AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 7899–7916.
DOI:10.3934/math.2023398
Received: 20 October 2022
Revised: 02 January 2023
Accepted: 12 January 2023
Published: 31 January 2023

Research article

Controllability and observability of discretized satellite magnetic attitude control system

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Abstract: In this paper, two different discrete schemes of the second-order linear time-varying system represented by the linearized satellite magnetic attitude control motion equation are obtained by Euler method. Then, the controllability and observability conditions of a new discrete second-order linear time-varying system are proposed and the validity of these conditions is further verified by some numerical examples. Next, the theoretical results are applied to investigate the controllability and observability of the discretized satellite magnetic control system. Different periods τ are chosen to investigate the effect on the controllability and observability of the resulting discrete system. The corresponding conclusions are obtained.

Keywords: linear time-varying system; controllability; observability; discrete second-order system; satellite attitude control motion equation
Mathematics Subject Classification: 93B05, 34A25

Table 1. Nomenclature.

b	earth's magnetic induction field	b_0	earth's magnetic induction field in coordinate system
e_r	unit vector of orbital coordinate system	J_i	main central inertia moments, $i = 1, 2, 3$
M_c	magnetic control moment	m	intrinsic magnetic moment
r	orbital radius	ω	absolute angular velocity
ω_0	orbital angular velocity	θ_i	attitude Euler angles, $i = 1, 2, 3$
μ_E	earth's magnetic field constant	β	inclination angle of orbital plane to the equatorial plane
Θ	direction cosine matrix		

1. Introduction

Satellite attitude control system, which is an important guarantee system for satellite normal operation, affects the working performance and on-orbit lifetimes directly. Because of low cost and high reliability, magnetic control technology has been widely applied in the early stages of satellite development [1–3]. White, Shigemoyo and Bourquin firstly provided the concept of satellite attitude control, making it possible to achieve orientation of satellites with magnetictorquers. Soon, the first satellite magnetic attitude control method was proposed by Ergin and Wheeler [4], and they gave some advantages of magnetic control. Later, Renard [5] presented method based on averaged models to study the issue of attitude control with magnetic moments. Recently, wide application of periodic control system brings the research and development of magnetic control theory to a new height [6–8]. In order to predict the trajectory of a satellite with magnetic moments, numerous efforts have been made to investigate the nonlinear system's stability and controllability, which is based on the following Euler dynamic equation [9, 10]

$$J\frac{d\omega}{dt} + \omega \times J\omega = 3\omega_0^2(e_r \times Je_r) + M_c, \tag{1.1}$$

where $J = \mathrm{diag}(\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3)$, $\omega = [\omega_1, \omega_2, \omega_3]^T$, $\omega_0 = [\omega_1^0, \omega_2^0, \omega_3^0]^T$, the unit vector $e_r = [0, 0, -1]$,and $\times$ is the familiar operation of cross product. The control moment M_c is

$$M_c = m(t) \times b(t),$$

where $m(t) = [m_1, m_2, m_3]^T$ and $b(t) = \Theta b_0(t)$, $b_0(t)$ can be approximated by the direct magnetic dipole as [11],

$$b_0(t) = \frac{\mu_E}{r^3}\begin{bmatrix} \cos\omega_0 t \sin\beta \\ -\cos\beta \\ 2\sin\omega_0 t \sin\beta \end{bmatrix}.$$

Many scholars focused on controllability of satellite magnetic control system based on control theories for nonlinear and linear time-varying continuous systems [12–15]. However, exact solutions and fundamental solution matrix of these time-varying continuous systems are hard to obtain, although controllability conditions of systems are satisfied. Therefore, the trajectory of satellites can not be predicted correctly. Therefore, numerous studies have been done to deal with the nonlinear second-order time-varying system (1.1) by linearization method [13, 15]. With the development of computer technology, quite a number of research works regarding discrete-time systems have been reported in the literature [16–18] and the controllability and observability of discrete linear systems have attracted a lot of interest [19–23]. Witczak etc. provided a necessary and sufficient condition for the observability of first-order discrete time-varying linear systems. Mahmudov proposed the controllability and observability conditions of second-order discrete linear time-varying systems in a matrix form. The controllability and observability conditions for the problem of discrete satellite magnetic attitude control have not been presented to the best of our knowledge. Usually, the difference methods based on Taylor series are widely used for approximation discretization of the continuous-time systems [23]. A brief review of the possible approach to discretize linear and nonlinear time-varying systems so far has been presented in [24]. Here, we investigate controllability and observability property of the linearized form of equation (1.1) by transforming it into a discrete time-varying system with second derivation by the forward and backward Euler method. Then, the controllability and observability conditions of a new discrete second-order linear time-varying system are proposed, which are applied to investigate the controllability and observability of the discretized satellite magnetic control system. Different periods τ are chosen to investigate the effect on controllability and observability of the resulting discrete system.

