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Research article

A coupled system of Laplacian and bi-Laplacian equations with nonlinear dampings and source terms of variable-exponents nonlinearities: Existence, uniqueness, blow-up and a large-time asymptotic behavior

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Abstract: In this paper, we consider a coupled system of Laplacian and bi-Laplacian equations with nonlinear dampings and source terms of variable-exponents nonlinearities. This system is supplemented with initial and mixed boundary conditions. First, we establish the existence and uniqueness results of a weak solution, under suitable assumptions on the variable exponents. Second, we show that the solutions with positive-initial energy blow-up in a finite time. Finally, we establish the global existence as well as the energy decay results of the solutions, using the stable-set and the multiplier methods, under appropriate conditions on the variable exponents and the initial data.

Keywords: biharmonic equations; blow-up; coupled system; global existence; variable exponent; stability
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1. Introduction

The biharmonic equation, besides providing a benchmark problem for various analytical and numerical methods, arises in many practical applications. For example, the bending behavior of a thin elastic rectangular plate, as might be encountered in ship design and manufacture, or the equilibrium of an elastic rectangle, can be formulated in terms of the two-dimensional biharmonic equation, e.g., Timoshenko & Woinowsky-Krieger [1]. Also, Stokes flow of a viscous fluid in a rectangular cavity under the influence of the motion of the walls, can be described in terms of the solution of this equation, e.g., Pan and Acrivos (1967), Shankar [2], Srinivasan [3], Meleshko [4] or Shankar and Deshpande [5]. A more recent application of the biharmonic equation has been in the area of geometric and functional design, where it has been used as a mapping to produce efficient mathematical descriptions of surfaces in physical space, e.g., Sevant et al. [6] and Bloor and Wilson [7]. Interest in solutions of the biharmonic equation and their mathematical properties go back over 130 years, and comprehensive reviews of this work have been given by Meleshko [8, 9]. In his review article, he concentrates upon the method of superposition in which the solution is described in terms of a sum of separable solutions of the biharmonic equation. In another work, Meleshko [4] obtained some results for Stokes flow in a rectangular cavity in which the solution is based upon the sum of terms consisting of the product of exponential and sinusoidal functions, where the coefficients in the series are determined from the requirement that the prescribed boundary conditions are satisfied, and Meleshko [10] described the work which has been done in trying to solve this problem, e.g., Meleshko and Gomilko [11]. Other physical phenomena like flows of electro-rheological fluids, fluids with temperature dependent viscocity, filtration processes through a porous media, image processing and thermorheological fluids give rise to mathematical models of hyperbolic, parabolic and biharmonic equations with variable exponents of nonlinearity. More details can also be found in references [12, 13]. Recently, the hyperbolic equations with nonlinearities of variable exponents type had received a considerable amount of attention. We refer the reader to [14–17] and the references therein. Only few works concerning coupled systems of wave equations in the variable-exponents case have been found in the literature. For examples, Bouhoufani and Hamchi [18] obtained the global existence of a weak solution and established decay rates of the solutions, in a bounded domain, of a coupled system of nonlinear hyperbolic equations with variable-exponents. Messaoudi et al. [15] studied a system of wave equations with nonstandard nonlinearities and proved a theorem of existence and uniqueness of a weak solution, established a blow-up result for certain solutions with positive-initial energy and gave some numerical applications for their theoretical results. In [16], Messaoudi et al. considered the following system

$$\begin{aligned} u_{tt} - \Delta u + |u_t|^{m(x)-2} u_t + f_1(u,v) &= 0 \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + |v_t|^{r(x)-2} v_t + f_2(u,v) &= 0 \text{in } \Omega \times (0,T), \end{aligned} \tag{1.1}$$

with initial and Dirichlet-boundary conditions (here, f_1 and f_2 are the coupling terms introduced in (1.3). The authors proved the existence of global solutions, obtained explicit decay rate estimates under suitable assumptions on the variable exponents m, r and p and presented some numerical tests. In this work, we consider the following initial-boundary-value problem

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m(x)-2} u_t = f_1(u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + |v_t|^{r(x)-2} v_t = f_2(u,v) & \text{in } \Omega \times (0,T), \\ u = v = \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{in } \Omega, \\ v(0) = v_0 \text{ and } v_t(0) = v_1 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth and bounded domain of $\mathbb{R}^n, (n = 1, 2, 3)$, the exponents m and r are continuous functions on $\overline{\Omega}$ satisfying some conditions to be specified later, $\frac{\partial u}{\partial \eta}$ denotes the external normal derivatives of u on the boundary $\partial\Omega$ and the coupling terms f_1 and f_2 are given as follows: for all $x \in \overline{\Omega}$ and $(u, v) \in \mathbb{R}^2$,

$$f_1(x, u, v) = \frac{\partial}{\partial u} F(x, u, v) \text{ and } f_2(x, u, v) = \frac{\partial}{\partial v} F(x, u, v), \tag{1.3}$$

with

$$F(x, u, v) = a|u + v|^{p(x)+1} + 2b|uv|^{\frac{p(x)+1}{2}}, \tag{1.4}$$

where $a, b > 0$ are two positive constants and p is a given continuous function on $\overline{\Omega}$ satisfying the condition $(H.2)$ (below).

2. Preliminaries

This section presents some material needed to prove the main result. Let $q : \Omega \longrightarrow [1, \infty)$ be a continuous function. We define the Lebesgue space with a variable exponent by

$$L^{q(.)}(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ measurable in } \Omega : \varrho_{q(.)}(\lambda f) < +\infty, \text{ for some } \lambda > 0 \right\},$$

where

$$\varrho_{q(.)}(f) = \int_{\Omega} |f(x)|^{q(x)} dx.$$

Lemma 2.1. *[13, 19] If $1 < q^- \leq q(x) \leq q^+ < +\infty$ holds then, for any $f \in L^{q(.)}(\Omega)$,*

$$min\left\{ \|f\|_{q(.)}^{q^-}, \|f\|_{q(.)}^{q^+} \right\} \leq \varrho_{q(.)}(f) \leq max\left\{ \|f\|_{q(.)}^{q^-}, \|f\|_{q(.)}^{q^+} \right\},$$

where

$$q^- = ess \inf_{x\in\Omega} q(x) \text{ and } q^+ = ess \sup_{x\in\Omega} q(x).$$

Lemma 2.2. *(Embedding property [20]) Let $q : \overline{\Omega} \longrightarrow [1, \infty)$ be a measurable function and $k \geq 1$ be an integer. Suppose that r is a log-Hölder continuous function on Ω, such that, for all $x \in \Omega$, we have*

$$\begin{cases} k \leq q^- \leq q(x) \leq q^+ < \frac{nr(x)}{n-kr(x)}, & if\ r^+ < \frac{n}{k}, \\ k \leq q^- \leq q^+ < \infty, & if\ r^+ \geq \frac{n}{k}. \end{cases}$$

Then, the embedding $W_0^{k,r(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Throughout this paper, we denote by $\mathcal{V}$ the following space

$$\mathcal{V} = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega\} = H_0^2(\Omega).$$

So, $\mathcal{V}$ is a separable Hilbert space endowed with the inner product and norm, respectively,

$$(w, z)_{\mathcal{V}} = \int_{\Omega} \Delta w \Delta z dx \text{ and } \|w\|_{\mathcal{V}} = \|\Delta w\|_2,$$

where $\|\Delta w\|_k = \|\Delta w\|_{L^k(\Omega)}$.

We assume the following hypotheses:

$(H.1)$ The exponents m and r are continuous on $\overline{\Omega}$ such that

$$\begin{array}{ll} 2 \leq m(x), & \text{if } n = 1, 2, \\ 2 \leq m_1 \leq m(x) \leq m_2 \leq 6, & \text{if } n = 3 \end{array} \tag{2.1}$$

and

$$\begin{array}{ll} 2 \leq r(x), & \text{if } n = 1, 2, \\ 2 \leq r_1 \leq r(x) \leq r_2 \leq 6, & \text{if } n = 3, \end{array} \tag{2.2}$$

for all $x \in \overline{\Omega}$, where

$$m_1 = \inf_{x \in \overline{\Omega}} m(x), \ m_2 = \sup_{x \in \overline{\Omega}} m(x), \ r_1 = \inf_{x \in \overline{\Omega}} r(x) \text{ and } r_2 = \sup_{x \in \overline{\Omega}} r(x).$$

$(H.2)$ The variable exponent p is a given continuous function on $\overline{\Omega}$ such that

$$\begin{array}{ll} 3 \leq p^- \leq p(x) \leq p^+ < +\infty, & \text{if } n = 1, 2, \\ p(x) = 3, & \text{if } n = 3, \end{array} \tag{2.3}$$

for all $x \in \overline{\Omega}$.

3. Existence of weak solution

In this section, we prove the local existence of the solutions of (1.2). For this purpose, we introduce the definition of a weak solution for system (1.2). We multiply the first equation in (1.2) by $\Phi \in C_0^\infty(\Omega)$ and the second equation by $\Psi \in C_0^\infty(\Omega)$, integrate each result over Ω, use Green's formula and the boundary conditions to obtain the following definition:

Definition 3.1. *Let $(u_0, v_0) \in \mathcal{V} \times H_0^1(\Omega), (u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$. Any pair of functions (u, v), such that*

$$\begin{cases} u \in L^\infty([0,T); \mathcal{V}), v \in L^\infty\left([0,T); H_0^1(\Omega)\right), \\ u_t \in L^\infty\left([0,T); L^2(\Omega)\right) \cap L^{m(.)}(\Omega \times (0,T)), \\ v_t \in L^\infty\left([0,T); L^2(\Omega)\right) \cap L^{r(.)}(\Omega \times (0,T)), \end{cases} \tag{3.1}$$

is called a weak solution of (1.2) *on* $[0, T)$, *if*

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t \Phi dx + \int_\Omega \Delta u \Delta \Phi dx + \int_\Omega |u_t|^{m(x)-2} u_t \Phi dx \\ = \int_\Omega f_1 \Phi dx, \\ \frac{d}{dt}\int_\Omega v_t \Psi dx + \int_\Omega \nabla v \nabla \Psi dx + \int_\Omega |v_t|^{r(x)-2} v_t \Psi dx \\ = \int_\Omega f_2 \Psi dx, \\ u(0) = u_0, u_t(0) = u_1, v(0) = v_0, v_t(0) = v_1, \end{cases}$$

for a.e. $t \in (0, T)$ and all test functions $\Phi \in \mathcal{V}$ and $\Psi \in H_0^1(\Omega)$. Note that $C_0^\infty(\Omega)$ is dense in $\mathcal{V}$ and in $H_0^1(\Omega)$ as well. In addition, the spaces $\mathcal{V}$, $H_0^1(\Omega) \subset L^{m(.)}(\Omega) \cap L^{r(.)}(\Omega)$, under the conditions $(H.1)$ and $(H.2)$.

In order to establish an existence result of a local weak solution for the system (1.2); we, first, consider the following auxiliary problem:

$$\begin{cases} u_{tt} + \Delta^2 u + u_t |u_t|^{m(x)-2} = f(x,t) & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + v_t |v_t|^{r(x)-2} = g(x,t) & \text{in } \Omega \times (0,T), \\ u = v = \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, u_t(0) = u_1, v(0) = v_0, v_t(0) = v_1 & \text{in } \Omega, \end{cases} \tag{S}$$

for given $f, g \in L^2(\Omega \times (0,T))$ and $T > 0$.

We have the following theorem of existence and uniqueness for Problem (S).

Theorem 3.1. *Let $n = 1,2,3$ and $(u_0, v_0) \in \mathcal{V} \times H_0^1(\Omega), (u_1, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Assume that assumptions $(H.1)$ and $(H.2)$ hold. Then, the problem (S) admits a unique weak solution on* $[0,T)$.

