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Research article

New techniques on fixed point theorems for symmetric contraction mappings with its application

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Abstract: The target of this manuscript is to introduce new symmetric fractional α-β-η-Υ-contractions and prove some new fixed point results for such contractions in the setting of M_b-metric space. Moreover, we derive some results for said contractions on closed ball of mentioned space. The existence of the solution to a fractional-order differential equation with one boundary stipulation will be discussed.

Keywords: fixed point theorems; M_b-metric space; symmetric fractional α-β-η-Υ-contraction; fractional-order differential equation
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1. Introduction

The fixed point (FP) theory popularize in different ways by many image authentications which are proposed in the literature. Recently, new approach based on the FP theory is given in the literature. It has become an essential stanchion of nonlinear analysis, where it is used to study the existence and uniqueness of the solutions for many differential and nonlinear integral equations [1–9]. There were many generalizations of metric space (MS), for instance the first extension of MS was to partial metric space (PMS) [10] which was done by defining the self distance, another extension was to b-metric space (bMS) [11] by changing the triangle inequality. In [12] Asadi et al. introduced and extended PMS to M-metric space (MMS). Also, he showed that every PMS is an M-MS, but inverse is not true. In 2016, Mlaiki et al. [13] introduced the concept of M_b-metric space (MbMS) which is an extension of MMS and they gave an example of an MbMS which is not an MMS with proving some FP results. BCP [14] was appeared in 1922, to be the base of functional analysis and plays a main role in several branches of mathematics and applied sciences, which asserts that every contraction mapping defined in complete MS has an FP. In many directions this principle has been extended and generalized either by relaxing the contractive stipulations or imposing some more stipulations on space. One of these generalizations and interesting approaches is interpolative Kannan type contraction which was introduced by Karapinar [15] and established new FP results on complete MS. In [16] Karapinar et al., discussed the interpolative Reich-Rus-Ciric type contractions in complete PMSs and deduced new FPs results. In 2020, Hussain [17] gave a proper extension of [15, 16] by presenting the notion of fractional convex Reich-type and Kannan type α-η-contractions and established some FP theorems in the setting of F-complete F-MS. Newly, the notion of fractional symmetric α-η-contraction was introduced in [18–20] with discussing of applications for solving fractional-order differential equations, they studied four types of said contraction and obtained FP results in the setting of F-complete F-MS. In 2022, Nazam et al. [29] introduced (Ψ, Φ)-orthogonal interpolative contractions with showing the existence of FPs of set-valued (Ψ, Φ)-orthogonal interpolative contractions. In this research article, we are going to give a splendid generalization of Hussain et al. [19] by introducing four new types of symmetric fractional α-β-η-Υ-contractions and prove some new FP results in the complete MbMS. As addition of our main results, we will show existence of FPs for such contractions on closed ball of mentioned space. As an application, we will investigate existence of solving of fractional-order differential equations.

2. Preliminaries

In this portion, some elementary discussions about MbMSs will be given. It should be noted that Mlaiki et al. [13] introduced the notion of MbMS and inaugurated the advanced Banach Contraction Principle on MbMS. So, the notion of $m_{b_{\xi,\mu}}$ and $M_{b_{\xi,\mu}}$ are defined as follows:

$$m_{b_{\xi,\mu}} = \min\left\{m_b(\xi,\xi), m_b(\mu,\mu)\right\},$$

and

$$M_{b_{\xi,\mu}} = \max\left\{m_b(\xi,\xi), m_b(\mu,\mu)\right\}.$$

Definition 2.1. An MbMS on a non-empty set Δ is a function $m_b : \Delta^2 \to R^+$ that fulfills the assumptions below, for all $\xi, \mu, \kappa \in \Delta$,

(Mb_1) $m_b(\xi,\xi) = m_b(\mu,\mu) = m_b(\xi,\mu)$ iff $\xi = \mu$;
(Mb_2) $m_{b_{\xi,\mu}} \leq m_b(\xi,\mu)$;
(Mb_3) $m_b(\xi,\mu) = m_b(\mu,\xi)$;
(Mb_4) There is a coefficient $s \geq 1$ so that for all $\xi, \mu, \kappa \in \Delta$, we have

$$m_b(\xi,\mu) - m_{b_{\xi,\mu}} \leq s\left[\left(m_b(\xi,\kappa) - m_{b_{\xi,\kappa}}\right) + \left(m_b(\kappa,\mu) - m_{b_{\kappa,\mu}}\right)\right] - m_b(\kappa,\kappa).$$

Then the pair (Δ, m_b) is called an MbMS.

Example 2.2. Let $\Delta = [0,\infty)$ and $p > 1$ be a constant. Define $m_b : \Delta^2 \longrightarrow [0,\infty)$ by

$$m_b(\xi,\mu) = \left(\max\{\xi,\mu\}\right)^p + |\xi - \mu|^p, \ \forall \xi,\mu \in \Delta.$$

Then (Δ, m_b) is an MbMS (with coefficient $s = 2^p$) and not MMS.

Definition 2.3. Let (Δ, m_b) be an MbMS. Then

- A sequence $\{\xi_n\}$ in Δ converges to a point ξ if and only if

$$\lim_{n\to\infty}\left(m_b(\xi_n,\xi)-m_{b_{\xi_n,\xi}}\right)=0.$$

- A sequence $\{\xi_n\}$ in Δ is called m_b-Cauchy sequence iff

$$\lim_{n,m\to\infty}\left(m_b(\xi_n,\xi_m)-m_{b_{\xi_n,\xi_m}}\right) \text{ and } \lim_{n,m\to\infty}\left(M_{b_{\xi_n,\xi_m}}-m_{b_{\xi_n,\xi_m}}\right)$$

exist and finite.
- An MbMS is called m_b-complete if every m_b-Cauchy sequence $\{\xi_n\}$ converges to a point ξ so that

$$\lim_{n\to\infty}\left(m_b(\xi_n,\xi)-m_{b_{\xi_n,\xi}}\right)=0 \text{ and } \lim_{n\to\infty}\left(M_{b_{\xi_n,\xi}}-m_{b_{\xi_n,\xi}}\right)=0.$$

Theorem 2.4. *Let (Δ, m_b) be an MbMS with coefficient $s \geq 1$ and Γ be a self-mapping on Δ. If there is $k \in [0,1)$ so that*

$$m_b(\Gamma\xi,\Gamma\mu)\leq k m_b(\xi,\mu),\ \forall \xi,\mu\in\Delta.$$

Then Γ has a unique FP ς in Δ.

Example 2.5. [22] Let $\Delta = [0,1]$ and $m_b : \Delta\times\Delta \longrightarrow [0,\infty)$ be defined by

$$m_b(\xi,\mu)=\left(\frac{\xi+\mu}{2}\right)^2,\ \forall \xi,\mu\in\Delta.$$

Then (Δ, m_b) is an MbMS (with coefficient $s = 2$) which is not an MMS.

The concept of cyclic (α,β)-admissible mapping is showed in the work of [22] as follows:

Definition 2.6. Let $\Delta \neq \emptyset$, $\alpha,\beta : \Delta \to [0,\infty)$ be two functions. A mapping $\Gamma : \Delta \to \Delta$ is called cyclic (α,β)-admissible if for some $\xi \in \Delta$,

$$\alpha(\xi)\geq 1 \Rightarrow \beta(\Gamma\xi)\geq 1,$$

and

$$\beta(\xi)\geq 1 \Rightarrow \alpha(\Gamma\xi)\geq 1.$$

Mudhesh et al. [23] extended this work to η-cyclic (α,β)-admissible mappings as following:

Definition 2.7. Let $\Delta \neq \emptyset$, $\alpha,\beta,\eta : \Delta \to [0,\infty)$ be given functions. The mapping $\Gamma : \Delta \to \Delta$ is called η-cyclic (α,β)-admissible if for some $\xi \in \Delta$,

$$\alpha(\xi)\geq\eta(\xi) \Rightarrow \beta(\Gamma\xi)\geq\eta(\Gamma\xi),$$

and

$$\beta(\xi)\geq\eta(\xi) \Rightarrow \alpha(\Gamma\xi)\geq\eta(\Gamma\xi).$$

Definition 2.8. [24] Assume that Γ is a self-mapping on a nonempty set Δ, $A \subseteq \Delta$ and let $\alpha, \eta : \Delta \times \Delta \to [0, \infty)$ be given functions. We say that Γ is semi α-admissible with respect to (wrt) η; if for some $\xi, \mu \in A \subseteq \Delta$, we have

$$\alpha(\xi,\mu) \geq \eta(\xi,\mu) \Rightarrow \alpha(\Gamma\xi,\Gamma\mu) \geq \eta(\Gamma\xi,\Gamma\mu).$$

It should be noted that if $A = \Delta$, then Γ is called α-admissible wrt η.

The following results are well known in the literature:

Let Ψ_s, where $s \geq 1$; denotes the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

- (ψ_1) $\sum_{n=1}^{\infty} s^n \psi^n(t) < +\infty$ for all $t > 0$;
- (ψ_2) $s\psi(t) < t$ for all $t > 0$;
- (ψ_3) $s^{n+1}\psi^{n+1}(t) < s^n s\psi^n\psi(t) < s^n\psi^n(t)$,where ψ^n is the n^{th} iterate of ψ.

Let Ψ, denotes the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that$\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n stands for the n^{th} iterate of ψ.

Lemma 2.9. *Let $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(t))_{n\in N}$ *converges to* 0 *as* $n \to \infty \forall t \in (0,\infty)$;
(ii) $\psi(t) < t$ *for each* $t > 0$;
(iii) $\psi(t) = 0$ *iff* $t = 0$.

The coming results are very useful in our study which are taken and proved as in [27, 28].
Let (Δ, m_b) be an $MbMS$. For all $\xi \in \Delta$ and $\varepsilon > 0$, the open ball with the center ξ and the radius ε is

$$B(\xi,\varepsilon) = \{\mu \in \Delta : m_b(\xi,\mu) - m_{b_{\xi,\mu}} < \varepsilon\}.$$

Notice that we have $\xi \in B(\xi,\varepsilon)$ for all $\varepsilon > 0$. Indeed, we get

$$m_b(\xi,\xi) - m_{b_{\xi,\xi}} = m_b(\xi,\xi) - m_b(\xi,\xi) = 0 < \varepsilon.$$

Similarly, the closed ball with the center ξ and the radius ε is

$$B[\xi,\varepsilon] = \{\mu \in \Delta : m_b(\xi,\mu) - m_{b_{\xi,\mu}} \leq \varepsilon\}.$$

Lemma 2.10. *Let (Δ, m_b) be an $MbMS$, $\xi \in \Delta$ and $\varepsilon > 0$. The collection of all open balls on $\Delta, \beta_{m_b} = \{B(\xi,\varepsilon)\}_{\xi\in\Delta}^{\varepsilon>0}$ forms a basis on Δ.*

Lemma 2.11. *The following inequality holds for all $\xi, \mu \geq 2$ and $r \geq 1$,*

$$(\xi + \mu)^r \leq (\xi\mu)^r.$$

3. Symmetric fractional α-β-η-Υ-contraction pattern-I

In this portion, we reset FP results for symmetric fractional α-β-η-Υ-contraction of pattern-I in complete MbMS.