The rest of this paper is structured as follows. In Section 2, two different discrete schemes of the second-order linear time-varying system represented by the linearized satellite magnetic attitude control motion equation are obtained by Euler method. The linearized satellite magnetic attitude control system is changed into a discrete second-order time-varying system. In Section 3, the controllability conditions of a new discrete second-order linear time-varying system are proposed, which are applied to investigate the controllability of the discretized satellite magnetic control system. Section 4 investigates the observability of the discrete second-order satellite magnetic control system based on corresponding observability conditions. We give concluding remarks in the final section.

2. Transformation of satellite attitude control system

In this section, the nonlinear second-order time-varying system represented by satellite attitude magnetic control motion equation (1.1) is linearized. We obtain two different discrete schemes of the second-order linear time-varying system by Euler method and transform the linearized satellite magnetic attitude control motion equation into a discrete second-order time-varying linear system.

2.1. Linearized equation of satellite magnetic control

We assume the mass center of the satellite moving in earth's gravitational field and in a circular orbit. To satellite magnetic attitude control motion equation, the coordinate systems are described as follows:

(1) orbital system (X, Y, Z). The origin is satellite mass center. The Z-axis points in the direction of the radius vector; the Y-axis is normal to the satellite orbit plane; and the X-axis forms the right-hand trial.

(2) satellite body frame (x, y, z) . The axes are assumed to coincide with the body's principle inertia axes and their origin are still at the center of satellite mass.

The attitude of system (x, y, z) relative to the orbital system (X, Y, Z) is given by Euler angles $\theta_1,\ \theta_2,\ \theta_3$. Then the components of ω have form [13]

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = C_{\theta_3\theta_2} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + \Theta \begin{bmatrix} \omega_1^0 \\ \omega_2^0 \\ \omega_3^0 \end{bmatrix}, \tag{2.1}$$

where

$$C_{\theta_3\theta_2} = \begin{bmatrix} \cos\theta_2\cos\theta_3 & \sin\theta_3 & 0 \\ \cos\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ \sin\theta_2 & 0 & 1 \end{bmatrix}, \tag{2.2}$$

and the direction cosine matrix $\Theta = [\Theta_{ij}]$ is represented using a 1-2-3 Euler angle rotation sequence as follows

$$\begin{cases} \Theta_{11} = \cos\theta_2 \cos\theta_3, \\ \Theta_{12} = \cos\theta_1 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ \Theta_{13} = \sin\theta_1 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3, \\ \Theta_{21} = -\cos\theta_2 \sin\theta_3, \\ \Theta_{22} = \cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3, \\ \Theta_{23} = \sin\theta_1 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3, \\ \Theta_{31} = \sin\theta_2, \\ \Theta_{32} = \sin\theta_1 \cos\theta_2, \\ \Theta_{33} = \cos\theta_1 \cos\theta_2. \end{cases} \tag{2.3}$$

In [15], Morozov and Kalenova gave the following linearized special second-order time-varying system of equation (1.1)

$$\ddot{x}(t) + D\dot{x}(t) + Kx(t) = B(t)u(t), \tag{2.4}$$

where

$$D = \begin{bmatrix} 0 & 0 & \frac{J_1+J_{32}}{J_1}\omega_0 \\ 0 & 0 & 0 \\ \frac{J_{21}-J_3}{J_3}\omega_0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -d_1\omega_0 \\ 0 & 0 & 0 \\ d_3\omega_0 & 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} \frac{-4\omega_0^2 J_{32}}{J_1} & 0 & 0 \\ 0 & \frac{3\omega_0^2 J_{13}}{J_2} & 0 \\ 0 & 0 & \frac{\omega_0^2 J_{21}}{J_3} \end{bmatrix} = \begin{bmatrix} -k_1\omega_0^2 & 0 & \\ 0 & -k_2\omega_0^2 & 0 \\ 0 & 0 & -k_3\omega_0^2 \end{bmatrix},$$