Proof. Let $\{\omega_j\}_{j=1}^{\infty}$ be an orthogonal basis of $\mathcal{V}$ and define, for all $k \geq 1$, (u^k, v^k) a sequence in $\mathcal{V}_k = span\{\omega_1, \omega_2, ..., \omega_k\} \subset \mathcal{V}$, given by

$$u^k(x,t) = \Sigma_{j=1}^k a_j(t)\omega_j(x) \text{ and } v^k(t) = \Sigma_{j=1}^k b_j(t)\omega_j(x)$$

for all $x \in \Omega$ and $t \in (0,T)$ and solves the following approximate problem:

$$\begin{cases} \int_\Omega u_{tt}^k(x,t)\omega_j dx + \int_\Omega \Delta u^k(x,t)\Delta\omega_j dx + \int_\Omega \left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t)\omega_j dx \\ = \int_\Omega f(x,t)\omega_j, \\ \int_\Omega v_{tt}^k(x,t)\omega_j dx + \int_\Omega \nabla v^k(x,t)\nabla\omega_j dx + \int_\Omega \left|v_t^k(x,t)\right|^{r(x)-2} v_t^k(x,t)\omega_j dx \\ = \int_\Omega g(x,t)\omega_j, \end{cases} \tag{S_k}$$

for all $j = 1,2,...,k$, with

$$\begin{aligned} u^k(0) = u_0^k = \Sigma_{i=1}^k \langle u_0, \omega_i\rangle \omega_i, \ u_t^k(0) = u_1^k = \Sigma_{i=1}^k \langle u_1, \omega_i\rangle \omega_i \\ v^k(0) = v_0^k = \Sigma_{i=1}^k \langle v_0, \omega_i\rangle \omega_i, \ v_t^k(0) = v_1^k = \Sigma_{i=1}^k \langle v_1, \omega_i\rangle \omega_i, \end{aligned} \tag{3.2}$$

such that

$$\begin{aligned} u_0^k \longrightarrow u_0 \text{ and } v_0^k \longrightarrow v_0 \ \ in\ H_0^1(\Omega), \\ u_1^k \longrightarrow u_1 \text{ and } v_1^k \longrightarrow v_1 \ \ in\ L^2(\Omega). \end{aligned} \tag{3.3}$$

For any $k \geq 1$, problem (S_k) generates a system of k nonlinear ordinary differential equations. The ODE standard existence theory assures the existence of a unique local solution (u^k, v^k) for (S_k) on $[0, T_k)$, with $0 < T_k \leq T$. Next, we have to show that $T_k = T, \forall k \geq 1$. Multiplying $(S_k)_1$ and $(S_k)_2$ by $a_j'(t)$ and $b_j'(t)$, respectively, and then summing each result over $j = 1, ..., k$, we obtain, for all $0 < t \leq T_k$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(|u_t^k(x,t)|^2 + (\Delta u^k)^2(x,t)\right) dx\right] + \int_\Omega \left|u_t^k(x,t)\right|^{m(x)} dx \\ &= \int_\Omega f(x,t)u_t^k(x,t)dx \end{aligned} \tag{3.4}$$

and

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(|v_t^k(x,t)|^2+|\nabla v^k|^2(x,t)\right)dx\right]+\int_\Omega \left|v_t^k(x,t)\right|^{r(x)}dx$$
$$=\int_\Omega g(x,t)v_t^k(x,t)dx. \tag{3.5}$$

The addition of (3.4) and (3.5), and then the integration of the result, over $(0,t)$, lead to

$$\begin{aligned}&\frac{1}{2}\left[\|u_t^k(t)\|_2^2+\|u^k(t)\|_{\mathcal{V}}^2+\|v_t^k(t)\|_2^2+\|\nabla v^k(t)\|_2^2\right]\\&+\int_0^t\int_\Omega\left(\left|u_t^k(x,s)\right|^{m(x)}+\left|v_t^k(x,s)\right|^{r(x)}\right)dxds\\&=\frac{1}{2}\left[\|u_1^k\|_2^2+\|u_0^k\|_{\mathcal{V}}^2+\|v_1^k\|_2^2+\|\nabla v_0^k\|_2^2\right]\\&+\int_0^t\int_\Omega\left[f(x,s)u_t^k(x,s)+g(x,s)v_t^k(x,s)\right]dxds.\end{aligned} \tag{3.6}$$

Using Young's inequality and the convergence (3.3), then Eq (3.6) becomes, for some $C>0$,

$$\begin{aligned}&\frac{1}{2}\left[\|u_t^k(t)\|_2^2+\|v_t^k(t)\|_2^2+\|u^k(t)\|_{\mathcal{V}}^2+\|\nabla v^k(t)\|_2^2\right]\\&+\int_0^{T_k}\int_\Omega\left(\left|u_t^k(x,s)\right|^{m(x)}+\left|v_t^k(x,s)\right|^{r(x)}\right)dxds\\&\le C+\varepsilon\int_0^{T_k}\left(\left\|u_t^k(s)\right\|_2^2+\left\|v_t^k(s)\right\|_2^2\right)ds\\&+C_\varepsilon\int_0^T\int_\Omega\left(|f(x,s)|^2+|g(x,s)|^2\right)dxds.\end{aligned}$$

Using the fact that $f,g\in L^2(\Omega\times(0,T))$ and choosing $\varepsilon=\frac{1}{4T}$, we infer

$$\begin{aligned}&\frac{1}{2}\sup_{(0,T_k)}\left[\|u_t^k\|_2^2+\|v_t^k\|_2^2+\|u^k\|_{\mathcal{V}}^2+\|\nabla v^k\|_2^2\right]+\int_0^{T_k}\int_\Omega\left(\left|u_t^k(x,s)\right|^{m(x)}+\left|v_t^k(x,s)\right|^{r(x)}\right)dxds\\&\le C_\varepsilon+T\varepsilon\sup_{(0,T_k)}\left(\|u_t^k\|_2^2+\|v_t^k\|_2^2\right)\\&\le C_T,\end{aligned} \tag{3.7}$$

where $C_T>0$ is a constant depending on T only. Consequently, the solution (u^k,v^k) can be extended to $(0,T)$, for any $k\ge 1$. In addition, we have

$$\begin{cases}(u^k)\text{ is bounded in }L^\infty((0,T),\mathcal{V}),\\(v^k)\text{ is bounded in }L^\infty((0,T),H_0^1(\Omega)),\\(u_t^k)\text{ is bounded in }L^\infty((0,T),L^2(\Omega))\cap L^{m(.)}(\Omega\times(0,T)),\\(v_t^k)\text{ is bounded in }L^\infty((0,T),L^2(\Omega))\cap L^{r(.)}(\Omega\times(0,T)).\end{cases}$$

Therefore, we can extract two subsequences, denoted by (u^l) and (v^l), respectively, such that, when $l \to \infty$, we have

$$\begin{cases} u^l \to u \text{ weakly * in } L^\infty((0,T),\mathcal{V}), \\ v^l \to v \text{ weakly * in } L^\infty((0,T),H_0^1(\Omega)), \\ u_t^l \to u_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{m(.)}(\Omega\times(0,T)), \\ v_t^l \to v_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{r(.)}(\Omega\times(0,T)). \end{cases}$$

Under the assumptions $(H.1)$ and $(H.2)$ and using similar ideas and arguments as in [[15], Theorem 3.2, p.6], one can see that

$$| u_t^l |^{m(.)-2} u_t^l \to | u_t |^{m(.)-2} u_t \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega\times(0,T)),$$

$$| v_t^l |^{r(.)-2} v_t^l \to | v_t |^{r(.)-2} v_t \text{ weakly in } L^{\frac{r(.)}{r(.)-1}}(\Omega\times(0,T))$$

and establish that (u, v) satisfies the two differential equations in (S), on $\Omega\times(0,T)$.

To handle the initial conditions, we follow the same procedures as in [15], and we easily conclude that (u, v) satisfies the initial conditions. For the uniqueness, Assume that (S) has two weak solutions (u_1, v_1) and (u_2, v_2), in the sense of Definition 3.1. Let $(\Phi,\Psi) = (u_{1t} - u_{2t}, v_{1t} - v_{2t})$, then $(u, v) = (u_1 - u_2, v_1 - v_2)$ satisfies the following identities, for all $t \in (0,T)$,

$$\begin{aligned} &\frac{d}{dt}\left[\int_\Omega \left(|u_t|^2 + (\Delta u)^2\right) dx\right] \\ &+ 2\int_\Omega \left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right)(u_{1t} - u_{2t})dx = 0 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\frac{d}{dt}\left[\int_\Omega \left(|v_t|^2 + |\nabla v|^2\right) dx\right] \\ &+ 2\int_\Omega \left(|v_{1t}|^{r(x)-2} v_{1t} - |v_{2t}|^{r(x)-2} v_{2t}\right)(v_{1t} - v_{2t})dx = 0. \end{aligned} \tag{3.9}$$

Integrating (3.8) and (3.9) over $(0, t)$, with $t \le T$, we obtain

$$\|u_t\|_2^2 + \|u\|_{\mathcal{V}}^2 + 2\int_0^t \int_\Omega \left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right)(u_{1t} - u_{2t})dxd\tau = 0 \tag{3.10}$$

and

$$\|v_t\|_2^2 + \|\nabla v\|_2^2 + 2\int_0^t \int_\Omega \left(|v_{1t}|^{r(x)-2} v_{1t} - |v_{2t}|^{r(x)-2} v_{2t}\right)(v_{1t} - v_{2t})dxd\tau = 0. \tag{3.11}$$

But we have, for all $x \in \Omega$, $Y, Z \in \mathbb{R}$ and $q(x) \ge 2$,

$$\left(|Y|^{q(x)-2} Y - |Z|\, Z^{q(x)-2}\right)(Y - Z) \ge 0, \tag{3.12}$$

then, estimates (3.10) and (3.11) yield

$$\|u_t\|^2 + \|u\|_{\mathcal{V}}^2 = \|v_t\|^2 + \|\nabla v\|_2^2 = 0.$$

Thus, $u_t(.,t) = v_t(.,t) = 0$ and $u(.,t) = v(.,t) = 0$, for all $t \in (0,T)$. Thanks to the boundary conditions, we conclude $u = v = 0$ on $\Omega \times (0,T)$, which proves the uniqueness of the solution. Therefore, (u,v) is the unique local solution of (S), in the sense of Definition 3.1, having the regularity (3.1).

Lemma 3.1. *Let* $y \in L^\infty((0,T),\mathcal{V})$ *and* $z \in L^\infty\left((0,T),H_0^1(\Omega)\right)$. *Then*

$$f_1(y,z), f_2(y,z) \in L^2(\Omega \times (0,T)). \tag{3.13}$$

Proof. From (1.3) and (1.4), we have, for all $(u,v) \in \mathbb{R}^2$,

$$f_1(u,v) = (p(x)+1)\left[a\,|u+v|^{p(x)-1}(u+v) + bu\,|u|^{\frac{p(x)-3}{2}}\,|v|^{\frac{p(x)+1}{2}}\right] \tag{3.14}$$

and

$$f_2(u,v) = (p(x)+1)\left[a\,|u+v|^{p(x)-1}(u+v) + bv\,|v|^{\frac{p(x)-3}{2}}\,|u|^{\frac{p(x)+1}{2}}\right]. \tag{3.15}$$

Let $y \in L^\infty((0,T),\mathcal{V})$ and $z \in L^\infty\left((0,T),H_0^1(\Omega)\right)$. Applying Young's inequality and the Sobolev embedding, we obtain, for all $t \in (0,T)$ and some $C_1, C_2 > 0$, the following results:

$$\begin{aligned}\int_\Omega |f_1(y,z)|^2\,dx &\le 2\left[a^2\int_\Omega |y+z|^{2p(x)}\,dx + b^2\int_\Omega |y|^{p(x)-1}\,|z|^{p(x)+1}\,dx\right]\\ &\le C_0\left[\int_\Omega |y+z|^{2p^+}\,dx + \int_\Omega |y+z|^{2p^-}\,dx + \int_\Omega |y|^{3(p(x)-1)}\,dx + \int_\Omega |z|^{\frac{3}{2}(p(x)+1)}\,dx\right],\end{aligned} \tag{3.16}$$

where $C_0 = 2\max\left\{a^2, 3b^2\right\} > 0$. By the embeddings, we have for $n = 1,2$,

- $$1 < \frac{3}{2}(p^-+1) \le \frac{3}{2}(p^++1) \le 2p^+ \le 3(p^+-1) < \infty,$$

  since $3 \le p^- \le p(x) \le p^+ < \infty$. Therefore, estimate (3.16) leads to

  $$\begin{aligned}&\int_\Omega |f_1(y,z)|^2\,dx\\ &\le C_1\left[\|\nabla(y+z)\|_2^{2p^+} + \|\nabla(y+z)\|_2^{2p^-} + \|\Delta y\|_2^{3(p^+-1)} + \|\Delta y\|_2^{3(p^--1)}\right]\\ &+ C_1\left[\|\nabla z\|_2^{\frac{3}{2}(p^++1)} + \|\nabla z\|_2^{\frac{3}{2}(p^-+1)}\right] < +\infty,\end{aligned} \tag{3.17}$$

  where $C_1 = C_0 C_e$.
- For $n = 3$, we use the embedding $H_0^1(\Omega)$ in $L^6(\Omega)$ to obtain (3.17), since $p \equiv 3$ on $\overline{\Omega}$.

So, under the assumption $(H.2)$, we have

$$\int_\Omega |f_1(y,z)|^2\, dx < \infty,$$

and similarly

$$\int_\Omega |f_2(y,z)|^2\, dx < \infty,$$

for all $t \in (0,T)$. Which completes the proof.

Corollary 3.1. *There exists a unique* (u,v) *solution of the problem:*

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m(x)-2}\, u_t = f_1(y,z), & in\ \Omega \times (0,T)\,, \\ v_{tt} - \Delta v + |v_t|^{r(x)-2}\, v_t = f_2(y,z), & in\ \Omega \times (0,T)\,, \\ u = v = \frac{\partial u}{\partial \eta} = 0 & on\ \partial\Omega \times (0,T)\,, \\ u\,(0) = u_0 \text{ and } u_t\,(0) = u_1 & in\ \Omega, \\ v\,(0) = v_0 \text{ and } v_t\,(0) = v_1, & in\ \Omega, \end{cases} \tag{R}$$

in the sense of Definition 3.1 *and having the regularity* 3.1.

Proof. A combination of Theorem 3.1 and Lemma 3.1 implies this corollary.