Definition 3.1. Let $\Gamma : \Delta \to \Delta$ be a mapping on an MbMS (Δ, m_b), $\alpha, \beta, \eta : \Delta \to [0, \infty)$ be three functions and $\Upsilon \in \Psi$. We say that Γ is a symmetric fractional α-β-η-Υ-contraction of pattern-I, if there exist constants $s \geq 1, a, b, c \in (0,1)$ and $\lambda = \left(sm_{b_{\xi,\mu}}\right)^{\frac{-1}{(c-a)(c-b)}} \in [0, \infty)$ such that $\forall \xi, \mu \in \Delta \setminus \text{Fix}(\Gamma)$, whenever $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, we have

$$s^2 m_b(\Gamma\xi, \Gamma\mu) \leq \Upsilon\left[\lambda\left(R_1(\xi, \mu)\right)\right], \tag{3.1}$$

where

$$\begin{aligned} R_1(\xi,\mu) &= m_b(\xi,\mu).\left[m_b(\xi,\Gamma\xi)\right]^{\frac{1}{(a-b)(a-c)}}.\left[m_b(\mu,\Gamma\mu)\right]^{\frac{1}{(a-b)(a-c)}}. \\ &\quad \left[m_b(\xi,\Gamma\xi)+m_b(\mu,\Gamma\mu)\right]^{\frac{1}{(b-a)(b-c)}}.\left[m_b(\xi,\Gamma\mu)+m_b(\mu,\Gamma\xi)\right]^{\frac{1}{(c-a)(c-b)}}. \end{aligned}$$

Example 3.2. Let $\Delta = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ and $m_b : \Delta \times \Delta \to R$ be defined by $m_b(\xi,\mu) = \left(\frac{\xi+\mu}{2}\right)^2$. Then (Δ, m_b) is a complete MbMS with $s = 2$. Define $\Gamma : \Delta \to \Delta$ by

$$\Gamma 0 = \Gamma\frac{1}{3} = \Gamma\frac{2}{3} = \Gamma 1 = 0, \ \Gamma\frac{1}{2} = \frac{1}{2},$$

and $\alpha, \beta, \eta : \Delta \to [0, \infty)$ by

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 1 & \text{if } \xi \in \Delta, \\ 0 & \text{otherwise} \end{cases} \text{ and } \eta(\xi) = \begin{cases} \frac{1}{2}, & \text{if } \xi \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Upsilon(t) = \frac{3}{4}t$. If $\xi, \mu \in \Delta$. Clearly $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, such that

$$\begin{aligned} s^2 m_b\left(\Gamma\frac{1}{3}, \Gamma\frac{2}{3}\right) = 0 &\leq \Upsilon[\lambda m_b\left(\frac{1}{3}, \frac{2}{3}\right).m_b\left(\frac{1}{3}, \Gamma\frac{1}{3}\right)^{\frac{1}{(a-b)(a-c)}}.m_b\left(\frac{2}{3}, \Gamma\frac{2}{3}\right)^{\frac{1}{(a-b)(a-c)}}. \\ &\quad [m_b\left(\frac{1}{3}, \Gamma\frac{1}{3}\right) + m_b\left(\frac{2}{3}, \Gamma\frac{2}{3}\right)]^{\frac{1}{(b-a)(b-c)}}.[m_b\left(\frac{1}{3}, \Gamma\frac{2}{3}\right) + m_b\left(\frac{2}{3}, \Gamma\frac{1}{3}\right)]^{\frac{1}{(c-a)(c-b)}}] \\ &= \Upsilon[\frac{\lambda}{4}\left(\frac{1}{36}\right)^{\frac{1}{(a-b)(a-c)}}.\left(\frac{1}{9}\right)^{\frac{1}{(a-b)(a-c)}}.[\left(\frac{1}{36}\right) + \left(\frac{1}{9}\right)]^{\frac{1}{(b-a)(b-c)}}. \\ &\quad [\left(\frac{1}{36}\right) + \left(\frac{1}{9}\right)]^{\frac{1}{(c-a)(c-b)}}] \\ &= \Upsilon\left[\frac{\lambda}{4}\left(\frac{1}{36} \times \frac{4}{36}\right)^{\frac{1}{(a-b)(a-c)}}.\left(\frac{5}{36}\right)^{\frac{1}{(b-a)(b-c)}}.\left(\frac{5}{36}\right)^{\frac{1}{(c-a)(c-b)}}\right] \\ &\leq \Upsilon\left[\frac{\lambda}{4}\left(\frac{1}{36} + \frac{4}{36}\right)^{\frac{1}{(a-b)(a-c)}}.\left(\frac{5}{36}\right)^{\frac{1}{(b-a)(b-c)}}.\left(\frac{5}{36}\right)^{\frac{1}{(c-a)(c-b)}}\right] \\ &= \Upsilon\left[\frac{\lambda}{4}\left(\frac{5}{36}\right)^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}}\right] \\ &= \frac{3\lambda}{16} \in [0, \infty). \end{aligned}$$

By taking any value of constants $\lambda \in [0, \infty)$ and $a, b, c \in (0, 1)$. Clearly, (3.1) holds for all $\xi, \mu \in \Delta \setminus$ Fix (Γ) . Thus Γ has two FPs of 0 and $\frac{1}{2}$.

Now we state and prove our main theorem.

Theorem 3.3. *Let (Δ, m_b) be a complete MbMS with coifficient $s \geq 1$ and Γ is a symmetric fractional α-β-η-Υ-contraction pattern-I satisfies the following statements:*

(i) Γ is an η-cyclic (α,β)-admissible mapping;
(ii) either there is $\xi_0 \in \Delta$ so that $\alpha(\xi_0) \geq \eta(\xi_0)$ or there is $\mu_0 \in \Delta$ so that $\beta(\mu_0) \geq \eta(\mu_0)$;
(iii) Γ is continuous.

Then Γ has an FP $\xi^* \in \Delta$.

Proof. Let $\xi_0 \in \Delta$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$, and $\beta(\xi_0) \geq \eta(\xi_0)$. Define a sequence $\{\xi_n\}$ in Δ by $\xi_n = \Gamma\xi_{n-1} \forall n \in N$. If $\exists$ some $n_0 \in \mathbb{N}$ for which $\Gamma\xi_{n_0} = \xi_{n_0}$, then ξ_{n_0} is an FP of Γ and the proof is done. Asume that $m_b(\xi_{n_0}, \Gamma\xi_{n_0}) > 0$, by (i) $\exists$ $\xi_1 \in \Delta$ such that

$$\alpha(\xi_0) \geq \eta(\xi_0) \Rightarrow \beta(\xi_1) = \beta(\Gamma\xi_0) \geq \eta(\xi_1) = \eta(\Gamma\xi_0),$$

and

$$\beta(\xi_0) \geq \eta(\xi_0) \Rightarrow \alpha(\xi_1) = \alpha(\Gamma\xi_0) \geq \eta(\xi_1) = \eta(\Gamma\xi_0).$$

Continuing in this way, we get

$$\alpha(\xi_n) \geq \eta(\xi_n) \Rightarrow \beta(\xi_{n+1}) \geq \eta(\xi_{n+1}).$$

Similarlly

$$\beta(\xi_n) \geq \eta(\xi_n) \Rightarrow \alpha(\xi_{n+1}) \geq \eta(\xi_{n+1}).$$

And hence, For all $n \in \mathbb{N}$

$$\alpha(\xi_n)\beta(\xi_{n+1}) \geq \eta(\xi_n)\eta(\xi_{n+1}). \tag{3.2}$$

If $\xi_{n+1} = \xi_n$ for sone $n \in \mathbb{N}$, then $\xi_n = \xi^*$, and the proof is done. So, we assume that for all $n \in \mathbb{N}, \xi_{n+1} \neq \xi_n$ accompanied by

$$m_b(\Gamma\xi_{n-1}, \Gamma\xi_n) = m_b(\xi_n, \Gamma\xi_n) > 0.$$

From (3.1) and for all $n \in \mathbb{N}$, we have

$$\begin{aligned} m_b(\xi_n, \xi_{n+1}) &\leq s^2 m_b(\Gamma\xi_{n-1}, \Gamma\xi_n) \\ &\leq \Upsilon\left[\lambda\left(R_1(\xi_{n-1}, \xi_n)\right)\right]. \end{aligned} \tag{3.3}$$

Where

$$\begin{aligned} R_1(\xi_{n-1}, \xi_n) &= \left[\begin{array}{c} m_b(\xi_{n-1}, \xi_n).m_b(\xi_{n-1}, \Gamma\xi_{n-1})^{\frac{1}{(a-b)(a-c)}}.m_b(\xi_n, \Gamma\xi_n)^{\frac{1}{(a-b)(a-c)}} \\ .[m_b(\xi_{n-1}, \Gamma\xi_{n-1}) + m_b(\xi_n, \Gamma\xi_n)]^{\frac{1}{(b-a)(b-c)}} \\ .[m_b(\xi_{n-1}, \Gamma\xi_n) + m_b(\xi_n, \Gamma\xi_{n-1})]^{\frac{1}{(c-a)(c-b)}} \end{array}\right] \\ &= \left[\begin{array}{c} m_b(\xi_{n-1}, \xi_n).m_b(\xi_{n-1}, \xi_n)^{\frac{1}{(a-b)(a-c)}}.m_b(\xi_n, \xi_{n+1})^{\frac{1}{(a-b)(a-c)}} \\ .[m_b(\xi_{n-1}, \xi_n) + m_b(\xi_n, \xi_{n+1})]^{\frac{1}{(b-a)(b-c)}} \\ .[m_b(\xi_{n-1}, \xi_{n+1}) + m_b(\xi_n, \xi_n)]^{\frac{1}{(c-a)(c-b)}} \end{array}\right] \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&\leq \left[\begin{array}{c} m_b(\xi_{n-1},\xi_n)^{1+\frac{1}{(a-b)(a-c)}}.m_b(\xi_n,\xi_{n+1})^{\frac{1}{(a-b)(a-c)}} \\ .[m_b(\xi_{n-1},\xi_n)+m_b(\xi_n,\xi_{n+1})]^{\frac{1}{(b-a)(b-c)}} \\ .[s\left(m_b(\xi_{n-1},\xi_n)-m_{b_{\xi_{n-1},\xi_n}}+m_b(\xi_n,\xi_{n+1})-m_{b_{\xi_n,\xi_{n+1}}}\right) \\ +m_{b_{\xi_{n-1},\xi_{n+1}}}]^{\frac{1}{(c-a)(c-b)}} \end{array}\right] \\
&\leq \left[\begin{array}{c} m_b(\xi_{n-1},\xi_n)^{1+\frac{1}{(a-b)(a-c)}}.m_b(\xi_n,\xi_{n+1})^{\frac{1}{(a-b)(a-c)}} \\ .[m_b(\xi_{n-1},\xi_n)+m_b(\xi_n,\xi_{n+1})]^{\frac{1}{(b-a)(b-c)}} \\ .[s\left(m_b(\xi_{n-1},\xi_n)+m_b(\xi_n,\xi_{n+1})\right)+m_{b_{\xi_{n-1},\xi_{n+1}}}]^{\frac{1}{(c-a)(c-b)}} \end{array}\right] \\
&\leq \left[\begin{array}{c} m_b(\xi_{n-1},\xi_n)^{1+\frac{1}{(a-b)(a-c)}}.m_b(\xi_n,\xi_{n+1})^{\frac{1}{(a-b)(a-c)}} \\ .[m_b(\xi_{n-1},\xi_n).m_b(\xi_n,\xi_{n+1})]^{\frac{1}{(b-a)(b-c)}} \\ .[s\left(m_b(\xi_{n-1},\xi_n).m_b(\xi_n,\xi_{n+1})\right).m_{b_{\xi_{n-1},\xi_{n+1}}}]^{\frac{1}{(c-a)(c-b)}} \end{array}\right] \\
&= \left[\begin{array}{c} s^{\frac{1}{(c-a)(c-b)}}m_b(\xi_{n-1},\xi_n)^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}} \\ .m_b(\xi_n,\xi_{n+1})^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}} \\ .\left(m_{b_{\xi_{n-1},\xi_{n+1}}}\right)^{\frac{1}{(c-a)(c-b)}} \end{array}\right] m_b\left(\xi_{n-1},\xi_n\right) \\
&= \left(sm_{b_{\xi_{n-1},\xi_{n+1}}}\right)^{\frac{1}{(c-a)(c-b)}} m_b\left(\xi_{n-1},\xi_n\right).
\end{aligned}
$$

Now fron (3.3) and (3.4), we obtain that

$$
\begin{aligned}
m_b\left(\xi_n,\xi_{n+1}\right) &\leq s^2 m_b\left(\Gamma\xi_{n-1},\Gamma\xi_n\right) \\
&\leq \Upsilon\left(\lambda\left[sm_{b_{\xi_{n-1},\xi_{n+1}}}\right]^{\frac{1}{(c-a)(c-b)}} m_b\left(\xi_{n-1},\xi_n\right)\right) \\
&= \Upsilon\left(m_b\left(\xi_{n-1},\xi_n\right)\right) \\
&< m_b\left(\xi_{n-1},\xi_n\right).
\end{aligned} \tag{3.5}
$$

From (3.5), we conclude that $m_b\left(\xi_{n-1},\xi_n\right)$ is a decreasing sequence with non-negative terms. Thus, there is a constant $\varrho \geq 0$ such that $\lim_{n\to\infty} m_b\left(\xi_{n-1},\xi_n\right) = \varrho$. Presume that $\varrho > 0$. From (3.5), we can write

$$
\begin{aligned}
m_b\left(\xi_n,\xi_{n+1}\right) &\leq s^2 m_b\left(\Gamma\xi_{n-1},\Gamma\xi_n\right) \\
&\leq \Upsilon\left[m_b\left(\xi_{n-1},\xi_n\right)\right] \\
&\leq \Upsilon^2\left[m_b\left(\xi_{n-2},\xi_{n-1}\right)\right] \\
&\leq \Upsilon^3\left[m_b\left(\xi_{n-3},\xi_{n-2}\right)\right] \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \Upsilon^n\left[m_b\left(\xi_0,\xi_1\right)\right].
\end{aligned} \tag{3.6}
$$

Taking limit as $n \to \infty$ in (3.6), and from properties of Υ, we obtain

$$
0 \leq \lim_{n\to+\infty} m_b\left(\xi_n,\xi_{n+1}\right) \leq \lim_{n\to+\infty} \Upsilon^n\left[m_b\left(\xi_0,\xi_1\right)\right] = 0.
$$

Which yield that

$$\lim_{n\to+\infty} m_b(\xi_n,\xi_{n+1}) = 0. \tag{3.7}$$

Now, we prove that $\{\xi_n\}$ is an M_b-Cauchy sequence in (Δ, m_b). Recall that from $(Mb2)$ and for all $n \in \mathbb{N}$, we have

$$0 \le m_{b_{\xi_n,\xi_{n+1}}} \le m_b(\xi_n,\xi_{n+1}).$$

Since from (3.7), we have

$$\lim_{n\to\infty} m_{b_{\xi_n,\xi_{n+1}}} = 0, \tag{3.8}$$

which denotes that

$$\lim_{n\to\infty} m_b(\xi_n,\xi_n) = 0, or \lim_{n\to\infty} m_b(\xi_{n+1},\xi_{n+1}) = 0. \tag{3.9}$$

