S. We write $S^{(\beta)}(S_{(\beta)})$ to indicate the direct sum of β copies of S and $S^{\alpha}(S_{\alpha})$ to indicate the direct product of α copies of S. Elements in $S^{(\beta)}$ are regarded as "row vectors", elements in $S_{(\beta)}$ are regarded as "column vectors", and elements in $S^{\alpha}(S_{\alpha})$ are regarded similarly. For the left R-module N, elements in $N^{(\beta)}(N_{(\beta)}, N^{\alpha}, N_{\alpha})$ have similar meanings.

Thus, we define left R-homomorphisms θ: $N \to N^{(\beta)}$ and σ: $N^{(\beta)} \to N$ as follows:

$$(x)\theta = (x)(\theta_1, \theta_2, \ldots, \theta_\beta), = ((x)\theta_1, (x)\theta_2, \ldots, (x)\theta_\beta),$$

and

$$(x_1, x_2, \ldots, x_\beta)\sigma = (x_1, x_2, \ldots, x_\beta)\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_\beta \end{pmatrix} = x_1\sigma_1 + x_2\sigma_2 + \cdots + x_\beta\sigma_\beta,$$

where $x \in N$, $(x_1, x_2, \ldots, x_\beta) \in N^{(\beta)}$, $\theta = (\theta_1, \theta_2, \ldots, \theta_\beta) \in S^{(\beta)}$, and $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_\beta \end{pmatrix} \in S_\beta$.

For any $A \in S^{\beta\times\alpha}$, we define

$$l_{S^{(\beta)}}(A) = \{s \in S^{(\beta)} \mid sA = 0\},$$

$$l_{N^{(\beta)}}(A) = \{u \in N^{(\beta)} \mid uA = 0\}.$$

If

$$A = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\alpha} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2\alpha} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{\beta1} & \sigma_{\beta2} & \cdots & \sigma_{\beta\alpha} \end{pmatrix} \in S^{\beta\times\alpha},$$

we can define a left R-homomorphism f: $N^{(\beta)} \to N^{(\beta)}A(N^\alpha)$ as follows:

$$\begin{aligned}(x_1, x_2, \ldots, x_\beta)f &= (x_1, x_2, \ldots, x_\beta)A \\ &= ((x_1)\sigma_{11} + (x_2)\sigma_{21} + \cdots + (x_\beta)\sigma_{\beta1}, (x_1)\sigma_{12} \\ &+ (x_2)\sigma_{22} + \cdots + (x_\beta)\sigma_{\beta2}, \ldots, (x_1)\sigma_{1\alpha} \\ &+ (x_2)\sigma_{2\alpha} + \cdots + (x_\beta)\sigma_{\beta\alpha}),\end{aligned}$$

for any $(x_1, x_2, \ldots, x_\beta) \in N^{(\beta)}$. Conversely, if φ: $N^{(\beta)} \to N^\alpha$ is a left R-homomorphism, there exists

$$A = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\alpha} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2\alpha} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{\beta1} & \sigma_{\beta2} & \cdots & \sigma_{\beta\alpha} \end{pmatrix} \in S^{\beta\times\alpha},$$

such that

$$(x_1, x_2, \ldots, x_\beta)\varphi = (x_1, x_2, \ldots, x_\beta)A,$$

for any $(x_1, x_2, \ldots, x_\beta) \in N^{(\beta)}$.

2. Flatness relative to some matrices

Let $\alpha \geq 1$ and $\beta \geq 1$ be two fixed cardinal numbers.

Definition 2.1. Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S^{\beta\times\alpha}$. N_S is called flat relative to A in case, for each $x \in l_{N^{(\beta)}}(A)$, there are a positive integer k, $y \in N^k$ and $C \in \mathrm{RFM}_{k\times\beta}(S)$ such that $CA = 0$ and $x = yC$.

Theorem 2.2. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S^{\beta\times\alpha}$, and the following conditions are equivalent.*

(1) *N_S is flat relative to A.*

(2) *The canonical map μ: $N \otimes_S S^{(\beta)}A \to N^\alpha$ ($\mu(x \otimes bA) = xbA$, $\forall x \in N$ and $b \in S^{(\beta)}$) is a monomorphism.*

(3) *The canonical map υ: $N \otimes_S S^{(\beta)}A \to \mathrm{Hom}(AS_{(\alpha)}, N)$ ($\upsilon(x \otimes bA)(Ad) = xbAd$, $\forall x \in N$, $b \in S^{(\beta)}$ and $d \in S_{(\alpha)}$) is a monomorphism.*

(4) *$l_{N^{(\beta)}}(A)$ is generated by N.*

Proof. (1) $\Rightarrow$ (2) Suppose $A = (a_{ij})_{\beta\times\alpha}$ and the t-th row of A is $a_t = (a_{tj})$. Note that

$$\mu(\sum_i(x_i \otimes a_i)) = \sum_i x_ia_i(\sum_i(x_i \otimes a_i) \in N \otimes_S S^{(\beta)}A).$$

If $\sum\limits_i x_ia_i = 0$, then

$$(x_1, x_2, \ldots, x_\beta)A = 0.$$

It follows that $(x_1, x_2, \ldots, x_\beta) \in l_{N^{(\beta)}}(A)$. By (1), there are a positive integer

$$k, y = (y_1, y_2, \ldots, y_k) \in N^k$$

and $C \in \mathrm{RFM}_{k\times\beta}(S)$ such that $CA = 0$ and

$$(x_1, x_2, \ldots, x_\beta) = yC.$$

Hence, $\sum\limits_i(x_i \otimes a_i) = 0$. Thus, (2) holds.

(2) $\Rightarrow$ (1) Let $A = (a_{ij})_{\beta\times\alpha}$. Suppose the t-th row of A is $a_t = (a_{tj})$ and $e_t \in S^{(\beta)}$ with Id_N in the t-th position and 0 elsewhere, $t = 1, 2, \ldots, \beta$. Define φ: $S^{(\beta)} \to S^{(\beta)}A$ such that $\varphi(e_t) = a_t$, $t = 1, 2, \ldots, \beta$. Then, φ is an epimorphism of left S-modules with $\mathrm{Ker}(\varphi) = l_{S^{(\beta)}}(A)$. Let τ: $l_{S^{(\beta)}}(A) \to S^{(\beta)}$ be the inclusion map. Consider the following exact sequence:

$$N \otimes_S l_{S^{(\beta)}}(A) \xrightarrow{N\otimes\tau} N \otimes_S S^{(\beta)} \xrightarrow{N\otimes\varphi} N \otimes_S S^{(\beta)}A \longrightarrow 0.$$

For any

$$x = (x_1, x_2, \ldots, x_\beta) \in l_{N^{(\beta)}}(A),$$

we get that $\sum\limits_i(x_i \otimes a_i) = 0$ by (2). Thus,

$$0 = \sum_i(x_i \otimes a_i) = \sum_i(x_i \otimes \varphi(e_i))$$

in $N \otimes S^{(\beta)}A$. It follows that

$$\sum_i (x_i \otimes e_i) \in \mathrm{Ker}(N \otimes \varphi) = \mathrm{Im}(N \otimes \tau).$$

Hence, there are a positive integer

$$k, y = (y_1, y_2, \ldots, y_k) \in N^k$$

and

$$c_1 = (c_{11}, c_{12}, \ldots, c_{1\beta}),\ c_2 = (c_{21}, c_{22}, \ldots, c_{2\beta}), \ldots, c_k = (c_{k1}, c_{k2}, \ldots, c_{k\beta}) \in l_{S^{(\beta)}}(A)$$

such that

$$\sum_i (x_i \otimes e_i) = y_1 \otimes c_1 + y_2 \otimes c_2 + \cdots + y_k \otimes c_k.$$

Set $C = (c_{ij})$. Then, C is a $k \times \beta$ row-finite matrix over S such that $CA = 0$. It is easy to see that

$$c_1 = \sum_i c_{1i} e_i,\ c_2 = \sum_i c_{2i} e_i, \ldots, c_k = \sum_i c_{ki} e_i.$$

Then,

$$\begin{aligned}
\sum_i (x_i \otimes e_i) &= y_1 \otimes c_1 + y_2 \otimes c_2 + \cdots + y_k \otimes c_k \\
&= y_1 \otimes \sum_i c_{1i} e_i + y_2 \otimes \sum_i c_{2i} e_i + \cdots + y_k \otimes \sum_i c_{ki} e_i \\
&= \sum_i y_1 c_{1i} \otimes e_i + \sum_i y_2 c_{2i} \otimes e_i + \cdots + \sum_i y_k c_{ki} \otimes e_i \\
&= \sum_i [(y_1 c_{1i} + y_2 c_{2i} + \cdots + y_k c_{ki}) \otimes e_i].
\end{aligned}$$

It follows that

$$x_i = y_1 c_{1i} + y_2 c_{2i} + \cdots + y_k c_{ki},$$

that is, $x = yC$. Thus, (1) holds.

