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Research article

New diverse types of soliton solutions to the Radhakrishnan-Kundu-Lakshmanan equation

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Abstract: The main purpose of this study was to produce abundant new types of soliton solutions for the Radhakrishnan-Kundu-Lakshmanan equation that represents unstable optical solitons that emerge from optical propagations through the use of birefringent fibers. These new types of soliton solutions have behaviors that are bright, dark, W-shaped, M-shaped, periodic trigonometric, and hyperbolic and were not realized before by any other method. These new forms have been detected by using four different techniques, which are, the extended simple equation method, the Paul-Painlevé approach method, the Ricatti-Bernoulli-sub ODE, and the solitary wave ansatz method. These new solitons will be arranged to create a soliton catalog with new impressive behaviors and they will contribute to future studies not only for this model but also for the optical propagations through birefringent fiber.

Keywords: Radhakrishnan-Kundu-Lakshmanan equation; extended simple equation method; Paul-Painlevé approach method; Ricatti-Bernoulli-sub ODE; solitary wave ansatz method; optical solitons
Mathematics Subject Classification: 35C07, 35Q51, 83C15

1. Introduction

This work focuses on using the extended simple equation method (ESEM) [1–3], the solitary wave ansatz method (SWAM) [4–6], the Paul-Painlevé approach method (PPAM) [7–10] and Ricatti-Bernoulli-sub ODE method (RBSODM) [2,11,12] to solve the Radhakrishnan-Kundu-Lakshmanan equation (RKLE) [13–17] and derive a soliton catalog for this model. The suggested model is one of the important models which perfectly treat unstable optical solitons. Recently, this has been discussed by some authors who achieved an abundance of results through a group of published articles [18–27]. Especially, more concentrated studies have been established through a few published articles considered; for example, Ghanbari and Gómez-Aguilar [28] applied the generalized exponential rational function method to obtain analytical solutions for the nonlinear RKLE. Additionally, Rehman and Ahmad [29] retrieved the optical solitons in birefringent fibers modeled by the RKLE in coupled vector form by using the extended rational sine-cosine and sinh-cosh techniques. Yıldırım et al. [30] handled dispersive solitons in birefringent fibers by using several numerical schemas, i.e., the Riccati function principle, the sine-Gordon function principle, the functional variable principle, and the F-expansion principle. Yıldırım et al. [31] recovered the bright, dark, and singular solitons of this type of model by using the modified simple function principles.

Most of the achieved results have been limited to polarization-preserving fibers; we will develop and construct new solitons to give extended study to this model. Similarly, there are other effective approaches to extract the solitary wave solutions for fractional nonlinear partial differential equations, see for example, He [32] who used the modified Riemann-Liouville derivative, fractional complex transform, exp-function method to extract the solitary wave solution to the nonlinear dispersive equations and phi-four equation. Tian and Liu [33] proposed a novel exponential rational function method to find the exact solutions for the time fractional Cahn-Allen equation and the time fractional phi-four equation. Ji et al. [34] established an approximate Hamilton principle for the transverse vibration of a reinforced concrete pillar by considering the dissipation energy, obtained a generalized Boussinesq equation, used the exp-function method to solve the equation, and discussed the solution properties.

Moreover, there are other important studies to explain how the solitary waves are affected by the unsmooth boundaries; see, for example He et al. [35] who used the fractal Korteweg-de Vries equation as an example to show the solution properties of a solitary wave traveling along an unsmooth boundary, established a fractal variational principle in a fractal space and obtained its solitary wave solution. He et al. [36] explained how the morphology of a shallow-water wave is affected by the unsmooth boundary while its peak is rarely changed they also revealed the basic properties of solitary waves in fractal space and studied the traveling solitary solution to the Boussinesq equation through the application of its fractal variational principle. Liu [37] studied the periodic solution to the fractal phi-four equation due to its strong nonlinearity. He and Yusry [38] derived the traveling wave transformation of the time-fractional Kundu-Mukherjee-Naskar equation by using the modified homotopy perturbation method. He [39] obtained both the solitary solutions and periodic solutions of the time-fractional Kundu-Mukherjee-Naskar equation by using the semi-inverse method.

Additionally, He et al. [40] suggested a Hamiltonian-based formulation to quickly determine the frequency property of the nonlinear oscillator of complex mechanical systems. Moreover, there exist recent studies to extract the soliton solutions in vector form arising in quantum space; see, for example, Zhao et al. [41] who proposed a scheme to generate stable vector spatiotemporal solitons through the use of a Rydberg electromagnetically induced transparency (EIT) system and found 3D vector monopole and vortex solitons for three nonlocal degrees. The obtained results indicate that these solitons are generated with low energy and can be propagated stably along the axes. Xu et al. [42] studied the 3D self-trapped modes in spinor Bose-Einstein condensates with a spin-orbit coupling that is described by coupled Gross-Pitaevskii equations that include beyond-mean-field Lee-Huang-Yang terms; they also checked the stability of all 3D states by using linear stability analysis and direct simulations to prove that they are stable against small perturbations in propagation on a limited scale.

Huang et al. [43] investigated the squeezing of two-component quantum optical solitons slowly moving in a tripod-type atomic system with double (EIT), finding that the quantum squeezing of vector soliton pairs is generated by a giant Kerr nonlinearity, which is provided by EIT and that the outcome of the squeezing can be optimized by the selection of the propagation distance and angle; they also obtained the atomic spin squeezing for short propagation distances.

The RKLE with the Kerr-Law of nonlinearity in the absence of the four-wave mixing (4WM) represents the basic case of fiber nonlinearity; this model according to [14,16,28–31], can be written in the following form:

$$iQ_t + aQ_{xx} + b|Q^2|Q = i\lambda(|Q|^2Q)_x - i\gamma Q_{xxx}. \tag{1.1}$$

Here, Q is the complex-valued function in terms of two independent variables x and t that represent time independence and limited components respectively. The first term denotes limited evolution, while a and b, respectively, denote the group velocity dispersion and coefficients of the Kerr-Law of nonlinearity; the parameters λ and γ which appear on the right-hand side of Eq (1.1) measure the third order dispersion. In the absence of 4WM for birefringent fibers, the RKLE with Kerr-Law of nonlinearity [30,31] (and references therein) by dividing Eq (1.1), it can be converted to the following two equations:

$$iU_t + a_1U_{xx} + [b_1|U|^2 + c_1|V|^2]U = i[\lambda_1(|U|^2U)_x + \theta_1(|V|^2U)_x] - i\gamma_1U_{xxx}, \tag{1.2}$$

$$iV_t + a_2V_{xx} + [b_2|V|^2 + c_2|U|^2]V = i[\lambda_2(|V|^2V)_x + \theta_2(|U|^2V)_x] - i\gamma_1V_{xxx}. \tag{1.3}$$

When Eq (1.2) theorizes the transformations $U(\eta) = R_j(\eta)e^{i\psi_1(x,t)}$ and Eq (1.3) theorizes the transformations $V(\eta) = R_{\bar{j}}(\eta)e^{i\psi_2(x,t)}$ with $\eta = x - vt$ and $\psi = -kx + wt + \theta_0$, where k, θ_0, and w are the frequency, phase constant, and wave number, respectively, and R_j denotes the soliton amplitude, the following real and imaginary parts will emerge:

$$(a_j + 3k\beta_j)R_j'' - (w + a_jk^2 + \beta_jk^3]R_j + (b_j - k\lambda_j)R_j^3 + c_jR_jR_{\bar{j}}^2 - k\gamma_jR_{\bar{j}}^3 = 0, \tag{1.4}$$

$$\beta_jR_j'' - (v + 2ka_j + 3\beta_jk^2]R_j' - 3\lambda_jR_j'R_j^2 - 3\gamma_jR_{\bar{j}}^2R_j' = 0. \tag{1.5}$$

When $R_j = R_{\bar{j}}$, Eqs (1.4) and (1.5) respectively become

$$(a_j + 3k\beta_j)R_j'' - (w + a_jk^2 + \beta_jk^3]R_j + (b_j - k\lambda_j + c_j - k\gamma_j)R_j^3 = 0, \tag{1.6}$$

$$\beta_jR_j'' - (v + 2ka_j + 3\beta_jk^2]R_j - (\lambda_j + \gamma_j)R_j^3 = 0. \tag{1.7}$$

Generally, these two equations are the same if and only if

$$b_i = -\frac{2k\beta_j\gamma_j + 2k\beta_j\lambda_j + a_j\gamma_j + a_j\lambda_j + \beta_jc_i}{\beta_j},$$

$$w = \frac{8k^3\beta_j^2 + 8k^2a_j\beta_j + 3kv\beta_j + 2ka_j^2 + va_j}{\beta_j}. \tag{1.8}$$

Now, we aim to concentrate on solving Eqs (1.6) and (1.7).

2. ESEM

To describe the ESEM, let us first propose the general form of a nonlinear partial differential equation which is

$$\Psi\left(R_j, \left(R_j\right)_x, \left(R_j\right)_t, \left(R_j\right)_{xx}, \left(R_j\right)_{tt}, \cdots\right) = 0. \tag{2.1}$$

Here, Ψ is in terms of $R(x,t)$ and its partial derivatives that include the highest order derivatives and nonlinear terms; when Eq (2.1) theorizes the transformation $R_j(x,t) = R_j(\eta)$ with $\eta = x - vt$ it will be converted to the following ordinary differential equation (ODE):

$$\Phi\left(R_j', R_j'', R_j''', \cdots\right) = 0. \tag{2.2}$$

Here, Φ is in terms of $R_j(\eta)$ and its total derivatives.