Therefore,

$$\lim_{m,n\to\infty} m_{b_{\xi_n,\xi_m}} = \lim_{m,n\to\infty} \min\{m_b(\xi_n,\xi_n), m_b(\xi_m,\xi_m)\} = 0. \tag{3.10}$$

Hence,

$$\lim_{m,n\to\infty}\left(M_{b_{\xi_m,\xi_n}} - m_{b_{\xi_m,\xi_n}}\right) = \lim_{m,n\to\infty} \mid m_b(\xi_n,\xi_n) - m_b(\xi_m,\xi_m) \mid = 0.$$

Next, we shall prove that $\lim_{m,n\to\infty}\left(m_b(\xi_m,\xi_n) - m_{b_{\xi_m,\xi_n}}\right) = 0$. Suppose on the contrary that

$$\lim_{m,n\to\infty}\left(m_b(\xi_m,\xi_n) - m_{b_{\xi_m,\xi_n}}\right) \neq 0,$$

then there exist $\varepsilon > 0$ and subsequence $\{\varsigma_k\} \subset \mathbb{N}$ such that

$$m_b(\xi_{\varsigma_k},\xi_{n_k}) - m_{b_{\xi_{\varsigma_k},\xi_{n_k}}} \ge \varepsilon. \tag{3.11}$$

Suppose that ς_k is the smallest integer which satisfies (3.11) such that

$$m_b(\xi_{\varsigma_k-1},\xi_{n_k}) - m_{b_{\xi_{\varsigma_k-1},\xi_{n_k}}} < \varepsilon. \tag{3.12}$$

By $(Mb4)$ in (3.11) and using (3.12), we get

$$\begin{aligned} \varepsilon &\le m_b(\xi_{\varsigma_k},\xi_{n_k}) - m_{b_{\xi_{\varsigma_k},\xi_{n_k}}} \\ &\le s\left[\left(m_b(\xi_{\varsigma_k},\xi_{\varsigma_k-1}) - m_{b_{\xi_{\varsigma_k},\xi_{\varsigma_k-1}}}\right) + \left(m_b(\xi_{\varsigma_k-1},\xi_{n_k}) - m_{b_{\xi_{\varsigma_k-1},\xi_{n_k}}}\right)\right] \\ &\quad -m_b(\xi_{\varsigma_k-1},\xi_{\varsigma_k-1}) \\ &\le s\varepsilon + s\left[m_b(\xi_{\varsigma_k},\xi_{\varsigma_k-1}) - m_{b_{\xi_{\varsigma_k},\xi_{\varsigma_k-1}}}\right] - m_b(\xi_{\varsigma_k-1},\xi_{\varsigma_k-1}). \end{aligned} \tag{3.13}$$

Letting $k \to \infty$ in (3.13), using (3.7)–(3.9), then

$$\varepsilon \le \lim_{k\to\infty}\left(m_b(\xi_{\varsigma_k},\xi_{n_k}) - m_{b_{\xi_{\varsigma_k},\xi_{n_k}}}\right) \le s\varepsilon. \tag{3.14}$$

Utilizing (3.10) and from (3.14), we have

$$\varepsilon \le \lim_{k\to\infty} m_b(\xi_{\varsigma_k},\xi_{n_k}) \le s\varepsilon. \tag{3.15}$$

Similarly from (Mb4) and (3.11), we obtain

$$\begin{aligned}
\varepsilon &\leq m_b\left(\xi_{\varsigma_k}, \xi_{n_k}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{n_k}}} \qquad (3.16)\\
&\leq s\left[\left(m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k+1}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{\varsigma_k+1}}}\right) + \left(m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{n_k}}}\right)\right] \\
&\quad - m_b\left(\xi_{\varsigma_k+1}, \xi_{\varsigma_k+1}\right) \\
&\leq s\left[\begin{array}{c}
\left(m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k+1}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{\varsigma_k+1}}}\right) \\
+s\left[\left(m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{n_k+1}}}\right) + \left(m_b\left(\xi_{n_k+1}, \xi_{n_k}\right) - m_{b_{\xi_{n_k+1}, \xi_{n_k}}}\right)\right] \\
-m_b\left(\xi_{n_k+1}, \xi_{n_k+1}\right)
\end{array}\right] \\
&\quad - m_b\left(\xi_{\varsigma_k+1}, \xi_{\varsigma_k+1}\right) \\
&= \left[\begin{array}{c}
s\left(m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k+1}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{\varsigma_k+1}}}\right) + s^2\left(m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{n_k+1}}}\right) \\
+s^2\left(m_b\left(\xi_{n_k+1}, \xi_{n_k}\right) - m_{b_{\xi_{n_k+1}, \xi_{n_k}}}\right) - s m_b\left(\xi_{n_k+1}, \xi_{n_k+1}\right) - m_b\left(\xi_{\varsigma_k+1}, \xi_{\varsigma_k+1}\right)
\end{array}\right].
\end{aligned}$$

Similar to (3.13), we find that

$$\begin{aligned}
\varepsilon &\leq m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{n_k+1}}} \qquad (3.17)\\
&\leq \left[\begin{array}{c}
s\left(m_b\left(\xi_{\varsigma_k+1}, \xi_{\varsigma_k}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{\varsigma_k}}}\right) + s^2\left(m_b\left(\xi_{\varsigma_k}, \xi_{n_k}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{n_k}}}\right) \\
+s^2\left(m_b\left(\xi_{n_k}, \xi_{n_k+1}\right) - m_{b_{\xi_{n_k}, \xi_{n_k+1}}}\right) - s m_b\left(\xi_{n_k+1}, \xi_{n_k+1}\right) - m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k}\right)
\end{array}\right].
\end{aligned}$$

Utilizing (3.16) and (3.17), then

$$\begin{aligned}
\varepsilon &\leq m_b\left(\xi_{\varsigma_k}, \xi_{n_k}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{n_k}}} \qquad (3.18)\\
&\leq \left[\begin{array}{c}
s\left(m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k+1}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{\varsigma_k+1}}}\right) + s^2\left(m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{n_k+1}}}\right) \\
+s^2\left(m_b\left(\xi_{n_k+1}, \xi_{n_k}\right) - m_{b_{\xi_{n_k+1}, \xi_{n_k}}}\right) - s m_b\left(\xi_{n_k+1}, \xi_{n_k+1}\right) - m_b\left(\xi_{\varsigma_k+1}, \xi_{\varsigma_k+1}\right)
\end{array}\right] \\
&\leq \left[\begin{array}{c}
s\left(m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k+1}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{\varsigma_k+1}}}\right) + s^2\left(\begin{array}{c}
s[m_b\left(\xi_{\varsigma_k+1}, \xi_{\varsigma_k}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{\varsigma_k}}}] \\
+s^2[m_b\left(\xi_{\varsigma_k}, \xi_{n_k}\right) - m_{b_{\xi_{\varsigma_k}, \xi_{n_k}}}] \\
+s^2[m_b\left(\xi_{n_k}, \xi_{n_k+1}\right) - m_{b_{\xi_{n_k}, \xi_{n_k+1}}}] \\
-s m_b\left(\xi_{n_k+1}, \xi_{n_k+1}\right) - m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k}\right)
\end{array}\right) \\
+s^2\left(m_b\left(\xi_{n_k+1}, \xi_{n_k}\right) - m_{b_{\xi_{n_k+1}, \xi_{n_k}}}\right) - s m_b\left(\xi_{n_k+1}, \xi_{n_k+1}\right) - m_b\left(\xi_{\varsigma_k+1}, \xi_{\varsigma_k+1}\right)
\end{array}\right].
\end{aligned}$$

Taking limit as $k \to \infty$ in (3.18) , and using (3.7)–(3.9) and (3.14), we get

$$\varepsilon \leq \lim_{k\to\infty} s^2\left(m_b\left(\xi_{\varsigma_k+1,\xi_{n_k+1}}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{n_k+1}}}\right) \leq s^5\varepsilon.$$

Therefore

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty}\left(m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) - m_{b_{\xi_{\varsigma_k+1}, \xi_{n_k+1}}}\right) \leq s^3\varepsilon. \qquad (3.19)$$

From (3.10), we have

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) \leq s^3\varepsilon. \tag{3.20}$$

Now, from (3.1), we obtain

$$\begin{aligned} s^2 m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) &= s^2 m_b\left(\Gamma\xi_{\varsigma_k}, \Gamma\xi_{n_k}\right) \\ &\leq \Upsilon\left[\lambda\left(R_1\left(\xi_{\varsigma_k}, \xi_{n_k}\right)\right)\right], \end{aligned}$$

where

$$\begin{aligned} R_1\left(\xi_{\varsigma_k}, \xi_{n_k}\right) &= m_b\left(\xi_{\varsigma_k}, \xi_{n_k}\right).m_b\left(\xi_{\varsigma_k}, \Gamma\xi_{\varsigma_k}\right)^{\frac{1}{(a-b)(a-c)}}.m_b\left(\xi_{n_k}, \Gamma\xi_{n_k}\right)^{\frac{1}{(a-b)(a-c)}} \\ &\quad .\left[m_b\left(\xi_{\varsigma_k}, \Gamma\xi_{\varsigma_k}\right) + m_b\left(\xi_{n_k}, \Gamma\xi_{n_k}\right)\right]^{\frac{1}{(b-a)(b-c)}} \\ &\quad .\left[m_b\left(\xi_{\varsigma_k}, \Gamma\xi_{n_k}\right) + m_b\left(\xi_{n_k}, \Gamma\xi_{\varsigma_k}\right)\right]^{\frac{1}{(c-a)(c-b)}} \\ &= m_b\left(\xi_{\varsigma_k}, \xi_{n_k}\right).m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k+1}\right)^{\frac{1}{(a-b)(a-c)}}.m_b\left(\xi_{n_k}, \xi_{n_k+1}\right)^{\frac{1}{(a-b)(a-c)}} \\ &\quad .\left[m_b\left(\xi_{\varsigma_k}, \xi_{\varsigma_k+1}\right) + m_b\left(\xi_{n_k}, \xi_{n_k+1}\right)\right]^{\frac{1}{(b-a)(b-c)}} \\ &\quad .\left[m_b\left(\xi_{\varsigma_k}, \xi_{n_k+1}\right) + m_b\left(\xi_{n_k}, \xi_{\varsigma_k+1}\right)\right]^{\frac{1}{(c-a)(c-b)}}. \end{aligned}$$

By taking limit as $k \to \infty$ in the above equation and using (3.7) and (3.8), we obtain

$$0 \leq \lim_{k\to\infty} R_1\left(\xi_{\varsigma_k}, \xi_{n_k}\right) \leq 0 \Rightarrow \lim_{k\to\infty} R_1\left(\xi_{\varsigma_k}, \xi_{n_k}\right) = 0. \tag{3.21}$$

Thence, it follows from (3.20), (3.21) and (iii) of Lemma 2.9 that

$$\begin{aligned} \varepsilon &= s^2\left(\frac{\varepsilon}{s^2}\right) \leq s^2 \lim_{k\to\infty} m_b\left(\xi_{\varsigma_k+1}, \xi_{n_k+1}\right) = s^2 \lim_{k\to\infty} m_b\left(\Gamma\xi_{\varsigma_k}, \Gamma\xi_{n_k}\right) \\ &\leq \Upsilon\left[\lambda \lim_{k\to\infty} R_1\left(\xi_{\varsigma_k}, \xi_{n_k}\right)\right] \\ &< \Upsilon\left[\lim_{k\to\infty} R_1\left(\xi_{\varsigma_k}, \xi_{n_k}\right)\right] \\ &= \Upsilon[0] \\ &= 0. \end{aligned}$$

Hence, we conclude that $\varepsilon < 0$ which is a contradiction. Thus, $\lim_{m,n\to\infty}\left(m_b\left(\xi_m, \xi_n\right) - m_{b_{\xi_m,\xi_n}}\right) = 0$, therefore $\{\xi_n\}$ is an M_b-Cauchy sequence in Δ. Since Δ is complete, there exist some $\xi^* \in \Delta$ such that $\xi_n \to \xi^*$ as $n \to \infty$. Since Γ is continuous then $\lim_{n\to\infty} \Gamma\xi_n = \Gamma\xi^*$, therefore we have

$$\lim_{n\to\infty}\left(m_b\left(\xi_{n+1}, \xi^*\right) - m_{b_{\xi_{n+1},\xi^*}}\right) = 0. \; and \lim_{n\to\infty}\left(M_{b_{\xi_{n+1},\xi^*}} - m_{b_{\xi_{n+1},\xi^*}}\right) = 0. \tag{3.22}$$