$$B(t) = \omega_0^2 \epsilon \begin{bmatrix} 0 & 2b_1 \sin\omega_0 t & b_4 \\ -2b_2 \sin\omega_0 t & 0 & b_2 \cos\omega_0 t \\ -b_5 & -b_3 \cos\omega_0 t & 0 \end{bmatrix},$$

and

$x(t) = [\theta_1,\ \theta_2,\ \theta_3]^T$ *is the state vector*,
$\epsilon = \frac{\mu_E}{\omega_0^2 r^3}$, $\quad d = J_2 - J_1 + J_3$, $\quad d_i = \frac{d}{J_i}\ (i = 1, 3)$, $\quad J_{ij} = J_i - J_j\ (i, j = 1, 2, 3)$,
$k_1 = \frac{-4J_{32}}{J_1}$, $\quad k_2 = \frac{3J_{31}}{J_2}$, $\quad k_3 = \frac{J_{12}}{J_3}$,
$b_j = \frac{\sin\beta}{J_j}\ (j = 1, 2, 3)$, $\quad b_4 = \frac{\cos\beta}{J_1}$, $\quad b_5 = \frac{\cos\beta}{J_3}$
$u(t) = m(t) = [m_1,\ m_2,\ m_3]^T$ *is control vector, which allows the satellite attitude position to be stabilized.*

The complete linearized derivation of equation (1.1) can be found in [13].

With certain assumptions, the equation of the measurement yielded by the magnetometer is represented as [25]

$$y(t) = C(t)x(t) \tag{2.5}$$

where

$$C(t) = \begin{bmatrix} \alpha_2 & 0 & -\alpha_1 \sin \omega_0 t - \alpha_3 \cos \omega_0 t \\ 0 & -\alpha_2 & \alpha_1 \cos \omega_0 t - \alpha_3 \sin \omega_0 t \end{bmatrix}$$

and α_j $(j = 1, 2, 3)$ are constant quantities determining the position of the orbit in space, $x(t) \in R^3$, $y(t) \in R^2$ are state vector and output vector respectively.

Then we have the following linear satellite magnetic attitude control system with measurement, corresponding coefficient matrices are in accordance with matrices in equation (2.4) and (2.5)

$$\begin{cases} \ddot{x}(t) + D\dot{x}(t) + Kx(t) = B(t)u(t), \\ y(t) = C(t)x(t). \end{cases} \tag{2.6}$$

2.2. Discretization of the linearized satellite control system

Firstly, letting $t = k\tau$ $(k = 0, 1, 2...)$, we realize an approximate form of $\dot{x}(t)$ and $\ddot{x}(t)$ using Taylor forward expansion as

$$\dot{x}(t) \approx \frac{x((k+1)\tau) - x(k\tau)}{\tau} \tag{2.7}$$

and

$$\ddot{x}(t) \approx \frac{x((k+1)\tau) - 2x(k\tau) + x((k-1)\tau)}{\tau^2}. \tag{2.8}$$

Substituting equation (2.7) and (2.8) into system (2.6), we have

$$\begin{cases} \frac{x((k+1)\tau)-2x(k\tau)+x((k-1)\tau)}{\tau^2} + D\frac{x((k+1)\tau)-x(k\tau)}{\tau} + Kx(k\tau) = B(k\tau)u(k\tau), \\ y(k\tau) = C(k\tau)x(k\tau). \end{cases} \tag{2.9}$$

Then we can get

$$\begin{cases} (I_3 + \tau D)x_{k+1} + (\tau^2 K - \tau D - 2I_3)x_k + x_{k-1} = \tau^2 B_k u_k, \\ y_k = C_k x_k, \end{cases} \tag{2.10}$$

where I_3 denotes the identity matrix of 3 dimension and

$$x_k = x(k\tau), \quad u_k = u(k\tau), \quad y_k = y(k\tau), \quad C_k = C(k\tau), \quad B_k = B(k\tau). \tag{2.11}$$