(2) $\Leftrightarrow$ (3) It is easy to see that $\upsilon(x \otimes bA) = 0$ if and only if $xbA = 0$.

(1) $\Rightarrow$ (4) For any

$$x = (x_1, x_2, \cdots, x_\beta) \in l_{N^{(\beta)}}(A),$$

there are a positive integer $k, y = (y_1, y_2, \cdots, y_k) \in N^k$ and $C \in \mathrm{RFM}_{k\times\beta}(S)$ such that $CA = 0$ and $x = yC$ by (1). Let $C = (c_{ij})_{k\times\beta}$. Define left R-homomorphisms

$$f_i = (c_{i1}, c_{i2}, \cdots, c_{i\beta}) : N \to N^{(\beta)}$$

via

$$(b)f_i = ((b)c_{i1}, (b)c_{i2}, \cdots, (b)c_{i\beta}),\ \forall b \in N,\ i = 1, 2, \ldots, k.$$

Note that $CA = 0$, and then $\mathrm{Im}(f_i) \subseteq l_{N^{(\beta)}}(A)$. Thus, we regard f_i: $N \to l_{N^{(\beta)}}(A)$, $i = 1, 2, \ldots, k$. It follows that

$$x = (x_1, x_2, \cdots, x_\beta) = (y_1, y_2, \cdots, y_k)C = (y_1)f_1 + (y_2)f_2 + \cdots + (y_k)f_k.$$

Therefore, $l_{N^{(\beta)}}(A)$ is generated by N by [7, Corollary 8.13(1)].

(4) $\Rightarrow$ (1) Let

$$x = (x_1, x_2, \cdots, x_\beta) \in l_{N^{(\beta)}}(A).$$

By (4), there exist left R-homomorphisms

$$f_i = (c_{i1}, c_{i2}, \cdots, c_{i\beta}) : N \to l_{N^{(\beta)}}(A)$$

and $y_i \in N$, $i = 1, 2, \ldots, k$, such that

$$\begin{aligned} x &= (x_1, x_2, \cdots, x_\beta) \\ &= (y_1)f_1 + (y_2)f_2 + \cdots + (y_k)f_k \\ &= \sum_{i=1}^{k}((y_i)c_{i1}, (y_i)c_{i2}, \cdots, (y_i)c_{i\beta}). \end{aligned}$$

Set $C = (c_{ij})_{k\times\beta} \in S^{k\times\beta}$. Then,

$$x = (y_1, y_2, \cdots, y_k)C.$$

Note that f_i: $N \to l_{N^{(\beta)}}(A)$, $i = 1, 2, \ldots, k$. This means that $CA = 0$. Thus, (1) holds. $\square$

Example 2.3. (1) Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S$. If A is an isomorphism or $A = 0$, then N is flat relative to A by Theorem 2.2 (4).

(2) Let $N =_R R$. Then, R is flat relative to any matrix $A \in S^{\beta\times\alpha}$ by Theorem 2.2 (4) since $S = R = \mathrm{End}(_RN)$.

(3) Let $R = \mathcal{Z}$ be an integer ring and $N = \mathcal{Z}_6$ an R-module. Define an R-homomorphism A: $\mathcal{Z}_6 \to \mathcal{Z}_6$ via $A(x) = 2x, \forall x \in \mathcal{Z}_6$. Then, there are two exact sequences

$$0 \longrightarrow \mathcal{Z}_3 \longrightarrow \mathcal{Z}_6 \xrightarrow{A} \mathcal{Z}_6\ ,\ \mathcal{Z}_6 \xrightarrow{\sigma} \mathcal{Z}_3 \longrightarrow 0,$$

where $\sigma(x) = 3x, \forall x \in \mathcal{Z}_6$. Thus, N is flat relative to A by Theorem 2.2 (4).

If A is a row-finite $\beta \times \alpha$ matrix, we have the following.

Corollary 2.4. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in \mathrm{RFM}_{\beta\times\alpha}(S)$, and the following conditions are equivalent:*

(1) *N_S is flat relative to A.*

(2) *The sequence $0 \to N \otimes_S S^{(\beta)}A \to N \otimes_S S^{(\alpha)}$ is exact.*

(3) $\mathrm{Tor}_1^S(N, S^{(\alpha)}/S^{(\beta)}A) = 0$.

(4) *The canonical map μ: $N \otimes_S S^{(\beta)}A \to N^{(\alpha)}$ ($\mu(x \otimes aA) = xaA$, for each $x \in N$ and $a \in S^{(\beta)}$) is a monomorphism.*

Proof. (2) $\Leftrightarrow$ (3) is trivial.

(1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (4) follow from Theorem 2.2 since $A \in \mathrm{RFM}_{\beta\times\alpha}(S)$.

If A is an $n \times \alpha$ matrix (n is a positive integer), we have the following.

Corollary 2.5. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S^{n\times\alpha}$, and the following conditions are equivalent.*

(1) *N_S is flat relative to A.*

(2) *For any right S-homomorphism f: $S_n/AS_{(\alpha)} \to N$, there is a finitely generated free right S-module F such that $f = f_2f_1$ for some f_1: $S_n/AS_{(\alpha)} \to F$ and f_2: $F \to N$.*

Proof. Let $f \in \mathrm{Hom}_S(S_n/AS_{(\alpha)}, N)$. Then, there exists $x \in l_{N^n}(A)$ such that

$$(b + AS_{(\alpha)})f = xb,\ \forall b \in S_n.$$

By (1), there are a positive integer $k, y = (y_1, y_2, \cdots, y_k) \in N^k$ and $C \in \mathrm{RFM}_{k\times n}(S)$ such that $CA = 0$ and $x = yC$. Define right S-homomorphisms f_1: $S_n/AS_{(\alpha)} \to S_k$ by

$$(b + AS_{(\alpha)})f_1 = Cb(\forall b \in S_n),$$

f_2: $S_k \to N$ by

$$(a)f_2 = ya(\forall a \in S_k).$$

Thus, $f = f_1 f_2$, and so (2) holds.

Conversely, it is easy to check that (2) $\Rightarrow$ (1).

Definition 2.6. Let N be a left R-module with $S = \mathrm{End}(_RN)$. N is called (α,β)-flat over its endomorphism ring if N_S is flat relative to all $A \in S^{\beta\times\alpha}$.

Example 2.7. Let N be a left R-module with $S = \mathrm{End}(_RN)$.

(1) Recall that a right S-module P is said to be (m, n)-presented [9] if there exists an exact sequence $0 \to K \to S^m \to P \to 0$, where K is n-generated of right S-modules. A right S-module N is said to be (m, n)-flat [1, 9] if $\mathrm{Tor}_1(P, N) = 0$ for any (m, n)-presented right S-module P. Thus, N is (m, n)-flat over its endomorphism ring if it is flat relative to all matrices $A \in S^{n\times m}$ by Corollary 2.4.

(2) It is easy to see that N is flat over its endomorphism ring if and only if it is $(1, n)$-flat, for all positive integers n, if and only if it is (m, n)-flat, for all positive integers m and n, if and only if it is flat relative to all $A \in \mathrm{RFM}(S)$ by Corollary 2.4.