Since from (3.9) and (3.22), we get

$$\begin{aligned} \lim_{n\to\infty}\left(m_b\left(\xi_{n+1}, \xi^*\right) - m_{b_{\xi_{n+1},\xi^*}}\right) &= 0 \\ &= \lim_{n\to\infty} m_b\left(\xi_{n+1}, \xi^*\right) \\ &= \lim_{n\to\infty} m_b\left(\Gamma\xi_n, \xi^*\right) \\ &= m_b\left(\Gamma\xi^*, \xi^*\right). \end{aligned} \tag{3.23}$$

So, that is $\Gamma\xi^* = \xi^*$ and ξ^* is an FP of Γ. $\square$

Theorem 3.4. *Let (Δ, m_b) be a complete MbMS with coefficient $s \geq 1$ and Γ is a symmetric fractional α-β-η-Υ-contraction pattern-I fulfilling the affirmations below:*

(i) Γ *is an* η*-cyclic* (α,β)*-admissible mapping;*
(ii) either there is $\xi_0 \in \Delta$ *so that* $\alpha(\xi_0) \geq \eta(\xi_0)$ *or there is* $\mu_0 \in \Delta$ *so that* $\beta(\mu_0) \geq \eta(\mu_0)$*;*
(iii) if $\{\xi_n\}$ *is a sequence in* Δ *such that* $\xi_n \to \xi^*$ *as* $n \to \infty$*, and* $\beta(\xi_n) \geq \eta(\xi_n)$ *for all* $n \in \mathbb{N}$*, then* $\beta(\xi^*) \geq \eta(\xi^*)$*.*

Then Γ *has an FP* $\xi^* \in \Delta$*.*

Proof. In the definitive lines of the proof of Theorem 3.3, we acquire $\beta(\xi^*) \geq \eta(\xi^*)$. Now we show that $m_b(\Gamma\xi^*, \xi^*) = 0$. $\xi_n \to \xi^*$ as $n \to \infty$, from (Mb4), we have

$$\begin{aligned} 0 &\leq \left|\left(m_b(\xi_{n+1}, \Gamma\xi^*) - m_{b_{\xi_{n+1},\Gamma\xi^*}}\right) - \left(m_b(\xi^*, \Gamma\xi^*) - m_{b_{\xi^*,\Gamma\xi^*}}\right)\right| \\ &\leq \left|\begin{array}{c} s\left(\left(m_b(\xi_{n+1}, \xi^*) - m_{b_{\xi_{n+1},\xi^*}}\right) - \left(m_b(\xi^*, \Gamma\xi^*) - m_{b_{\xi^*,\Gamma\xi^*}}\right)\right) - m_b(\xi^*, \xi^*) \\ -\left(s\left(\left(m_b(\xi^*, \xi^*) - m_{b_{\xi^*,\xi^*}}\right) - \left(m_b(\xi^*, \Gamma\xi^*) - m_{b_{\xi^*,\Gamma\xi^*}}\right)\right) - m_b(\xi^*, \xi^*)\right) \end{array}\right|. \end{aligned} \tag{3.24}$$

So taking limit as $n \to \infty$ in (3.24) and using of (3.9) and (3.22), we get

$$0 \leq \lim_{n\to\infty} \left|\left(m_b(\xi_{n+1}, \Gamma\xi^*) - m_{b_{\xi_{n+1},\Gamma\xi^*}}\right) - \left(m_b(\xi^*, \Gamma\xi^*) - m_{b_{\xi^*,\Gamma\xi^*}}\right)\right| \leq 0,$$

this implies that

$$\lim_{n\to\infty} \left(m_b(\xi_{n+1}, \Gamma\xi^*) - m_{b_{\xi_{n+1},\Gamma\xi^*}}\right) = m_b(\xi^*, \Gamma\xi^*) - m_{b_{\xi^*,\Gamma\xi^*}} = m_b(\xi^*, \Gamma\xi^*). \tag{3.25}$$

Now from (3.3) and (3.25), we have

$$\begin{aligned} m_b(\xi_{n+1}, \Gamma\xi^*) - m_{b_{\xi_{n+1},\Gamma\xi^*}} &\leq s^2 m_b(\Gamma\xi_n, \Gamma\xi^*) - m_{b_{\Gamma\xi_n,\Gamma\xi^*}} \\ &\leq \Upsilon\left[\lambda\left(R_1(\xi_n, \xi^*)\right)\right] \\ &\leq \Upsilon\left(\begin{array}{c} \lambda m_b(\xi_n, \xi^*) . m_b(\Gamma\xi^*, \xi^*)^{\frac{1}{(a-b)(a-c)}} . m_b(\xi_n, \Gamma\xi_n)^{\frac{1}{(a-b)(a-c)}} \\ .[m_b(\Gamma\xi^*, \xi^*) + m_b(\xi_n, \Gamma\xi_n)]^{\frac{1}{(b-a)(b-c)}} \\ .[m_b(\Gamma\xi_n, \xi^*) + m_b(\xi_n, \Gamma\xi^*)]^{\frac{1}{(c-a)(c-b)}} \end{array}\right). \end{aligned} \tag{3.26}$$

By taking limit as $n \to \infty$ in (3.26) and since $\Upsilon \in \Psi$, we get

$$\lim_{n\to\infty} m_b(\xi_{n+1}, \Gamma\xi^*) - m_{b_{\xi_{n+1},\Gamma\xi^*}} = 0. \tag{3.27}$$

Therefore, from (3.25) and (3.27), we get $m_b(\xi^*, \Gamma\xi^*) = 0$ and ξ^* is an FP of Γ. □

The example below supports Theorems 3.3 and 3.4.

Example 3.5. Let $\Delta = [0, 1]$ and $m_b : \Delta \times \Delta \longrightarrow [0, \infty)$ defined by

$$m_b(\xi, \mu) = \left(\frac{\xi + \mu}{2}\right)^2, \ \forall \xi, \mu \in \Delta.$$

Clearly, (Δ, m_b) is an MbMS with $s = 2$. Define $\Gamma : \Delta \to \Delta$ by

$$\Gamma\xi = \begin{cases} \frac{\xi^2}{9}, & \text{if } \xi \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Describe the functions $\alpha, \beta, \eta : \Delta \to [0, \infty)$ as,

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 2, & \text{if } \xi \in (0,1], \\ 0, & \text{otherwise} \end{cases}, \ \eta(\xi) = \begin{cases} 1, & \text{if } \xi \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly for all $\xi, \mu \in (0, 1]$, $\alpha(\xi) = 2 \geq 1 = \eta(\xi) \Rightarrow \beta(\Gamma\xi) = 2 \geq 1 = \eta(\Gamma\xi)$, and $\beta(\xi) = 2 \geq 1 = \eta(\xi) \Rightarrow \alpha(\Gamma\xi) \geq \eta(\Gamma\xi)$. So, Γ is η-cyclic (α,β)-admissible mapping. Now if $\{\xi_n\}$ is a sequence in Δ such that $\xi_n \to \xi^*$ as $n \to \infty$ and $\beta(\xi_n) \geq \eta(\xi_n)$. Then $\beta(\xi^*) \geq \eta(\xi^*)$ whenever, $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, such that

$$\begin{aligned} s^2 m_b(\Gamma\xi, \Gamma\mu) &= 4m_b\left(\frac{\xi^2}{9}, \frac{\mu^2}{9}\right) = 4\left(\frac{\xi^2+\mu^2}{18}\right)^2 \leq \left(\frac{\xi+\mu}{18}\right)^2 = \frac{4}{9}\left[\frac{1}{9}\left(\frac{\xi+\mu}{2}\right)^2\right] \\ &\leq \frac{4}{9}\left[\frac{1}{9}\left(\begin{array}{c} \left(\frac{\xi+\mu}{2}\right)^2 .(\frac{3\xi}{4})^{2\frac{1}{(a-b)(a-c)}} .(\frac{3\mu}{4})^{2\frac{1}{(a-b)(a-c)}} . \\ \left((\frac{3\xi}{4})^2 + (\frac{3\mu}{4})^2\right)^{\frac{1}{(b-a)(b-c)}} . \left((\frac{2\xi+\mu}{4})^2 + (\frac{\xi+2\mu}{4})^2\right)^{\frac{1}{(c-a)(c-b)}} \end{array}\right)\right] \\ &= \Upsilon\left[\lambda\left(R_1(\xi,\mu)\right)\right]. \end{aligned}$$

That is achieved when we take $\Upsilon(t) = \frac{4t}{9}$ and constants $\lambda = \frac{1}{9} \in [0, \infty)$, $a, b, c \in (0, 1)$, for all $\xi, \mu \in \Delta\backslash\text{Fix}(\Gamma)$. Otherwise, for $\xi = \mu = 0$, we obtain that Γ is η-cyclic (α,β)-admissible mapping, whenever $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, and

$$s^2 m_b(\Gamma\xi, \Gamma\mu) = 0 \leq \Upsilon\left[\lambda\left(R_1(\xi,\mu)\right)\right].$$

Therefore, all affirmations of Theorems 3.3 and 3.4 are satisfied. Hence Γ has an FP $\xi^* = 0 \in \Delta$. (Note that 9 is an another FP of Γ, but it does not belong to Δ.

4. Symmetric fractional α-β-η-Υ-contraction pattern-II

In this portion, we devote our efforts to introduce the notion of symmetric fractional α-β-η-Υ-contraction pattern-II and some FP results are obtained via a complete MbMS.

Definition 4.1. Let $\Gamma : \Delta \to \Delta$ be a mapping on an MbMS (Δ, m_b), $\alpha, \beta, \eta : \Delta \to [0, \infty)$ be three functions and $\Upsilon \in \Psi$. We say that Γ is a symmetric fractional α-β-η-Υ-contraction of pattern-II provided that there are constants $s \geq 1, a, b, c \in (0, 1)$ and $\lambda = \left(sm_{b_{\xi,\mu}}\right)^{\frac{-c}{(c-a)(c-b)}} \in [0, \infty)$ such that $\forall \xi, \mu \in \Delta\setminus$ Fix (Γ), whenever $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, we have

$$s^2 m_b(\Gamma\xi, \Gamma\mu) \leq \Upsilon\left[\lambda\left(R_2(\xi,\mu)\right)\right], \tag{4.1}$$

where

$$\begin{aligned} R_2(\xi,\mu) &= m_b(\xi,\mu).\left[m_b(\xi,\Gamma\xi)\right]^{\frac{a}{(a-b)(a-c)}} .\left[m_b(\mu,\Gamma\mu)\right]^{\frac{a}{(a-b)(a-c)}} . \\ &\quad \left[m_b(\xi,\Gamma\xi) + m_b(\mu,\Gamma\mu)\right]^{\frac{b}{(b-a)(b-c)}} .\left[m_b(\xi,\Gamma\mu) + m_b(\mu,\Gamma\xi)\right]^{\frac{c}{(c-a)(c-b)}} . \end{aligned}$$

Now we show and demonstrate our next theorem.

Theorem 4.2. *Let (Δ, m_b) be a complete MbMS and Γ be a symmetric fractional α-β-η-Υ-contraction of pattern-II fulfilling the same affirmations of Theorem 3.3:*

Then Γ has an FP in Δ.

Proof. By the same steps as in proof of Theorem 3.3, we deduce that Γ has an FP $\xi^* \in \Delta$. □

Theorem 4.3. *Let (Δ, m_b) be a complete MbMS and Γ be a symmetric fractional α-β-η-Υ-contraction of pattern-II fulfilling the same affirmations of Theorem 3.4:*

Then Γ has an FP in Δ.

Proof. Similar to the same steps as in proof of Theorem 3.4, we conclude that ξ^* is an FP of Γ. □

5. Symmetric fractional α-β-η-Υ-contraction pattern-III

In this segment, the notion of symmetric fractional α-β-η-Υ-contraction pattern-III and some FP results are established via a complete MbMS:

Definition 5.1. Let (Δ, m_b) be an MbMS with a self-map $\Gamma : \Delta \to \Delta$, $\alpha, \beta, \eta : \Delta \to [0, \infty)$ be three functions and $\Upsilon \in \Psi$. We say that Γ is a symmetric fractional α-β-η-Υ-contraction pattern-III along with constants $s \geq 1, a, b, c \in (0, 1)$ and $\lambda = \left(sm_{b_{\xi,\mu}}\right)^{\frac{-c^2}{(c-a)(c-b)}} \in [0, \infty)$ such that $\forall \xi, \mu \in \Delta \setminus \text{Fix}(\Gamma)$, whenever $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, we have

$$s^2 m_b(\Gamma\xi, \Gamma\mu) \leq \Upsilon\left[\lambda\left(R_3(\xi, \mu)\right)\right], \tag{5.1}$$

where

$$R_3(\xi, \mu) = \max \left\{ \begin{array}{c} m_b(\xi, \mu), m_b(\xi, \Gamma\xi)^{\frac{a^2}{(a-b)(a-c)}} .m_b(\mu, \Gamma\mu)^{\frac{a^2}{(a-b)(a-c)}} \\ .[m_b(\xi, \Gamma\xi) + m_b(\mu, \Gamma\mu)]^{\frac{b^2}{(b-a)(b-c)}} \\ .[m_b(\xi, \Gamma\mu) + m_b(\mu, \Gamma\xi)]^{\frac{c^2}{(c-a)(c-b)}} \end{array} \right\}.$$

Now, we declare and demonstrate our next theorem.