The form of the solution in the framework of this method is as follows:

$$R_j(\eta) = \sum_{j=-M}^{M} A_j \varphi^j(\eta). \tag{2.3}$$

The integer M appearing in Eq (2.3) can be calculated by theorizing Eq (2.3) to the homogeneous balance between the highest order derivative term and the nonlinear term, while the arbitrary constants A_j can be defined later; the function $\varphi(\eta)$ realizes the following equation:

$$\varphi'(\zeta) = a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3. \tag{2.4}$$

Here, a_0, a_1, a_2, and a_3 are arbitrary parameters through which the following two cases [44–46] are extracted:

Case 1. If $a_1 = a_3 = 0$, Eq (2.4) becomes the Riccati equation, which has the following solutions:

$$\varphi(\eta) = \frac{\sqrt{a_0 a_2}}{a_2}\tan\left(\sqrt{a_0 a_2}(\eta + \eta_0)\right), a_0 a_2 > 0, \tag{2.5}$$

$$\varphi(\eta) = \frac{\sqrt{-a_0 a_2}}{a_2}\tanh\left(\sqrt{-a_0 a_2}\eta - \frac{\rho \ln \eta_0}{2}\right), a_0 a_2 < 0, \eta > 0, \rho = \mp 1. \tag{2.6}$$

Case 2. If $a_0 = a_3 = 0$, Eq (2.4) becomes the Bernoulli equation, which has the following solutions:

$$\varphi(\eta) = \frac{a_1 \operatorname{Exp}\left(a_1(\eta + \eta_0)\right)}{1 - a_1 \operatorname{Exp}\left(a_1(\eta + \eta_0)\right)}, a_1 > 0, \tag{2.7}$$

$$\varphi(\eta) = \frac{-a_1 \operatorname{Exp}\left(a_1(\eta + \eta_0)\right)}{1 + a_2 \operatorname{Exp}\left(a_1(\eta + \eta_0)\right)}, a_1 < 0. \tag{2.8}$$

Thus, the general solution to the ansatz Eq (2.4), is as follows:

$$\varphi(\eta) = -\frac{1}{a_2}\left(a_1 - \sqrt{4a_1 a_2 - a_1^2}\tan\left(\frac{\sqrt{4a_1 a_2 - a_1^2}}{2}(\eta + \eta_0)\right)\right),\ 4a_1 a_2 > a_1^2,\ a_2 > 0. \tag{2.9}$$

$$\varphi(\eta) = \frac{1}{a_2}\left(a_1 + \sqrt{4a_1 a_2 - a_1^2}\tanh\left(\frac{\sqrt{4a_1 a_2 - a_1^2}}{2}(\eta + \eta_0)\right)\right),\ 4a_1 a_2 > a_1^2,\ a_2 < 0. \tag{2.10}$$

Here, η_0 denotes the integration constancy.

Finally, by combining Eqs (2.3) and (2.4) we can determine the unknown constants through the extracted system by equating the coefficients of various powers of φ^j to zero. Then, the target solution can be obtained by substituting these achieved constants into Eq (2.4).

Now, we will apply the ESEM to extract new types of solitons for the RKLE of Eq (1.6) as follows:

$$\left(a_j+3k\beta_j\right)R_j''-\left(w+a_jk^2+\beta_jk^3\right]R_j+\left(b_j-k\lambda_j+c_j-k\gamma_j\right)R_j^3=0; \tag{2.11}$$

by implementing the balance between R_j'' and R_j^3 appearing in Eq (2.11) leads to $3M=2M+1$ implying that $M=1$; hence, according to the suggested method, the solution of Eq (1.6) is as follows:

$$R_j(\zeta)=\frac{A_{-1}}{\varphi}+A_0+A_1\varphi. \tag{2.12}$$

Here, $\varphi'=a_0+a_1\varphi+a_2\varphi^2+a_3\varphi^3$.

Case 1. Regarding the first family in which $a_1=a_3=0\Rightarrow\varphi'=a_0+a_2\varphi^2$, we have the following:

$$R_j'=-\frac{a_0A_{-1}}{\varphi^2}+A_1a_0+A_1a_2\varphi^2-a_2A_{-1}. \tag{2.13}$$

$$R_j''=\frac{2a_0^2A_{-1}}{\varphi^3}+\frac{2a_0a_2A_{-1}}{\varphi}+2A_1a_0a_2\varphi+2A_1a_2^2\varphi^3. \tag{2.14}$$

$$R_j^2=A_1^2\varphi^2+2A_0A_1\varphi+\left(A_0^2+2A_{-1}A_1\right)+\frac{A_{-1}^2}{\varphi^2}+\frac{2A_{-1}A_0}{\varphi}. \tag{2.15}$$

$$R_j^3=A_1^3\varphi^3+3A_0A_1^2\varphi^2+3\left(A_1A_0^2+A_{-1}A_1^2\right)\varphi+\left(A_0^3+6A_{-1}A_0A_1\right)+\frac{A_{-1}^3}{\varphi^3}+\frac{3A_0A_{-1}^2}{\varphi^2}+3\,\frac{A_{-1}A_0^2+A_1A_{-1}^2}{\varphi}. \tag{2.16}$$

By combining Eqs (2.11)–(2.16) and collecting and equating the coefficients of various powers of φ^i to zero, we derive the system of equations, and, by solving this system, we get the following eight acceptable results:

$$A_0=0, a_0=-iA_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}, a_2=iA_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}},$$
$$w=a_jk^2+4b_jA_{-1}A_1+4c_jA_{-1}A_1-k^3\beta_j-4kA_1A_{-1}-4kA_1A_{-1}\gamma_j. \tag{2.17}$$

$$A_0=0, a_0=-iA_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}, a_2=-iA_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}},$$
$$w=a_jk^2+2b_jA_{-1}A_1+2c_jA_{-1}A_1-k^3\beta_j-2kA_1A_{-1}-2kA_1A_{-1}\gamma_j. \tag{2.18}$$

$$A_0=0, a_0=iA_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}, a_2=iA_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}},$$
$$w=a_jk^2+2b_jA_{-1}A_1+2c_jA_{-1}A_1-k^3\beta_j-2kA_1A_{-1}-2kA_1A_{-1}\gamma_j. \tag{2.19}$$

$$A_0=0, a_0=-iA_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}, a_2=-iA_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}},$$
$$w=a_jk^2+4b_jA_{-1}A_1+4c_jA_{-1}A_1-k^3\beta_j-4kA_1A_{-1}-4kA_1A_{-1}\gamma_j. \tag{2.20}$$

$$A_0=A_1=0, a_0=-iA_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}},$$
$$w=-a_jk^2-k^3\beta_j-2ika_2a_jA_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}-6ika_2\beta_jA_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}. \tag{2.21}$$

$$A_0 = A_1 = 0, a_0 = iA_{-1}\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{2a_j + 6k\beta_j}},$$

$$w = -a_j k^2 - k^3\beta_j + 2ika_2 a_j A_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}} + 6ika_2\beta_j A_{-1}\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}. \quad (2.22)$$

$$A_0 = A_{-1} = 0, a_2 = iA_1\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{2a_j + 6k\beta_j}},$$

$$w = -a_j k^2 - k^3\beta_j + 2ika_0 a_j A_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}} + 6ika_0\beta_j A_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}. \quad (2.23)$$

$$A_0 = A_{-1} = 0, a_2 = -iA_1\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{2a_j + 6k\beta_j}},$$

$$w = -a_j k^2 - k^3\beta_j - 2ika_0 a_j A_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}} - 6ika_0\beta_j A_1\sqrt{\frac{b_j+c_j-k\gamma_j-k\lambda_j}{2a_j+6k\beta_j}}. \quad (2.24)$$

These eight different results will generate eight different solutions; for simplicity, we will study only the first and the second results.

Consider Eq (2.17), i.e., the first result:

$$A_0 = 0, a_0 = -iA_{-1}\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{2a_j + 6k\beta_j}}, a_2 = iA_1\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{2a_j + 6k\beta_j}},$$

$$w = a_j k^2 + 4b_j A_{-1}A_1 + 4c_j A_{-1}A_1 - k^3\beta_j - 4kA_1A_{-1} - 4kA_1A_{-1}\gamma_j.$$

This result can be simplified to be

$$A_0 = 0, v = A_1 = A_{-1} = a_j = b_j = c_j = \gamma_j = \lambda_j = 1, k = -1,$$

$$a_0 = 1,\ a_2 = -1,\ w = -3,\ \theta_0 = 0.1,\ w = 18. \quad (2.25)$$

According to the proposed method, the solution is as follows:

$$\varphi(\eta) = \frac{\sqrt{-a_0 a_2}}{a_2}\tanh\left(\sqrt{-a_0 a_2}\eta - \frac{\rho\ln\eta_0}{2}\right), a_0 a_2 < 0, \eta > 0, \rho = \mp 1,$$

$$\varphi(\eta) = -\tanh(x - t + 0.3), \quad (2.26)$$

$$R_j(\eta) = \frac{A_{-1}}{\varphi} + A_0 + A_1\varphi,$$

$$R_j(\eta) = -(\coth[x - t + 0.3] + \tanh[x - t + 0.3]). \quad (2.27)$$

The solution of the original equation is as follows:

$$v(x,t) = R_j(\eta)e^{i\Psi(x,t)}, \eta = x - vt, \Psi = -kx + wt + \theta_0,$$

$$v(x,t) = -(\coth[x - t + 0.3] + \tanh[x - t + 0.3])e^{i(x+18t+0.1)}, \quad (2.28)$$

$$ReV(x,t) = -(\coth[x - t + 0.3] + \tanh[x - t + 0.3])\cos(x + 18t + 0.1), \quad (2.29)$$

$$ImV(x,t) = -(\coth[x - t + 0.3] + \tanh[x - t + 0.3])\sin(x + 18t + 0.1). \quad (2.30)$$

As a result Figures 1 and 2 show the solitons for the RKLE of Eqs (2.29) and (2.30), respectively, in 2D and 3D.