Now, consider the following Banach spaces

$$A_T = \left\{ w \in L^\infty((0,T),\mathcal{V})/w_t \in L^\infty((0,T),L^2(\Omega)) \right\},$$

equipped with the norm:

$$\|w\|^2_{A_T} = \sup_{(0,T)} \|w\|^2_{\mathcal{V}} + \sup_{(0,T)} \|w_t\|^2_2$$

and

$$B_T = \left\{ w \in L^\infty((0,T),H^1_0(\Omega))/w_t \in L^\infty((0,T),L^2(\Omega)) \right\},$$

equipped with the norm:

$$\|w\|^2_{B_T} = \sup_{(0,T)} \|\nabla w\|^2_2 + \sup_{(0,T)} \|w_t\|^2_2$$

and define a map $F : A_T \times B_T :\longrightarrow A_T \times B_T$ by $F(y,z) = (u,v)$.

Lemma 3.2. F *maps* $D(0,d)$ *into itself where*

$$D(0,d) = \{(w,w) \in A_T \times B_T \text{ such that } \|(w,w)\|_{A_T\times B_T} \le d\}.$$

Proof. Let (y,z) be in $D(0,d)$ and (u,v) be the corresponding solution of problem (R) (i.e., $F(y,z) = (u,v)$). Taking $(\Phi,\Psi) = (u_t,v_t)$ in Definition 3.1 and integrating each identity over $(0,t)$, we obtain, for all $t \le T$,

$$\frac{1}{2}\left[\|u_t\|^2_2 - \|u_1\|^2_2 + \|\Delta u\|^2_2 - \|\Delta u_0\|^2_2\right] + \int_0^t \int_\Omega |u_t(x,s)|^{m(x)}\, dxds$$
$$= \int_0^t \int_\Omega u_t f_1(y,z) dxds \tag{3.18}$$

and

$$\frac{1}{2}\left[\|v_t\|_2^2 - \|v_1\|_2^2 + \|\nabla v\|_2^2 - \|\nabla v_0\|_2^2\right] + \int_0^t \int_\Omega |v_t(x,s)|^{r(x)}\,dxds$$
$$= \int_0^t \int_\Omega v_t f_2(y,z)dxds. \tag{3.19}$$

The addition of (3.18) and (3.19) lead to

$$\frac{1}{2}\left[\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla v\|_2^2\right]$$
$$\leq \frac{1}{2}\left[\|u_1\|_2^2 + \|v_1\|_2^2 + \|\Delta u_0\|_2^2 + \|\nabla v_0\|_2^2\right]$$
$$+ \int_0^t \left(\left|\int_\Omega u_t f_1(y,z)dx\right| + \left|\int_\Omega v_t f_2(y,z)dx\right|\right)ds.$$

for all $t \in (0,T)$. Therefore,

$$\sup_{0\leq t\leq T}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{\mathcal{V}}^2 + \|\nabla v\|_2^2\right)$$
$$\leq \gamma + 2\sup_{0\leq t\leq T}\int_0^t \left(\left|\int_\Omega u_t f_1(y,z)dx\right| + \left|\int_\Omega v_t f_2(y,z)dx\right|\right)d\tau, \tag{3.20}$$

where $\gamma = \|u_1\|_2^2 + \|v_1\|_2^2 + \|u_0\|_{\mathcal{V}}^2 + \|\nabla v_0\|_2^2$. Under the assumption (2.3) and applying Young's inequality and the Sobolev embedding (Lemma 2.2), we obtain for all $t \in (0,T)$,

$$\left|\int_\Omega u_t f_1(y,z)dx\right| \leq (p^+ + 1)\left[a\int_\Omega |u_t|\,|y+z|^{p(x)}\,dx + b\int_\Omega |u_t|\,.\,|y|^{\frac{p(x)-1}{2}}\,|z|^{\frac{p(x)+1}{2}}\,dx\right]$$
$$\leq (p^+ + 1)\left[\frac{\varepsilon(a+b)}{2}\int_\Omega |u_t|^2\,dx + \frac{2a}{\varepsilon}\int_\Omega |y+z|^{2p(x)}\,dx + \frac{2b}{\varepsilon}\int_\Omega |y|^{p(x)-1}\,|z|^{p(x)+1}\,dx\right]$$
$$\leq c_1\left[\frac{\varepsilon}{2}\|u_t\|_2^2 + C_\varepsilon\left(\int_\Omega |y+z|^{2p^+} + \int_\Omega |y+z|^{2p^-} + \int_\Omega |y|^{3(p(x)-1)} + \int_\Omega |z|^{\frac{3}{2}(p(x)+1)}\right)\right]$$
$$\leq c_2\left[\varepsilon\|u_t\|_2^2 + \|\Delta y\|_2^{2p^-} + \|\nabla z\|_2^{2p^-} + \|\Delta y\|_2^{2p^+} + \|\nabla z\|_2^{2p^+}\right]$$
$$+ c_2\left[\|\Delta y\|_2^{3(p^- -1)} + \|\Delta y\|_2^{3(p^+ -1)} + \|\nabla z\|_2^{\frac{3}{2}(p^- +1)} + \|\nabla z\|_2^{\frac{3}{2}(p^+ +1)}\right], \tag{3.21}$$

where ε, c_1, c_2 are positive constants. Likewise, we get

$$\left|\int_\Omega v_t f_2(y,z)dx\right| \leq (p^+ + 1)\left[a\int_\Omega |v_t|\,|y+z|^{p(x)}\,dx + b\int_\Omega |v_t|\,.\,|z|^{\frac{p(x)-1}{2}}\,|y|^{\frac{p(x)+1}{2}}\,dx\right]$$
$$\leq c_2\left[\varepsilon\|v_t\|_2^2 + \|\Delta y\|_2^{2p^-} + \|\nabla z\|_2^{2p^-} + \|\Delta y\|_2^{2p^+} + \|\nabla z\|_2^{2p^+}\right]$$
$$+ c_2\left[\|\nabla z\|_2^{3(p^- -1)} + \|\nabla z\|_2^{3(p^+ -1)} + \|\Delta y\|_2^{\frac{3}{2}(p^- +1)} + \|\Delta y\|_2^{\frac{3}{2}(p^+ +1)}\right]. \tag{3.22}$$

Combining (3.21) and (3.22), yields

$$\sup_{(0,T)}\int_0^t \left(\left|\int_\Omega u_t f_1(y,z)dx\right| + \left|\int_\Omega v_t f_2(y,z)dx\right|\right)ds \leq \varepsilon T c_2\,\|(u,v)\|_{A_T\times B_T}^2$$

$$+ 2Tc_2\left(\|(y,z)\|_{A_T\times B_T}^{2p^-} + \|(y,z)\|_{A_T\times B_T}^{2p^+}\right)$$
$$+ Tc_2\left(\|(y,z)\|_{A_T\times B_T}^{3(p^--1)} + \|(y,z)\|_{A_T\times B_T}^{3(p^+-1)} + \|(y,z)\|_{A_T\times B_T}^{\frac{3}{2}(p^-+1)} + \|(y,z)\|_{A_T\times B_T}^{\frac{3}{2}(p^++1)}\right). \tag{3.23}$$

By substituting (3.23) into (3.20), we obtain, for some $c_3 > 0$,

$$\frac{1}{2}\|(u,v)\|_{A_T\times B_T}^2 \le \gamma_0 + \varepsilon Tc_3\|(u,v)\|_{A_T\times B_T}^2$$
$$+ 2Tc_3\left(\|(y,z)\|_{A_T\times B_T}^{2p^-} + \|(y,z)\|_{A_T\times B_T}^{2p^+}\right)$$
$$+ Tc_3\left(\|(y,z)\|_{A_T\times B_T}^{3(p^--1)} + \|(y,z)\|_{A_T\times B_T}^{3(p^+-1)} + \|(y,z)\|_{A_T\times B_T}^{\frac{3}{2}(p^-+1)} + \|(y,z)\|_{A_T\times B_T}^{\frac{3}{2}(p^++1)}\right). \tag{3.24}$$

Choosing ε such that $\varepsilon Tc_3 = \frac{1}{4}$ and recalling that $\|(y,z)\|_{A_T\times B_T} \le d$, for some $d > 1$ (large enough), inequality (3.24) implies

$$\|(u,v)\|_{A_T\times B_T}^2 \le 4\gamma_0 + 8Tc_3\left(\|(y,z)\|_{A_T\times B_T}^{2p^-} + \|(y,z)\|_{A_T\times B_T}^{2p^+}\right)$$
$$+ 4Tc_3\left(\|(y,z)\|_{A_T\times B_T}^{3(p^--1)} + \|(y,z)\|_{A_T\times B_T}^{3(p^+-1)} + \|(y,z)\|_{A_T\times B_T}^{\frac{3}{2}(p^-+1)} + \|(y,z)\|_{A_T\times B_T}^{\frac{3}{2}(p^++1)}\right)$$
$$\le 4\gamma_0 + Tc_4 d^{3(p^+-1)},\ c_4 > 0,$$

So, if we take d such that $d^2 >> 4\gamma_0$ and $T \le T_0 = \frac{d^2-4\gamma_0}{c_4 d^{3(p^+-1)}}$, we find

$$4\gamma_0 + Tc_4 d^{3(p^+-1)} \le d^2.$$

Therefore,

$$\|(u,v)\|_{A_T\times B_T}^2 \le d^2.$$

Thus, F maps $D(0,d)$ to $D(0,d)$.

Lemma 3.3. *$F : D(0,d) \longrightarrow D(0,d)$ is a contraction.*

Proof. Let (y_1,z_1) and (y_2,z_2) be in $D(0,d)$ and set $(u_1,v_1) = F(y_1,z_1)$ and $(u_2,v_2) = F(y_2,z_2)$. Clearly, $(U,V) = (u_1-u_2, v_1-v_2)$ is a weak solution of the following system

$$\begin{cases} U_{tt} + \Delta^2 U + |u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t} & \\ = f_1(y_1,z_1) - f_1(y_2,z_2) & \text{in } \Omega\times(0,T), \\ V_{tt} - \Delta V + |v_{1t}|^{r(x)-2}v_{1t} - |v_{2t}|^{r(x)-2}v_{2t} & \\ = f_2(y_1,z_1) - f_2(y_2,z_2) & \text{in } \Omega\times(0,T), \\ U = V = 0 & \text{on } \partial\Omega\times(0,T), \\ (U(0),V(0)) = (U_t(0),V_t(0)) = (0,0) & \text{in } \Omega, \end{cases}$$

in the sense of Definition 3.1. So, taking $(\Phi,\Psi) = (U_t,V_t)$, in this definition, using Green's formula together with the boundary conditions and then, integrating each result over $(0,t)$, we obtain, for a.e. $t \le T$,

$$\frac{1}{2}\left(\|U_t\|_2^2 + \|\Delta U\|_2^2\right) + \int_0^t\int_\Omega \left(u_{1t}|u_{1t}|^{m(x)-2} - u_{2t}|u_{2t}|^{m(x)-2}\right)U_t\,dxds$$

$$\leq \int_0^t \int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|\,|U_t| dxds$$

and

$$\begin{aligned}
&\frac{1}{2}\left(\|V_t\|_2^2 + \|\nabla V\|_2^2\right) + \int_0^t \int_\Omega \left(v_{1t}\,|v_{1t}|^{r(x)-2} - v_{2t}\,|v_{2t}|^{r(x)-2}\right) V_t dxds \\
&\leq \int_0^t \int_\Omega |f_2\,(y_1,z_1) - f_2\,(y_2,z_2)|\,|V_t| dxds.
\end{aligned}$$

Under the condition $(H.2)$, using Hölder's inequality and inequality (3.12), these two estimates give, for $n = 1, 2, 3$,

$$\|U_t\|_2^2 + \|U\|_V^2 \leq 4 \int_0^t \|U_t\|_2 \|f_1(y_1,z_1) - f_1(y_2,z_2)\|_2 ds \tag{3.25}$$

and

$$\|V_t\|_2^2 + \|\nabla V\|_2^2 \leq 4 \int_0^t \|V_t\|_2 \|f_2\,(y_1,z_1) - f_2\,(y_2,z_2)\,\|_2 ds. \tag{3.26}$$

The addition of (3.25) and (3.26) imply

$$\begin{aligned}
\|U_t\|_2^2 + \|V_t\|_2^2 + \|U\|_V^2 + \|\nabla V\|_2^2 &\leq 4 \int_0^t \|U_t\|_2 \|f_1(y_1,z_1) - f_1(y_2,z_2)\|_2 ds \\
+ 4 \int_0^t \|V_t\|_2 \|f_2\,(y_1,z_1) - f_2\,(y_2,z_2)\,\|_2 ds, &
\end{aligned} \tag{3.27}$$

for all $t \in (0, T)$. Now, we estimate the terms:

$$\|f_1(y_1,z_1) - f_1(y_2,z_2)\|_2 \text{ and } \|f_2\,(y_1,z_1) - f_2\,(y_2,z_2)\,\|_2.$$