(3) Recall from [8], N is said to be n-projective in case for each right S-epimorphism π: $M_S \to N_S$ and each n-generated submodule N_0 of N_S there exists $g \in \mathrm{Hom}_S(N_0, M)$ such that $(x)g\pi = (x)i$, $\forall x \in N_0$, where i: $N_0 \to N$ is the inclusion. It is easy to check that N is n-projective over its endomorphism ring if and only if it is (α, n)-flat (for all cardinal numbers α) by Corollary 2.5. Thus, let $_RN =_R R$. We get [6, Proposition 4.1].

(4) Recall from [3], N is finite-projective if and only if it is n-projective for all positive integers n. Thus, N is finite-projective over its endomorphism ring if and only if it is (α, n)-flat, for all cardinal numbers α and positive integers n by (3) and Corollary 2.5.

Definition 2.8. A left R-module K is called N-(α,β)-copresented if there exists an exact sequence $0 \to K \to N^{(\beta)} \to N^\alpha$ of left R-modules.

Corollary 2.9. *Let N be a left R-module with $S = \mathrm{End}(_RN)$.*

(1) [1, Lemma 2.6] *N is (m, n)-flat over its endomorphism ring if and only if every N-(m, n)-copresented left R-module is generated by N.*

(2) *N is flat over its endomorphism ring if and only if every N-(m, n)-copresented left R-module is generated by N, for all positive integers m, n.*

(3) *N is n-projective over its endomorphism ring if and only if N-(α, n)-copresented left R-module is generated by N, for all cardinal numbers α.*

(4) *N is finite-projective over its endomorphism ring if and only if N-(α, n)-copresented left R-module is generated by N, for all cardinal numbers α and positive integers n.*

Lemma 2.10. *Suppose N is a left R-module with $S = \mathrm{End}({}_RN)$ and N_S is flat relative to a matrix $A \in S^{\beta\times\alpha}$. If $x \in l_{N^{(\beta)}}(A)$ and*

$$\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_\beta \end{pmatrix} \in r_{S^\beta} l_{S^{(\beta)}}(A),$$

then $x \in Ker(\sigma)$.

Proof. Set

$$x = (x_1, x_2, \cdots, x_\beta) \in l_{N^{(\beta)}}(A).$$

Since N_S is flat relative to A, there are a positive integer k, $y = (y_1, y_2, \cdots, y_k) \in N^k$ and $C \in \mathrm{RFM}_{k\times\beta}(S)$ such that $CA = 0$ and $x = yC$. Suppose $C = (c_{ij})_{k\times\beta}$. Then, $(c_{i1}, c_{i2}, \cdots, c_{i\beta})A$=0, that is,

$$(c_{i1}, c_{i2}, \cdots, c_{i\beta}) \in l_{S^{(\beta)}}(A),\ i = 1, 2, \ldots, k.$$

If

$$\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_\beta \end{pmatrix} \in r_{S^\beta} l_{S^{(\beta)}}(A),$$

then

$$(c_{i1}, c_{i2}, \cdots, c_{i\beta})\sigma = 0,\ i = 1, 2, \ldots, k.$$

Note that

$$x = y_1(c_{11}, c_{12}, \cdots, c_{1\beta}) + y_2(c_{21}, c_{22}, \cdots, c_{2\beta}) + \cdots + y_k(c_{k1}, c_{k2}, \cdots, c_{k\beta}).$$

Thus,

$$\begin{aligned}(x)\sigma &= (y_1(c_{11}, c_{12}, \cdots, c_{1\beta}) + y_2(c_{21}, c_{22}, \cdots, c_{2\beta}) + \cdots + y_k(c_{k1}, c_{k2}, \cdots, c_{k\beta}))\sigma \\ &= y_1(c_{11}, c_{12}, \cdots, c_{1\beta})\sigma + y_2(c_{21}, c_{22}, \cdots, c_{2\beta})\sigma + \cdots + y_k(c_{k1}, c_{k2}, \cdots, c_{k\beta})\sigma \\ &= 0.\end{aligned}$$

It follows that $x \in Ker(\sigma)$.

Definition 2.11. Let N be a left R-module with $S = \mathrm{End}({}_RN)$ and $A \in \mathrm{RFM}_{\beta\times\alpha}(S)$.

(1) N is called quasi-injective relative to A in the case that for every $h \in \mathrm{Hom}_R(N^{(\beta)}A, N)$, there exists $g \in \mathrm{Hom}_R(N^{(\alpha)}, N)$ such that $h = \eta g$, i.e., the following diagram commutes:

$$\begin{tikzcd} N^{(\beta)}A \arrow[r, "\eta"] \arrow[d, "h"] & N^{(\alpha)} \arrow[dl, dotted, "g"] \\ N & \end{tikzcd}$$

where η: ${}_RN^{(\beta)}A \to_R N^{(\alpha)}$ is the canonical inclusion.

(2) N is called quasi-(β, α)-injective [6, Definition 3.8] if it is quasi-injective relative to all $A \in \mathrm{RFM}_{\beta\times\alpha}(S)$.

(3) S is called quasi-injective relative to A (i.e., S is left injective relative to A [6]) in the case that for every $h \in \mathrm{Hom}_S(S^{(\beta)}A, S)$, there exists $g \in \mathrm{Hom}_S(S^{(\alpha)}, S)$ such that $h = \eta g$, i.e., the following diagram commutes:

$$\begin{tikzcd} S^{(\beta)}A \arrow[r, "\eta"] \arrow[d, "h"] & S^{(\alpha)} \arrow[dl, dotted, "g"] \\ S & \end{tikzcd}$$

where η: ${}_SS^{(\beta)}A \rightarrow_S S^{(\alpha)}$ is the canonical inclusion.

Lemma 2.12. *Let N be a left R-module with $S = \mathrm{End}({}_RN)$, $A \in \mathrm{RFM}_{\beta\times\alpha}(S)$ and S left injective relative to A. Then,*

(1) *If σ: $N^{(\beta)} \rightarrow N$ is a left R-homomorphism with $l_{N^{(\beta)}}(A) \subseteq Ker(\sigma)$, then $\sigma \in AS_\alpha$.*

(2) *N is quasi-injective relative to A.*

Proof. (1) Let σ: $N^{(\beta)} \rightarrow N$ be a left R-homomorphism with $l_{N^{(\beta)}}(A) \subseteq Ker(\sigma)$. If

$$(t_1, t_2, \ldots, t_\beta) \in l_{S^{(\beta)}}(A),$$

then

$$(a)(t_1, t_2, \ldots, t_\beta)A = 0,$$

for any $a \in N$. Thus,

$$(a)(t_1, t_2, \ldots, t_\beta) \in l_{N^{(\beta)}}(A).$$

Since

$$l_{N^{(\beta)}}(A) \subseteq Ker(\sigma),\ (a)(t_1, t_2, \ldots, t_\beta)\sigma = 0,$$

for any $a \in N$, it follows that

$$(t_1, t_2, \ldots, t_\beta)\sigma = 0,$$

that is, $\sigma \in r_{S_\beta} l_{S^{(\beta)}}(A)$. Note that S is left injective relative to A. Then, $r_{S_\beta} l_{S^{(\beta)}}(A) = AS_\alpha$ by Definition 2.11 (3) and [6, Theorem 3.10]. Therefore, $\sigma \in AS_\alpha$.

(2) Let η: ${}_RN^{(\beta)}A \rightarrow_R N^{(\alpha)}$ be the canonical inclusion, π: ${}_RN^{(\beta)} \rightarrow_R N^{(\beta)}A$ be the canonical epimorphism, and ψ: $N^{(\beta)}A \rightarrow N$ be any left R-homomorphism. Clearly, $l_{N^{(\beta)}}(A) \subseteq Ker(\pi\psi)$. By (1), there exists a left R-homomorphism $\tau \in S_\alpha$ such that $\pi\psi = A\tau$. Thus,

$$\begin{aligned} ((x_1, x_2, \cdots, x_\beta)A)\psi &= ((x_1, x_2, \cdots, x_\beta)\pi)\psi \\ &= (x_1, x_2, \cdots, x_\beta)(\pi\psi) \\ &= (x_1, x_2, \cdots, x_\beta)(A\tau) \\ &= (x_1, x_2, \cdots, x_\beta)(\pi\eta\tau) \\ &= ((x_1, x_2, \cdots, x_\beta)A)\eta\tau, \end{aligned}$$

for any $(x_1, x_2, \cdots, x_\beta) \in N^{(\beta)}$. It follows that $\psi = \eta\tau$.