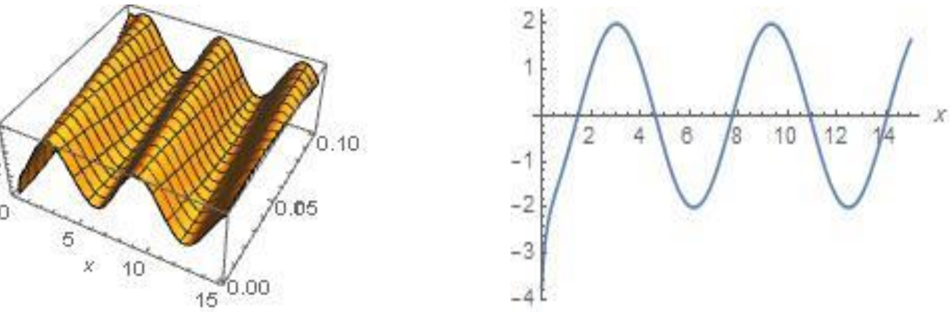

Figure 1. Soliton for the RKLE of Eq (2.29) in 2D and 3D with the following values: $A_0 = 0,\ \nu = A_1 = A_{-1} = a_j = b_j = c_j = \gamma_j = \lambda_j = 1,\ k = -1,\ a_0 = 1,\ a_2 = -1,$ $\theta_0 = 0.1,\ w = 18,\ \rho = -1,\ \eta_0 = 2.$

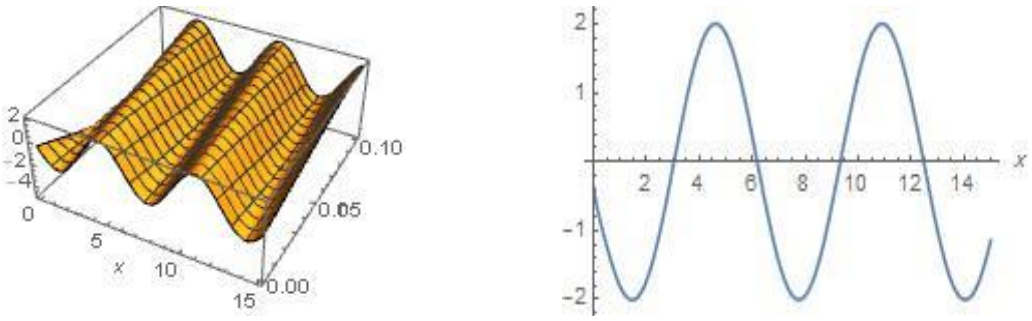

Figure 2. Soliton for the RKLE of Eq (2.30) in 2D and 3D with the following values: $A_0 = 0,\ \nu = A_1 = A_{-1} = a_j = b_j = c_j = \gamma_j = \lambda_j = 1,\ k = -1,\ a_0 = 1,\ a_2 = -1,$ $\theta_0 = 0.1,\ w = 18,\ \rho = -1,\ \eta_0 = 2.$

Consider Eq (2.18) as follows:

$$A_0 = 0, a_0 = -iA_{-1}\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{2a_j + 6k\beta_j}}, a_2 = -iA_1\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{2a_j + 6k\beta_j}},$$

$$w = a_j k^2 + 2b_j A_{-1}A_1 + 2c_j A_{-1}A_1 - k^3\beta_j - 2kA_1A_{-1} - 2kA_1A_{-1}\gamma_j. \tag{2.31}$$

This result can be simplified to be

$$A_0 = 0, \nu = A_1 = A_{-1} = a_j = b_j = c_j = \gamma_j = \lambda_j = 1,$$

$$k = -1, a_0 = a_2 = 1,\ \theta_0 = 0.1,\ w = 10. \tag{2.32}$$

The solution in the framework of this result is as follows:

$$\varphi(\eta) = \frac{\sqrt{a_0a_2}}{a_2}\tan\left(\sqrt{a_0a_2}(\eta + \eta_0)\right), a_0a_2 > 0,$$

$$\varphi(\eta) = \tan[x - t + 1], \tag{2.33}$$

$$R_j(\zeta) = \tan[x - t + 1] + \cot[x - t + 1], \tag{2.34}$$

$$V(x,t) = (\tan[x - t + 1] + \cot[x - t + 1])e^{i(x+10t+0.1)}, \tag{2.35}$$

$$ReV(x,t) = (\tan[\, x - t + 1] + \cot[\, x - t + 1]) \cos(\, x + 10t + 0.1), \tag{2.36}$$

$$ImV(x,t) = (\tan[\, x - t + 1] + \cot[\, x - t + 1]) \sin(x + 10t + 0.1). \tag{2.37}$$

As a result Figures 3 and 4 show the solitons for the RKLE of Eqs (2.36) and (2.37), respectively, in 2D and 3D.

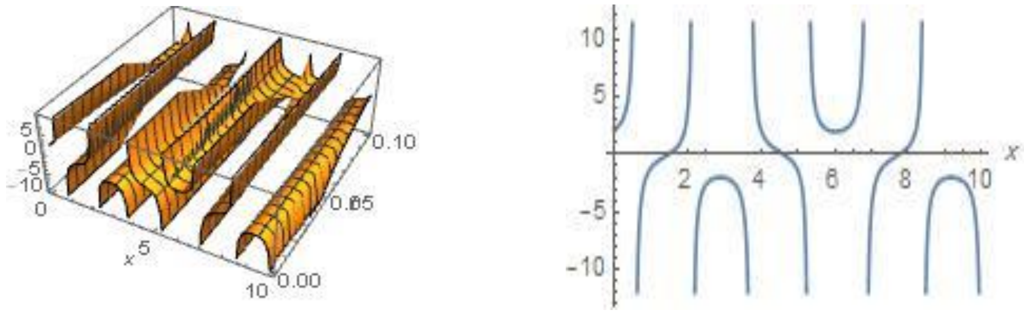

Figure 3. Soliton for the RKLE of Eq (2.36) in 2D and 3D with the following values: $A_0 = 0,\ \nu = A_1 = A_{-1} = a_j = b_j = c_j = \gamma_j = \lambda_j = 1,\ k = -1,\ a_0 = a_2 = 1,\ \theta_0 = 0.1,\ w = 10,\ \eta_0 = 1.$

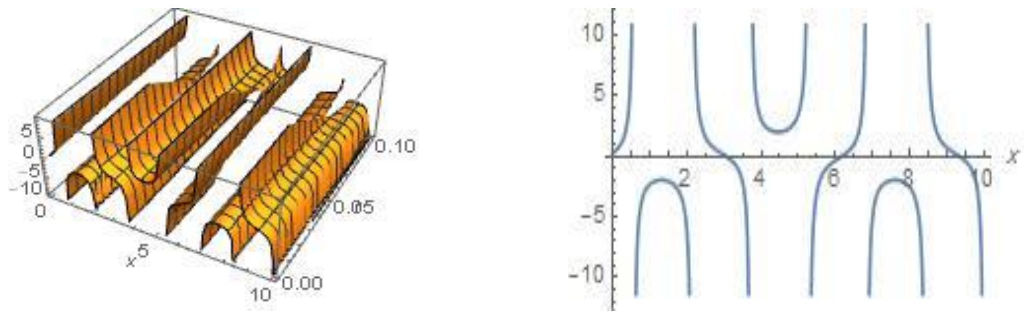

Figure 4. Soliton for the RKLE of Eq (2.37) in 2D and 3D with the following values: $A_0 = 0,\ \nu = A_1 = A_{-1} = a_j = b_j = c_j = \gamma_j = \lambda_j = 1,\ k = -1,\ a_0 = a_2 = 1,\ \theta_0 = 0.1,\ w = 10,\ \eta_0 = 1.$

In the same manner, we can design and extract the solutions corresponding to the remaining results.

Case 2. Regarding the second family in which $a_0 = a_3 = 0 \Rightarrow \varphi' = a_1\varphi + a_2\varphi^2$, we have

$$R_j' = A_1 a_2 \varphi^2 + a_1 A_1 \varphi - \frac{A_{-1}a_1}{\varphi} - A_{-1}a_2, \tag{2.38}$$

$$R_j'' = 2A_1 a_2^2 \varphi^3 + 3A_1 a_1 a_2 \varphi^2 + A_1 a_1^2 \varphi + A_{-1} a_1 a_2 + \frac{a_1^2 A_{-1}}{\varphi}, \tag{2.39}$$

$$R_j^2 = A_1^2 \varphi^2 + 2A_0 A_1 \varphi + (A_0^2 + 2A_{-1}A_1) + \frac{A_{-1}^2}{\varphi^2} + \frac{2A_{-1}A_0}{\varphi}, \tag{2.40}$$

$$R_j^3 = A_1^3 \varphi^3 + 3A_0 A_1^2 \varphi^2 + 3(A_1 A_0^2 + A_{-1} A_1^2)\varphi + (A_0^3 + 6A_{-1}A_0A_1) + \frac{A_{-1}^3}{\varphi^3} + \frac{3A_0 A_{-1}^2}{\varphi^2} + \frac{3\left(A_{-1}A_0^2 + A_1 A_{-1}^2\right)}{\varphi}. \tag{2.41}$$

By combining Eqs (2.14) and (2.38)–(2.41) and collecting and equating the coefficients of various powers of φ^i to zero, we derive the system of equations and by solving this system we get only two acceptable results while the remaining results are refused given either $a_1 = 0,\ A_1 = A_{-1} = 0$ or both.

$$a_1 = i\sqrt{2}A_0\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{a_j + 3k\beta_j}}, a_2 = -\frac{A_1(b_j + c_j - k\gamma_j - k\lambda_j)\sqrt{a_j + 3k\beta_j}}{i\sqrt{b_j + c_j - k\gamma_j - k\lambda_j}(\sqrt{2}a_j + 3\sqrt{2}k\beta_j)},$$

$$A_{-1} = 0,\ w = -a_jk^2 + b_jA_0^2 + c_jA_0^2 - k^3\beta_j - kA_0^2\gamma_j - kA_0^2\lambda_j. \tag{2.42}$$

$$a_1 = -i\sqrt{2}A_0\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{a_j + 3k\beta_j}}, a_2 = \frac{A_1(b_j + c_j - k\gamma_j - k\lambda_j)\sqrt{a_j + 3k\beta_j}}{i\sqrt{b_j + c_j - k\gamma_j - k\lambda_j}(\sqrt{2}a_j + 3\sqrt{2}k\beta_j)},$$

$$A_{-1} = 0, w = -a_jk^2 + b_jA_0^2 + c_jA_0^2 - k^3\beta_j - kA_0^2\gamma_j - kA_0^2\lambda_j. \tag{2.43}$$

These two different results will produce two different solutions; for similarity and simplicity, we will study only one of them.