Using appropriate algebraic inequalities (see [21]), we obtain for two constants $C_1, C_2 > 0$ and for all $x \in \Omega$ and $t \in (0, T)$,

$$\int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|^2 \, dx \leq I_1 + I_2 + I_3 + I_4, \tag{3.28}$$

where

$$\begin{aligned}
I_1 &= C_1 \int_\Omega |y_1 - y_2|^2 (|y_1|^{2(p(x)-1)} + |z_1|^{2(p(x)-1)}) dx \\
&\quad + C_1 \int_\Omega |y_1 - y_2|^2 (|y_2|^{2(p(x)-1)} + |z_2|^{2(p(x)-1)}) dx, \\
I_2 &= C_1 \int_\Omega |z_1 - z_2|^2 (|y_1|^{2(p(x)-1)} + |z_1|^{2(p(x)-1)}) dx \\
&\quad + C_1 \int_\Omega |z_1 - z_2|^2 (|y_1|^{2(p(x)-1)} + |z_2|^{2(p(x)-1)}) dx,
\end{aligned}$$

$$I_3 = C_2 \int_\Omega |z_1 - z_2|^2 |y_1|^{p(x)-1} \left(|z_1|^{p(x)-1} + |z_2|^{p(x)-1}\right) dx,$$
$$I_4 = C_2 \int_\Omega |y_1 - y_2|^2 |z_2|^{p(x)+1} \left(|y_1|^{p(x)-3} + |y_2|^{p(x)-3}\right) dx.$$

By using Hölder's and Young's inequalities and the Sobolev embedding (Lemma 2.2), we get the following estimate for a typical term in I_1 and I_2,

$$\begin{aligned}
&\int_\Omega |y_1 - y_2|^2 \, |y_1|^{2(p(x)-1)} \, dx \le 2 \left(\int_\Omega |y_1 - y_2|^6 dx\right)^{\frac{1}{3}} \left(\int_\Omega |y_1|^{3(p(x)-1)}\right)^{\frac{2}{3}} \\
&\le C\|y_1 - y_2\|_6^2 \left[\left(\int_\Omega |y_1|^{3(p^+-1)} dx\right)^{\frac{2}{3}} + \left(\int_\Omega |y_1|^{3(p^--1)} dx\right)^{\frac{2}{3}}\right] \\
&\le C\|\Delta(y_1 - y_2)\|_2^2 \left(\|y_1\|_{3(p^+-1)}^{2(p^+-1)} + \|y_1\|_{3(p^--1)}^{2(p^--1)}\right) \\
&\le C\|\Delta Y\|_2^2 \left(\|\Delta y_1\|_2^{2(p^+-1)} + \|\Delta y_1\|_2^{2(p^--1)}\right) \\
&\le C\|\Delta Y\|_2^2 \left(\|(y_1, z_1)\|_{A_T \times B_T}^{2(p^+-1)} + \|(y_1, z_1)\|_{A_T \times B_T}^{2(p^--1)}\right),
\end{aligned} \tag{3.29}$$

since

- $1 \le 3(p^- - 1) \le 3(p^+ - 1) < \infty$, when $n = 1, 2$.
- $1 \le 3(p^- - 1) = 3(p^+ - 1) = 6 = \frac{2n}{n-2}$, when $n = 3$.

Likewise, we obtain

$$\int_\Omega |z_1 - z_2|^2 \, |y_2|^{2(p(x)-1)} \, dx \le C\|\nabla Z\|_2^2 \left(\|(y_2, z_2)\|_{A_T \times B_T}^{2(p^+-1)} + \|(y_2, z_2)\|_{A_T \times B_T}^{2(p^--1)}\right). \tag{3.30}$$

Since $(y_1, z_1), (y_2, z_2) \in D(0, d)$ and $d > 1$, estimates (3.29) and (3.30) lead to

$$I_1 \le C\|\Delta Y\|_2^2 \;\; d^{2(p^+-1)} \text{ and } \; I_2 \le C\|\nabla Z\|_2^2 \;\; d^{2(p^+-1)}.$$

Hence,

$$I_1 + I_2 \le C d^{2(p^+-1)} \left(\|\Delta Y\|_2^2 + \|\nabla Z\|_2^2\right). \tag{3.31}$$

Similarly, a typical term in I_3 can be handled as follows

$$\begin{aligned}
&\int_\Omega |z_1 - z_2|^2 \, |y_1|^{p(x)-1} \, |z_1|^{p(x)-1} \, dx \\
&\le 2 \left(\int_\Omega |z_1 - z_2|^6 dx\right)^{\frac{1}{3}} \left(\int_\Omega |y_1|^{\frac{3}{2}(p(x)-1)} |z_1|^{\frac{3}{2}(p(x)-1)}\right)^{\frac{2}{3}} \\
&\le C\|z_1 - z_2\|_6^2 \left[\left(\int_\Omega |y_1|^{\frac{3}{2}(p(x)-1)} dx\right)^{\frac{2}{3}} + \left(\int_\Omega |z_1|^{\frac{3}{2}(p(x)-1)} dx\right)^{\frac{2}{3}}\right] \\
&\le C\|\nabla(z_1 - z_2)\|_2^2 \left(\|y_1\|_{\frac{3}{2}(p^+-1)}^{(p^+-1)} + \|y_1\|_{\frac{3}{2}(p^--1)}^{(p^--1)} + \|z_1\|_{\frac{3}{2}(p^+-1)}^{(p^+-1)} + \|z_1\|_{\frac{3}{2}(p^--1)}^{(p^--1)}\right) \\
&\le C\|\nabla(z_1 - z_2)\|_2^2 \left(\|\Delta y_1\|_2^{(p^+-1)} + \|\Delta y_1\|_2^{(p^--1)} + \|\nabla z_1\|_2^{(p^+-1)} + \|\nabla z_1\|_2^{(p^--1)}\right) \\
&\le 2C\|\nabla Z\|_2^2 \left(\|(y_1, z_1)\|_{A_T \times B_T}^{(p^+-1)} + \|(y_1, z_1)\|_{A_T \times B_T}^{(p^--1)}\right),
\end{aligned}$$

since

- $1 \leq \frac{3}{2}(p^- - 1) \leq \frac{3}{2}(p^+ - 1) < \infty$, when $n = 1, 2$.
- $1 \leq \frac{3}{2}(p^- - 1) = \frac{3}{2}(p^+ - 1) = 6 = \frac{2n}{n-2}$, when $n = 3$.

Therefore,

$$I_3 \leq Cd^{p^+-1} \|\nabla Z\|_2^2, \tag{3.32}$$

since $(y_1, z_1), (y_2, z_2) \in D(0, d)$. Using the same arguments, a typical term in I_4, can be estimated as follows:

Case 1: If $n = 1, 2$, we have $3 \leq p^- \leq p^+ < \infty$. So,

$$\begin{aligned}
&\int_\Omega |y_1 - y_2|^2 \, |z_2|^{p(x)+1} \, |y_1|^{p(x)-3} \, dx \\
&\leq 2 \left(\int_\Omega |y_1 - y_2|^3 dx \right)^{\frac{2}{3}} \left(\int_\Omega |z_2|^{3(p(x)+1)} |y_1|^{3(p(x)-3)} \right)^{\frac{1}{3}} \\
&\leq C \|y_1 - y_2\|_3^2 \left[\left(\int_\Omega |z_2|^{6(p(x)+1)} dx \right)^{\frac{1}{3}} + \left(\int_\Omega |y_1|^{6(p(x)-3)} dx \right)^{\frac{1}{3}} \right] \\
&\leq C \|\Delta Y\|_2^2 \left(\|\nabla z_2\|_2^{2(p^++1)} + \|\nabla z_2\|_2^{2(p^-+1)} + \|\Delta y_1\|_2^{2(p^+-3)} + \|\Delta y_1\|_2^{2(p^--3)} \right) \\
&\leq 4C \|\Delta Y\|_2^2 \; d^{2(p^++1)},
\end{aligned}$$

since $(y_1, z_1), (y_2, z_2) \in D(0, d)$ and $d > 1$.

Case 2: If $n = 3$, then $p \equiv 3$ on $\overline{\Omega}$. Hence,

$$\begin{aligned}
\int_\Omega |y_1 - y_2|^2 \, |z_2|^{p(x)+1} \, |y_1|^{p(x)-3} \, dx &= \int_\Omega |y_1 - y_2|^2 \, |z_2|^4 \, dx \\
&\leq C \left(\int_\Omega |y_1 - y_2|^6 \, dx \right)^{\frac{1}{3}} \left(\int_\Omega |z_2|^6 \, dx \right)^{\frac{2}{3}} \\
&\leq C \|y_1 - y_2\|_6^2 . \|z_2\|_6^4 \\
&\leq C \|\Delta Y\|_2^2 . \|(y_2, z_2)\|_{A_T \times B_T}^4 .
\end{aligned}$$

So, for all $t \in (0, T)$, we deduce that

$$I_4 \leq C \|\Delta Y\|_2^2 d^{2(p^++1)}. \tag{3.33}$$

Finally, by substituting (3.31)–(3.33) in (3.28), the following can be obtained

$$\int_\Omega |f_1(y_1, z_1) - f_1(y_2, z_2)|^2 \, dx \leq Cd^{2(p^++1)} \left(\|\Delta Y\|_2^2 + \|\nabla Z\|_2^2 \right), \tag{3.34}$$

for all $t \in (0, T)$. Similarly, we get

$$\int_\Omega |f_2(y_1, z_1) - f_2(y_2, z_2)|^2 \, dx \leq Cd^{2(p^++1)} \left(\|\Delta Y\|_2^2 + \|\nabla Z\|_2^2 \right). \tag{3.35}$$

Now, we use (3.34) and (3.35) in (3.27) to obtain

$$\|(u, v)\|_{A_T \times B_T}^2 \leq Cd^{2(p^++1)} \sup_{(0,T)} \int_0^t \left(\|\Delta Y(s)\|_2^2 + \|\nabla Z(s)\|_2^2 \right) ds$$

$$\leq Cd^{2(p^{+}+1)}T\,\|(Y,Z)\|^{2}_{A_T\times B_T}.$$

Hence, if we take T small enough, we get for, $0<\gamma<1$,

$$\|(u,v)\|^{2}_{A_T\times B_T}\leq \gamma\,\|(Y,Z)\|^{2}_{A_T\times B_T}.$$

Thus,

$$\|K(y_1,z_1)-K(y_2,z_2)\|^{2}_{A_T\times B_T}\leq \gamma\,\|(y_1,z_1)-(y_2,z_2)\|^{2}_{A_T\times B_T}.$$

This proves that $F: D(0,d)\longrightarrow D(0,d)$ is a contraction.

Theorem 3.2. *Let* $n=1,2,3$. *Under the assumptions* $(H.1)$ *and* $(H.2)$ *and for any* $(u_0,v_0)\in\mathcal{V}\times H_0^1(\Omega)$, $(u_1,v_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *the problem* (1.2) *admits a unique weak solution* (u,v), *in the sense of Definition* 3.1, *having the regularity* (3.1), *for T small enough.*

Proof. The above Lemmas and the Banach-fixed-point theorem guarantee the existence of a unique $(u,v)\in D(0,d)$, such that $F(u,v)=(u,v)$, which is a local weak solution of (1.2).

Remark 3.1. *From the definitions* (1.3) *and* (1.4), *one can easily see that, for all* $(u,v)\in\mathbb{R}^2$,

$$u\,f_1(x,u,v)+vf_2(x,u,v)=(p(x)+1)F(x,u,v). \tag{3.36}$$

We, also, have the following results.

Lemma 3.1. *[22] There exist* $C_1,C_2>0$ *such that, for all* $x\in\overline{\Omega}$ *and* $(u,v)\in\mathbb{R}^2$, *we have*

$$C_1(|u|^{p(x)+1}+|v|^{p(x)+1})\leq F(x,u,v)\leq C_2(|u|^{p(x)+1}+|v|^{p(x)+1}). \tag{3.37}$$

Corollary 3.2. *For all* $x\in\overline{\Omega}$ *and* $(u,v)\in\mathbb{R}^2$, *we have*

$$C_1(\zeta(u)+\zeta(v))\leq\int_\Omega F(x,u,v)dx\leq C_2(\zeta(u)+\zeta(v)), \tag{3.38}$$

where

$$\zeta(u)=\int_\Omega |u|^{p(x)+1}\,dx \text{ and } \zeta(v)=\int_\Omega |v|^{p(x)+1}\,dx.$$

Now, we introduce the energy functional associated with our problem

$$E(t)=\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2+\|\Delta u\|_2^2+\|\nabla v\|_2^2\right)-\int_\Omega F(x,u,v)\,dx, \tag{3.39}$$

for all $t\in[0,T)$. A direct computation implies, for *a.e.* $t\in(0,T)$,

$$E'(t)=-\int_\Omega |u_t|^{m(x)}\,dx-\int_\Omega |v_t|^{r(x)}\,dx\leq 0. \tag{3.40}$$

4. Blow-up result

In this section, our goal is to prove that any solution of Problem (1.2) blows-up in some finite time T^*, if

$$\max\{m^+, r^+\} < p^- \text{ and } 0 < E(0) < E_1, \tag{4.1}$$

where

$$E_1 = \left(\frac{1}{2} - \frac{1}{p^- + 1}\right)\gamma_1^2, \quad \gamma_1 = \left(d_* \left(p^- + 1\right)\right)^{\frac{1}{1-p^-}}, \tag{4.2}$$

$$d_* = \left(\sqrt{2^{(p^-+1)}}a + 2b\right) c_*^{p^-+1}$$

and c_* is a positive constant, which comes from the Sobolev embedding.