Theorem 2.13. *Let N be a left R-module with $S = \mathrm{End}({}_RN)$, $A \in \mathrm{RFM}_{\beta\times\alpha}(S)$ and N_S flat relative to A. Then, the following conditions are equivalent.*

(1) *S is left injective relative to A.*

(2) *If σ: $N^{(\beta)} \rightarrow N$ is a left R-homomorphism with $l_{N^{(\beta)}}(A) \subseteq Ker(\sigma)$, then $\sigma \in AS_\alpha$.*

(3) *N is quasi-injective relative to A.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from Lemma 2.12.

(3) $\Rightarrow$ (1) Let η: ${}_RN^{(\beta)}A \to_R N^{(\alpha)}$ be the canonical inclusion and $\sigma \in r_{S_\beta} l_{S^{(\beta)}}(A)$. Define a left R-homomorphism ψ: $N^{(\beta)}A \to N$ via

$$((x_1, x_2, \dots, x_\beta)A)\psi = ((x_1, x_2, \dots, x_\beta))\sigma,$$

for any $(x_1, x_2, \dots, x_\beta) \in N^{(\beta)}$. Since N_S is flat relative to A, $l_{N^{(\beta)}}(A) \subseteq Ker(\sigma)$ by Lemma 2.10. This means that ψ is well-defined. By (3), there exists a left R-homomorphism τ: $N^{(\alpha)} \to N$ such that $\psi = \eta\tau$. Set

$$A = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\alpha} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2\alpha} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{\beta 1} & \sigma_{\beta 2} & \cdots & \sigma_{\beta\alpha} \end{pmatrix}.$$

Then, for any

$$\begin{aligned}(x_1, x_2, \dots, x_\beta) \in N^{(\beta)}, ((x_1, x_2, \dots, x_\beta))\sigma &= ((x_1, x_2, \dots, x_\beta)A)\psi = ((x_1, x_2, \dots, x_\beta)A)\eta\tau \\ &= (x_1\sigma_{11} + x_2\sigma_{21} + \cdots + x_\beta\sigma_{\beta 1}, x_1\sigma_{12} + x_2\sigma_{22} + \cdots \\ &\quad + x_\beta\sigma_{\beta 2}, \dots, x_1\sigma_{1\alpha} + x_2\sigma_{2\alpha} + \cdots + x_\beta\sigma_{\beta\alpha})\tau \\ &= ((x_1, x_2, \dots, x_\beta))A\tau,\end{aligned}$$

which implies that $\sigma = A\tau$, it follows that S is left injective relative to A by Definition 2.11 (3) and [6, Theorem 3.10].

Corollary 2.14. *Let N be a left R-module with $S = \mathrm{End}({}_RN)$ and N_S flat. Then, the following conditions are equivalent.*

(1) *S is a left (α,β)-injective ring.*

(2) *N is quasi-(α,β)-injective.*

Remark 2.15. From Remark 2.7, Corollary 2.14 and Theorem 2.2, we get that the condition "$\beta < \infty$ or ${}_RN$ is finitely generated" in [6, Proposition 3.11(2)] is superfluous.

Corollary 2.16. *Let N be a left R-module with $S = \mathrm{End}(N)$ and N_S (m,n)-flat. Then, the following conditions are equivalent.*

(1) *S is a left (m,n)-injective ring.*

(2) *N is quasi-(m,n)-injective.*

3. Coherence relative to some matrices

Definition 3.1. Let N be a left R-module with $S = \mathrm{End}({}_RN)$ and $A \in S^{\beta\times\alpha}$. S is called left coherent relative to A if $\mathrm{Ker}({}_SS^{(\beta)} \to_S S^{(\beta)}A)$ is finitely generated.

Example 3.2. (1) Let N be a left R-module with $S = \mathrm{End}({}_RN)$ and $A \in S$. If A is an isomorphism or $A = 0$, then S is coherent relative to A by the definition.

(2) Let $N =_R R$ and $S = \mathrm{End}({}_RN) \cong R$. [12, Example 2.2] shows that S is left coherent relative to any $A \in S$ but not right coherent relative to any $A \in S$.

Recall that a right S-module N_S is said to be S-Mittag Leffler (S-ML) [13] in the case the canonical map $\mu_{N,I}$: $N \otimes S^I \to N^I$ defined via $\mu_{N,I}(x \otimes (s_i)) = (xs_i)$ ($\forall x \in N$ and $(s_i) \in S^I$) is a monomorphism for every set I. It is well known that N_S is finitely presented if and only if N is finitely generated and S-ML.

Remark 3.3. In [5, Theorem 4] and [6, Chapter 5], a ring S is said to be left coherent relative to $A \in S^{\beta\times\alpha}$ if $S^{(\beta)}A$ is a left S-ML module. Note that $S^{(\beta)}A$ is a β-generated submodule of S^α in Definition 3.1. Hence, if β is finite, $S^{(\beta)}A$ is finitely presented if and only if $S^{(\beta)}A$ is S-ML. Therefore, the left coherence relative to A here coincides with the definition in [5, 6] when β is finite.

Definition 3.4. Let N be a left R-module with $S = \mathrm{End}(_RN)$. S is called left (α, n)-coherent if it is left coherent relative to all matrices $A \in S^{n\times\alpha}$.

Remark 3.5. Let N be a left R-module with $S = \mathrm{End}(_RN)$.

(1) S is left (m, n)-coherent (a ring S is called left (m, n)-coherent [9] if every n-generated left submodule of the free left S-module S^m is finitely presented) if it is left coherent relative to all $A \in S^{n\times m}$.

(2) It is easy to see that S is left coherent if and only if it is left $(1, n)$-coherent for all positive integers n if and only if it is left (m, n)-coherent for all positive integers m and n.

(3) S is left π-coherent (a ring S is called left π-coherent [11] if every finitely generated torsionless left S-module is finitely present) if and only if it is left (α,β)-coherent for all $\beta \in \mathbb{N}$ and all cardinal numbers α.

The following example is taken from [14].

Example 3.6. Let S be a commutative ring and C be a cyclic S-module generated by an element a. Let R be the upper triangular matrix ring

$$R = \begin{pmatrix} S & C \\ 0 & S \end{pmatrix}$$

and $M = Re$, with

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $ReR = M$ and $\mathrm{End}(_RM) \cong eRe \cong S$. If S is coherent but not Noetherian (e.g., the polynomial ring on a denumerable set of indeterminates over a field), then $\mathrm{End}(_RM)$ is a coherent ring. Let $x, y_1, y_2, \cdots$ be indeterminates over a field K, $S = K[x^2, x^3, y_i, xy_i]$ (see [15, p. 110]). Then, S is coherent relative to any $A \in S$ (i.e., S is (1,1)-coherent) but not coherent relative to

$$A = \begin{pmatrix} x^2 \\ x^3 \end{pmatrix}.$$

In fact, $\mathrm{Ker}(_SS^{(2)} \to_S S^{(2)}A)$ is generated by (x_4, x_3) and (xy_i, y_i). Moreover, there is a submodule X of M such that

$$X \cong R/\begin{pmatrix} l_S(a) & C \\ 0 & S \end{pmatrix}$$

by [14]. Thus, X is not generated by M since

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e = 0$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \neq 0, \forall 0 \neq x \in X.$$

Let $\mathscr{C}$ be a class of left R-modules and M be a left R-module. Following [16], we say that a homomorphism φ: $C \to M$ is a $\mathscr{C}$-precover of M if $C \in \mathscr{C}$ and the abelian group homomorphism $\mathrm{Hom}(C', \varphi)$: Hom $(C', C) \to$ Hom (C', M) is surjective for each $C' \in \mathscr{C}$. A $\mathscr{C}$-precover φ: $C \to M$ is called a $\mathscr{C}$-cover if every endomorphism f: $C \to C$ such that $f\varphi = \varphi$ is an isomorphism. $\mathscr{C}$-covers may not exist in general, but if they exist, they are unique up to isomorphisms.