Consider the first result, which is as follows:

$$a_1 = i\sqrt{2}A_0\sqrt{\frac{b_j + c_j - k\gamma_j - k\lambda_j}{a_j + 3k\beta_j}}, a_2 = -\frac{A_1(b_j + c_j - k\gamma_j - k\lambda_j)\sqrt{a_j + 3k\beta_j}}{i\sqrt{b_j + c_j - k\gamma_j - k\lambda_j}(\sqrt{2}a_j + 3\sqrt{2}k\beta_j)},$$

$$A_{-1} = 0,\ w = -a_jk^2 + b_jA_0^2 + c_jA_0^2 - k^3\beta_j - kA_0^2\gamma_j - kA_0^2\lambda_j. \tag{2.44}$$

This result can be simplified to be

$$A_{-1} = 0, \nu = A_0 = A_1 = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1,$$

$$k = -1,\ a_1 = 2,\ a_2 = 1,\ \theta_0 = 0.1,\ w = 4,\ \eta_0 = 1. \tag{2.45}$$

The solution in the framework of this result is as follows:

$$\varphi(\eta) = \frac{2\mathrm{Exp}[2x-2t+2]}{1-\mathrm{Exp}[2x-2t+2]}, \tag{2.46}$$

$$R_j(\zeta) = 1 + \frac{2\mathrm{Exp}[2x-2t+2]}{1-\mathrm{Exp}[2x-2t+2]}, \tag{2.47}$$

$$V(x,t) = \left\{1 + \frac{2\mathrm{Exp}[2x-2t+2]}{1-\mathrm{Exp}[2x-2t+2]}\right\}e^{i(x+4t+0.1)}, \tag{2.48}$$

$$ReV(x,t) = \left\{1 + \frac{2\mathrm{Exp}[2x-2t+2]}{1-\mathrm{Exp}[2x-2t+2]}\right\}\cos(x + 4t + 0.1), \tag{2.49}$$

$$ImV(x,t) = \left\{1 + \frac{2\mathrm{Exp}[2x-2t+2]}{1-\mathrm{Exp}[2x-2t+2]}\right\}\sin(x + 4t + 0.1). \tag{2.50}$$

As a result Figures 5 and 6 show the solitons for the RKLE of Eqs (2.49) and (2.50), respectively, in 2D and 3D.

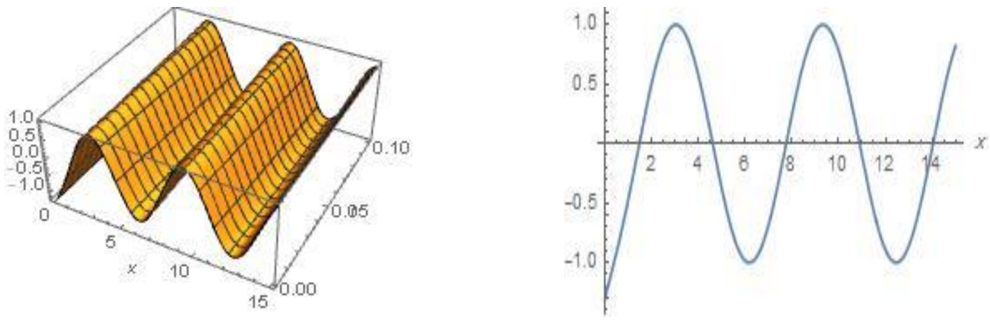

Figure 5. Soliton for the RKLE of Eq (2.49) in 2D and 3D with the following values: $A_{-1} = 0,\ \nu = A_0 = A_1 = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1,\ k = -1,\ a_1 = 2,\ a_2 = 1,$ $\theta_0 = 0.1,\ w = 4,\ \eta_0 = 1.$

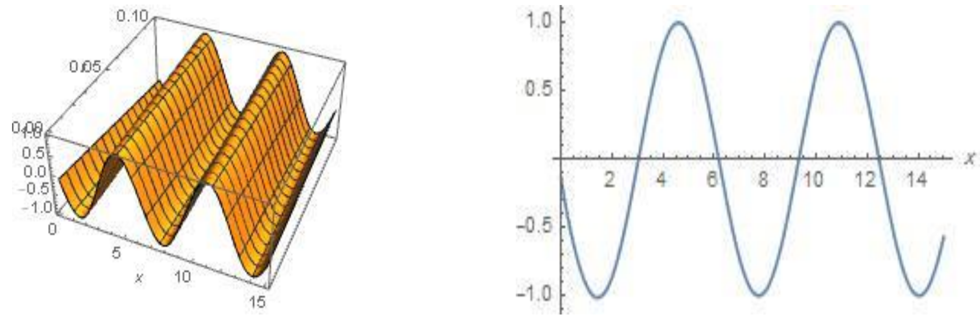

Figure 6. Soliton for the RKLE of Eq (2.50) in 2D and 3D with the following values: $A_{-1} = 0,\ \nu = A_0 = A_1 = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1,\ k = -1,\ a_1 = 2,\ a_2 = 1,$ $\theta_0 = 0.1,\ w = 4,\ \eta_0 = 1.$

3. PPAM

The exact solution for Eq (2.2) in the framework of the PPAM [7–10] can be offered as follows:

$$R_j(\eta) = A_0 + A_1 S e^{-N\eta}, \chi = \phi(\eta) = C_1 - \frac{e^{-N\eta}}{N}. \tag{3.1}$$

Here, $S(\chi)$ in Eq (3.1) realizes the Riccati-equation and takes the form $S_\chi - AS^2 = 0$, which has the solution $S(\chi) = \frac{1}{A\chi + \chi_0}$. Anyhow, simple calculations of Eq (3.1) imply the following relations:

$$R_j^{'} = -NA_1 e^{-N\eta} S - AA_1 e^{-2N\eta} S^2, \tag{3.2}$$

$$R_j^{''} = N^2 A_1 e^{-N\eta} S + 3NAA_1 e^{-2N\eta} S^2 + 2A^2 A_1 e^{-3N\eta} S^3, \tag{3.3}$$

$$R_j^3 = A_0^3 + 3A_0^2 A_1 e^{-N\eta} S + 3A_0 A_1^2 e^{-2N\eta} S^2 + A_1^3 e^{-3N\eta} S^3. \tag{3.4}$$

Now, we will apply the PPAM to construct the solitons for the RKLE of Eq (1.6) whose balance is $M = 1$. Anyhow, by inserting $R_j^{''}$, R_j, and R_j^3 into Eq (1.6), we get

$$\left(a_j + 3k\beta_j\right)(N^2 A_1 e^{-N\eta} S + 3NAA_1 e^{-2N\eta} S^2 + 2A^2 A_1 e^{-3N\eta} S^3)$$
$$-(w + a_j k^2 + \beta_j k^3](A_0 + A_1 S e^{-N\eta}) + (b_j - k\lambda_j + c_j - k\gamma_j)(A_0 + A_1 S e^{-N\eta})^3 = 0. \tag{3.5}$$

The equivalence of various powers of $Se^{-N\eta}$ in Eq (3.5) leads to the following results:

$$A_0=\sqrt{\frac{-v-k^2a_j+k^3\beta_j}{-b_j-c_j+k\gamma_j+k\lambda_j}},\ A=iA_1\frac{\sqrt{b_j+c_j-k\gamma_j-k\lambda_j}}{\sqrt{2a_j+6k\beta_j}},\ N=\frac{A_0A_1(-b_j-c_j+k\gamma_j+k\lambda_j)}{A(a_j+3k\beta_j)}, \tag{3.6}$$

$$A_0=\sqrt{\frac{-v-k^2a_j+k^3\beta_j}{-b_j-c_j+k\gamma_j+k\lambda_j}},\ A=-iA_1\frac{\sqrt{b_j+c_j-k\gamma_j-k\lambda_j}}{\sqrt{2a_j+6k\beta_j}},\ N=-\frac{A_0A_1(-b_j-c_j+k\gamma_j+k\lambda_j)}{A(a_j+3k\beta_j)}, \tag{3.7}$$

$$A_0=-\sqrt{\frac{-v-k^2a_j+k^3\beta_j}{-b_j-c_j+k\gamma_j+k\lambda_j}},\ A=iA_1\frac{\sqrt{b_j+c_j-k\gamma_j-k\lambda_j}}{\sqrt{2a_j+6k\beta_j}},\ N=\frac{A_0A_1(b_j+c_j-k\gamma_j-k\lambda_j)}{A(a_j+3k\beta_j)}, \tag{3.8}$$

$$A_0=-\sqrt{\frac{-v-k^2a_j+k^3\beta_j}{-b_j-c_j+k\gamma_j+k\lambda_j}},\ A=-iA_1\frac{\sqrt{b_j+c_j-k\gamma_j-k\lambda_j}}{\sqrt{2a_j+6k\beta_j}},\ N=-\frac{A_0A_1(b_j+c_j-k\gamma_j-k\lambda_j)}{A(a_j+3k\beta_j)}. \tag{3.9}$$