Noting that

$$I_3 + \tau D = \begin{bmatrix} 1 & 0 & -d_1\omega_0\tau \\ 0 & 1 & 0 \\ d_3\omega_0\tau & 0 & 1 \end{bmatrix} \tag{2.12}$$

and

$$det(I_3 + \tau D) = 1 + d_1 d_3 \omega_0^2 \tau^2 > 0, \tag{2.13}$$

which means the matrix $I_3 + \tau D$ is invertible. Therefore, we can rewrite the system (2.10) as the following discrete system

$$\begin{cases} x_{k+1} = A_0 x_{k-1} + A_1 x_k + \widetilde{B}_k u_k, \\ y_k = C_k x_k, \end{cases} \tag{2.14}$$

where

$$A_0 = -(I_3 + \tau D)^{-1} = \begin{bmatrix} -\frac{1}{d_1 d_3 \omega_0^2 \tau^2 + 1} & 0 & -\frac{d_1 \omega_0 \tau}{d_1 d_3 \omega_0^2 \tau^2 + 1} \\ 0 & -1 & 0 \\ \frac{d_3 \omega_0 \tau}{d_1 d_3 \omega_0^2 \tau^2 + 1} & 0 & -\frac{1}{d_1 d_3 \omega_0^2 \tau^2 + 1} \end{bmatrix},$$

$$A_1 = (I_3 + \tau D)^{-1}(\tau D + 2I_3 - \tau^2 K) = \begin{bmatrix} 1 + \frac{k_1 \omega_0^2 \tau^2 + 1}{d_1 d_3 \omega_0^2 \tau^2 + 1} & 0 & \frac{d_1 \tau \omega_0 (k_3 \omega_0^2 \tau^2 + 1)}{d_1 d_3 \omega_0^2 \tau^2 + 1} \\ 0 & k_2 \omega_0^2 \tau^2 + 2 & 0 \\ \frac{-d_3 \omega_0 \tau (k_1 \omega_0^2 \tau^2 + 1)}{d_1 d_3 \omega_0^2 \tau^2 + 1} & 0 & 1 + \frac{k_3 \omega_0^2 \tau^2 + 1}{d_1 d_3 \omega_0^2 \tau^2 + 1} \end{bmatrix},$$

$$\widetilde{B}_k = (I_3 + \tau D)^{-1} \tau^2 B_k = \tau^2 \omega_0^2 \epsilon \begin{bmatrix} -\frac{b_5 d_1 \omega_0 \tau}{d_1 d_3 \omega_0^2 \tau^2 + 1} & \frac{2b_1 \sin \omega_0 k\tau - b_3 d_1 \omega_0 \tau \cos \omega_0 k\tau}{d_1 d_3 \omega_0^2 \tau^2 + 1} & \frac{b_4}{d_1 d_3 \omega_0^2 \tau^2 + 1} \\ -2b_2 \sin \omega_0 k\tau & 0 & b_2 \cos \omega_0 k\tau \\ -\frac{b_5}{d_1 d_3 \omega_0^2 \tau^2 + 1} & \frac{-(b_3 \cos \omega_0 k\tau + 2b_1 d_3 \omega_0 \tau \sin \omega_0 k\tau)}{d_1 d_3 \omega_0^2 \tau^2 + 1} & \frac{-b_4 d_3 \omega_0 \tau}{d_1 d_3 \omega_0^2 \tau^2 + 1} \end{bmatrix}.$$

Next, by analogy, the system (2.6) can also be discretized with backward Euler method. We can obtain an approximate form of $\dot{x}(t)$ and $\ddot{x}(t)$ using Taylor backward expansion as

$$\dot{x}(t) \approx \frac{x(k\tau) - x((k-1)\tau)}{\tau} \tag{2.15}$$

and

$$\ddot{x}(t) \approx \frac{x((k+1)\tau) - 2x(k\tau) + x((k-1)\tau)}{\tau^2}. \tag{2.16}$$

Substituting equation (2.15) and (2.16) into system (2.6), letting $t = k\tau$ $(k = 0, 1, 2...)$,

$$\begin{cases} \frac{x((k+1)\tau) - 2x(k\tau) + x((k-1)\tau)}{\tau^2} + D\frac{x((k\tau) - x((k-1)\tau)}{\tau} + Kx(k\tau) = B(k\tau)u(k\tau), \\ y(k\tau) = C(k\tau)x(k\tau), \end{cases} \tag{2.17}$$

Based on equation (2.11) and (2.17) we get

$$\begin{cases} x_{k+1} + (K\tau^2 + D\tau - 2I_3)x_k + (I_3 - D\tau)x_{k-1} = \tau^2 B_k u_k, \\ y_k = C_k x_k. \end{cases} \tag{2.18}$$