Theorem 3.7. *Let N be a left R-module with $S = \mathrm{End}({}_RN)$ and $A \in S^{n\times\alpha}$. Then, the following are equivalent.*

(1) *S is left coherent relative to A.*

(2) $\mathrm{Hom}_R(N, l_{N^n}(A))$ *is a finitely generated left S-module.*

(3) $l_{N^n}(A)$ *has an* $\mathrm{add}(N)$*-precover.*

Proof. Consider the exact sequence of left R-modules

$$0 \longrightarrow l_{N^n}(A) \longrightarrow N^n \xrightarrow{\varphi} N^nA,$$

which leads to the exact sequence of left S-modules

$$0 \longrightarrow \mathrm{Hom}_R(N, l_{N^{(\beta)}}(A)) \longrightarrow \mathrm{Hom}_R(N, N^n) \xrightarrow{\varphi_*} \mathrm{Hom}_R(N, N^nA).$$

Thus, $\mathrm{Hom}_R(N, N^n) \cong S^n$, and $\mathrm{im}(\varphi_*) \cong S^nA$.

(1) $\Rightarrow$ (2) Since S is left coherent relative to A, $\mathrm{Hom}_R(N, l_{N^n}(A))$ is finitely generated by the definition.

(2) $\Rightarrow$ (3) Since $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated left S-module, there is a generating set $\{f_i \in \mathrm{Hom}_R(N, l_{N^n}(A)) : 1 \leq i \leq k\}$ of $\mathrm{Hom}_R(N, l_{N^n}(A))$. Define a left R-homomorphism f: $N^k \to l_{N^n}(A)$ via

$$((x_1, x_2, \ldots, x_k))f = \sum_{i=1}^{k} (x_i)f_i, \forall x_i \in N.$$

Thus, f is an $\mathrm{add}(N)$-precover of $l_{N^n}(A)$. In fact, let m be any positive integer and ϕ: $N^m \to l_{N^n}(A)$ be any left R-morphism. Suppose

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_m \end{pmatrix},$$

where $\phi_i \in \mathrm{Hom}_R(N, l_{N^n}(A))$, $i = 1, 2, \ldots, m$. Then, there exist left R-endomorphisms of N: $g_{ij} \in S (1 \leq i \leq m, 1 \leq j \leq k)$ such that $\phi_i = \sum\limits_{j=1}^{k} g_{ij}f_j$. Define a left R-homomorphism

$$g = (g_{ij})_{m\times k} : N^m \to N^k.$$

It is easy to check that $\phi = gf$. So, f is an add(N)-precover of $l_{N^n}(A)$.

(3) ⇒ (1) By (3), $l_{N^n}(A)$ has an add(N)-precover $N^k \to l_{N^n}(A)$. Then, there exists an exact sequence of left S-homomorphisms

$$\mathrm{Hom}_R(N, N^k) \to \mathrm{Hom}_R(N, l_{N^n}(A)) \to 0.$$

Note that $\mathrm{Hom}_R(N, N^k) \cong S^k$. It follows that $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated left S-module. Hence, S is a left coherent ring relative to A.

Proposition 3.8. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S^{n\times\alpha}$. Then, the following are equivalent.*

(1) *S is left coherent relative to A, and N_S is flat relative to A.*

(2) *$l_{N^n}(A)$ has an epic* add(N)*-precover.*

Proof. (1) ⇒ (2) Since N_S is flat relative to A, there exists a left R-epimorphism g: $N^{(I)} \to l_{N^n}(A)$ by Theorem 2.2. For any $y \in l_{N^n}(A)$, there is an element $x \in N^{(I)}$ such that $(x)g = y$. Let π_i: $N^{(I)} \to N$ be the canonical projection and η_i: $N \to N^{(I)}$ be the canonical injection. Thus,

$$\sum_{i=1}^{I}(x)\pi_i\eta_i = x$$

by [7, Proposition 6.21]. Since S is left coherent relative to A, $l_{N^n}(A)$ has an add(N)-precover f: $N^k \to l_{N^{(\beta)}}(A)$ by Theorem 3.7. Hence, there exist left R-homorphisms θ_i: $N \to N^k$ such that $\eta_i g = \theta_i f$. It follows that

$$\pi_i\eta_i g = \pi_i\theta_i f.$$

So,

$$y = (x)g = (\sum_{i=1}^{I}(x)\pi_i\eta_i)g = (\sum_{i=1}^{I}(x)\pi_i\eta_i g) = (\sum_{i=1}^{I}(x)\pi_i\theta_i)f.$$

Thus, f is an epic *add*(N)-precover of $l_{N^n}(A)$, and so (2) holds.

(2) ⇒ (1) follows by Theorem 3.7 and Theorem 2.2.

Let $_RN =_R R$.

Corollary 3.9. *Let $A \in R^{n\times\alpha}$. Then, the following are equivalent.*

(1) *R is left coherent relative to A.*

(2) *$l_{R^n}(A)$ is a finitely generated left R-module.*

Corollary 3.10. (1) *R is left coherent if and only if each finitely generated left ideal of R is finitely presented.*

(2) *R is left (m, n)-coherent if and only if each n-generated submodule of R^m is finitely presented.*

(3) *R is left π-coherent if and only if each finitely generated left R-module is finitely presented.*

Recall that a $\mathscr{C}$-cover φ: $C \to X$ of X is said to have the unique mapping property [17] if for any homomorphism, f: $C' \to X$ with $C' \in \mathscr{C}$, there is a unique homomorphism g: $C' \to C$ such that $g\varphi = f$.

Theorem 3.11. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S^{n\times\alpha}$. The following are equivalent.*

(1) $l_{N^n}(A)$ *has an* add(N)*-cover with the unique mapping property.*

(2) $\mathrm{Hom}_R(N, l_{N^n}(A))$ *is a finitely generated projective left* S*-module.*

(3) $l_{N^n}(A)$ *has an* add(N)*-cover* $f\colon H \to l_{N^n}(A)$ *such that* $\mathrm{Hom}_R(N, kefr(f)) = 0$.

(4) S *is left coherent relative to A, and* $\mathrm{Hom}_R(N, l_{N^n}(A))$ *is a projective left* S*-module.*

Moreover, $N \otimes_S \mathrm{Hom}_R(N, l_{N^n}(A)) \to l_{N^n}(A)$ *is an* add(N)*-cover of* $l_{N^n}(A)$ *with the unique mapping property.*

Proof. (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are trivial by Theorem 3.7.

(1) $\Rightarrow$ (2) In view of (1), we get that $l_{N^n}(A)$ has an add(N)-precover $f\colon H \to l_{N^n}(A)(H \in \mathrm{add}(N))$ with the unique mapping property. Consider the following exact sequence of left R-modules $0 \to Ker(f) \to H \to l_{N^n}(A)$, which induces the following sequence of left S-modules:

$$0 \to \mathrm{Hom}_R(N, Ker(f)) \to \mathrm{Hom}_R(N, H) \to \mathrm{Hom}_R(N, l_{N^n}(A)) \to 0.$$

Note that f has the unique mapping property. Then, $\mathrm{Hom}_R(N, Ker(f)) = 0$. It follows that $\mathrm{Hom}_R(N, H) \cong \mathrm{Hom}_R(N, l_{N^n}(A))$. Thus, $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated projective left S-module because H is a direct summand of N^k, for some integer k.