Anyhow, we will achieve four different solutions; for simplicity, we will study only the first one considering the following values:

$$A=w=v=\chi_0=A_1=a_j=b_j=c_j=\gamma_j=\beta_j=\lambda_j=1,$$
$$A_0=0.9,\ \theta_0=0.1,\ k=-1,\ N=1.8. \tag{3.10}$$

The solution obtained via the PPAM approach is as follows:

$$R_j(\eta)=A_0+\frac{A_1e^{-N\eta}}{A\left(1-\frac{e^{-N\eta}}{N}\right)+\chi_0}, \tag{3.11}$$

$$R_j(\eta)=0.9+\frac{1.8e^{-1.8(x-vt)}}{2.8-e^{-1.8(x-vt)}}, \tag{3.12}$$

$$V(x,t)=R(\eta)e^{i\psi(x,t)},$$

$$V(x,t)=\left(0.9+\frac{1.8e^{-1.8(x-vt)}}{2.8-e^{-1.8(x-vt)}}\right)e^{i(-kx+wt+\theta_0)}, \tag{3.13}$$

$$ReV(x,t)=\left(0.9+\frac{1.8e^{-1.8(x-t)}}{2.8-e^{-1.8(x-t)}}\right)\cos(x+t+0.1), \tag{3.14}$$

$$ImV(x,t)=\left(0.9+\frac{1.8e^{-1.8(x-t)}}{2.8-e^{-1.8(x-t)}}\right)\sin(x+t+0.1). \tag{3.15}$$

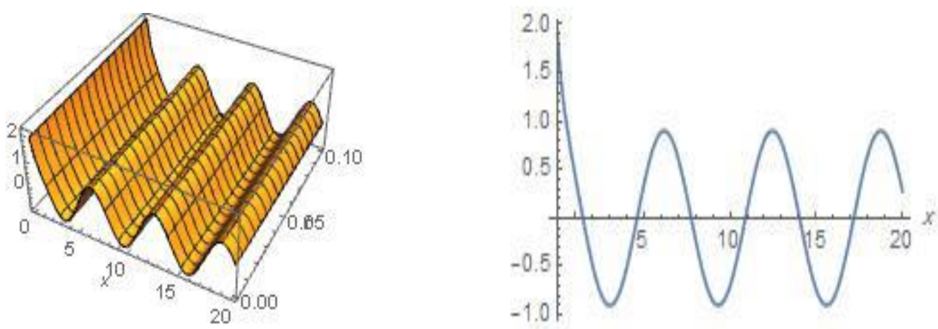

Figure 7. Soliton for the RKLE of Eq (3.14) in 2D and 3D with the following values: $A=w=v=\chi_0=A_1=a_j=b_j=c_j=\gamma_j=\beta_j=\lambda_j=1,\ A_0=0.9,\ \theta_0=0.1,\ k=-1, N=1.8$.

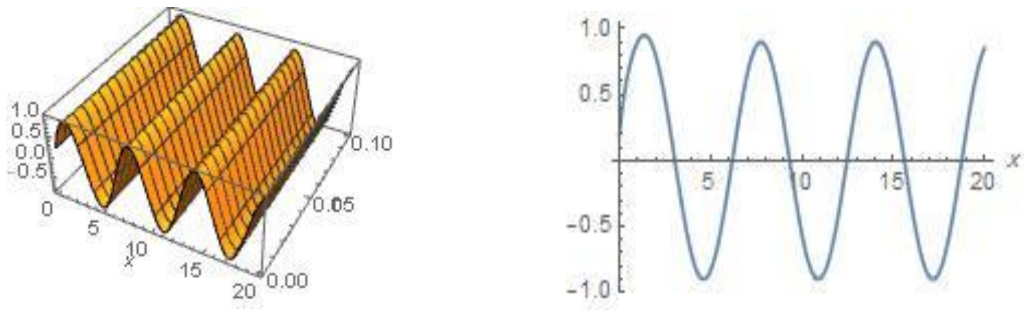

Figure 8. Soliton for the RKLE of Eq (3.15) in 2D and 3D with the following values: $A = w = \nu = \chi_0 = A_1 = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1,\ A_0 = 0.9,\ \theta_0 = 0.1,\ k = -1,$ $N = 1.8.$

As a result Figures 7 and 8 show the solitons for the RKLE of Eqs (3.14) and (3.15), respectively, in 2D and 3D.

4. RBSODM

This method can be summarized in the form of the following steps:

Step 1. Set

$$R_j' = aR_j^{2-s} + bR_j + cR_j^s, \tag{4.1}$$

in which $a, b, c,$ and s are constants to be determined.

Step 2. From Eq (4.1), we get

$$R_j'' = ab(3-s)R_j^{2-s} + a^2(2-s)R_j^{3-2s} + sc^2R_j^{2s-1} + bc(s+1)R_j^s + (2ac+b^2)R_j. \tag{4.2}$$

Remark 4.1. When $ac \neq 0$ and $s = 0$, Eq (4.1) tends to a Riccati, while if $a \neq 0,\ c = 0$ and $s \neq 1$; Eq (4.1) becomes a Bernoulli. Clearly, the Riccati and Bernoulli equations are singular cases of Eq (4.1). Since Eq (4.1) is initially planned, we call Eq (4.1) the RBSODM to avoid presenting a new terminology. Anyhow, Eq (4.1) admits the following forms of solutions (Here, $C_1 \in \mathbb{R}$):

Case 1. When $s = 1$, the solution is $R_j(\eta) = C_1 e^{(a+b+c)\eta}$.

Case 2. When $s \neq 1,\ b = 0$, and $c = 0$ the solution is $R_j(\eta) = \big(a(s-1)(\eta + C_1)\big)^{\frac{1}{(1-s)}}$.

Case 3. When $s \neq 1,\ b \neq 0$, and $c = 0$ the solution is $R_j(\eta) = \left(-\frac{a}{b} + C_1 e^{b(s-1)\eta}\right)^{\frac{1}{(s-1)}}$.

Case 4. When $s \neq 1,\ a \neq 0$, and $b^2 - 4ac < 0$ the solutions are

$$R_j(\eta) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\left(\frac{(1-s)\sqrt{4ac-b^2}}{2}\right)(\eta + C_1)\right)^{\frac{1}{(1-s)}},$$

$$R_j(\eta) = \left(\frac{-b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\cot\left(\frac{(1-s)\sqrt{4ac-b^2}}{2}\right)(\eta + C_1)\right)^{\frac{1}{(1-s)}}.$$

Case 5. When $s \neq 1,\ a \neq 0$, and $b^2 - 4ac > 0$ the solutions are

$$R_j(\eta)=\left(\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\coth\left(\frac{(1-s)\sqrt{b^2-4ac}}{2}\right)(\eta+C_1)\right)^{\frac{1}{(1-s)}},$$

$$R_j(\eta)=\left(\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\tanh\left(\frac{(1-s)\sqrt{b^2-4ac}}{2}\right)(\eta+C_1)\right)^{\frac{1}{(1-s)}}.$$

Case 6. When $s\neq 1$, $a\neq 0$, and $b^2-4ac=0$, the solution is

$$R_j(\eta)=\left(\frac{1}{A(s-1)(\eta+C_1)}-\frac{B}{2A}\right)^{\frac{1}{1-s}}. \tag{4.3}$$

Step 3. By inserting the derivatives of R into Eq (4.1), we get an algebraic equation in P. Using the symmetry of the right-hand side of Eq (4.1) and setting the highest power exponents P, we can determine the value(s) of s. Anyhow, the equivalence of coefficients of P^i implies a set of algebraic equations in a, b, c, and C_1. By solving this system and utilizing $\eta=x-\nu t$ and using one of the above forms of solutions of Eq (4.1), the traveling wave solutions of Eq (2.2) will be achieved.