Theorem 5.2. *Let (Δ, m_b) be a complete MbMS and Γ be a symmetric fractional α-β-η-Υ-contraction of pattern-III that satisfies the same assertions of Theorem 3.3:*

Then Γ has an FP in Δ.

Proof. Let $\xi_0 \in \Delta$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$ and $\beta(\xi_0) \geq \eta(\xi_0)$. For all $n \in \mathbb{N}$, we build an iteration $\{\xi_n\}_{n=1}^{\infty}$ such that $\xi_1 = \Gamma\xi_0, \xi_2 = \Gamma\xi_1 = \Gamma^2\xi_0$. By proceeding in this manner, we obtain $\xi_{n+1} = \Gamma\xi_n = \Gamma^{n+1}\xi_0$. Now from (i), we can conclude that for all $n \in \mathbb{N}$,

$$\alpha(\xi_n)\beta(\xi_{n+1}) \geq \eta(\xi_n)\eta(\xi_{n+1}). \tag{5.2}$$

If $\xi_{n+1} = \xi_n$ for some $n \in \mathbb{N}$, then $\xi_n = \xi^*$ is an FP of Γ. So, we assume that $\xi_n \neq \xi_{n+1}$ accompanied by $m_b(\Gamma\xi_{n-1}, \Gamma\xi_n) = m_b(\xi_n, \Gamma\xi_n)$ for every $n \in \mathbb{N}$.

From (5.1), we own

$$m_b\left(\xi_n,\xi_{n+1}\right)\leq s^2 m_b\left(\Gamma\xi_{n-1},\Gamma\xi_n\right)\leq \Upsilon\left[\lambda\left(R_3\left(\xi_{n-1},\xi_n\right)\right)\right], \tag{5.3}$$

where by the same steps in (3.4), we deduce that

$$R_3\left(\xi_{n-1},\xi_n\right)=\max\{m_b\left(\xi_{n-1},\xi_n\right),\left[sm_{b_{\xi_{n-1},\xi_{n+1}}}\right]^{\frac{c^2}{(c-a)(c-b)}}.m_b\left(\xi_n,\xi_{n+1}\right)\}.$$

Now if

$$R_3\left(\xi_{n-1},\xi_n\right)=\left[sm_{b_{\xi_{n-1},\xi_{n+1}}}\right]^{\frac{c^2}{(c-a)(c-b)}}.m_b\left(\xi_n,\xi_{n+1}\right). \tag{5.4}$$

Then, from (5.3) and (5.4), we get

$$\begin{aligned} m_b\left(\xi_n,\xi_{n+1}\right) &\leq s^2 m_b\left(\Gamma\xi_{n-1},\Gamma\xi_n\right)\\ &\leq \Upsilon\left(\lambda\left[sm_{b_{\xi_{n-1},\xi_{n+1}}}\right]^{\frac{c^2}{(c-a)(c-b)}}.m_b\left(\xi_n,\xi_{n+1}\right)\right)\\ &= \Upsilon\left(m_b\left(\xi_n,\xi_{n+1}\right)\right)\\ &< m_b\left(\xi_n,\xi_{n+1}\right), \end{aligned}$$

which gives a contradiction, thus

$$R_3\left(\xi_{n-1},\xi_n\right)=m_b\left(\xi_{n-1},\xi_n\right). \tag{5.5}$$

Now, from (5.3) and (5.5), we conclude that

$$\begin{aligned} m_b\left(\xi_n,\xi_{n+1}\right) &\leq s^2 m_b\left(\Gamma\xi_{n-1},\Gamma\xi_n\right)\\ &\leq \Upsilon\left[\lambda\left(R_3\left(\xi_{n-1},\xi_n\right)\right)\right]\\ &< m_b\left(\xi_{n-1},\xi_n\right), \end{aligned} \tag{5.6}$$

The rest of the proof follows along the same lines as the proof of Theorem 3.3. So, we find that Γ has an FP $\xi^*\in\Delta$. □

Theorem 5.3. *Let* (Δ,m_b) *be a complete MbMS and* Γ *be a symmetric fractional* α-β-η-Υ*-contraction of pattern-III fulfilling the same affirmations of Theorem 3.4:*

Then Γ has an FP in Δ.

Proof. In the same style of the proof of Theorem 3.4, we obtain that ξ^* is an FP of Γ. □

The example below supports Theorems 5.2 and 5.3.

Example 5.4. Let $\Delta=[0,1]$ and $m_b:\Delta\times\Delta\longrightarrow[0,\infty)$ defined by

$$m_b(\xi,\mu)=\left(\frac{\xi+\mu}{2}\right)^2,\ \forall\xi,\mu\in\Delta.$$

Clearly, (Δ,m_b) is an MbMS with $s=2$. Define $\Gamma:\Delta\to\Delta$ by

$$\Gamma\xi=\begin{cases}\frac{1}{15}, & \text{if } \xi\in[0,1),\\ 1, & \text{if } \xi=1.\end{cases}$$

Describe the functions $\alpha, \beta, \eta : \Delta \to [0, \infty)$ as,

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 2, & \text{if } \xi \in [0,1), \\ 0, & \text{otherwise}, \end{cases}, \quad \eta(\xi) = \begin{cases} 1, & \text{if } \xi \in [0,1), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly Γ is an η-cyclic (α, β)-admissible mapping. Now if $\{\xi_n\}$ is a sequence in Δ such that $\xi_n \to \xi^*$ as $n \to \infty$ and $\beta(\xi_n) \geq \eta(\xi_n)$. Then $\beta(\xi^*) \geq \eta(\xi^*)$ whenever, $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, and for $\xi \in [0,1), \mu = 1$, we have

$$\begin{aligned} s^2 m_b(\Gamma\xi, \Gamma\mu) &= 2^2 m_b(\Gamma\xi, \Gamma 1) = 4\left(\frac{\frac{1}{15}+1}{2}\right)^2 = 4\left(\frac{8}{15}\right)^2 \\ &= 4\left(\frac{8}{3\times 5}\right)^2 = \frac{4}{5}\left[\frac{64}{5}\left(\frac{1}{3}\right)^2\right] \\ &\leq \Upsilon\left[\lambda\left(R_3(\xi, 1)\right)\right]. \end{aligned}$$

That is satisfied when we define $\Upsilon : [0, \infty) \to [0, \infty)$ by $\Upsilon(t) = \frac{4t}{5}$. and we choose the constants $\lambda = \frac{64}{5} \in [0, \infty)$, $a, b, c \in (0, 1)$. Therefore, all affirmations of Theorems 5.2 and 5.3 are satisfied. Hence Γ has two FPs $\frac{1}{15}$ and $1 \in \Delta$.

6. Symmetric fractional α-β-η-Υ-contraction pattern-IV

This portion is consecrated to presenting a symmetric fractional α-β-η-Υ-contraction of pattern-IV in the framework of complete MbMS. Furthermore, new fixed point results are obtained in the said space.

Definition 6.1. Let (Δ, m_b) be an MbMS with a self-map $\Gamma : \Delta \to \Delta$, $\alpha, \beta, \eta : \Delta \to [0, \infty)$ be three functions and $\Upsilon \in \Psi$. We say that Γ is a symmetric fractional α-β-η-Υ-contraction pattern-IV along with constants $s \geq 1, a, b, c \in (0, 1)$ and $\lambda = \left(s m_{b_{\xi,\mu}}\right)^{\frac{-c^3}{(c-a)(c-b)}} \in [0, \infty)$ with $a + b + c < 1$ such that $\forall \xi, \mu \in \Delta \setminus \operatorname{Fix}(\Gamma)$, whenever $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, we have

$$s^2 m_b(\Gamma\xi, \Gamma\mu) \leq \Upsilon\left[\lambda\left(R_4(\xi, \mu)\right)\right], \tag{6.1}$$

where

$$\begin{aligned} R_4(\xi, \mu) &= \left[m_b(\xi, \mu)\right]^{\frac{a^3}{(a-b)(a-c)}} . \left[m_b(\xi, \Gamma\xi)\right]^{\frac{a^3}{(a-b)(a-c)}} . \\ &\quad \left[m_b(\xi, \Gamma\xi) + m_b(\mu, \Gamma\mu)\right]^{\frac{b^3}{(b-a)(b-c)}} . \left[m_b(\xi, \Gamma\mu) + m_b(\mu, \Gamma\xi)\right]^{\frac{c^3}{(c-a)(c-b)}} . \end{aligned}$$

Now, we declare and demonstrate our next theorem.

Theorem 6.2. *Let (Δ, m_b) be a complete MbMS and Γ be a symmetric fractional α-β-η-Υ-contraction of pattern-IV that satisfies the same assertions of Theorem 3.3:*

Then Γ has an FP in Δ.

Proof. Take any $\xi_0 \in \Delta$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$ and $\beta(\xi_0) \geq \eta(\xi_0)$. For all $n \in \mathbb{N}$, we build an iteration $\{\xi_n\}_{n=1}^{\infty}$ such that $\xi_1 = \Gamma\xi_0, \xi_2 = \Gamma\xi_1 = \Gamma^2\xi_0$. By proceeding in this manner, we obtain $\xi_{n+1} = \Gamma\xi_n = \Gamma^{n+1}\xi_0$. Now from (i), we can conclude that for all $n \in \mathbb{N}$,

$$\alpha(\xi_n)\beta(\xi_{n+1}) \geq \eta(\xi_n)\eta(\xi_{n+1}). \tag{6.2}$$

If $\xi_{n+1} = \xi_n$ for some $n \in \mathbb{N}$, then $\xi_n = \xi^*$ is an FP of Γ. So, we assume that $\xi_n \neq \xi_{n+1}$ accompanied by $m_b(\Gamma\xi_{n-1}, \Gamma\xi_n) = m_b(\xi_n, \Gamma\xi_n)$ for every $n \in \mathbb{N}$. Now, from (6.1), we have

$$m_b(\xi_n, \xi_{n+1}) \leq s^2 m_b(\Gamma\xi_{n-1}, \Gamma\xi_n) \leq \Upsilon[\lambda R_4(\xi_{n-1}, \xi_n)],$$

where by the same steps in (3.4), we deduce that

$$\begin{aligned} R_4(\xi_{n-1}, \xi_n) &\leq (m_b(\xi_{n-1}, \xi_n).m_b(\xi_n, \xi_{n+1}))^{a+b+c} \\ &\leq \max\{m_b(\xi_{n-1}, \xi_n), m_b(\xi_n, \xi_{n+1})\}. \end{aligned}$$

If $\max\{m_b(\xi_{n-1}, \xi_n), m_b(\xi_n, \xi_{n+1})\} = m_b(\xi_n, \xi_{n+1})$, then

$$\begin{aligned} m_b(\xi_n, \xi_{n+1}) &\leq \Upsilon[m_b(\xi_n, \xi_{n+1})] \\ &\leq m_b(\xi_n, \xi_{n+1}), \end{aligned}$$

which is a contradiction, thus

$$\begin{aligned} m_b(\xi_n, \xi_{n+1}) &\leq \Upsilon[m_b(\xi_{n-1}, \xi_n)] \\ &\leq m_b(\xi_{n-1}, \xi_n), \end{aligned} \tag{6.3}$$

The rest of the proof follows along the same lines as the proof of Theorem 3.3. So, we find that Γ has an FP $\xi^* \in \Delta$. □

Theorem 6.3. *Consider a complete MbMS (Δ, m_b) and let Γ be a symmetric fractional α-β-η-Υ-contraction of pattern-IV fulfilling the same affirmations of Theorem 3.4:*

Then Γ has an FP in Δ.

Proof. By the same way of the proof of Theorem 3.4, we canclud that Γ has an FP in Δ. □

The example below supports Theorem 6.2.