(2) $\Rightarrow$ (1) Since $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated left S-module, $l_{N^n}(A)$ has an add(N)-precover f: $N^t \to l_{N^n}(A)$ by Theorem 3.7. Let i: $Ker(f) \to N^t$ be the inclusion. This induces the following exact sequence of left S-modules:

$$0 \longrightarrow \mathrm{Hom}_R(N, Ker(f)) \xrightarrow{i_*} \mathrm{Hom}_R(N, N^t) \xrightarrow{f_*} \mathrm{Hom}_R(N, l_{N^{(n)}}(A)) \longrightarrow 0.$$

Note that $\mathrm{Hom}_R(N, l_{N^n}(A))$ is projective, and then i_* is a split monomorphism. Suppose that

$$e_1 = (Id_N, 0, 0, \ldots, 0), e_2 = (0, Id_N, 0, \ldots, 0), \ldots, e_t = (0, 0, 0, \ldots, Id_N)$$

are the generated elements of $\mathrm{Hom}_R(N, N^t)$. If $\rho \in \mathrm{Hom}_R(N, N^t)$, then there exist $s_i \in S$ $(i = 1, 2, \ldots, t)$ such that

$$\rho = s_1e_1 + s_2e_2 + \cdots + s_te_t.$$

Thus,

$$a \otimes \rho = a \otimes (s_1e_1 + s_2e_2 + \cdots + s_te_t) = (a)s_1 \otimes e_1 + (a)s_2 \otimes e_2 + \cdots + (a)s_t \otimes e_t,$$

for any $a \otimes \rho \in N \otimes_S \mathrm{Hom}_R(N, N^t)$. We suppose that $\mathrm{Hom}_R(N, Ker(f))$ is a submodule of $\mathrm{Hom}_R(N, N^t)$. There is a left S-module $Q \cong \mathrm{Hom}_R(N, l_{N^n}(A))$ such that $\mathrm{Hom}_R(N, N^t) = \mathrm{Hom}_R(N, Ker(f)) \oplus Q$. So, there are elements $g_i \in \mathrm{Hom}_R(N, Ker(f))$ and $h_i \in Q$ such that $e_i = g_i \oplus h_i$, $i = 1, 2, \ldots, t$. For any $\tau \in \mathrm{Hom}_R(N, Ker(f))$, there exist $\xi_i \in S$ $(i = 1, 2, \ldots, t)$ such that

$$\tau = \xi_1e_1 + \xi_2e_2 + \cdots + \xi_te_t,$$

which implies that

$$\tau = \xi_1(g_1 + h_1) + \xi_2(g_2 + h_2) + \cdots + \xi_t(g_t + h_t) = \xi_1g_1 + \xi_2g_2 + \cdots + \xi_tg_t + \xi_1h_1 + \xi_2h_2 + \cdots + \xi_th_t.$$

It follows that

$$\tau = \xi_1e_1 + \xi_2e_2 + \cdots + \xi_te_t = \xi_1g_1 + \xi_2g_2 + \cdots + \xi_tg_t.$$

It is easy to check that

$$b\otimes\tau = b\otimes(\xi_1 g_1+\xi_2 g_2+\cdots+\xi_t g_t) = (b)\xi_1\otimes g_1+(b)\xi_2\otimes g_2+\cdots+(b)\xi_t\otimes g_t,\ \forall b\otimes\tau \in N\otimes_S \mathrm{Hom}_R(N, Ker(f)).$$

Consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N \otimes_S \mathrm{Hom}_R(N, Ker(f)) \arrow[r, "Id_N\otimes i_*"] \arrow[d, "\sigma''"] & N \otimes_S \mathrm{Hom}_R(N, N^t) \arrow[r, "Id_N\otimes f_*"] \arrow[d, "\sigma"] & N \otimes_S \mathrm{Hom}_R(N, l_{N^n}(A)) \arrow[r] \arrow[d, "\sigma'"] & 0 \\
0 \arrow[r] & Ker(f) \arrow[r, "i"] & N^t \arrow[r, "f"] & l_{N^n}(A) &
\end{tikzcd}$$

where $(a \otimes e_i)\sigma = (a)e_i$ and $(a \otimes g_i)\sigma'' = (a)g_i$, for any $a \in N$. Thus, σ and σ'' are both isomorphisms. Clearly, the first row in the diagram above is split.

Let $\varphi \in \mathrm{Hom}_R(N, N \otimes \mathrm{Hom}(N, N^t))$. If $x \in N$, we write

$$(x)\varphi = a_{1x} \otimes e_1 + a_{2x} \otimes e_2 + \cdots + a_{tx} \otimes e_t.$$

Suppose that $s_i\colon N \to N$ are left R-morphisms via $(x)s_i = a_{ix}$. Thus, s_i is well-defined. It follows that

$$(x)\varphi = x \otimes (s_1e_1 + s_2e_2 + \cdots + s_te_t),$$

for any $x \in N$. Let $\phi \in \mathrm{Hom}_R(N, N \otimes \mathrm{Hom}(N, Ker(f)))$. Similarly, we can get that

$$(x)\phi = x \otimes (s_1g_1 + s_2g_2 + \cdots + s_tg_t),$$

for any $x \in N$ (We may regard $\mathrm{Hom}_R(N, N \otimes \mathrm{Hom}(N, Ker(f))$ as a direct summand of $\mathrm{Hom}_R(N, N \otimes \mathrm{Hom}(N, N^t)))$. Since $Id_N \otimes i_*$ is a split monomorphism and $\mathrm{Hom}_R(N, l_{N^n}(A))$ is projective, we have the following diagram with split exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(N, N \otimes_S \mathrm{Hom}_R(N, Ker(f))) \arrow[r, "(Id_N\otimes i_*)_*"] \arrow[d, "(\sigma'')_*"] & \mathrm{Hom}_R(N, N \otimes_S \mathrm{Hom}_R(N, N^t)) \arrow[r, "(Id_N\otimes f_*)_*"] \arrow[d, "\sigma_*"] & \mathrm{Hom}_R(N, N \otimes_S \mathrm{Hom}_R(N, l_{N^n}(A))) \arrow[r] \arrow[d, "(\sigma')_*"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(N, Ker(f)) \arrow[r, "i_*"] & \mathrm{Hom}_R(N, N^k) \arrow[r, "f_*"] & \mathrm{Hom}_R(N, l_{N^n}(A)) \arrow[r] & 0.
\end{tikzcd}$$

Note that $(\sigma'')_*$ and σ_* are isomorphisms. By the five lemma, $(\sigma')_*$ is also an isomorphism. Since $\mathrm{Hom}_R(N, l_{N^n}(A))$ is a finitely generated projective left S-module, $N\otimes\mathrm{Hom}_R(N, l_{N^n}(A)) \in \mathrm{add}(N)$. Thus, $N \otimes_S \mathrm{Hom}_R(N, l_{N^n}(A)) \to l_{N^n}(A)$ is an $\mathrm{add}(N)$-cover with the unique mapping property.

Theorem 3.12. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S^{n\times\alpha}$. The following are equivalent.*

(1) *$l_{N^n}(A)$ has a monic $\mathrm{add}(N)$-cover.*

(2) *S is left coherent relative to A, and any left R-homomorphism $f\colon L_1 \to l_{N^n}(A)$ factors through a left R-module in $\mathrm{add}(N)$, where L_1 is a quotient-module of any left R-module $L \in \mathrm{add}(N)$.*

Proof. (1) ⇒ (2) S is left coherent relative to A by Theorem 3.7. By (1), $l_{N^n}(A)$ has a monic $\mathrm{add}(N)$-cover $g\colon P \to l_{N^n}(A)(P \in \mathrm{add}(N))$. Let L_1 be a quotient-module of any left R-module $L \in \mathrm{add}(N)$.

For any $f \in \mathrm{Hom}(L_1, l_{N^n}(A))$, there is a left R-homomorphism ψ: $P \to L$ such that $\psi g = \pi f$, where π: $L \to L_1$ is the canonical epimorphism. Consider the following diagram:

$$\begin{tikzcd} & L \arrow[r, "\pi"] \arrow[d, dotted, "\psi"'] & L_1 \arrow[r] \arrow[d, "f"] \arrow[dl, dotted, "s"] & 0. \\ 0 \arrow[r] & P \arrow[r, "g"] & l_{N^n}(A) & \end{tikzcd}$$

Define a left R-homomorphism s: $L_1 \to P$ via $((x)\pi)s = (x)\psi$, for any $x \in L$. Clearly, s is well-defined. In fact, if $(x)\pi = 0$, then

$$(x)\psi g = ((x)\pi)f = 0.$$

It follows that $(x)\psi = 0$ since g is monic. It is easy to see that $f = sg$, i.e., f factors through a left R-module $P \in \mathrm{add}(N)$.