Remark 4.2. The Bäcklund transformation concerning the RBSODM will generate an infinite sequence of solutions as follows [2]:

$$R_s(\eta)=\left(\frac{-cD_1+aD_2(R_{s-1}(\eta))^{1-\varepsilon}}{bD_1+aD_2+aD_1(R_{s-1}(\eta))^{1-\varepsilon}}\right)^{\frac{1}{1-\varepsilon}}. \tag{4.4}$$

Here, we will apply the RBSODM to extract the solitons of the RKLE by inserting Eq (4.2) into Eq (1.6). Anyhow, we get

$$\left(a_j+3k\beta_j\right)\left(ab(3-s)R_j^{2-s}+a^2(2-s)R_j^{3-2s}+sc^2R_j^{2s-1}+bc(s+1)R_j^s+(2ac+b^2)R_j\right)$$
$$-(w+a_jk^2+\beta_jk^3]R_j+(b_j-k\lambda_j+c_j-k\gamma_j)R_j^3=0. \tag{4.5}$$

Substituting $s=0$, we obtain

$$\left(a_j+3k\beta_j\right)\left(3abR_j^2+2a^2R_j^3+bc+(2ac+b^2)R_j\right)$$
$$-(w+a_jk^2+\beta_jk^3)R_j+(b_j-k\lambda_j+c_j-k\gamma_j)R_j^3=0. \tag{4.6}$$

By achieving the equivalence process between the various powers of R_j^i, we get the following system:

$$2a^2(a_j+3k\beta_j)+(b_j-k\lambda_j+c_j-k\gamma_j)=0,$$
$$3ab(a_j+3k\beta_j)=0,$$
$$(a_j+3k\beta_j)(2ac+b^2)-(w+a_jk^2+\beta_jk^3)=0,$$
$$3bc(a_j+3k\beta_j)=0. \tag{4.7}$$

The second and the fourth parts of Eq (4.7) imply that $b=0$, while the first and the third parts lead to the following results:

$$2a^2(a_j+3k\beta_j)+(b_j-k\lambda_j+c_j-k\gamma_j)=0$$

$$(a_j + 3k\beta_j)(2ac + b^2) - (w + a_j k^2 + \beta_j k^3) = 0. \tag{4.8}$$

In another formation, one can write

$$a = \pm\sqrt{\frac{k\lambda_j + k\gamma_j - b_j - c_j}{2(a_j + 3k\beta_j)}},\ c = \frac{w + a_j k^2 + \beta_j k^3}{2a(a_j + 3k\beta_j)},\ b = 0. \tag{4.9}$$

This generates the following different solutions:

$$a = \sqrt{\frac{k\lambda_j + k\gamma_j - b_j - c_j}{2(a_j + 3k\beta_j)}},\ c = \frac{w + a_j k^2 + \beta_j k^3}{2a(a_j + 3k\beta_j)}, b = 0. \tag{4.10}$$

$$a = \sqrt{\frac{k\lambda_j + k\gamma_j - b_j - c_j}{2(a_j + 3k\beta_j)}},\ c = -\frac{w + a_j k^2 + \beta_j k^3}{2a(a_j + 3k\beta_j)}, b = 0. \tag{4.11}$$

$$a = -\sqrt{\frac{k\lambda_j + k\gamma_j - b_j - c_j}{2(a_j + 3k\beta_j)}},\ c = \frac{w + a_j k^2 + \beta_j k^3}{2a(a_j + 3k\beta_j)}, b = 0. \tag{4.12}$$

$$a = -\sqrt{\frac{k\lambda_j + k\gamma_j - b_j - c_j}{2(a_j + 3k\beta_j)}},\ c -= \frac{w + a_j k^2 + \beta_j k^3}{2a(a_j + 3k\beta_j)}, b = 0. \tag{4.13}$$

For simplicity, we choose only the first result and derive the following corresponding solution:

$$a = \sqrt{\frac{k\lambda_j + k\gamma_j - b_j - c_j}{2(a_j + 3k\beta_j)}}, c = \frac{w + a_j k^2 + \beta_j k^3}{2a(a_j + 3k\beta_j)}, b = 0.$$

Anyhow, one can find the following results:

$$a = \pm 1, c = \pm\frac{1}{4}, \qquad b = 0, w = \nu = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1, \\ \theta_0 = 0.1, k = -1. \tag{4.14}$$

From Eq (4.14), we have four sub-results and among them, we will study the following case:

$$a = 1, c = 0.25, b = 0, w = \nu = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1, \\ \theta_0 = 0.1,\ k = -1. \tag{4.15}$$

This result will achieve two forms of solutions of Eq (1.4) as follows:

$$R_j(\eta) = \frac{-b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}\tan\left(\frac{\sqrt{4ac - b^2}}{2}\right)(\eta + C_1),$$

$$R_j(\eta) = \frac{-b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}\cot\left(\frac{\sqrt{4ac - b^2}}{2}\right)(\eta + C_1).$$

The first part can be written as follows:

$$R_j(\eta) = 0.5\tan 0.5(\eta + C_1), \tag{4.16}$$

$$V(x,t) = \left(0.5\tan 0.5(x - t + 1)\right)e^{i(x+t+0.1)}, \tag{4.17}$$

$$ReV(x,t) = \left(0.5\tan 0.5(x - t + 1)\right)\cos(x + t + 0.1), \tag{4.18}$$

$$ImV(x,t) = (0.5\tan 0.5(x - t + 1))\sin(x + t + 0.1). \tag{4.19}$$

As a result Figures 9 and 10 show the solitons for the RKLE of Eqs (4.17) and (4.18), respectively, in 2D and 3D.

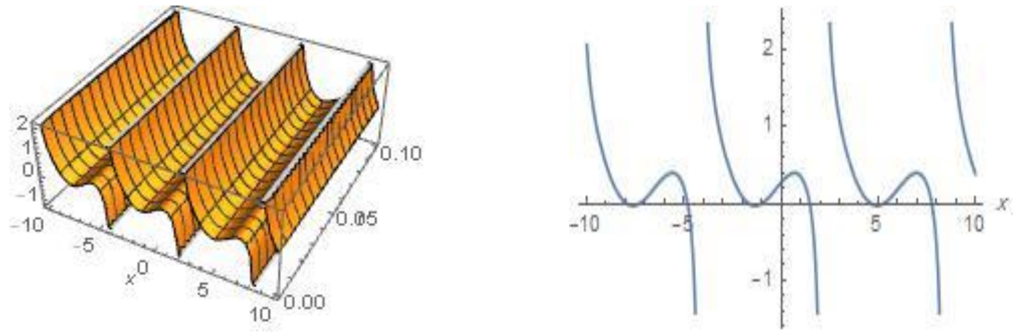

Figure 9. Soliton for the RKLE of Eq (4.17) in 2D and 3D with the following values: $a = 1, c = 0.25, b = 0, C_1 = w = v = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1, \theta_0 = 0.1, k = -1$.

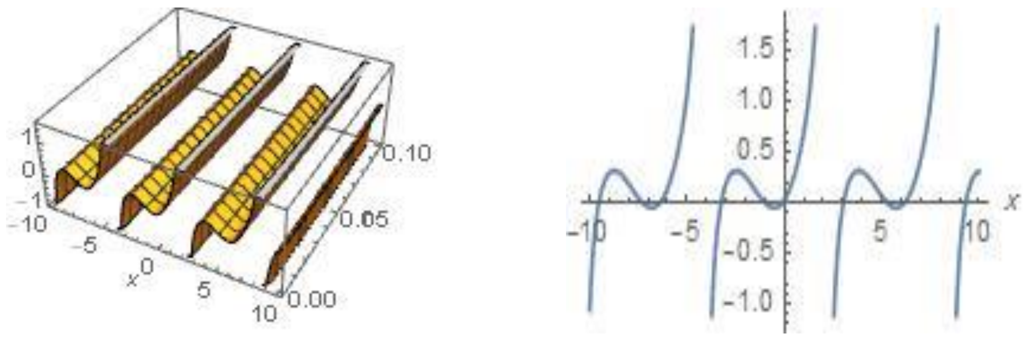

Figure 10. Soliton for the RKLE of Eq (4.18) in 2D and 3D with the following values: $a = 1, c = 0.25, b = 0, C_1 = w = v = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1, \theta_0 = 0.1, k = -1$.

Similarly, the second part can be written as follows:

$$R_j(\eta) = -0.5\cot[0.5(\eta + C_1)], \tag{4.20}$$

$$V(x,t) = \left(-0.5\cot 0.5(x - t + 1)\right)e^{i(x+t+0.1)}, \tag{4.21}$$

$$ReV(x,t) = \left(-0.5\,cot\,0.5(x - t + 1)\right)\cos(x + t + 0.1), \tag{4.22}$$

$$ImV(x,t) = (-0.5\cot 0.5(x - t + 1))\sin(x + t + 0.1). \tag{4.23}$$

As a result Figures 11 and 12 show the solitons for the RKLE of Eqs (4.22) and (4.23), respectively, in 2D and 3D.

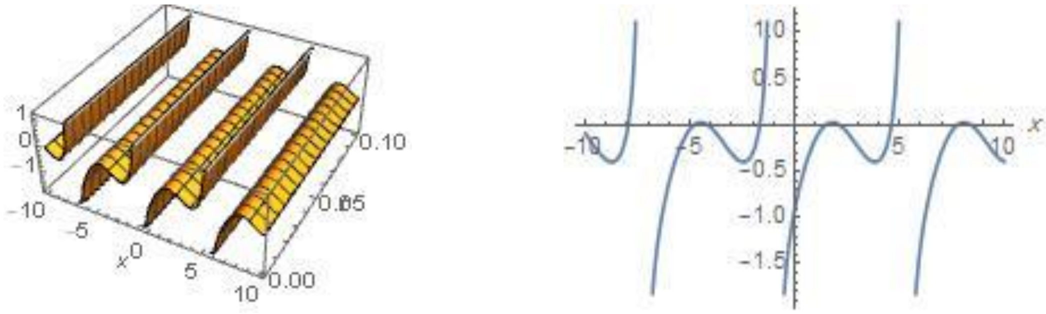

Figure 11. Soliton for the RKLE of Eq (4.22) in 2D and 3D with the following values: $a = 1, c = 0.25, b = 0, C_1 = w = \nu = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1, \theta_0 = 0.1, k = -1.$