Remark 4.1. *The following well-known inequalities are needed in the proof of the lemmas.*

(1) For $A, B \geq 0$ *and* $d \geq 1$*, we have*

$$(A + B)^d \leq 2^{d-1}\left(A^d + B^d\right). \tag{4.3}$$

(2) For $z \geq 0,\ 0 < \delta \leq 1$ *and* $a > 0$*, we have*

$$z^\delta \leq z + 1 \leq \left(1 + \frac{1}{a}\right)(z + a). \tag{4.4}$$

(3) For $X,\ Y \geq 0,\ \ \delta > 0$ *and* $\frac{1}{\lambda} + \frac{1}{\beta} = 1$*, Young's inequality gives*

$$XY \leq \frac{\delta^\lambda}{\lambda} X^\lambda + \frac{\delta^{-\beta}}{\beta} Y^\beta. \tag{4.5}$$

(4) The embedding Lemma 2.2, Hölder's and Young's inequalities and (4.3) *imply that*

$$\|u + v\|_{p(.)+1} \leq \sqrt{2}c_* \left[\left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right)\right]^{1/2} \tag{4.6}$$

and

$$\|uv\|_{\frac{p(.)+1}{2}} \leq c_*^2\left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right). \tag{4.7}$$

Lemma 4.1. *For any solution* (u, v) *of the system* (1.2)*, with initial energy*

$$E(0) < E_1 \tag{4.8}$$

and

$$\gamma_1 < \left(\|\Delta u_0\|_2^2 + \|\nabla v_0\|_2^2\right)^{1/2} \leq \frac{1}{\sqrt{2}c_*}, \tag{4.9}$$

there exists $\gamma_2 > \gamma_1$ *such that*

$$\gamma_2 \leq \left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right)^{1/2},\ \forall t \in [0, T). \tag{4.10}$$

Proof. Let $\gamma = \left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right)^{1/2}$, then using (3.39), we have

$$E(t) \geq \frac{1}{2}\gamma^2 - \int_\Omega F(x,u,v)\,dx. \tag{4.11}$$

The use of Lemma 2.1, (4.6) and (4.7) leads to

$$\begin{aligned}\int_\Omega F(x,u,v)\,dx &= a\int_\Omega |u+v|^{p(x)+1}\,dx + 2b\int_\Omega |uv|^{\frac{p(x)+1}{2}}\,dx\\ &\leq a\max\left\{\|u+v\|_{p(.)+1}^{p^-+1}, \|u+v\|_{p(.)+1}^{p^++1}\right\}\\ &\quad + 2b\max\left\{\|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^-+1}{2}}, \|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^++1}{2}}\right\}\\ &\leq a\max\left\{\left(\sqrt{2}c_*\gamma\right)^{p^-+1}, \left(\sqrt{2}c_*\gamma\right)^{p^++1}\right\}\\ &\quad + 2b\max\left\{(c_*\gamma)^{p^-+1}, (c_*\gamma)^{p^++1}\right\}.\end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we obtain

$$\begin{aligned}E(t) \geq &\frac{1}{2}\gamma^2 - a\max\left\{\left(\sqrt{2}c_*\gamma\right)^{p^-+1}, \left(\sqrt{2}c_*\gamma\right)^{p^++1}\right\}\\ &- 2b\max\left\{(c_*\gamma)^{p^-+1}, (c_*\gamma)^{p^++1}\right\}.\end{aligned} \tag{4.13}$$

For γ in $\left[0, \frac{1}{\sqrt{2}c_*}\right]$, one can easily check that

$$c_*^2\gamma^2 \leq 2c_*^2\gamma^2 \leq 1.$$

Consequently, we have

$$\left(\sqrt{2}c_*\gamma\right)^{p^-+1} \geq \left(\sqrt{2}c_*\gamma\right)^{p^++1} \text{ and } (c_*\gamma)^{p^-+1} \geq \left(\sqrt{2}c_*\gamma\right)^{p^++1}.$$

Thus, (4.13) reduces to

$$E(t) \geq \frac{1}{2}\gamma^2 - \left(\sqrt{2^{(p^-+1)}}a + 2b\right)c_*^{p^-+1}\gamma^{p^-+1}.$$

If we set

$$h(\gamma) = \frac{1}{2}\gamma^2 - k\gamma^{p^-+1}, \text{where } k = \left(\sqrt{2^{(p^-+1)}}a + 2b\right)c_*^{p^-+1},$$

then

$$E(t) \geq h(\gamma), \text{ for all } \gamma \in \left[0, \frac{1}{\sqrt{2}c_*}\right]. \tag{4.14}$$

It is clear that h is strictly increasing on $[0,\gamma_1)$ and strictly decreasing on $[\gamma_1,+\infty)$. Since $E(0) < E_1$ and $E_1 = h(\gamma_1)$, then, we can find $\gamma_2 > \gamma_1$ such that $h(\gamma_2) = E(0)$. But,

$$\alpha_0 = \left(\|\Delta u_0\|_2^2 + \|\nabla v_0\|_2^2\right)^{1/2},$$

therefore, by (4.14), we get

$$h(\gamma_2) = E(0) \geq h(\gamma_0).$$

This implies that $\gamma_0 \geq \gamma_2$. Hence, $\gamma_2 \in \left(\gamma_1, \frac{1}{\sqrt{2}c_*}\right]$. To prove (4.10), we assume that there is a $t_0 \in [0, T)$ such that

$$\left(\|\Delta u(., t_0)\|_2^2 + \|\nabla v(., t_0)\|_2^2\right)^{1/2} < \gamma_2.$$

Since $\left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right)^{1/2}$ is continuous and $\gamma_2 > \gamma_1$, t_0 can be selected so that

$$\left[\|\Delta u(., t_0)\|_2^2 + \|\nabla v(., t_0)\|_2^2\right]^{1/2} > \gamma_1.$$

Using (4.14) and the fact that h is decreasing on $\left[\gamma_1, \frac{1}{\sqrt{2}c_*}\right]$, we obtain

$$E(t_0) \geq h\left(\left(\|\Delta u(., t_0)\|_2^2 + \|\nabla v(., t_0)\|_2^2\right)^{1/2}\right) > h(\gamma_2) = E(0),$$

which contradicts the fact that $E(t) \leq E(0)$, for all $t \in [0.T)$. Thus, (4.10) is established.

Lemma 4.2. *Let $\mathcal{H}(t) = E_1 - E(t)$, for all $t \in [0, T)$. Then, we have*

$$0 < \mathcal{H}(0) \leq \mathcal{H}(t) \leq \int_\Omega F(x, u, v)\, dx, \text{ for all } t \in [0, T) \tag{4.15}$$

and

$$\int_\Omega F(x, u, v)\, dx \geq d_* \gamma_2^{p^-+1}. \tag{4.16}$$

Proof. Using (3.40), (4.8) and (4.11), we have

$$0 < E_1 - E(0) = H(0) \leq H(t) \leq E_1 - \frac{1}{2}\gamma^2 + \int_\Omega F(x, u, v)\, dx. \tag{4.17}$$

From the fact that $h(\gamma_1) = \frac{1}{2}\gamma_1^2 - d_* \gamma_1^{p^-+1} = E_1$, we have

$$E_1 - \frac{1}{2}\gamma_1^2 = -d_* \gamma_1^{p^-+1},$$

then since $\gamma \geq \gamma_2 > \gamma_1$, we obtain

$$\mathcal{H}(t) \leq -d_* \gamma_1^{p^-+1} + \int_\Omega F(x, u, v)\, dx \leq \int_\Omega F(x, u, v)\, dx.$$

Thus, (4.15) is established. To establish (4.16), we use (4.15) to obtain

$$E(0) \geq \frac{1}{2}\gamma^2 - \int_\Omega F(x, u, v)\, dx,$$

which implies,

$$\int_\Omega F(x, u, v)\, dx \geq \frac{1}{2}\gamma^2 - E(0).$$

But $E(0) = h(\gamma_2)$ and $\gamma \geq \gamma_2$, so

$$\int_\Omega F(x, u, v)\, dx \geq \frac{1}{2}\gamma_2^2 - h(\gamma_2) = d_* \gamma_2^{p^-+1}.$$

Lemma 4.3. *There exist $C_3, C_4, C_5 > 0$ such that any solution of (1.2) satisfies*

$$\|u\|_{p^-+1}^{p^-+1} + \|v\|_{p^-+1}^{p^-+1} \le C_3\left(\zeta(u) + \zeta(v)\right), \tag{4.18}$$

$$\int_\Omega |u|^{m(x)}\,dx \le C_4\left[\left(\zeta(u) + \zeta(v)\right)^{\frac{m^+}{p^-+1}} + \left(\zeta(u) + \zeta(v)\right)^{\frac{m^-}{p^-+1}}\right] \tag{4.19}$$

and

$$\int_\Omega |v|^{r(x)}\,dx \le C_5\left[\left(\zeta(u) + \zeta(v)\right)^{\frac{r^+}{p^-+1}} + \left(\zeta(u) + \zeta(v)\right)^{\frac{r^-}{p^-+1}}\right], \tag{4.20}$$

where $\zeta(u)$ and $\zeta(v)$ are defined in Corollary 3.2.

Proof. We define the following partition of Ω

$$\Omega_+ = \{x \in \Omega \;/\; |u(x,t)| \ge 1\} \text{ and } \Omega_- = \{x \in \Omega \;/\; |u(x,t)| < 1\}.$$

The properties of $p(.)$ and Hölder's inequality imply that, for some $c_1 > 0$,

$$\begin{aligned}\zeta(u) &= \int_{\Omega_+} |u|^{p(x)+1}\,dx + \int_{\Omega_-} |u|^{p(x)+1}\,dx\\ &\ge \int_{\Omega_+} |u|^{p^-+1}\,dx + \int_{\Omega_-} |u|^{p^++1}\,dx\\ &\ge \int_{\Omega_+} |u|^{p^-+1}\,dx + c_1\left(\int_{\Omega_-} |u|^{p^-+1}\,dx\right)^{\frac{p^++1}{p^-+1}}.\end{aligned}$$

Hence,

$$\zeta(u) \ge \int_{\Omega_+} |u|^{p^-+1}\,dx \text{ and } \left(\frac{\zeta(u)}{c_1}\right)^{\frac{p^-+1}{p^++1}} \ge \int_{\Omega_-} |u|^{p^-+1}\,dx. \tag{4.21}$$

Use (4.21) to obtain, for some $c_2 > 0$.

$$\begin{aligned}\|u\|_{p^-+1}^{p^-+1} &\le \zeta(u) + c_2\left(\zeta(u)\right)^{\frac{p^-+1}{p^++1}}\\ &\le \zeta(u) + \zeta(v) + c_2\left(\zeta(u) + \zeta(v)\right)^{\frac{p^-+1}{p^++1}}\\ &= \left(\zeta(u) + \zeta(v)\right)\left[1 + c_2\left(\zeta(u) + \zeta(v)\right)^{\frac{p^--p^+}{p^++1}}\right].\end{aligned}$$

Recalling (3.38) and (4.15), we deduce that

$$0 < \mathcal{H}(0) \le \mathcal{H}(t) \le C_2\left(\zeta(u) + \zeta(v)\right). \tag{4.22}$$

Therefore,

$$\|u\|_{p^-+1}^{p^-+1} \le \left(\zeta(u) + \zeta(v)\right)\left[1 + c_2\left(\mathcal{H}(0)/C_2\right)^{\frac{p^--p^+}{p^++1}}\right] \le c\left(\zeta(u) + \zeta(v)\right).$$

Similarly, we arrive at

$$\|v\|_{p^-+1}^{p^-+1} \le c\left(\zeta(u) + \zeta(v)\right).$$

Therefore, (4.18) is established. To establish (4.19), we recall that $p^- \geq \max\{m^+, r^+\}$, to conclude that

$$\begin{aligned}\int_\Omega |u|^{m(x)}\,dx &\leq \int_{\Omega_+} |u|^{m^+}\,dx + \int_{\Omega_-} |u|^{m^-}\,dx \\ &\leq c\left(\int_{\Omega_+} |u|^{p^-+1}\,dx\right)^{\frac{m^+}{p^-+1}} + c\left(\int_{\Omega_-} |u|^{p^-+1}\,dx\right)^{\frac{m^-}{p^-+1}} \\ &\leq c\left(\|u\|_{p^-+1}^{m^+} + \|u\|_{p^-+1}^{m^-}\right),\ c > 0.\end{aligned}$$

Using similar calculations as above, we obtain (4.19) and (4.20).