(2) $\Rightarrow$ (1) Since S is left coherent relative to A, $l_{N^n}(A)$ has an add(N)-precover f: $F \to l_{N^n}(A)$ by Theorem 3.7. Let $F_1 = F/Ker(f)$, θ: $F_1 \to l_{N^n}(A)$ be the induced monomorphism of f and π: $F \to F_1$ be the canonical epimorphism. By (2), there is a left R-module $H \in \mathrm{add}(N)$ and left R-homomorphisms g: $H \to l_{N^n}(A)$ and h: $F_1 \to H$ such that $\theta = hg$. Note that f is a precover. There exists a left R-homomorphism φ: $H \to F$ such that $g = \varphi f$. Thus,

$$\theta = hg = h\varphi f = h\varphi\pi\theta,$$

and so $Id_{F_1} = h\varphi\pi$ since θ is monic. Hence, $F_1 \in \mathrm{add}(N)$. It is easy to see that θ is a monic add(N)-cover of $l_{N^n}(A)$.

Corollary 3.13. *Let N be a left R-module with $S = \mathrm{End}(_RN)$.*

(1) [1, Theorem 3.1] *S is left (m, n)-coherent if and only if each N-(m, n)-copresented left R-module has an* add(N)*-precover.*

(2) [1, Corollary 3.3] *S is left (m, n)-coherent, and N_S is (m, n)-flat if and only if each N-(m, n)-copresented left R-module has an epic* add(N)*-precover.*

(3) *S is left (m, n)-coherent, and* $\mathrm{Hom}_R(N, K)$ *is a projective left S-module, for each N-(m, n)-copresented left R-module K, if and only if each N-(m, n)-copresented left R-module has an* add(N)*-cover with the unique mapping property.*

(4) *S is left (m, n)-coherent, and any left R-homomorphism f: $L_1 \to K$ factors through a module in* add(N)*, where K is any N-(m, n)-copresented left R-module and L_1 is a quotient-module of any left R-module $L \in add(N)$, if and only if each N-(m, n)-copresented left R-module has a monic* add(N)*-cover.*

Proposition 3.14. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in S^{n\times\alpha}$. The following are equivalent.*

(1) *$l_{N^n}(A)$ has a monic* add(N)*-cover, and N_S is flat relative to A.*

(2) *$l_{N^n}(A) \in$* add(N)*.*

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) By (1), we get that $l_{N^n}(A)$ has a monic add(N)-precover f: $H \to l_{N^{(n)}}(A) (H \in \mathrm{add}(N))$. Since N_S is flat relative to A, f is epic by Theorem 3.7 and Corollary 3.8. Thus, f is an isomorphism and (1) holds.

Corollary 3.15. *Let N be a left R-module with $S = \mathrm{End}(_RN)$, $A \in S^{n\times\alpha}$ and N_S flat (as a right S-module). Then, the following are equivalent.*

(1) $l_{N^n}(A)$ *has an* $\mathrm{add}(N)$-*cover with the unique mapping property.*

(2) $l_{N^n}(A)$ *has a monic* $\mathrm{add}(N)$-*cover.*

(3) $l_{N^n}(A) \in \mathrm{add}(N)$.

(4) $\mathrm{Hom}_R(N, l_{N^n}(A))$ *is a finitely generated projective left S-module.*

Proof. (2) ⇒ (1) is trivial.

(2) ⇔ (3) follows by the above Corollary.

(1) ⇔ (4) follows by Proposition 3.11.

(1) ⇒ (2) By the definition of having unique mapping property, we get that $l_{N^n}(A)$ has an $\mathrm{add}(N)$-cover $f\colon H \to l_{N^n}(A)$ such that $\mathrm{Hom}_R(N, Ker(f)) = 0$. Note that $H \in \mathrm{add}(N)$. Let $\eta\colon l_{N^n}(A) \to N^n$ be the canonical inclusion and $\pi\colon N^k \to H$ be the canonical projection. It follows that $Ker(\pi f\eta)$ is generated by N since N_S is flat by Theorem 2.2 and Remark 2.7. Thus, $Ker(\pi f)$ and $Ker(f)$ are both generated by N. However, $\mathrm{Hom}_R(N, Ker(f)) = 0$, and then $Ker(f) = 0$. It follows that f is a monomorphism.

Let $_RN = _R R$.

Corollary 3.16. *Let $A \in R^{n\times\alpha}$. Then, the following are equivalent.*

(1) $l_{R^n}(A)$ *has a finitely generated projective cover with the unique mapping property.*

(2) $l_{R^n}(A)$ *has a monic finitely generated projective cover.*

(3) $l_{R^n}(A)$ *is a finitely generated projective left R-module.*

Corollary 3.17. *Let $a \in R$. Then, the following are equivalent.*

(1) $l_R(a)$ *has a finitely generated projective cover with the unique mapping property.*

(2) $l_R(a)$ *has a monic finitely generated projective cover.*

(3) $l_R(a)$ *is a finitely generated projective left R-module.*

Theorem 3.18. *Let N be a left R-module with $S = \mathrm{End}(_RN)$ and $A \in \mathrm{RFM}_{n\times\alpha}(S)$. The following are equivalent.*

(1) S^nA *is a projective left S-module, and N_S is flat relative to A.*

(2) $l_{N^n}(A)$ *is a direct summand of N^n.*

Proof. Let $K = l_{N^n}(A)$. Then, there is an exact sequence

$$0 \longrightarrow K \xrightarrow{i} N^n \xrightarrow{\varphi} N^nA.$$

Note that $A \in \mathrm{RFM}_{n\times\alpha}(S)$. Then, $0 \longrightarrow N^nA \longrightarrow N^{(\alpha)}$ is exact. These yield the following diagram of left S-modules:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(N,K) \arrow[r, "i_*"] & \mathrm{Hom}_R(N,N^n) \arrow[rr, "\varphi_*"] \arrow[dr] & & \mathrm{Hom}_R(N,N^nA) \\
& & 0 \arrow[r] & S^nA \arrow[ur] \arrow[r] \arrow[dr] & S^{(\alpha)} \\
& & 0 \arrow[ur] & & 0,
\end{tikzcd}$$

where $\mathrm{Hom}_R(N, N^n) \cong S^n$ and $\mathrm{im}(\varphi_*) \cong S^n A$.

(2) ⇒ (1) Since K is a direct summand of N^n, i_* is split by the diagram above. It follows that $S^n A$ is a projective left S-module. According to (2), we get that K is finitely generated by N. Thus, N is flat relative to A by Theorem 2.2.

(1) ⇒ (2) Since $S^n A$ is projective, i_* is a spilt monomorphism by the diagram above. Thus, $Id_N \otimes i_*$ is a split monomorphism. Since N_S is flat relative to A, we have the following exact sequence by Corollary 2.4:

$$0 \longrightarrow N \otimes_S \mathrm{Hom}_R(N, K) \xrightarrow{Id_N \otimes i_*} N \otimes_S S^n \longrightarrow N \otimes_S S^{(\alpha)}.$$

Consider the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & N \otimes_S \mathrm{Hom}_R(N, K) \arrow[r, "Id_N \otimes i_*"] \arrow[d, "\sigma"] & N \otimes_S S^n \arrow[r] \arrow[d, "\cong"] & N \otimes_S S^{(\alpha)} \arrow[d, "\cong"] \\ 0 \arrow[r] & K \arrow[r, "i"] & N^n \arrow[r] & N^{(\alpha)}. \end{tikzcd}$$

By the five lemma, σ is an isomorphism. Note that $Id_N \otimes i_*$ is a spilt monomorphism. Then, i is split by the above diagram. Hence, $K = l_{N^n}(A)$ is a direct summand of N^n.

Recall that R is a left n-semihereditary ring [18,19] if every n-generated right ideal of R is projective, or equivalently, if every n-generated submodule of a projective right R-module is projective.

Corollary 3.19. [1, Theorem 3.6] *Let N be a left R-module with $S = \mathrm{End}(_R N)$. The following are equivalent.*

(1) *S is a left n-semihereditary ring, and N_S is (m, n)-flat over its endomorphism.*

(2) *K is a direct summand of N^n, for any $K = Ker(_R N^n \to_R N^m)$.*

Proof. It is trivial by Remark 3.5 (1) and Theorem 3.18.