Example 6.4. Let $\Delta = [0, \infty)$, $p > 1$ and $m_b : \Delta \times \Delta \longrightarrow [0, \infty)$ defined by

$$m_b(\xi, \mu) = \max\{\xi, \mu\}^p + |\xi - \mu|^p, \ \forall \xi, \mu \in \Delta.$$

Clearly, (Δ, m_b) is an MbMS with $s = 2^p$. Define $\Gamma : \Delta \to \Delta$ by

$$\Gamma\xi = \begin{cases} \frac{\xi+0.5}{128}, & \text{if } \xi \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Describe the functions $\alpha, \beta, \eta : \Delta \to [0, \infty)$ as,

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 2, & \text{if } \xi \in (0,1], \\ 0, & \text{otherwise,} \end{cases}, \ \eta(\xi) = \begin{cases} 1, & \text{if } \xi \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly for all $\xi,\mu \in (0,1]$, Γ is an η-cyclic (α,β)-admissible mapping, whenever $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, we have

$$\begin{aligned}
s^2 m_b(\Gamma\xi,\Gamma\mu) &= 2^{2p} m_b\left(\frac{\xi+0.5}{128},\frac{\mu+0.5}{128}\right) \leq 2^{2p} m_b\left(\frac{\xi}{64},\frac{\mu}{64}\right)\\
&= 2^{2p}\left(\max\{\frac{\xi}{64},\frac{\mu}{64}\}^p + |\frac{\xi}{64}-\frac{\mu}{64}|^p\right)\\
&= \frac{2^{2p}}{64^p}\left(\max\{\xi,\mu\}^p + |\xi-\mu|^p\right) = \frac{2^{2p}}{2^{6p}} m_b(\xi,\mu) = \frac{1}{2^{4p}} m_b(\xi,\mu)\\
&\leq \frac{1}{2^{2p}}\left[\frac{1}{2^p}\left(R_3(\xi,\mu)\right)\right]\\
&= \Upsilon\left[\lambda\left(R_3(\xi,\mu)\right)\right].
\end{aligned}$$

That is achieved when we take $\Upsilon(t) = \frac{t}{2^{2p}}$ and constants $\lambda = \frac{1}{2^p} \in [0,\infty)$, $a,b,c \in (0,1)$, for all $\xi,\mu \in \Delta\backslash\mathrm{Fix}(\Gamma)$. Otherwise, we can obtain that Γ is η-cyclic (α,β)-admissible mapping, whenever $\alpha(\xi)\beta(\mu) \geq \eta(\xi)\eta(\mu)$, and

$$s^2 m_b(\Gamma\xi,\Gamma\mu) = 0 \leq \Upsilon\left[\lambda\left(R_1(\xi,\mu)\right)\right].$$

Therefore, all affirmations of Theorem 6.2 are satisfied. Hence Γ has two FPs 0 and $\frac{1}{254} \in \Delta$.

By taking $\eta(\xi) = \eta(\mu) = 1$, in Theorems 3.3, 3.4, 4.2 and 4.3, we derive the following corollaries.

Corollary 6.5. *Let (Δ, m_b) be a complete MbMS and Γ be a symmetric fractional α-β-Υ-contraction of pattern-I fulfilling the accompanying affirmations:*

(i) Γ is a cyclic (α,β)-admissible mapping;
(ii) there is an $\xi_0 \in \Delta$ so that $\alpha(\xi_0) \geq 1$ or there is a $\mu_0 \in \Delta$ so that $\beta(\mu_0) \geq 1$;
(iii) Γ is continuous.

Then Γ has an FP in Δ.

Corollary 6.6. *Let (Δ, m_b) be a complete MbMS and Γ be a symmetric fractional α-β-Υ-contraction of pattern-I fulfilling the accompanying affirmations:*

(i) Γ is a cyclic (α,β)-admissible mapping;
(ii) there is an $\xi_0 \in \Delta$ so that $\alpha(\xi_0) \geq 1$ or there is a $\mu_0 \in \Delta$ so that $\beta(\mu_0) \geq 1$;
(iii) if $\{\xi_n\}$ is a sequence in Δ such that $\xi_n \to \xi^*$ as $n \to \infty$, and $\beta(\xi_n) \geq 1$ $\forall n \in \mathbb{N}$, then $\beta(\xi^*) \geq 1$.

Hence, Γ has an FP in Δ.

Corollary 6.7. *Let (Δ, m_b) be a complete MbMS, and Γ be a symmetric fractional α-β-Υ-contraction of pattern-II fulfilling the same affirmations in Corollary 6.5.*

Then Γ has an FP in Δ.

Corollary 6.8. *Let (Δ, m_b) be a complete MbMS, and Γ be a symmetric fractional α-β-Υ-contraction of pattern-II fulfilling the same affirmations in Corollary 6.6.*

Then Γ has an FP in Δ.

Note. In a similar action, we can deduce the Corollaries 6.5–6.8 for symmetric fractional α-β-Υ-contractions of pattern III and IV respectively.

7. Symmetric fractional α-η-Υ-contraction on closed ball

In this portion, we derive some fixed point results for symmetric fractional contraction mappings on a closed ball of MbMS.

Theorem 7.1. *Let (Δ, m_b) be a complete MbMS, ξ_0 be an arbitrary point in a closed ball $B[\xi_0, \varepsilon]$, α, η:$\Delta \times \Delta \to [0, \infty)$ be semi α-admissible mappings wrt η on $B[\xi_0, \varepsilon]$ with $\alpha(\xi_0, \xi_1) \geq \eta(\xi_0, \xi_1)$ and $\Upsilon \in \Psi$. Let $\Gamma : \Delta \to \Delta$ be a continuous semi α-admissible mapping satisfying (3.1) for all $\xi, \mu \in B[\xi_0, \varepsilon] \subseteq \Delta \backslash Fix(\Gamma)$, $\alpha(\xi, \mu) \geq \eta(\xi, \mu)$. Moreover, for all $\varepsilon > 0$*

$$m_b\left(\xi_0, \xi_1\right) - m_{b_{\xi_0, \xi_1}} \leq \sum_{i=0}^{n} s^{i+1} \Upsilon^i \left[m_b\left(\xi_0, \xi_1\right)\right] \leq \varepsilon. \tag{7.1}$$

Then Γ has an FP in $B[\xi_0, \varepsilon] \subseteq \Delta$.

Proof. Since $\xi_0 \in B[\xi_0, \varepsilon]$ there exists $\xi_1 \in \Delta$ such that $\xi_1 = \Gamma\xi_0$ and $\xi_2 \in \Delta$ such that $\xi_2 = \Gamma\xi_1$. Continuing in this process, we construct a sequence $\{\xi_n\}$ of points in Δ such that, $\xi_n = \Gamma\xi_n$. As $\alpha(\xi_0, \xi_1) \geq \eta(\xi_0, \xi_1)$ and it is semi α-admissible mapping wrt η, we have $\alpha(\Gamma\xi_0, \Gamma\xi_1) \geq \eta(\Gamma\xi_0, \Gamma\xi_1)$ from which we deduce that $\alpha(\xi_1, \xi_2) \geq \eta(\xi_1, \xi_2)$ which also implies that $\alpha(\Gamma\xi_1, \Gamma\xi_2) \geq \eta(\Gamma\xi_1, \Gamma\xi_2)$. Continuing in this way, we obtain $\alpha(\Gamma\xi_{n-1}, \Gamma\xi_n) \geq \eta(\Gamma\xi_{n-1}, \Gamma\xi_n)$. which leads to $\alpha(\xi_n, \xi_{n+1}) \geq \eta(\xi_n, \xi_{n+1})$. for all $n \in \mathbb{N}$. Now, we show that $\xi_n \in B[\xi_0, \varepsilon]$ for all $n \in \mathbb{N}$. Utilizing inequality (7.1), we have

$$m_b\left(\xi_0, \xi_1\right) - m_{b_{\xi_0, \xi_1}} \leq \sum_{i=0}^{n} s^{i+1} \Upsilon^i \left[m_b\left(\xi_0, \xi_1\right)\right] \leq \varepsilon, \forall \varepsilon > 0.$$

That is $\xi_1 \in B[\xi_0, \varepsilon]$. Let $\xi_2, \xi_3, ..., \xi_j \in B[\xi_0, \varepsilon]$ for some $j \in \mathbb{N}$. Now, we can write

$$m_b\left(\xi_j, \xi_{j+1}\right) \leq s^2 m_b\left(\Gamma\xi_{j-1}, \Gamma\xi_j\right) \leq \Upsilon\left[\lambda\left(R_1\left(\xi_{j-1}, \xi_j\right)\right)\right], \tag{7.2}$$

where, by the same steps in (3.4), we deduce that

$$R_1\left(\xi_{j-1}, \xi_j\right) \leq \left(s m_{b_{\xi_{j-1}, \xi_{j+1}}}\right)^{\frac{1}{(c-a)(c-b)}} m_b\left(\xi_{j-1}, \xi_j\right). \tag{7.3}$$

Therefore, from (7.2), (7.3) and similar to (3.3) and (3.4), we conclude that

$$m_b\left(\xi_j, \xi_{j+1}\right) < \Upsilon^j\left[m_b\left(\xi_0, \xi_1\right)\right], \forall j \in \mathbb{N}. \tag{7.4}$$

Using (Mb4) and (7.4), we have

$$\begin{aligned}
m_b\left(\xi_0, \xi_{j+1}\right) - m_{b_{\xi_0, \xi_{j+1}}} &\leq s\left[\left(m_b\left(\xi_0, \xi_1\right) - m_{b_{\xi_0, \xi_1}}\right) + \left(m_b\left(\xi_1, \xi_{j+1}\right) - m_{b_{\xi_0, \xi_{j+1}}}\right)\right] - m_b\left(\xi_1, \xi_1\right) \\
&\leq s\left[m_b\left(\xi_0, \xi_1\right) + s m_b\left(\xi_1, \xi_2\right) + s^2 m_b\left(\xi_2, \xi_3\right) + ... + s^j m_b\left(\xi_j, \xi_{j+1}\right)\right] \\
&= s m_b\left(\xi_0, \xi_1\right) + s^2 m_b\left(\xi_1, \xi_2\right) + s^3 m_b\left(\xi_2, \xi_3\right) + ... + s^{j+1} m_b\left(\xi_j, \xi_{j+1}\right) \\
&< s m_b\left(\xi_0, \xi_1\right) + s^2 \Upsilon\left(m_b\left(\xi_0, \xi_1\right)\right) + s^3 \Upsilon\left(m_b\left(\xi_0, \xi_1\right)\right) + ... + s^{j+1} \Upsilon^j m_b\left(\xi_0, \xi_1\right)
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{i=0}^{j} s^{i+1}\Upsilon^{i}\left(m_b\left(\xi_0,\xi_1\right)\right) \\
&< \varepsilon.
\end{aligned}
$$

Thus $\xi_{j+1} \in B[\xi_0, \varepsilon]$. Hence by induction, we get $\xi_n \in B[\xi_0, \varepsilon]$ $\forall n \in \mathbb{N}$, therefore $\{\xi_n\}$ is a sequence in $B[\xi_0, \varepsilon]$. As Γ is simi α-admissible wrt η on $B[\xi_0, \varepsilon]$, so $\alpha\left(\xi_n, \xi_{n+1}\right) \geq \eta\left(\xi_n, \xi_{n+1}\right)$. Also inequality (7.4) can be written as

$$
m_b\left(\xi_n, \xi_{n+1}\right) < \Upsilon^{n}\left[m_b\left(\xi_0, \xi_1\right)\right] \forall n \in \mathbb{N}. \tag{7.5}
$$

As $\sum_{i=1}^{\infty} s^{i}\Upsilon^{i}(t) < \infty$, then for some $k \in \mathbb{N}$ the series $\sum_{i=1}^{\infty} s^{i}\Upsilon^{i}\left[\Upsilon^{k-1}\left(m_b\left(\xi_0, \xi_1\right)\right]\right.$, converges. Fix $\varepsilon > 0$, then there exists $k(\varepsilon) \in \mathbb{N}$, such that

$$
\sum_{i=1}^{\infty} s^{i}\Upsilon^{i}\left[\Upsilon^{n-1}\left(m_b\left(\xi_0, \xi_1\right)\right] < \varepsilon.\right. \tag{7.6}
$$

Let $n, m \in \mathbb{N}$ with $n > m > k(\varepsilon)$ and from (Mb4), (7.5), (7.6) and (Υ3), we get

$$
\begin{aligned}
m_b\left(\xi_n, \xi_m\right) - m_{b_{\xi_n,\xi_m}} &\leq \sum_{i=n}^{m-1} s^{i-n+1} m_b\left(\xi_i, \xi_{i+1}\right) \\
&\leq \sum_{i=n}^{m-1} s^{i-n+1}\Upsilon^{i}\left[m_b\left(\xi_i, \xi_{i+1}\right)\right] \\
&< \sum_{i=n}^{m-n} s^{i-n}\Upsilon^{i-n}\left[\Upsilon^{n-1}\left(m_b\left(\xi_0, \xi_1\right)\right)\right] \\
&< \sum_{i=n}^{m-n} s^{i}\Upsilon^{i}\left[\Upsilon^{k(\varepsilon)-1}\left(m_b\left(\xi_0, \xi_1\right)\right)\right] \\
&< \varepsilon.
\end{aligned}
$$

The convergence of the series $\sum_{i=n}^{m-n} s^{i}\Upsilon^{i}\left[\Upsilon^{k-1}\left(m_b\left(\xi_0, \xi_1\right)\right)\right]$ leads to

$$
\lim_{n,m\to\infty}\left(m_b\left(\xi_n, \xi_m\right) - m_{b_{\xi_n,\xi_m}}\right) = 0.
$$

By the same way, we can show that $\lim_{n,m\to\infty}\left(M_{b_{\xi_n,\xi_m}} - m_{b_{\xi_n,\xi_m}}\right) = 0$. Therefore, $\{\xi_n\}$ is an m_b-Cauchy sequence in $B[\xi_0, \varepsilon]$. Since every closed set in a complete MbMS is complete. So, there exists $\xi^* \in B[\xi_0, \varepsilon]$ such that $\xi_n \to \xi^*$ as $n \to \infty$. Since Γ is continuous then $\lim_{n\to\infty}\Gamma\xi_n = \Gamma\xi^*$ and

$$
\lim_{n\to\infty}\left(m_b\left(\xi_n, \xi^*\right) - m_{b_{\xi_n,\xi^*}}\right) = 0. \tag{7.7}
$$