Lemma 4.4. *Let $\mathcal{G}(t) = \mathcal{H}^{1-\sigma}(t) + \varepsilon\int_\Omega (uu_t + vv_t)\,dx, t > 0$, where $\varepsilon > 0$ to be fixed later. Then, there exists $\rho > 0$, such that*

$$\mathcal{G}'(t) \geq \varepsilon\rho\left(\mathcal{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \zeta(u) + \zeta(v)\right) \tag{4.23}$$

and hence,

$$\mathcal{G}(t) \geq \mathcal{G}(0) > 0, \text{ for } t > 0,$$

where

$$0 < \sigma \leq \min\left\{\frac{p^- - m^+ + 1}{(p^- + 1)(m^+ - 1)}, \frac{p^- - r^+ + 1}{(p^- + 1)(r^+ - 1)}, \frac{p^- - 1}{2(p^- + 1)}\right\}. \tag{4.24}$$

Proof. Differentiate $\mathcal{G}$ and use (1.2) to have

$$\begin{aligned}\mathcal{G}'(t) &= (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &+ \varepsilon\int_\Omega (uf_1(x,u,v) + vf_2(x,u,v))\,dx - \varepsilon\left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right) \\ &- \varepsilon\int_\Omega \left(|u_t|^{m(x)-2}u_t u + |v_t|^{r(x)-2}v_t v\right)dx.\end{aligned} \tag{4.25}$$

By the definition of $\mathcal{H}$ and E, we get

$$\|\Delta u\|_2^2 + \|\nabla v\|_2^2 = 2\int_\Omega F(x,u,v)\,dx - \|u_t\|_2^2 - \|v_t\|_2^2 + 2E_1 - 2\mathcal{H}(t). \tag{4.26}$$

Combining (3.36), (4.25) and (4.26), we obtain

$$\begin{aligned}\mathcal{G}'(t) &\geq (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + 2\varepsilon\mathcal{H}(t) \\ &- 2\varepsilon E_1 + \varepsilon(p^- - 1)\int_\Omega F(x,u,v)\,dx \\ &- \varepsilon\int_\Omega \left(|u|\,|u_t|^{m(x)-1} + |v|\,|v_t|^{r(x)-1}\right)dx.\end{aligned} \tag{4.27}$$

A combination of (4.16) and (4.27) leads to

$$\begin{aligned}\mathcal{G}'(t) &\geq (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \qquad (4.28) \\ &+ \varepsilon\left(p^- - 1 - 2\left(d_*\gamma_2^{p^-+1}\right)^{-1}E_1\right)\int_\Omega F(x,u,v)\,dx\end{aligned}$$

$$+2\varepsilon\mathcal{H}(t)-\varepsilon\int_{\Omega}\left(|u|\,|u_t|^{m(x)-1}+|v|\,|v_t|^{r(x)-1}\right)dx, \tag{4.29}$$

where $p^{-}-1-2\left(d_*\alpha_2^{p^{-}+1}\right)^{-1}E_1>0$, since $\gamma_2>\gamma_1$.

Now, the last two terms of (4.29) can be estimated by applying (4.5) with $X=|u|$, $Y=|u_t|^{m(x)-1}$, $\lambda=m(x)$, $\beta=\frac{m(x)}{m(x)-1}$, as follows:

$$\begin{aligned}\int_{\Omega}|u|\,|u_t|^{m(x)-1}\,dx&\le\int_{\Omega}\frac{\delta^{m(x)}}{m(x)}|u|^{m(x)}\,dx\\&\quad+\int_{\Omega}\frac{m(x)-1}{m(x)}\delta^{-m(x)/(m(x)-1)}|u_t|^{m(x)}\,dx.\end{aligned}\tag{4.30}$$

Let $\tilde{k}$ be a positive constant to be selected later and take $\delta=\left[\tilde{k}\mathcal{H}^{-\sigma}(t)\right]^{\frac{1-m(x)}{m(x)}}$ to obtain

$$\begin{aligned}\int_{\Omega}|u|\,|u_t|^{m(x)-1}\,dx&\le\frac{\tilde{k}^{1-m^{-}}}{m^{-}}\int_{\Omega}[\mathcal{H}(t)]^{\sigma(m(x)-1)}|u|^{m(x)}\,dx\\&\quad+\frac{m^{+}-1}{m^{-}}\tilde{k}\mathcal{H}^{-\sigma}(t)\int_{\Omega}|u_t|^{m(x)}\,dx.\end{aligned}\tag{4.31}$$

The properties of $m(x)$ and $\mathcal{H}(t)$ give

$$\begin{aligned}\int_{\Omega}[\mathcal{H}(t)]^{\sigma(m(x)-1)}|u|^{m(x)}\,dx&=\int_{\Omega}\left[\frac{\mathcal{H}(t)}{\mathcal{H}(0)}\right]^{\sigma(m(x)-1)}[\mathcal{H}(0)]^{\sigma(m(x)-1)}|u|^{m(x)}\,dx\\&\le\tilde{c_2}\,[\mathcal{H}(t)]^{\sigma\left(m^{+}-1\right)}\int_{\Omega}[\mathcal{H}(0)]^{\sigma(m(x)-1)}|u|^{m(x)}\,dx,\end{aligned}$$

where $\tilde{c_2}=1/[\mathcal{H}(0)]^{\sigma\left(m^{+}-1\right)}$. But $[\mathcal{H}(0)]^{\sigma(m(x)-1)}\le c_3$, for all $x\in\Omega$, where $c_3>0$. So, for some $c_4>0$, we get

$$\int_{\Omega}[\mathcal{H}(t)]^{\sigma(m(x)-1)}|u|^{m(x)}\,dx\le c_4\,[\mathcal{H}(t)]^{\sigma\left(m^{+}-1\right)}\int_{\Omega}|u|^{m(x)}\,dx. \tag{4.32}$$

Combining (4.31) and (4.32) to obtain

$$\begin{aligned}\int_{\Omega}|u|\,|u_t|^{m(x)-1}\,dx&\le\frac{c_4\tilde{k}^{1-m^{-}}}{m^{-}}[\mathcal{H}(t)]^{\sigma\left(m^{+}-1\right)}\int_{\Omega}|u|^{m(x)}\,dx\\&\quad+\frac{m^{+}-1}{m^{-}}\tilde{k}\mathcal{H}^{-\sigma}(t)\int_{\Omega}|u_t|^{m(x)}\,dx.\end{aligned}\tag{4.33}$$

Applying Similar calculations, we arrive at

$$\begin{aligned}\int_{\Omega}|v_t|^{r(x)-1}v\,dx&\le\frac{c_5\tilde{k}^{1-r^{-}}}{r^{-}}[\mathcal{H}(t)]^{\sigma\left(r^{+}-1\right)}\int_{\Omega}|v|^{r(x)}\,dx\\&\quad+\frac{r^{+}-1}{r^{-}}\tilde{k}\mathcal{H}^{-\sigma}(t)\int_{\Omega}|v_t|^{r(x)}\,dx.\end{aligned}\tag{4.34}$$

Adding (4.33) and (4.34), we have

$$\begin{aligned}\int_{\Omega}\left(|u|\,|u_t|^{m(x)-1}+|v|\,|v_t|^{r(x)-1}\right)dx \leq & \frac{c_4\tilde{k}^{1-m^-}}{m^-}\left[\mathcal{H}(t)\right]^{\sigma\left(m^+-1\right)}\int_{\Omega}|u|^{m(x)}\,dx\\ &+\frac{c_5\tilde{k}^{1-r^-}}{r^-}\left[\mathcal{H}(t)\right]^{\sigma\left(r^+-1\right)}\int_{\Omega}|v|^{r(x)}\,dx\\ &+\tilde{\alpha}\mathcal{H}^{-\sigma}(t)\left(\int_{\Omega}|u_t|^{m(x)}\,dx+\int_{\Omega}|v_t|^{r(x)}\,dx\right),\end{aligned}\tag{4.35}$$

where $\tilde{\alpha}=\max\{\frac{m^+-1}{m^-}\tilde{k},\frac{r^+-1}{r^-}\tilde{k}\}$. Using (3.40), we have

$$\mathcal{H}'(t)=\int_{\Omega}|u_t|^{m(x)}\,dx+\int_{\Omega}|v_t|^{r(x)}\,dx.$$

Hence, (4.35) becomes

$$\begin{aligned}\int_{\Omega}\left(|u|\,|u_t|^{m(x)-1}+|v|\,|v_t|^{r(x)-1}\right)dx \leq & \frac{c_4\tilde{k}^{1-m^-}}{m^-}\left[\mathcal{H}(t)\right]^{\sigma\left(m^+-1\right)}\int_{\Omega}|u|^{m(x)}\,dx\\ &+\frac{c_5\tilde{k}^{1-r^-}}{r^-}\left[\mathcal{H}(t)\right]^{\sigma\left(r^+-1\right)}\int_{\Omega}|v|^{r(x)}\,dx\\ &+\tilde{\alpha}\mathcal{H}^{-\sigma}(t)\,\mathcal{H}'(t).\end{aligned}\tag{4.36}$$

Using (3.38) and (4.15), we have

$$\left[\mathcal{H}(t)\right]^{\sigma\left(m^+-1\right)}\leq c\left(\zeta(u)+\zeta(v)\right)^{\sigma(m^+-1)}.$$

Using the last inequality and (4.19), it can be concluded that

$$\begin{aligned}\left[\mathcal{H}(t)\right]^{\sigma\left(m^+-1\right)}\int_{\Omega}|u|^{m(x)}\,dx \leq & c_6\left(\zeta(u)+\zeta(v)\right)^{\sigma(m^+-1)+\frac{m^+}{p^-+1}}\\ &+c_6\left(\zeta(u)+\zeta(v)\right)^{\sigma(m^+-1)+\frac{m^-}{p^-+1}},\end{aligned}\tag{4.37}$$

Applying (4.4) with $z=\zeta(u)+\zeta(v)$, $a=\mathcal{H}(0)$, $\delta=\sigma(m^+-1)+\frac{m^+}{p^-+1}$ and then with $\delta=\sigma(m^+-1)+\frac{m^-}{p^-+1}$, respectively, we get

$$\begin{aligned}\left(\zeta(u)+\zeta(v)\right)^{\sigma\left(m^+-1\right)+\frac{m^+}{p^-+1}} &\leq \left[1+\frac{1}{\mathcal{H}(0)}\right]\left(\zeta(u)+\zeta(v)+\mathcal{H}(0)\right)\\ &\leq \alpha\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right)\end{aligned}\tag{4.38}$$

and

$$\left(\zeta(u)+\zeta(v)\right)^{\sigma\left(m^+-1\right)+\frac{m^-}{p^-+1}}\leq \alpha\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right)\tag{4.39}$$

where $\alpha=1+\frac{1}{\mathcal{H}(0)}$.

A combination of (4.37)–(4.39) implies that, for some $c_7>0$,

$$\left[\mathcal{H}(t)\right]^{\sigma(m^+-1)}\int_{\Omega}|u|^{m(x)}\,dx\leq c_7\left(\zeta(u)+\zeta(v)+\mathcal{H}(t)\right).\tag{4.40}$$

Similar calculations give, for some $c_8 > 0$,

$$[\mathcal{H}(t)]^{\sigma(r^+-1)} \int_\Omega |v|^{r(x)}\, dx \le c_8 \left(\zeta(u) + \zeta(v) + \mathcal{H}(t)\right). \tag{4.41}$$

Using (4.35), (4.40) and (4.41), we obtain, for $c_9, c_{10} > 0$,

$$\begin{aligned} \int_\Omega \left(|u|\,|u_t|^{m(x)-1} + |v|\,|v_t|^{r(x)-1}\right) dx \le\; & \frac{\tilde{k}^{1-m^-}}{m^-} c_9 \left(\zeta(u) + \zeta(v) + \mathcal{H}(t)\right) \\ & + \frac{\tilde{k}^{1-r^-}}{r^-} c_{10} \left(\zeta(u) + \zeta(v) + \mathcal{H}(t)\right) \\ & + \frac{r^+ - 1}{r^-} \tilde{k} \mathcal{H}^{-\sigma}(t) \mathcal{H}'(t). \end{aligned} \tag{4.42}$$

Inserting (4.42) into (4.29), we have

$$\begin{aligned} \mathcal{G}'(t) \ge & \left(1 - \sigma - \varepsilon \tilde{R}\right) \mathcal{H}^{-\sigma}(t)\, \mathcal{H}'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \varepsilon \left(2 - \frac{\tilde{k}^{1-m^-}}{m^-} c_9 - \frac{\tilde{k}^{1-r^-}}{r^-} c_{10}\right) \mathcal{H}(t) \\ & + \varepsilon \left(c_{11} - \frac{\tilde{k}^{1-m^-}}{m^-} c_9 - \frac{\tilde{k}^{1-r^-}}{r^-} c_{10}\right) \left(\zeta(u) + \zeta(v)\right). \end{aligned}$$

where $c_{11} > 0$ and $\tilde{R} = \tilde{k}\left(\frac{m^+-1}{m^-} + \frac{r^+-1}{r^-}\right)$. Now, we select $\tilde{k}$ large enough so that

$$\begin{aligned} \mathcal{G}'(t) \ge & \left(1 - \sigma - \varepsilon \tilde{R}\right) \mathcal{H}^{-\sigma}(t)\, \mathcal{H}'(t) \\ & + \varepsilon c_{12} \left(\|u_t\|_2^2 + \|v_t\|_2^2 + \mathcal{H}(t) + \zeta(u) + \zeta(v)\right), \end{aligned}$$

where $c_{12} > 0$. Once $\tilde{k}$ is fixed, we select ε small enough so that

$$1 - \sigma - \varepsilon \tilde{R} \ge 0 \text{ and } \mathcal{G}(0) = \mathcal{H}^{1-\sigma}(0) + \varepsilon \int_\Omega (u_0 u_1 + v_0 v_1)\, dx > 0.$$

Using the fact that $\mathcal{H}$ is a non-decreasing function, therefore (4.23) is established.