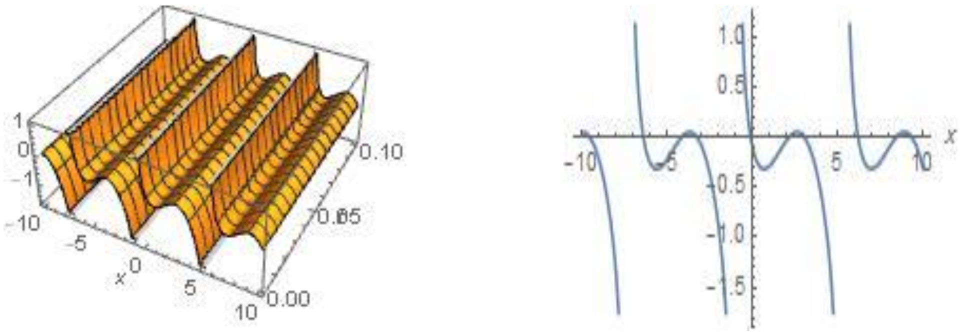

Figure 12. Soliton for the RKLE of Eq (4.23) in 2D and 3D with the following values: $a = 1, c = 0.25, b = 0, C_1 = w = \nu = a_j = b_j = c_j = \gamma_j = \beta_j = \lambda_j = 1, \theta_0 = 0.1, k = -1.$

5. Quickly overview of SWAM

To investigate the SWAM, let us first consider the following general form of a nonlinear partial differential equation:

$$G(U, U_x, U_t, U_{xx}, U_{tt}, \dots) = 0. \tag{5.1}$$

Here, the substituting $U(x,t) = U(\eta)$ with $\eta = x - \nu t$ in Eq (5.1) will lead to the following ODE:

$$\mathrm{H}(\mathrm{U}, \mathrm{U}', \mathrm{U}'', \mathrm{U}''', \dots) = 0. \tag{5.2}$$

Here, H is a function of $\mathrm{U}(\zeta)$ and its total derivatives.

The form of the solution utilizing the SWAM [1,3,4–6] can be written as follows:

$$U(x,t) = \psi(x,t)e^{iR(x,t)}, \ R(x,t) = kx - \Omega t. \tag{5.3}$$

Here, $\psi(x,t)$ and $R(x,t)$ respectively denote the portion largeness and phase slice of the soliton.

The following relations can be easily derived from Eq (5.3):

$$U_t = (\psi_t + i\psi R_t)e^{iR}, \tag{5.4}$$

$$U_x = (\psi_x + i\psi R_x)e^{iR}, \tag{5.5}$$

$$U_{xx} = (\psi_{xx} + 2i\psi_x R_x + i\psi R_{xx} - \psi R_x^2)e^{iR}, \tag{5.6}$$

$$U_{xxx} = (\psi_{xxx} + 3iR_x\psi_{xx} + 3iR_{xx}\psi_x + iR_{xxx}\psi - iR_x^3\psi - 3R_xR_{xx}\psi)e^{iR}. \tag{5.7}$$

In the framework of [14,16], the KRLE for birefringent fiber with the Kerr-Law of nonlinearity can be written in the following form:

$$iQ_t + \alpha Q_{xx} + \beta|Q^2|Q = i\lambda(|Q|^2Q)_x - i\delta Q_{xxx}. \tag{5.8}$$

Thus, considering the 4WM, Eq (5.8) can be written as follows:

$$iU_t + \alpha_1 U_{xx} + [\beta_1|U|^2 + \gamma_1|V|^2]U = i[\lambda_1(|U|^2U)_x + \theta_1(|V|^2U)_x] - i\delta_1 U_{xxx}, \tag{5.9}$$

$$iV_t + \alpha_2 V_{xx} + [\beta_2|V|^2 + \gamma_2|U|^2]V = i[\lambda_2(|V|^2V)_x + \theta_2(|U|^2V)_x] - i\delta_2 V_{xxx}. \tag{5.10}$$

These two equations are the same when $U = V$; hence, we will implement the suggested method for only one of them. By combining Eqs (5.3)–(5.7) with Eq (5.9), we get

$$i(\psi_t + i\psi R_t)e^{iR} + \alpha_1(\psi_{xx} + 2i\psi_x R_x + i\psi R_{xx} - \psi R_x^2)e^{iR} + (\beta_1 + \gamma_1)\psi^3 e^{iR} = i(\lambda_1 + \theta_1)$$
$$\left(3\psi^2\psi_x e^{iR} + ik\psi^3 e^{iR}\right) - i\delta_1(\psi_{xxx} + 3iR_x\psi_{xx} + 3iR_{xx}\psi_x + iR_{xxx}\psi - iR_x^3\psi - 3R_xR_{xx}\psi)e^{iR}. \tag{5.11}$$

So, the real and imaginary parts can be simplified as follows:

$$Re\colon (a_1 - 3\delta_1 k)\psi_{xx} + [\beta_1 + \gamma_1 + k(\lambda_1 + \theta_1)]\psi^3 - (\Omega + \alpha_1 k^2 - \delta_1 k^3)\psi = 0. \tag{5.12}$$

$$Im\colon \psi_t + \delta_1\psi_{xxx} - 3(\lambda_1 + \theta_1)\psi^2\psi_x + 2\alpha_1 k\psi_x = 0. \tag{5.13}$$

Next, we will investigate the bright and dark soliton solutions individually.

5.1 Bright solitons

Put $t_1 = x - v_1 t$ and consider the following:

$$\psi(x,t) = A_1 \operatorname{sech}^P t_1, \tag{5.14}$$

$$\psi_t = -A_1 v_1 P \operatorname{sech}^P t_1 \tanh t_1, \tag{5.15}$$

$$\psi_x = A_1 P \operatorname{sech}^P t_1 \tanh t_1, \tag{5.16}$$

$$\psi_{xx} = A_1 P(1+P)\operatorname{sech}^{P+2} t_1 - A_1 P^2 \operatorname{sech}^P t_1, \tag{5.17}$$

$$\psi_{xxx} = A_1 P(P+1)(P+2)\operatorname{sech}^{P+2} t_1 \tanh t_1 - A_1 P^3 \operatorname{sech}^P t_1 \tanh t_1. \tag{5.18}$$

By inserting Eqs (5.15)–(5.19) into Eqs (5.12) and (5.13), we get

$$(\alpha_1 - 3\delta_1 k)A_1 P(1+P)\operatorname{sech}^{P+2} t_1 - A_1 P^2 \operatorname{sech}^P t_1$$
$$+[\beta_1 + \gamma_1 + k(\lambda_1 + \theta_1)]A_1^3 \operatorname{sech}^{3P} t_1 - (\Omega + \alpha_1 k^2 - \delta_1 k^3)A_1 \operatorname{sech}^P t_1 = 0. \tag{5.19}$$

$$-A_1 v_1 P \operatorname{sech}^P t_1 \tanh t_1 + \delta_1[A_1 P(P+1)(P+2)\operatorname{sech}^{P+2} t_1 \tanh t_1 - A_1 P^3 \operatorname{sech}^P t_1 \tanh t_1]$$
$$-3(\lambda_1 + \theta_1)A_1^3 P \operatorname{sech}^{3P} t_1 \tanh t_1 + 2\alpha_1 k A_1 P \operatorname{sech}^P t_1 \tanh t_1 = 0. \tag{5.20}$$

When the equivalence is implemented between the higher orders of $\operatorname{sech}^i t_1$ in Eq (5.19), we get $P = 1$. Substituting $P = 1$ into Eqs (5.19) and (5.20), we get the following results:

$$\Omega = \delta_1 k^3 - \alpha_1 k^2 - 1,\ A_1^2 = \frac{2\alpha_1}{2k(\lambda_1+\theta_1)-\beta_1-\gamma_1},\ v_1 = 2\alpha_1 k - \delta_1. \tag{5.21}$$

From Eq (5.21), we obtian $P = 1,\ k = \alpha_1 = \beta_1 = \gamma_1 = \lambda_1 = \theta_1 = \delta_1 = v_1 = 1,\ A_1 = \pm 1$, and $\Omega = -1$. Thus, the suggested solution is as follows:

$$U = \pm \operatorname{sech}(x - t) e^{i(x+t)}, \tag{5.22}$$

$$\operatorname{Re} U = \pm \operatorname{sech}(x - t) \cos(x + t), \tag{5.23}$$

$$\operatorname{Im} U = \pm \operatorname{sech}(x - t) \sin(x + t). \tag{5.24}$$

As a result Figures 13 and 14 show the bright solitons for the RKLE of Eqs (5.23) and (5.24), respectively, in 2D and 3D.

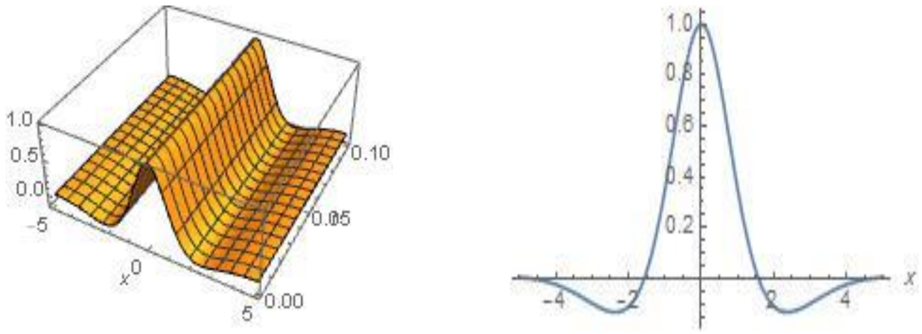

Figure 13. Bright soliton for the RKLE of Eq (5.23) in 2D and 3D with the following values: $P = 1, k = \alpha_1 = \beta_1 = \gamma_1 = \lambda_1 = \theta_1 = \delta_1 = v_1 = 1, A_1 = 1, \Omega = -1$.