We will show that $\Gamma\xi^* = \xi^*$. Suppose that $m_b\left(\xi^*, \Gamma\xi^*\right) - m_{b_{\xi^*,\Gamma\xi^*}} > 0$. So, by (Mb4), we have

$$
\begin{aligned}
m_b\left(\xi^*, \Gamma\xi^*\right) - m_{b_{\xi^*,\Gamma\xi^*}} \leq\ & s\left[\left(m_b\left(\xi^*, \xi_{n+1}\right) - m_{b_{\xi^*,\xi_{n+1}}}\right) + \left(m_b\left(\xi_{n+1}, \Gamma\xi^*\right) - m_{b_{\xi_{n+1},\Gamma\xi^*}}\right)\right] \\
& - m_b\left(\xi_{n+1}, \xi_{n+1}\right)
\end{aligned} \tag{7.8}
$$

$$
\begin{aligned}
&\le s\left(m_b(\xi^*,\xi_{n+1})-m_{b_{\xi^*,\xi_{n+1}}}\right)+s\left(m_b(\Gamma\xi_n,\Gamma\xi^*)-m_{b_{\xi_{n+1},\Gamma\xi^*}}\right)\\
&= s\left(m_b(\xi^*,\xi_{n+1})-m_{b_{\xi^*,\xi_{n+1}}}\right)+\frac{1}{s}\left(s^2m_b(\Gamma\xi_n,\Gamma\xi^*)\right)-sm_{b_{\xi_{n+1},\Gamma\xi^*}}\\
&\le s\left(m_b(\xi^*,\xi_{n+1})-m_{b_{\xi^*,\xi_{n+1}}}\right)+\frac{1}{s}\Upsilon\left[\lambda R_1(\xi_n,\xi^*)\right)-sm_{b_{\xi_{n+1},\Gamma\xi^*}}\\
&= s\left(m_b(\xi^*,\xi_{n+1})-m_{b_{\xi^*,\xi_{n+1}}}\right)+\\
&\quad\frac{1}{s}\Upsilon\left[\lambda\begin{array}{c}
m_b(\xi_n,\xi^*).m_b(\xi_n,\Gamma\xi_n)^{\overline{(a-b)(a-c)}^{\,1}}\\
.m_b(\xi^*,\Gamma\xi^*)^{\overline{(a-b)(a-c)}^{\,1}}\\
.[m_b(\xi_n,\Gamma\xi_n)+m_b(\xi^*,\Gamma\xi^*)]^{\overline{(b-a)(b-c)}^{\,1}}\\
.[m_b(\xi_n,\Gamma\xi^*)+m_b(\xi^*,\Gamma\xi_n)]^{\overline{(c-a)(c-b)}^{\,1}}
\end{array}\right]-sm_{b_{\xi_{n+1},\Gamma\xi^*}}.
\end{aligned}
\tag{7.9}
$$

Taking limit as $n\to\infty$ in (7.8) and utilizing (7.7), we get

$$
\begin{aligned}
m_b(\xi^*,\Gamma\xi^*)-m_{b_{\xi^*,\Gamma\xi^*}} &\le \frac{1}{s}\Upsilon\left[\lambda\begin{array}{c}
m_b(\xi^*,\Gamma\xi^*).m_b(\xi^*,\xi^*)^{\frac{1}{(a-b)(a-c)}}.m_b(\xi^*,\Gamma\xi^*)^{\frac{1}{(a-b)(a-c)}}\\
.\left[m_b(\xi^*,\xi^*)+m_b(\xi^*,\Gamma\xi^*)\right]^{\frac{1}{(b-a)(b-c)}}\\
.\left[m_b(\xi^*,\Gamma\xi^*)+sm_b(\xi^*,\xi^*)+m_{b_{\xi^*,\Gamma\xi_n}}\right]^{\frac{1}{(c-a)(c-b)}}
\end{array}\right]\\
&\quad -sm_{b_{\xi^*,\Gamma\xi^*}}\\
&\le \frac{1}{s^2}s\Upsilon\left[\lambda\begin{array}{c}
m_b(\xi^*,\Gamma\xi^*).m_b(\xi^*,\xi^*)^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}}\\
.\left[m_b(\xi^*,\Gamma\xi^*)\right]^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}}\\
.\left[sm_{b_{\xi^*,\Gamma\xi^*}}\right]^{\frac{1}{(c-a)(c-b)}}
\end{array}\right]\\
&\quad -sm_{b_{\xi^*,\Gamma\xi^*}}\\
&< \frac{1}{s^2}m_b(\xi^*,\Gamma\xi^*)-sm_{b_{\xi^*,\Gamma\xi^*}}\, by (\Upsilon 3)\\
&< m_b(\xi^*,\Gamma\xi^*)-m_{b_{\xi^*,\Gamma\xi^*}}.
\end{aligned}
$$

Which is a contradiction. Therefore $m_b(\xi^*,\Gamma\xi^*)-m_{b_{\xi^*,\Gamma\xi^*}}=0$ implies that $m_b(\xi^*,\Gamma\xi^*)=m_{b_{\xi^*,\Gamma\xi^*}}$. So, that is $\Gamma\xi^*=\xi^*$ and $\xi^*\in B[\xi_0,\varepsilon]$ is an FP of Γ. □

In a similar conductance, we can state and prove the same Theorems fulfill symmetric fractional α-β-Υ-contraction mappings (4.1), (5.1) and (6.1) on closed ball.

The example below supports Theorem 7.1.

Example 7.2. Let $\Delta=[0,\infty)$, $p>1$ and $m_b:\Delta\times\Delta\longrightarrow[0,\infty)$ is defined by

$$m_b(\xi,\mu)=\max\{\xi,\mu\}^p+|\xi-\mu|^p,\ \forall\xi,\mu\in\Delta.$$

Clearly, (Δ,m_b) is an MbMS with $s=2^p$. Define $\Gamma:\Delta\to\Delta$ by

$$\Gamma\xi=\begin{cases}\frac{\xi}{e^5}, & \text{if } \xi\in[0,10],\\ 4\xi-45, & \text{if } \xi\in(10,\infty).\end{cases}$$

Describe the functions $\alpha, \eta : \Delta \times \Delta \to [0,\infty)$ as,

$$\alpha(\xi,\mu) = \begin{cases} 2, & \text{if } \xi,\mu \in [0,10], \\ 1, & \text{if } \xi,\mu \in (10,\infty), \end{cases} \quad , \ \eta(\xi,\mu) = \begin{cases} 1, & \text{if } \xi,\mu \in [0,10], \\ 0, & \text{if } \xi,\mu \in [0,\infty]. \end{cases}$$

Considering $\xi_0 = 1$, $\varepsilon = 180$, then $B[\xi_0,\varepsilon] = [0,10]$ and $m_b(\xi_0,\Gamma\xi_0) = m_b(1,\Gamma 1) = m_b\left(1,\frac{1}{e^5}\right) = 1+(1-\frac{1}{e^5})^p$. Therefore, $\alpha(1,\Gamma 1) = 2 \geq \eta(1,\Gamma 1) = 1$. Now for all $\xi,\mu \in [0,10]$, Γ is a continuous semi α-admissible mapping wrt η, whenever $\alpha(\xi,\mu) \geq \eta(\xi,\mu)$, we have

$$\begin{aligned} s^2 m_b(\Gamma\xi,\Gamma\mu) &= 2^{2p} m_b\left(\frac{\xi}{e^5},\frac{\mu}{e^5}\right) = 2^{2p}\left(\max\{\frac{\xi}{e^5},\frac{\mu}{e^5}\}^p + |\frac{\xi}{e^5}-\frac{\mu}{e^5}|^p\right) \\ &= \frac{2^{2p}}{e^{5p}}\left(\max\{\xi,\mu\}^p + |\xi-\mu|^p\right) \leq \frac{2^p}{e^{5p}}\left[2^{2p} m_b(\xi,\mu)\right] \\ &\leq \Upsilon\left[\lambda\left(R_1(\xi,\mu)\right)\right]. \end{aligned}$$

That is achieved when we choose $\Upsilon(t) = \frac{2^p t}{e^{5p}}$ and constants $\lambda = 2^{2p} \in [0,\infty)$, $a,b,c \in (0,1)$, for all $\xi,\mu \in \Delta\backslash \mathrm{Fix}(\Gamma)$. Also, for all $n \geq 0$ and $p > 1$, we obtain

$$\begin{aligned} m_b(\xi_0,\xi_1) &\leq \sum_{i=0}^{n} s^{i+1}\Upsilon^i\left[m_b(\xi_0,\xi_1)\right] \\ &\leq \sum_{i=0}^{n} s^{i+1}\Upsilon^i\left[1+\left(1-\frac{1}{e^5}\right)^p\right] \\ &= 2^2\left[1+\left(1-\frac{1}{e^5}\right)^p\right]\sum_{i=0}^{n}\left(\frac{2^{2p}}{e^{5p}}\right)^i \leq 180 = \varepsilon. \end{aligned}$$

Note that for $20, 21 \in \Delta$ and for $p = 2$, we have $\alpha(20,21) \geq \eta(20,21)$ and we can calculate

$$s^2 m_b(\Gamma 20,\Gamma 21) > \Upsilon\left[\lambda R_1(20,21)\right].$$

So that condition (3.1) does not hold. Therefore, all affirmations of Theorem 7.1 are satisfied. Hence Γ has an FP $\xi^* = 0 \in B[\xi_0,\varepsilon]$. (Note that 15 is an another FP of Γ, which belongs to Δ but it does not belong to $B[\xi_0,\varepsilon]$.

8. An application to fractional-order differential equation

In this portion, we shall apply Theorem 4.2 to discuss the existence and uniqueness of the bounded solution to fractional order differential equations (FODE), which have recently proved to be significant tools in the modeling of many phenomena in numerious fields of science and building. Consider a function $f : (0,1) \to \mathbb{R}$. The conformable fractional derivative of order α of f at $t > 0$ is defined in [26] as follows:

$$D^\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}. \tag{8.1}$$

The conformable fractional integral associated with (8.1) is defined in [25, 26] as following:

$$I_0^{\alpha} f(t) = \int_0^t s^{\alpha-1} f(s)ds. \tag{8.2}$$

We consider the following boundary value problem (BVP) of a fractional order differential equation:

$$\begin{cases} D_t^{\alpha}\xi(t) = \lambda(t, \xi(t), D_t^{\alpha-1}\xi(t), & t, \alpha \in (0, 1) \\ \xi(0) = 0, & \xi(1) = \int_0^1 \xi(s)ds. \end{cases} \tag{8.3}$$

The BVP (8.3) can be expressed as the integral equation as follows:

$$\xi(t) = \lambda \int_0^1 G(t, s) f(s, \xi(s))ds. \tag{8.4}$$

Where $G(t, s)$ is defined as the Green function under the assumption of (8.1), which is given by

$$G(s, t) = \begin{cases} -2ts^{\alpha} + s^{\alpha-1}, & 0 \le s \le t \le 1 \\ -2ts^{\alpha}, & 0 \le t \le s \le 1 \end{cases} \tag{8.5}$$

and $\int_0^1 \xi(s)ds$ denotes the Riemann integrable of ξ with respect to s and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Consider

$$\xi(t) = c_0 + c_1 t + \lambda \int_0^1 s^{\alpha-1} f(s, \xi(s))ds, \tag{8.6}$$

when $\xi(0) = 0$, then $c_0 = 0$, $\xi(1) = c_1 + \lambda \int_0^1 s^{\alpha-1} f(s, \xi(s))ds$, and from the condition $\xi(1) = \int_0^1 \xi(s)ds$, we have

$$\begin{aligned} \int_0^1 \xi(s)ds &= \int_0^1 c_1 s ds + \lambda \int_0^1 \int_0^s v^{\alpha-1} f(v, \xi(v))dvds \\ &= \frac{1}{2}c_1 + \lambda \int_0^1 \int_v^1 v^{\alpha-1} f(v, \xi(v))dvds \\ &= \frac{1}{2}c_1 + \lambda \int_0^1 (1 - v)v^{\alpha-1} f(v, \xi(v))dvds \\ &= \frac{1}{2}c_1 + \lambda \int_0^1 (s^{\alpha-1} - s^{\alpha}) f(s, \xi(s))ds, \end{aligned}$$

this implies that

$$\begin{aligned} \frac{1}{2}c_1 &= \lambda \int_0^1 s^{\alpha-1} f(s, \xi(s))ds + \lambda \int_0^1 (s^{\alpha-1} - s^{\alpha}) f(s, \xi(s))ds \\ &= -\lambda \int_0^1 s^{\alpha} f(s, \xi(s))ds \end{aligned}$$

hence,

$$c_1 = -2\lambda \int_0^1 s^{\alpha} f(s, \xi(s))ds.$$

It follows from (8.6), c_0 and c_1 that

$$\begin{aligned}\xi(t) &= -2\lambda t\int_0^1 s^\alpha f(s,\xi(s))ds + \lambda\int_0^t s^{\alpha-1} f(s,\xi(s))ds\\ &= -2\lambda t\int_0^t s^\alpha f(s,\xi(s))ds - 2\lambda t\int_t^1 s^\alpha f(s,\xi(s))ds + \lambda\int_0^t s^{\alpha-1} f(s,\xi(s))ds\\ &= \lambda\int_0^t (-2ts^\alpha + s^{\alpha-1}) f(s,\xi(s))ds + \lambda\int_t^1 (-2ts^\alpha) f(s,\xi(s))ds\\ &= \lambda\int_0^t G(t,s) f(s,\xi(s))ds.\end{aligned}$$