Theorem 4.1. *Under the assumptions* (4.1) *and* (4.9)*, any solution of the system* (1.2) *blows-up in a finite time.*

Proof. Using (4.3) and the definition of $\mathcal{G}$, we have

$$\begin{aligned} \mathcal{G}^{1/(1-\sigma)}(t) \le & \left(\mathcal{H}^{1-\sigma}(t) + \varepsilon \int_\Omega |u u_t + v v_t|\, dx\right)^{1/(1-\sigma)} \\ \le & \; 2^{\sigma/(1-\sigma)} \left(\mathcal{H}(t) + \left(\varepsilon \int_\Omega (|u u_t| + |v v_t|)\, dx\right)^{1/(1-\sigma)}\right) \\ \le & \; c_{13} \left(\mathcal{H}(t) + \left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\, dx\right)^{1/(1-\sigma)}\right), \end{aligned} \tag{4.43}$$

where $c_{13} = 2^{\sigma/(1-\sigma)} \max\left\{1, \varepsilon^{1/(1-\sigma)}\right\}$.

The embedding Lemma 2.2, Lemma 4.2, Hölder's and Young's inequalities give

$$\begin{aligned}
&\left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\,dx\right)^{1/(1-\sigma)} \\
&\leq 2^{\sigma/(1-\sigma)}\left(\int_\Omega |u|\,|u_t|\,dx\right)^{1/(1-\sigma)} + 2^{\sigma/(1-\sigma)}\left(\int_\Omega |v|\,|v_t|\,dx\right)^{1/(1-\sigma)} \\
&\leq 2^{\sigma/(1-\sigma)}\left(\|u\|_2^{1/(1-\sigma)}\|u_t\|_2^{1/(1-\sigma)} + \|v\|_2^{1/(1-\sigma)}\|v_t\|_2^{1/(1-\sigma)}\right) \\
&\leq c_{14}\left(\|u\|_{p^-+1}^{1/(1-\sigma)}\|u_t\|_2^{1/(1-\sigma)} + \|v\|_{p^-+1}^{1/(1-\sigma)}\|v_t\|_2^{1/(1-\sigma)}\right) \\
&\leq c_{15}\left(\|u\|_{p^-+1}^{2/(1-2\sigma)} + \|u_t\|_2^2 + \|v\|_{p^-+1}^{2/(1-2\sigma)} + \|v_t\|_2^2\right) \\
&\leq c_{15}\left((\zeta(u) + \zeta(v))^\tau + \|u_t\|_2^2 + \|v_t\|_2^2\right),
\end{aligned} \tag{4.44}$$

where $\tau = 2/(p^- + 1)(1 - 2\sigma)$.

Using (4.15), (3.38) and since $\tau \leq 1$, we get, for some $c_{18} > 0$,

$$\left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\,dx\right)^{1/(1-\sigma)} \leq c_{16}\left(\zeta(u) + \zeta(v) + \|u_t\|_2^2 + \|v_t\|_2^2 + \mathcal{H}(t)\right).$$

Inserting the last estimate in (4.43), we obtain

$$\mathcal{G}^{1/(1-\sigma)}(t) \leq c_{17}\left(\zeta(u) + \zeta(v) + \mathcal{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2\right). \tag{4.45}$$

Combining (4.23) and (4.45), we deduce that

$$\mathcal{G}'(t) \geq \tilde{c}\mathcal{G}^{1/(1-\sigma)}(t), \text{ for all } t > 0.$$

where $\tilde{c} = \frac{\varepsilon\rho}{c_{16}}$. A simple integration over $(0, t)$ yields

$$\mathcal{G}^{\sigma/(1-\sigma)}(t) \geq \frac{1}{\mathcal{G}^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\sigma\tilde{c}t}{1-\sigma}},$$

which implies that $\mathcal{G}(t) \longrightarrow +\infty$, as $t \longrightarrow T^*$, where $T^* \leq \frac{1-\sigma}{\sigma\tilde{c}\left[\mathcal{G}^{\frac{\sigma}{(1-\sigma)}}(0)\right]}$. Consequently, the solution of Problem (1.2) blows-up in a finite time.

5. Global existence and decay-rate estimates

In this section, we establish the existence of global solutions for initial data in a certain stable set. Then, we show that the decay estimates of the solution energy are exponential or polynomial, depending on the $max\{m^+, r^+\}$.

5.1. Global existence

To state and prove our first result, we introduce the two functionals defined for all $t \in (0, T)$ by

$$I(t) = I(u(t)) = \|\Delta u\|_2^2 + \|\nabla v\|_2^2 - (p^+ + 1)\int_\Omega F(x, u, v)\,dx, \tag{5.1}$$

$$J(t) = J(u(t)) = \frac{1}{2}\left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right) - \int_\Omega F(x,u,v)\,dx \tag{5.2}$$

and give the following Lemma.

Lemma 5.1. *Under the assumptions* $(H.1)$ *and* $(H.2)$*, we suppose that*

$$I(0) > 0 \text{ and } \beta < 1,$$

where

$$\beta = C_2(p^+ + 1)max\left\{c_*^{p^-+1}\left(\frac{2(p^+ + 1)}{p^+ - 1}E(0)\right)^{\frac{p^- -1}{2}}, c_*^{p^+ +1}\left(\frac{2(p^+ + 1)}{p^+ - 1}E(0)\right)^{\frac{p^+ -1}{2}}\right\}.$$

Then,

$$I(t) > 0, \text{ for all } t \in (0,T). \tag{5.3}$$

Proof. From the continuity of I and the fact that $I(0) > 0$, there exists t_k in $]0,T)$ such that

$$I(t) \geq 0, \ \forall t \in (0,t_k). \tag{5.4}$$

We have to show that this inequality is strict.

Recalling (5.1) and (5.2), we have

$$J(t) = \frac{p^+ - 1}{2(p^+ + 1)}\left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{1}{p^+ + 1}I(t),$$

Combining with (5.4), this gives

$$J(t) \geq \frac{p^+ - 1}{2(p^+ + 1)}\left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2\right), \forall t \in (0,t_k). \tag{5.5}$$

From the definition of the energy, we have

$$E(t) = J(t) + \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right), \tag{5.6}$$

for all $t \in (0,T)$. Consequently,

$$\|\Delta u\|_2^2 + \|\nabla v\|_2^2 \leq \frac{2(p^+ + 1)}{(p^+ - 1)}E(t).$$

Thus, the decreasing property of E leads to

$$\max\left\{\|\Delta u\|_2^2, \|\nabla v\|_2^2\right\} \leq \frac{2(p^+ + 1)}{(p^+ - 1)}E(0), \forall t \in (0,t_k). \tag{5.7}$$

On the other hand, from Lemma 2.1 and the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{p(\cdot)+1}(\Omega)$, we have

$$\int_\Omega |u|^{p(x)+1}\,dx \leq max\{c_*^{p^-+1}\|\Delta u\|_2^{p^-+1}, c_*^{p^+ +1}\|\Delta u\|_2^{p^+ +1}\}$$

$$\leq max\, \{c_*^{p^-+1}\, \|\Delta u\|_2^{p^--1}\, , c_*^{p^++1}\, \|\Delta u\|_2^{p^+-1}\}\, \|\Delta u\|_2^2\, .$$

Combining with (5.7), this yields, for all $t \in (0, t_k)$,

$$\begin{aligned}
&\int_\Omega |u|^{p(x)+1}\, dx \\
&\leq max \left\{ c_*^{p^-+1} \left(\frac{2\,(p^+ + 1)}{(p^+ - 1)} E\,(0) \right)^{\frac{p^--1}{2}} , c_*^{p^++1} \left(\frac{2\,(p^+ + 1)}{(p^+ - 1)} E\,(0) \right)^{\frac{p^+-1}{2}} \right\} \|\Delta u\|_2^2\, .
\end{aligned}$$

Therefore,

$$\int_\Omega |u|^{p(x)+1}\, dx \leq \frac{\beta}{C_2\,(p^+ + 1)}\, \|\Delta u\|_2^2\, . \tag{5.8}$$

Similarly, we have

$$\int_\Omega |v|^{p(x)+1}\, dx \leq \frac{\beta}{C_2\,(p^+ + 1)}\, \|\nabla v\|_2^2\, . \tag{5.9}$$

The addition of (5.8) and (5.9) gives

$$\int_\Omega \left(|u|^{p(x)+1} + |v|^{p(x)+1} \right) dx \leq \frac{\beta}{C_2\,(p^+ + 1)} \left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2 \right). \tag{5.10}$$

Combining (5.10) with (3.38), we infer that

$$\begin{aligned}
\int_\Omega F\,(x, u, v)\, dx &\leq \frac{\beta}{p^+ + 1} \left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2 \right) \\
&< \frac{1}{p^+ + 1} \left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2 \right),
\end{aligned} \tag{5.11}$$

for all $t \in (0, t_k)$. From the definition of I, this leads to

$$I\,(t) > 0.\ \forall t \in (0, t_k)\, .$$

By repeating the above procedure and using the decreasing property of E, we can extend t_k to T and obtain (5.3).

Theorem 5.1. *Suppose that all assumptions of Lemma* 5.1 *are fulfilling. Then, the local solution* (u, v) *of the system* (1.2) *exists globally.*

Proof. Substituting (5.5) into (5.6) and thanks to (5.3), it yields

$$E\,(t) \geq \frac{p^+ - 1}{2\,(p^+ + 1)} \left(\|\Delta u\|_2^2 + \|\nabla v\|_2^2 \right) + \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right),$$

for all $t \in (0, T)$. Then, we have

$$\begin{aligned}
\|\Delta u\|_2^2 + \|\nabla v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2 &\leq C_3 E\,(t) \\
&\leq C_3 E(0),
\end{aligned} \tag{5.12}$$

for $C_3 = \max\{2, \frac{2(p^+ + 1)}{p^+ - 1}\}$. This means that the norm in (5.12) is bounded independently of t. Therefore, the solution (u, v) exists globally.

5.2. Decay-rate estimates

To prove the decay result, we need the following Lemma.

Lemma 5.2. *Suppose that the assumptions of Lemma* 5.1 *hold. Then, there exists a positive constant* C_4*, such that the global solution* (u, v) *satisfies*

$$\int_\Omega \left(|u(t)|^{m(x)} + |v(t)|^{r(x)}\right) dx \leq C_4 E(t) \text{ for all } t \geq 0. \tag{5.13}$$

Proof. The result is immediate by replacing p with m and r in (5.8) and (5.9), respectively, and by recalling (5.12).

Theorem 5.2. *Under the assumptions of Lemma* 5.1*, the solution of the system* (1.2) *satisfies the following decay estimates, for all* $t \geq 0$,

$$E(t) \leq \begin{cases} \frac{k}{(1+t)^{2/(\lambda^+-2)}}, & if\ \alpha > 2, \\ ke^{-\omega t}, & if\ \alpha = 2, \end{cases} \tag{5.14}$$

where $\alpha = max\{m^+, r^+\}$ *and* $k, w > 0$ *are two positive constants.*

Proof. Multiplying $(1.2)_1$ by $u(t)E^\eta(t)$ and $(1.2)_2$ by $v(t)E^\eta(t)$ and then, integrating each result over $\Omega \times (s, T)$, for $s \in (0, T)$ and $\eta \geq 0$ to be specified later, we arrive at

$$\int_s^T \int_\Omega E^\eta(t)\left[u(t)u_{tt}(t) + u(t)\Delta^2 u(t) + u(t)|u_t|^{m(x)-2}u_t(t)\right] dxdt$$
$$= \int_s^T \int_\Omega E^\eta(t)u(t)f_1(x, u, v)\, dxdt$$

and

$$\int_s^T \int_\Omega E^\eta(t)\left[v(t)v_{tt}(t) - v(t)\Delta v(t) + v(t)|v_t(t)|^{r(x)-2}v_t(t)\right] dxdt$$
$$= \int_s^T \int_\Omega E^\eta(t)v(t)f_2(x, u, v)\, dxdt.$$

Green's formula and the boundary conditions lead to

$$\int_s^T \int_\Omega E^\eta(t)\left[(u(t)u_t(t))_t - |u_t(t)|^2 + |\Delta u(t)|^2 + u(t)u_t(t)|u_t(t)|^{m(x)-2}\right] dxdt$$
$$= \int_s^T \int_\Omega E^\eta(t)u(t)f_1(x, u, v)\, dxdt, \tag{5.15}$$

and

$$\int_s^T \int_\Omega E^\eta(t)\left[(v(t)v_t(t))_t - |v_t(t)|^2 + |\nabla v(t)|^2 + v(t)v_t(t)|v_t(t)|^{r(x)-2}\right] dxdt$$
$$= \int_s^T \int_\Omega E^\eta(t)v(t)f_2(x, u, v)\, dxdt. \tag{5.16}$$

Adding and subtracting the following two terms

$$\left| \begin{array}{l} \int_s^T \int_\Omega E^\eta(t)\left[\beta|\Delta u(t)|^2+(1+\beta)|u_t(t)|^2\right]dxdt \\ \int_s^T \int_\Omega E^\eta(t)\left[\beta|\nabla v(t)|^2+(1+\beta)|v_t(t)|^2\right]dxdt, \end{array} \right.$$

to (5.15) and (5.16), respectively, and recalling (5.11), we arrive at

$$\begin{aligned} &(1-\beta)\int_s^T E^\eta(t)\int_\Omega\left(|\Delta u(t)|^2+|\nabla v(t)|^2+|u_t(t)|^2+|v_t(t)|^2\right)dxdt \\ &+\int_s^T E^\eta(t)\int_\Omega\left[(u(t)u_t(t)+v(t)v_t(t))_t-(2-\beta)\left(|u_t(t)|^2+|v_t(t)|^2\right)\right]dxdt \\ &+\int_s^T E^\eta(t)\int_\Omega\left(u(t)u_t(t)|u_t(t)|^{m(x)-2}+v(t)v_t(t)|v_t(t)|^{r(x)-2}\right)dxdt \\ &=-\int_s^T E^\eta(t)\int_\Omega\left[\beta\left(|\Delta u(t)|^2+|\nabla v(t)|^2\right)-(p(x)+1)F(x,u,v)\right]dxdt\leq 0. \end{aligned} \tag{5.17}$$