It is well known that S is a von Neumann regular ring if and only if every left (right) S-module is (1,1)-injective if and only if every left (right) S-module is (m, n)-injective if and only if every left (right) S-module is $(1, 1)$-flat if and only if every left (right) S-module is (m, n)-flat.

Corollary 3.20. *Let N be a quasi-(m, n)-injective left R-module with $S = \mathrm{End}(_R N)$. The following are equivalent.*

(1) *S is a left n-semihereditary ring, and N_S is (m, n)-flat.*

(2) *S is a von Neumann regular ring.*

Proof. (1) ⇒ (2) By (1) and Corollary 2.16, S is a left (m, n)-injective ring, and so is $S^{(I)}$, for any set I. Since S is a left n-semihereditary ring, each quotient-module of an (m, n)-injective left S-module is (m, n)-injective by [19, Theorem 3]. It follows that every left S-module is (m, n)-injective. Thus, S is a von Neumann regular ring.

(2) ⇒ (1) It is trivial. □

4. Conclusions

Remark 4.1. It would be interesting to extend the results to coherent [14, 20], n-coherent [18] and π-coherent [11] endomorphism rings. For example, a left R-module M is called finitely

N-copresented [20] if there exist positive integers m, n and an exact sequence $0 \to M \to N^m \to N^n$ of left R-modules. Let $WD(S)$ denote the weak global dimension of S. Then, S is left coherent if and only if every finitely N-copresented module has an add(N)-precover by Theorem 3.7. S is left coherent, and $WD(S) \leq 2$ if and only if every finitely N-copresented module has an add(N)-cover with the unique mapping property if and only if $\mathrm{Hom}_R(N, K)$ is a finitely generated projective left S-module (K is any finitely N-copresented module) by [20, Lemma 1.7] and Theorem 3.11. S is left coherent, N_S is flat over its endomorphism ring, and $WD(S) \leq 2$ if and only if every finitely N-copresented module belongs to add(N) by [20, Lemma 1.7] and Corollary 3.15. S is semihereditary (S is called semihereditary if every finitely generated right ideal of R is projective), and N_S is flat over its endomorphism ring if and only if every finitely N-copresented module belongs to add(N) by Theorem 3.18.

Remark 4.2. The computation of the endomorphism rings is an important problem in computational number theory as well as in cryptography. For instance, it is important in the computation of class polynomials, which play an important role in explicit class field theory. There are many calculations and much research on endomorphism rings, such as [21–24], etc. According to the calculation of endomorphism ring S, we can choose the specific matrix $A \in S^{n\times\alpha}$ and then discuss the flatness and coherence relative to A. Moreover, according to the corresponding matrix, we can study relative flatness such as [18, 25–27], etc.

Remark 4.3. Let N be a left R-module with the endomorphism ring $S = \mathrm{End}({}_RN)$. Duality to flat modules and add(N)-covers, injective modules and add(N)-envelopes may also be studied similarly.

Acknowledgments

Thanks for the referees' comments. Those comments are all valuable and very helpful for improving the paper. The work was supported by the National Natural Science Foundation of China (61772292), the Natural Science Foundation of Fujian Province, China (2020J01908), and the Educational Commission of Fujian Province, China (JAT190582).

Conflict of interest

The author declares that there is no conflict of interest.

References

1. L. X. Mao, (m, n)-injective and (m, n)-coherent endomorphism rings of modules, *J. Algebra Appl.*, **19** (2020), 993–1008. http://doi.org/10.1142/S0219498820500486

2. L. Angeleri-Hügel, Endocoherent modules, *Pac. J. Math.*, **212** (2003), 1–11. http://doi.org/10.2140/pjm.2003.212.1

3. G. Azumaya, Finite splitness and finite projectivity, *J. Algebra*, **106** (1987), 114–134. http://dx.doi.org/10.1016/0021-8693(87)90024-X

4. N. Q. Ding, J. L. Chen, Relative coherence and preenvelopes, *Manuscripta Math.*, **81** (1993), 243–262. http://doi.org/10.1007/BF02567857

5. X. X. Zhang, J. L. Chen, Homological properties of modules characterized by matrices, *J. Southeast Univ.*, **21** (2005), 239–243.

6. X. X. Zhang, J. L. Chen, Properties of modules and rings relative to some matrices, *Commun. Algebra*, **36** (2008), 3682–3707. http://dx.doi.org/10.1080/00927870802157996

7. F. W. Anderson, K. R. Fuller, *Rings and categries of modules*, Springer Verlag, 1974.

8. S. L. Zhu, On rings over which every flat left module is finitely projective, *J. Algebra*, **139** (1991), 311–321. http://doi.org/10.1016/0021-8693(91)90295-J

9. X. X. Zhang, J. L. Chen, J. Zhang, On (m,n)-injective modules and (m,n)-coherent rings, *Algebra Colloq.*, **12** (2005), 149–160. https://doi.org/10.1142/S1005386705000143

10. S. U. Chase, Direct products of modules, *Trans. Amer. Math. Soc.*, **97** (1960), 457–473. http://doi.org/10.2307/1993382

11. V. Camillo, Coherence for polynomial rings, *J. Algebra*, **132** (1990), 72–76. http://doi.org/10.1016/0021-8693(90)90252-J

12. L. X. Mao, N. Q. Ding, On divisible and torsionfree modules, *Commun. Algebra*, **36** (2008), 708–731. http://doi.org/10.1080/00927870701724201

13. T. G. Clarke, *On $\aleph$-projective modules*, Ph. D. thesis, Kent State University, Kent, 1976.

14. J. L. G. Pardo, J. M. Hernández, Coherence of endomorphism rings, *Arch. Math.*, **48** (1987), 40–52. http://doi.org/10.1007/BF01196353

15. S. Glaz, *Commutative coherent rings*, Springer Verlag, 1989.

16. E. E. Enochs, Injective and flat covers, envelopes and resolvents, *Isr. J. Math.*, **39** (1981), 189–209. http://doi.org/10.1007/BF02760849

17. N. Q. Ding, On envelopes with the unique mapping property, *Commun. Algebra*, **24** (1995), 1459–1470. http://dx.doi.org/10.1080/00927879608825646

18. A. Shamsuddin, n-injective and n-flat modules, *Commun. Algebra*, **29** (2001), 2039–2050. http://doi.org/10.1081/AGB-100002166

19. Z. M. Zhu, Z. S. Tan, On n-semihereditary rings, *Sci. Math. Jpn.*, **62** (2005), 511–515.

20. J. L. Garcia, M. Hernandez, P. L. Gomez Sanchez, Endomorphism rings and the category of finitely generated projective modules, *Commun. Algebra*, **28** (2000), 3837–3852. http://doi.org/10.1080/00927870008827061

21. A. Abyzov, R. Barati, P. V. Danchev, Rings close to periodic with applications to matrix, endomorphism and group rings, *ArXiv*, 2023. https://doi.org/10.48550/arXiv.2301.07948

22. S. Breaz, T. Brzeziński, The Baer-Kaplansky theorem for all abelian groups and modules, *Bull. Math. Sci.*, **12** (2022), 2150005. https://doi.org/10.1142/S1664360721500053

23. J. F. Carlson, Negative cohomology and the endomorphism ring of the trivial module, *J. Pure Appl. Algebra*, **226** (2022), 107046. https://doi.org/10.1016/j.jpaa.2022.107046

24. C. Fieker, T. Hofmann, S. P. Sanon, On the computation of the endomorphism rings of abelian surfaces, *J. Number Theory*, **229** (2021), 39–52. http://doi.org/10.1016/j.jnt.2021.04.024

25. F. Couchot, RD-flatness and RD-injectivity, *Commun. Algebra*, **34** (2006), 3675–3689. http://dx.doi.org/10.1080/00927870600860817

26. N. Q. Ding, Y. L. Li, L. X. Mao, J-coherent rings, *J. Algebra Appl.*, **8** (2009), 139–155. http://dx.doi.org/10.1142/S0219498809003229

27. A. Moradzadeh-Dehkordi, F. Couchot, RD-flatness and RD-injectivity of simple modules, *J. Pure Appl. Algebra*, **226** (2022), 107034. https://doi.org/10.1016/j.jpaa.2022.107034