Therefore, we can rewrite the system (2.17) as the following discrete system with backward Euler method,

$$\begin{cases} x_{k+1} = \widetilde{A}_0 x_{k-1} + \widetilde{A}_1 x_k + \widetilde{\widetilde{B}}_k u_k, \\ y_k = C_k x_k, \end{cases} \tag{2.19}$$

where

$$\widetilde{A}_0 = \tau D - I_3 = \begin{bmatrix} -1 & 0 & -d_1 \omega_0 \tau \\ 0 & -1 & 0 \\ d_3 \omega_0 \tau & 0 & -1 \end{bmatrix},$$

$$\widetilde{A}_1 = 2I_3 - \tau D - \tau^2 K = \begin{bmatrix} k_1\omega_0^2\tau^2 + 2 & 0 & d_1\omega_0\tau \\ 0 & k_2\omega_0^2\tau^2 + 2 & 0 \\ -d_3\omega_0\tau & 0 & k_3\omega_0^2\tau^2 + 2 \end{bmatrix},$$

$$\widetilde{\widetilde{B}}_k = \tau^2\omega_0^2\epsilon \begin{bmatrix} 0 & 2b_1\sin\omega_0 k\tau & b_4 \\ -2b_2\sin\omega_0 k\tau & 0 & b_2\cos\omega_0 k\tau \\ -b_5 & -b_3\cos\omega_0 k\tau & 0 \end{bmatrix}, \ k = 0, 1, 2... .$$

In general, when the sampling period τ is about one tenth of the minimum time constant of the system, the approximation is satisfactory enough. In fact, the error between the exact solution and the numerical solution will be lager if the period is too big, which makes the mathematical precision lower. On the contrary, if the period is too small, then step size increases and the computation is huge. Then, based on the similar form of discrete system (2.14) and (2.19), we can rewrite them as the following general discrete system

$$\begin{cases} x_{k+1} = \hat{A}_0 x_{k-1} + \hat{A}_1 x_k + \hat{B}_k u_k, \\ y_k = C_k x_k, \end{cases} \tag{2.20}$$

where $x_k \in R^n,\ y_k \in R^r,\ u_k \in R^m\ (m \le n)$ are the state vector, the output vector and the control vector respectively. $\hat{A}_0,\ \hat{A}_1 \in R^{n\times n},\ \hat{B}_k \in R^{n\times m},\ C_k \in R^{r\times n}$ are coefficient matrices.

3. Controllability of the discrete system

3.1. Controllability analysis

According to the definition of controllability of discrete linear time-varying systems in [26], we present the definition of controllability and uncontrollability of the second-order discrete time-varying linear system (2.20).

Definition 1. The second-order discrete time-varying linear system (2.20) is said to be controllable to final state $x_n = x_f$ in finite n steps if it exists an input sequence $U = \{u_0, u_1, ..., u_{n-1}\}$ which brings the initial state (x_{-1}, x_0) to a final state $x_n = x_f$ in the finite discrete time interval $[0, n]$. Otherwise the system (2.20) is uncontrollable.

Theorem 1. The linear discrete-time varying system (2.20) is controllable if and only if rank $\mathbb{C}$ = n, and the controllability matrix $\mathbb{C}$ is defined by

$$\mathbb{C} = [M_{n-1}^{(n)}\hat{B}_{n-1},\ M_{n-2}^{(n)}\hat{B}_{n-2}, ..., M_1^{(n)}\hat{B}_1, M_0^{(n)}\hat{B}_0], \tag{3.1}$$

where

$$\begin{aligned} &M_{i-2}^{(n)} = M_{i-1}^{(n)}\hat{A}_1 + M_i^{(n)}\hat{A}_0,\ \ i = 2, ..., n-1, \\ &M_{n-2}^{(n)} = \hat{A}_1, \quad M_{n-1}^{(n)} = I_n. \end{aligned} \tag{3.2}$$

Proof Based on the form of the state vector in the system (2.20), we derive the following set of equations

$$\begin{aligned} x_1 &= \hat{A}_0 x_{-1} + \hat{A}_1 x_0 + \hat{B}_0 u_0, \\ x_2 &= \hat{A}_0 x_0 + \hat{A}_1 x_1 + \hat{B}_1 u_1 = \hat{A}_0 x_1 + \hat{A}_1(\hat{A}_0 x_{-1} + \hat{A}_1 x_0 + \hat{B}_0 u_0) + \hat{B}_1 u_1 \\ &= \hat{A}_1\hat{A}_0 x_{-1} + (\hat{A}_0 + \hat{A}_1^2) x_0 + \hat{A}_1\hat{B}_0 u_0 + \hat{B}_1 u_1. \end{aligned} \tag{3.3}$$