Now, by exploiting the formula:

$$\begin{aligned} E^\eta(t)\int_\Omega(u(t)u_t(t)+v(t)v_t(t))_t\,dx =& \frac{d}{dt}\left(E^\eta(t)\int_\Omega(u(t)u_t(t)+v(t)v_t(t))\,dx\right) \\ &-\eta E^{\eta-1}(t)E^{'}(t)\int_\Omega(u(t)u_t(t)+v(t)v_t(t))\,dx, \end{aligned}$$

estimate (5.17) yields

$$\begin{aligned} 2(1-\beta)\int_s^T E^{\eta+1}(t)\,dt \leq& \eta\int_s^T E^{\eta-1}(t)E^{'}(t)\int_\Omega(u(t)u_t(t)+v(t)v_t(t))\,dxdt \\ &-\int_s^T\frac{d}{dt}\left(E^\eta(t)\int_\Omega(u(t)u_t(t)+v(t)v_t(t))\,dx\right)dt \\ &-\int_s^T E^\eta(t)\int_\Omega\left(u(t)u_t(t)|u_t(t)|^{m(x)-2}+v(t)v_t(t)|v_t(t)|^{r(x)-2}\right)dxdt \\ &+(2-\beta)\int_s^T E^\eta(t)\int_\Omega\left(|u_t(t)|^2+|v_t(t)|^2\right)dxdt \\ =& I_1+I_2+I_3+I_4. \end{aligned} \tag{5.18}$$

Next, we handle the terms $I_i, i=\overline{1,4}$ and denote by C a positive generic constant.

- First, applying Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} I_1 &= \eta\int_s^T E^{\eta-1}(t)E^{'}(t)\int_\Omega(u(t)u_t(t)+v(t)v_t(t))\,dxdt \\ &\leq \frac{\eta}{2}\int_s^T E^{\eta-1}(t)\left(-E^{'}(t)\right)\left[\|u(t)\|_2^2+\|u_t(t)\|_2^2+\|v(t)\|_2^2+\|v_t(t)\|_2^2\right]dt \end{aligned}$$

$$\leq C\int_s^T E^{\eta-1}(t)\left(-E'(t)\right)\left[\|\Delta u(t)\|_2^2+\|\nabla v(t)\|_2^2+\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\right]dt,$$

By (5.12), this gives

$$\begin{aligned} I_1 &\leq C\int_s^T E^{\eta}(t)\left(-E'(t)\right)dt \\ &\leq CE^{\eta+1}(s)-CE^{\eta+1}(T)\leq CE^{\eta}(0)E(s)\leq CE(s). \end{aligned} \tag{5.19}$$

- Concerning the second term, we have

$$\begin{aligned} I_2 &= -\int_s^T \frac{d}{dt}\left(E^{\eta}(t)\int_\Omega (u(t)u_t(t)+v(t)v_t(t))\,dx\right)dt \\ &= E^{\eta}(s)\left(\int_\Omega (u(x,s)u_t(x,s)+v(x,s)v_t(x,s))\,dx\right) \\ &- E^{\eta}(T)\left(\int_\Omega (u(x,T)u_t(x,T)+v(x,T)v_t(x,T))\,dx\right) \end{aligned}$$

Again, by (5.12) and the inequalities of Young and Poincaré, we get

$$\left|\int_\Omega u(x,s)u_t(x,s)\,dx\right|\leq C\left(\|\Delta u(s)\|_2^2+\|u_t(s)\|_2^2\right)\leq CE(s),$$
$$\left|\int_\Omega u(x,T)u_t(x,T)\,dx\right|\leq C\left(\|\Delta u(T)\|_2^2+\|u_t(T)\|_2^2\right)\leq CE(T)$$

and likewise

$$\left|\int_\Omega v(x,s)v_t(x,s)\,dx\right|\leq C\left(\|\nabla v(s)\|_2^2+\|v_t(s)\|_2^2\right)\leq CE(s)$$
$$\left|\int_\Omega v(x,T)v_t(x,T)\,dx\right|\leq C\left(\|\nabla v(T)\|_2^2+\|v_t(T)\|_2^2\right)\leq CE(T).$$

Therefore,

$$I_2\leq CE^{\eta+1}(s)\leq CE^{\eta}(0)E(s)\leq CE(s). \tag{5.20}$$

- For the third term, we apply Young's inequality (as in (4.30)) to obtain, for some $\varepsilon>0$,

$$\begin{aligned} I_3 &= -\int_s^T E^{\eta}(t)\int_\Omega\left(u(t)u_t(t)|u_t(t)|^{m(x)-2}+v(t)v_t(t)|v_t(t)|^{r(x)-2}\right)dxdt \\ &\leq \int_s^T E^{\eta}(t)\left(\frac{\varepsilon}{2}\int_\Omega |u(t)|^{m(x)}dx+\frac{1}{\varepsilon}\int_\Omega |u_t(t)|^{m(x)}dx\right)dt \\ &+\int_s^T E^{\eta}(t)\left(\frac{\varepsilon}{2}\int_\Omega |v(t)|^{r(x)}dx+\frac{1}{\varepsilon}\int_\Omega |v_t(t)|^{r(x)}dx\right)dt. \end{aligned}$$

Invoking Lemma 5.2 and recalling (3.40), yields

$$\begin{aligned} I_3 &\leq \frac{\varepsilon}{2}\int_s^T E^\eta(t)\int_\Omega\left(|u(t)|^{m(x)}+|v(t)|^{r(x)}\right)dxdt+\frac{1}{\varepsilon}\int_s^T E^\eta(t)\left(-E'(t)\right)dt \\ &\leq \varepsilon C\int_s^T E^{\eta+1}(t)\,dt+C_\varepsilon E(s). \end{aligned} \tag{5.21}$$

- Now, we handle I_4, as follows:

$$\begin{aligned} I_4 &= (2-\beta)\int_s^T E^\eta(t)\int_\Omega\left(|u_t(t)|^2+|v_t(t)|^2\right)dxdt \\ &= (2-\beta)\left[\int_s^T E^\eta(t)\int_\Omega |u_t(t)|^2\,dxdt+\int_s^T E^\eta(t)\int_\Omega |v_t(t)|^2\,dxdt\right] \\ &= (2-\beta)(J_1+J_2). \end{aligned}$$

We claim that

$$J_1, J_2 \leq \varepsilon C\int_s^T E^{\eta+1}(t)\,dt+C_\varepsilon E(s). \tag{5.22}$$

Since $2\leq\tilde{\alpha}\leq m(.)\leq\alpha$ on Ω, we obtain

$$\begin{aligned} J_1 &= \int_s^T E^\eta(t)\int_\Omega |u_t(t)|^2\,dxdt \\ &= \int_s^T E^\eta(t)\left[\int_{\Omega_-}|u_t(t)|^2\,dx+\int_{\Omega_+}|u_t(t)|^2\,dx\right]dt \\ &\leq C\int_s^T E^\eta(t)\left[\left(\int_{\Omega_-}|u_t(t)|^\alpha\,dx\right)^{2/\alpha}+\left(\int_{\Omega_+}|u_t(t)|^{\tilde{\alpha}}\,dx\right)^{2/\tilde{\alpha}}\right]dt \\ &\leq C\int_s^T E^\eta(t)\left[\left(\int_{\Omega_-}|u_t(t)|^{m(x)}\,dx\right)^{2/\alpha}+\left(\int_{\Omega_+}|u_t(t)|^{m(x)}\,dx\right)^{2/\tilde{\alpha}}\right]dt, \end{aligned}$$

where

$$\tilde{\alpha} = \ min\{m^-,r^-\},\ \alpha = \ max\{m^+,r^+\},$$

$$\Omega_+ = \{x\in\Omega : |u(x,t)|\geq 1\} \text{ and } \Omega_- = \{x\in\Omega : |u(x,t)|<1\}.$$

Therefore,

$$\begin{aligned} J_1 &\leq C\int_s^T E^\eta(t)\left(-E'(t)\right)^{2/\alpha}dt+C\int_s^T E^\eta(t)\left(-E'(t)\right)^{2/\tilde{\alpha}}dt \\ &= C(J_\alpha+J_{\tilde{\alpha}}). \end{aligned} \tag{5.23}$$

Three cases are possible:

(1) if $\alpha = \tilde{\alpha} = 2$ ($m(x) = r(x) = 2$, on Ω), then

$$J_1 \leq C \int_s^T E^{\eta}(t) \left(-E'(t)\right) dt$$
$$\leq CE(s) \leq \varepsilon C \int_s^T E^{\eta+1}(t)\, dt + CE(s).$$

(2) if $\alpha > 2$ and $\tilde{\alpha} = 2$, we exploit Young's inequality with

$$\delta = (\eta + 1)/\eta \text{ and } \delta' = \eta + 1$$

to find

$$J_\alpha = \int_s^T E^{\eta}(t) \left(-E'(t)\right)^{2/\alpha} dt$$
$$\leq \varepsilon C \int_s^T E^{\eta+1}(t)\, dt + C_\varepsilon \int_s^T \left(-E'(t)\right)^{2(\eta+1)/\alpha} dt.$$

So, for $\eta = \frac{\alpha}{2} - 1$, we get

$$J_\alpha \leq \varepsilon C \int_s^T E^{\eta+1}(t)\, dt + C_\varepsilon \int_s^T \left(-E'(t)\right) dt$$
$$\leq \varepsilon C \int_s^T E^{\eta+1}(t)\, dt + C_\varepsilon E(s). \tag{5.24}$$

Also, in this case, we have

$$J_{\tilde{\alpha}} = \int_s^T E^{\eta}(t) \left(-E'(t)\right) dt \leq CE(s). \tag{5.25}$$

By inserting (5.24) and (5.25) into (5.23), we infer that J_1 (and similarly J_2) satisfies (5.22).

(3) if $\alpha > \tilde{\alpha} > 2$, we apply Young's inequality with

$$\delta = \tilde{\alpha}/(\tilde{\alpha} - 2) \text{ and } \delta' = \tilde{\alpha}/2$$

to obtain

$$J_{\tilde{\alpha}} = \int_s^T E^{\eta}(t) \left(-E'(t)\right)^{2/\tilde{\alpha}} dt$$
$$\leq \varepsilon C \int_s^T E(t)^{\eta\tilde{\alpha}/(\tilde{\alpha}-2)} dt + C_\varepsilon E(s).$$

But $\eta\tilde{\alpha}/(\tilde{\alpha} - 2) = \eta + 1 + (\alpha - \tilde{\alpha})/(\tilde{\alpha} - 2)$, then

$$J_{\tilde{\alpha}} \leq \varepsilon C \left(E(s)\right)^{(\alpha-\tilde{\alpha})/(\alpha-2)} \int_s^T E^{\eta+1}(t)\, dt + C_\varepsilon E(s)$$
$$\leq \varepsilon C \int_s^T E^{\eta+1}(t)\, dt + C_\varepsilon E(s). \tag{5.26}$$

The addition of (5.24) and (5.26) leads to (5.22).

We conclude that the claim is true for any $\alpha \geq \tilde{\alpha} \geq 2$. Therefore,

$$I_4 \leq \varepsilon C \int_s^T E^{\eta+1}(t)\, dt + C_\varepsilon E(s). \tag{5.27}$$

Now, substituting (5.19)–(5.21) and (5.27) into (5.18), we get

$$2(1-\beta) \int_s^T E^{\eta+1}(t)\, dt \leq \varepsilon C \int_s^T E^{\eta+1}(t)\, dt + C_\varepsilon E(s),$$

with $\eta = \frac{\alpha}{2} - 1$. So,

$$2(1-\beta) \int_s^T E^{\frac{\alpha}{2}}(t)\, dt \leq \varepsilon C \int_s^T E^{\frac{\alpha}{2}}(t)\, dt + C_\varepsilon E(s).$$

Choosing ε small enough, we obtain

$$\int_s^T E^{\frac{\alpha}{2}}(t)\, dt \leq CE(s).$$

Letting $T \longrightarrow \infty$, it yields

$$\int_s^\infty E^{\frac{\alpha}{2}}(t)\, dt \leq CE(s), \forall s > 0.$$

Applying Komornik's lemma [23], we get the desired decay estimates.

6. Conclusions

We considered a coupled system of Laplacian and bi-Laplacian equations with nonlinear damping and source terms of variable-exponents nonlinearities. We gave a detailed proof of the local existence using Faedo-Galerkin method and Banach-fixed-point theorem. We also showed that the solutions with positive-initial energy blow-up in a finite time. Furthermore, we proved a global existence theorem, using the Stable-set method and established a decay estimate of the solution energy, by Komornik's integral approach.

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Conflict of interest

The authors declare that there is no conflict of interest.

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