Let $C^\alpha(I)$ be the space of all continuous functions de…ned on I, where $I = [0,1]$, $\alpha > 0$ and let $m_b(\xi,\mu) = \|\frac{\xi+\mu}{2}\|_\infty^2 = \max_{t\in I}\left(|\frac{\xi(t)+\mu(t)}{2}|\right)^2$ for all $\xi,\mu \in C^\alpha(I)$. Then $(C^\alpha(I), m_b)$ is a complete MbMS with a constant $\delta = 2$.
Now we consider the BVP (8.3) under the following stipulations:

(1) there exist a function $\omega : \mathbb{R} \to (0,1)$, $\delta \geq 1$ and $\rho, \sigma, \varrho : \mathbb{R} \to \mathbb{R}$ are three functions such that for all $t \in I$ and $\xi, \mu \in \mathbb{R}$ with $\rho(\xi)\sigma(\mu) \geq \varrho(\xi)\varrho(\mu)$, then

$$|f(s,\xi(s)) + f(s,\mu(s))| \leq \sqrt{\frac{4\omega(t)}{10\delta^2}\aleph(\xi(s),\mu(s))},$$

where, For all $a, b, c \in (0,1)$

$$\begin{aligned}\aleph(\xi(s),\mu(s)) &= |\frac{\xi(s)+\mu(s)}{2}|^2.|\frac{\xi(s)+\Gamma\xi(s)}{2}|^{\frac{2a}{(a-b)(a-c)}}.|\frac{\mu(s)+\Gamma\mu(s)}{2}|^{\frac{2a}{(a-b)(a-c)}}\\ &\quad.\left[|\frac{\xi(s)+\Gamma\xi(s)}{2}| + |\frac{\mu(s)+\Gamma\mu(s)}{2}|^2\right]^{\frac{b}{(b-a)(b-c)}}\\ &\quad.\left[|\frac{\xi(s)+\Gamma\mu(s)}{2}| + |\frac{\mu(s)+\Gamma\xi(s)}{2}|^2\right]^{\frac{c}{(c-a)(c-b)}},\end{aligned}$$

(2) there exists $\xi_1 \in C^\alpha(I)$ such that for all $t \in I$,

$$\rho(\xi_1(t))\,\sigma\left(\int_0^1 G(t,s)f(s,\xi_1(s))ds\right) \geq \varrho(\xi_1(t))\,\varrho\left(\int_0^1 G(t,s)f(s,\xi_1(s))ds\right),$$

(3) for all $t \in I$ and for all $\xi, \mu \in C^\alpha(I)$, there are $\xi_1, \mu_1 \in C^\alpha(I)$, such that

$$\rho(\xi(t)) \geq \varrho(\xi(t))$$

implies

$$\sigma\left(\int_0^1 G(t,s)f(s,\xi_1(s))ds\right) \geq \varrho\left(\int_0^1 G(t,s)f(s,\xi_1(s))ds\right)$$

and

$$\sigma(\mu(t)) \geq \varrho(\mu(t))$$

implies

$$\rho\left(\int_0^1 G(t,s)f(s,\mu_1(s))ds\right) \geq \varrho\left(\int_0^1 G(t,s)f(s,\mu_1(s))ds\right),$$

(4) for any cluster point ξ of a sequence $\{\xi_n\}$ of points in $C^\alpha(I)$ with $\rho(\xi_n)\sigma(\xi_{n+1}) \geq \varrho(\xi_n)\varrho(\xi_{n+1})$, such that

$$\liminf_{n\to\infty} \rho(\xi_n)\sigma(\xi) \geq \lim_{n\to\infty} \varrho(\xi_n)\varrho(\xi).$$

Now, we present our main theorem in this part.

Theorem 8.1. *Under the postulates (1)–(4), the BVP (8.3) has at least one solution $\xi^* \in C^\alpha(I)$.*

Proof. We known that $\xi^* \in C^\alpha(I)$ is a solution of (8.3) if and only if $\xi^* \in C^\alpha(I)$ is a solution of the fractional order integral equation:

$$\xi(t) = \lambda \int_0^1 G(t,s)f(s,\xi(s))ds, \forall\lambda, t \in I.$$

We define a map $\Gamma : C^\alpha(I) \to C^\alpha(I)$ by

$$\Gamma\xi(t) = \lambda \int_0^1 G(t,s)f(s,\xi(s))ds, \forall\lambda, t \in I.$$

Then, problem (8.3) is equivalent to find $\xi^* \in C^\alpha(I)$ that is a fixed point of Γ. Let $\xi, \mu \in C^\alpha(I)$, such that $\rho(\xi(t))\sigma(\xi(t)) \geq 0$, for all $t \in I$. Using stipulation (1), we get

$$\begin{aligned}
[\mid \Gamma\xi(t) + \Gamma\mu(t) \mid]^2 &= \left[\mid \lambda \mid\mid \int_0^1 G(t,s)f(s,\xi(s))ds + \int_0^1 G(t,s)f(s,\mu(s))ds \mid\right]^2 \\
&\leq \left[\mid \lambda \mid \int_0^1 G(t,s) \mid (s,\xi(s)) + f(s,\mu(s)) \mid ds\right]^2 \\
&\leq \left[\mid \lambda \mid \int_0^1 G(t,s)ds \sqrt{\frac{4\omega(t)}{10\delta^2}\aleph(\xi(s),\mu(s))}\right]^2 \\
&= \left[\mid \lambda \mid \int_0^1 G(t,s)ds\right]^2 \frac{4\omega(t)}{10\delta^2}\aleph(\xi(s),\mu(s)) \\
&\leq \left[\mid \lambda \mid \int_0^1 G(t,s)ds\right]^2 \frac{4\omega(t)}{10\delta^2}\left|\frac{\xi(s)+\mu(s)}{2}\right|^2 \\
&\quad .\left|\frac{\xi(s)+\Gamma\xi(s)}{2}\right|^{\frac{2a}{(a-b)(a-c)}} .\left|\frac{\mu(s)+\Gamma\mu(s)}{2}\right|^{\frac{2a}{(a-b)(a-c)}} \\
&\quad \cdot\left[\left|\frac{\xi(s)+\Gamma\xi(s)}{2}\right| + \left|\frac{\mu(s)+\Gamma\mu(s)}{2}\right|^2\right]^{\frac{b}{(b-a)(b-c)}} \\
&\quad \cdot\left[\left|\frac{\xi(s)+\Gamma\mu(s)}{2}\right| + \left|\frac{\mu(s)+\Gamma\xi(s)}{2}\right|^2\right]^{\frac{c}{(c-a)(c-b)}}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \max_{t\in I}\left[\int_0^1 G(t,s)ds\right]^2 \frac{4\omega(t)}{10\delta^2}\|\frac{\xi(s)+\mu(s)}{2}\|_\infty^2 \\
& .\|\frac{\xi(s)+\Gamma\xi(s)}{2}\|_\infty^{\frac{2a}{(a-b)(a-c)}}.\|\frac{\mu(s)+\Gamma\mu(s)}{2}\|_\infty^{\frac{2a}{(a-b)(a-c)}} \\
& .\left[\|\frac{\xi(s)+\Gamma\xi(s)}{2}\|_\infty^2+\|\frac{\mu(s)+\Gamma\mu(s)}{2}\|_\infty^2\right]^{\frac{b}{(b-a)(b-c)}} \\
& .\left[\|\frac{\xi(s)+\Gamma\mu(s)}{2}\|_\infty^2+\|\frac{\mu(s)+\Gamma\xi(s)}{2}\|_\infty^2\right]^{\frac{c}{(c-a)(c-b)}} \\
\leq\ & \frac{4\omega(t)}{10\delta^2}\|\frac{\xi(s)+\mu(s)}{2}\|_\infty^2 \\
& .\|\frac{\xi(s)+\Gamma\xi(s)}{2}\|_\infty^{\frac{2a}{(a-b)(a-c)}}.\|\frac{\mu(s)+\Gamma\mu(s)}{2}\|_\infty^{\frac{2a}{(a-b)(a-c)}} \\
& .\left[\|\frac{\xi(s)+\Gamma\xi(s)}{2}\|_\infty^2+\|\frac{\mu(s)+\Gamma\mu(s)}{2}\|_\infty^2\right]^{\frac{b}{(b-a)(b-c)}} \\
& .\left[\|\frac{\xi(s)+\Gamma\mu(s)}{2}\|_\infty^2+\|\frac{\mu(s)+\Gamma\xi(s)}{2}\|_\infty^2\right]^{\frac{c}{(c-a)(c-b)}}.
\end{aligned}
$$

Thus,

$$
\delta^2 m_b(\Gamma\xi,\Gamma\mu)\leq\Upsilon\left[\lambda R_2(\xi,\mu)\right],
$$

where, $\Upsilon(t)=\frac{\omega(t)}{5}t$, $\lambda=\frac{1}{2}$ and

$$
R_2(\xi,\mu)=\left\{\begin{array}{c} m_b(\xi,\mu).\left[m_b(\xi,\Gamma\xi)\right]^{\frac{a}{(a-b)(a-c)}}.\left[m_b(\mu,\Gamma\mu)\right]^{\frac{a}{(a-b)(a-c)}} \\ .\left[m_b(\xi,\Gamma\xi)+m_b(\mu,\Gamma\mu)\right]^{\frac{b}{(b-a)(b-c)}} \\ .\left[m_b(\xi,\Gamma\mu)+m_b(\mu,\Gamma\xi)\right]^{\frac{c}{(c-a)(c-b)}} \end{array}\right\}.
$$

For each $\xi,\mu\in C^\alpha(I)$ such that $\rho(\xi(t))\sigma(\mu(t))\geq\varrho(\xi(t))\varrho(\mu(t))$ for all $t\in I$. We define $\alpha,\beta,\eta: C^\alpha(I)\to[0,\infty)$ by

$$
\alpha(\xi)=\beta(\xi)=\begin{cases} 2, & \text{if } \rho(\xi(t))\sigma(\mu(t))\geq 0, t\in I \\ 0, & \text{otherwise,}\end{cases} \quad \text{and } \eta(\xi)=\begin{cases} \frac{1}{4}, & \text{if } \rho(\xi(t))\sigma(\mu(t))\geq 0, t\in I \\ 0, & \text{otherwise.}\end{cases}
$$

Then, for all $\xi,\mu\in C^\alpha(I)$, & $\alpha(\xi)\geq\eta(\xi)$ and $\beta(\mu)\geq\eta(\mu)$. If $\alpha(\xi)\geq\eta(\xi)$ and $\beta(\mu)\geq\eta(\mu)$. for each $\xi,\mu\in C(I)$, then $\rho(\xi(t))\geq\varrho(\xi(t))$ and $\sigma(\mu(t))\geq\varrho(\mu(t))$. From stipulation (3), we have $\rho(\Gamma\xi(t))\geq\varrho(\Gamma\xi(t))$ and $\sigma(\Gamma\mu(t))\geq\varrho(\Gamma\mu(t))$, and so $\alpha(\Gamma\xi)\geq\eta(\Gamma\xi)$ and $\beta(\Gamma\mu)\geq\eta(\Gamma\mu)$. Thus, Γ is η-cyclic α-admissible mapping. From stipulation (2) there subsist $\xi_1\in C^\alpha(I)$ parallel to $\alpha(\xi)\beta(\mu)\geq\eta(\xi)\eta(\mu)$. By stipulation (4), we have that for any cluster point ξ of a sequence $\{\xi_n\}$ of points in $C^\alpha(I)$ with $\rho(\xi_n)\sigma(\mu_{n+1})\geq\varrho(\xi_n)\varrho(\mu_{n+1})$ implies that $\alpha(\xi_n)\beta(\mu_{n+1})\geq\eta(\xi_n)\eta(\mu_{n+1})$ and $\lim_{n\to\infty}\inf\alpha(\xi_n)\beta(\mu_{n+1})\geq\lim_{n\to\infty}\inf\eta(\xi_n)\eta(\mu_{n+1})$. So, all affirmations of Theorem 4.2 are satisfied and then Γ has an FP $\xi^*\in C^\xi(I)$, which is a solution of the BVP (8.3). □

9. Conclusions

Four classes of symmetric fractional contractions are produced in this paper. The focus was on a new idea of symmetric fractional α-β-η-Υ-contraction pattern-I, pattern-II, pattern-III and pattern-IV in the setting of MbMS and the fifth class studied the same results on closed ball of the said space. The main results were suported by two nontrivial examples and an application for the existence and uniqueness of the bounded solution to (FODE). This paper generalized many results in the litrature.

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Conflict of interest

The authors declare no conflict of interest.

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