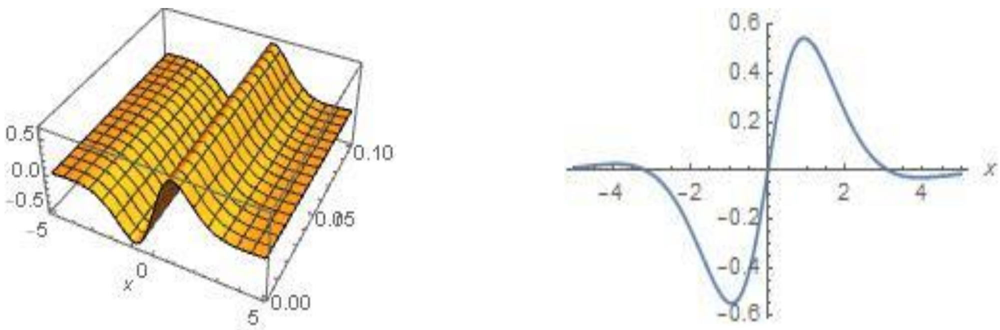

Figure 14. Bright soliton for the RKLE of Eq (5.24) in 2D and 3D with the following values: $P = 1, k = \alpha_1 = \beta_1 = \gamma_1 = \lambda_1 = \theta_1 = \delta_1 = v_1 = 1, A_1 = 1, \Omega = -1$.

5.2 Dark solitons

Put $t_2 = x - v_2 t$ and consider the following:

$$\psi(x,t) = A_2 \tanh^P t_2, \tag{5.25}$$

$$\psi_t = -A_2 P v_2 [\tanh^{P-1} t_2 - \tanh^{P+1} t_2], \tag{5.26}$$

$$\psi_x = A_2 P [\tanh^{P-1} t_2 - \tanh^{P+1} t_2], \tag{5.27}$$

$$\psi_{xx} = A_2 P(P-1) \tanh^{P-2} t_2 - 2A_2 P^2 \tanh^P t_2 + A_2 P(P+1) \tanh^{P+2} t_2, \tag{5.28}$$

$$\psi_{xxx} = A_2P(P-1)(P-2)\tanh^{P-3} t_2 - [A_2P(P-1)(P-2) + 2A_2P^3]\tanh^{P-1} t_2$$
$$+[A_2P(P+1)(P+2) + 2A_2P^3]\tanh^{P+1} t_2 - A_2P(P+1)(P+2)\tanh^{P+3} t_2. \qquad (5.29)$$

By inserting Eqs (5.25)–(5.29) into Eqs (5.12) and (5.13), we get

$$(\alpha_1 - 3\delta_1 k)[A_2P(P-1)\tanh^{P-2} t_2 - 2A_2P^2\tanh^P t_2 + A_2P(P+1)\tanh^{P+2} t_2]$$
$$+[\beta_1 + \gamma_1 + k(\lambda_1 + \theta_1)]A_2^3\tanh^{3P} t_2 - (\Omega + \alpha_1 k^2 - \delta_1 k^3)A_2\tanh^P t_2 = 0. \qquad (5.30)$$

$$-A_2Pv_2[\tanh^{P-1} t_2 - \tanh^{P+1} t_2]$$
$$+\delta_1\left\{\begin{array}{l} A_2P(P-1)(P-2)\tanh^{P-3} t_2 - [A_2P(P-1)(P-2) + 2A_2P^3]\tanh^{P-1} t_2 \\ +[A_2P(P+1)(P+2) + 2A_2P^3]\tanh^{P+1} t_2 - A_2P(P+1)(P+2)\tanh^{P+3} t_2 \end{array}\right\}$$
$$-3(\lambda_1 + \theta_1)A_2^3P[\tanh^{P+1} t_2 - \tanh^{P+3} t_2] + 2\alpha_1 kA_2P[\tanh^{P-1} t_2 - \tanh^{P+1} t_2] = 0. \qquad (5.31)$$

By equating the higher orders of $\tanh^i t_2$ in Eq (5.30) and substituting $P = 1$, the following relations will be obtained:

$$\Omega = \delta_1 k^3 + 6\delta_1 k - 2\alpha_1 - \alpha_1 k^2, 2(\alpha_1 - 3\delta_1 k) + [\beta_1 + \gamma_1 + k(\lambda_1 + \theta_1)]A_2^2 = 0. \qquad (5.32)$$

Substituting $P = 1$ into the imaginary part of Eq (5.31), we get

$$A_2^2 = \frac{2\delta_1}{(\lambda_1+\theta_1)}, v_2 = 2\alpha_1 k - 2\delta_1. \qquad (5.33)$$

Now, Eqs (5.32) and (5.33) lead to

$$v_2 = 2\alpha_1 k - 2\delta_1,\ \Omega = \delta_1 k^3 + 6\delta_1 k - 2\alpha_1 - \alpha_1 k^2,\ A_2^2 = \frac{2\alpha_1 - 4\delta_1 k}{\beta_1+\gamma_1}. \qquad (5.34)$$

After simplification, we get

$$P = \alpha_1 = \beta_1 = \gamma_1 = \lambda_1 = \theta_1 = \delta_1 = 1, A_2 = \pm 1.7,$$
$$k = -1, \Omega = -10, v_2 = -4. \qquad (5.35)$$

This result will generate two solutions according to the value of A_2. Anyhow, these two solutions in the framework of the suggested method are as follows:

$$U(x,t) = A_2 e^{iR(x,t)}\tanh^P t_2,$$
$$U(x,t) = \pm 1.7\tanh(x + 4t)e^{i(-x+10t)}, \qquad (5.36)$$

$$\operatorname{Re} U(x,t) = \pm 1.7\tanh(x + 4t)\cos(-x + 10t), \qquad (5.37)$$

$$\operatorname{Im} U(x,t) = \pm 1.7\tanh(x + 4t)\sin(-x + 10t). \qquad (5.38)$$

As a result Figures 15 and 16 show the dark solitons for the RKLE of Eqs (5.37) and (5.38), respectively, in 2D and 3D.

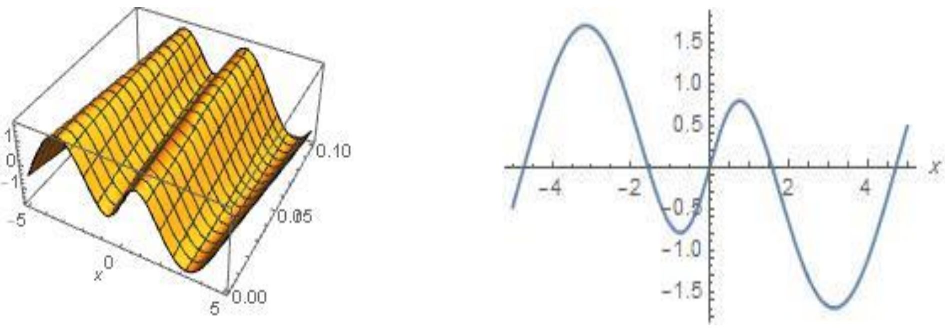

Figure 15. Dark soliton for the RKLE of Eq (5.37) in 2D and 3D with the following values: $P = \alpha_1 = \beta_1 = \gamma_1 = \lambda_1 = \theta_1 = \delta_1 = 1, A_2 = 1.7, k = -1, \Omega = -10, v_2 = -4$.

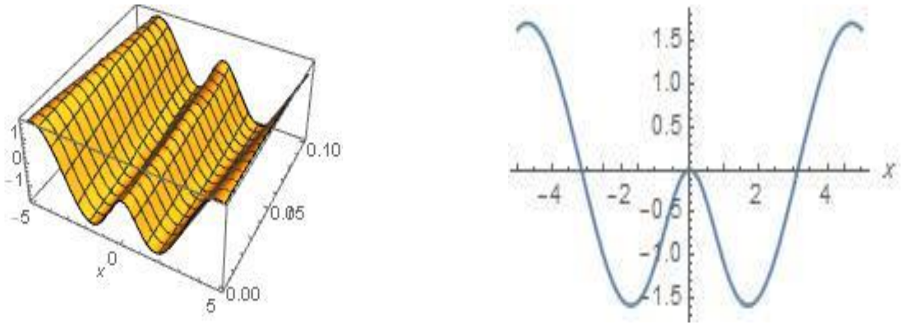

Figure 16. Dark soliton for the RKLE of Eq (5.38) in 2D and 3D with the following values: $P = \alpha_1 = \beta_1 = \gamma_1 = \lambda_1 = \theta_1 = \delta_1 = 1, A_2 = 1.7, k = -1, \Omega = -10, v_2 = -4$.

6. Conclusions

In this paper, various multiple types of soliton solutions of the RKLE have been constructed perfectly. The obtained solutions have single-velocity traveling wave solutions. Many types of solutions have been established as W-shaped, M-shaped, periodic trigonometric, hyperbolic, bright, and dark soliton solutions; other rational forms of solutions have been obtained by using the utlized methods. Our new impressive multiple soliton solutions that give a good description of the arising solitons for the PKLE will encourage modern studies of this model. These achieved new soliton solutions can be arranged to create something that we will name a catalog. The interesting aspect of our work is the fact that these are periodic solutions in which single solitons propagate in parallel as realized by Whitham [47] who is one of the scientists that had a serious paper dealing with the idea of parallel identical single solitons and gave the elementary arguments for various results in wave propagation from the perspective of the representation of periodic waves as sums of solitons for the Korteweg-de Vries, various modified Korteweg-de Vries, and Boussinesq equations. The novelty of our achieved soliton solutions appears when comparing the documented solitons in this catalog with those previously achieved before. Consequently, new important and impressive visions of the solitons of this model which were not constructed before have been demonstrated and will encourage extended future studies not only for this model but also for all related phenomena.

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Conflict of interest

The authors declare that they have no competing interest.

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