Thus, by iteration, it is finally not difficult to find that in equations (3.3) the $k-th$ instant state vector x_k starting from initial time $k=0$ and $k=1$ is

$$x_k = Q(k)x_{-1} + P(k)x_0 + \sum_{i=0}^{k-1} M_i^{(k)} \hat{B}_i u_i, \tag{3.4}$$

where $Q(k),\ P(k),\ M_i^{(k)}$ are polynomial functions consisted of matrices $\hat{A}_0,\ \hat{A}_1$, and they satisfy the following important iteration equations

$$\begin{array}{l} Q(k) = \hat{A}_0 Q(k-2) + \hat{A}_1 Q(k-1),\ \ Q(1) = O_{n\times n},\ \ Q(0) = I_n, \\ P(k) = \hat{A}_0 P(k-2) + \hat{A}_1 P(k-1),\ \ P(0) = O_{n\times n},\ \ P(1) = I_n,\ \ k = 2,3,\dots, \\ M_{i-2}^{(k)} = M_{i-1}^{(k)}\hat{A}_1 + M_i^{(k)}\hat{A}_0,\ \ i = 2,...,k-1,\ \ M_{k-2}^{(k)} = \hat{A}_1,\ \ \ M_{k-1}^{(k)} = I_n. \end{array} \tag{3.5}$$

In a general case, the state vector at final time n can be written as

$$x_n = Q(n)x_{-1} + P(n)x_0 + \sum_{i=0}^{n-1} M_i^{(n)} \hat{B}_i u_i. \tag{3.6}$$

Therefore

$$\begin{array}{l} x_n - Q(n)x_{-1} - P(n)x_0 = \sum_{i=0}^{n-1} M_i^{(n)} \hat{B}_i u_i \\ = M_0^{(n)}\hat{B}_0 u_0 + M_1^{(n)}\hat{B}_1 u_1 + \dots + M_{n-1}^{(n)}\hat{B}_{n-1}u_{n-1} \\ = [M_{n-1}^{(n)}\hat{B}_{n-1},\ M_{n-2}^{(n)}\hat{B}_{n-2}, ..., M_1^{(n)}\hat{B}_1, M_0^{(n)}\hat{B}_0] \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \dots \\ u_1 \\ u_0 \end{bmatrix}. \end{array} \tag{3.7}$$

If we denote

$$\mathbb{C} = [M_{n-1}^{(n)}\hat{B}_{n-1},\ M_{n-2}^{(n)}\hat{B}_{n-2}, ..., M_1^{(n)}\hat{B}_1, M_0^{(n)}\hat{B}_0]$$

as the controllability matrix in the sequel, then the equation (3.7) determines the input sequence which transfers the initial state (x_{-1}, x_0) to the desired state $x_f = x_n$ in n steps. Thus, these equations will have a solution for any given vector x_f if and only if the matrix has full rank, i.e. rank $\mathbb{C}$ = n, the discrete system (2.20) is controllable. □

In order to further verify the correctness of the theoretical results, some numerical examples are given.

Example.1 Let us investigate the controllability of the following system [20]:

$$\ddot{x}(t) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 4 & -5 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} u(t). \tag{3.8}$$

Firstly, it is noticed that there is no $\dot{x}(t)$ in system (3.8), which means the discrete forms with forward and backward method of system (3.8) are equivalent, that is

$$x_{k+1} = \hat{A}_0 x_{k-1} + \hat{A}_1 x_k + \hat{B} u_k \tag{3.9}$$

where

$$\hat{A}_1 = 2I_3 + \tau^2 A, \quad \hat{A}_0 = -I_3, \quad \hat{B} = \tau^2 B, \quad A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Letting $\tau = 0.1s$, according to Theorem 1, we have

$$n = 3, \quad M_2^{(3)} = I_3, \quad M_1^{(3)} = \hat{A}_1 = \begin{bmatrix} 2.01 & 0 & -0.01 \\ 0.02 & 2.03 & 0.01 \\ 0.04 & -0.05 & 2.02 \end{bmatrix},$$

$$M_0^{(3)} = M_1^{(3)}\hat{A}_1 + M_2^{(3)}\hat{A}_0 = \begin{bmatrix} 3.0397 & 0.0005 & -0.0403 \\ 0.0812 & 3.1204 & 0.0403 \\ 0.1602 & -0.2025 & 3.0795 \end{bmatrix}.$$

Then

$$\mathbb{C} = [M_2^{(3)}\hat{B},\ M_1^{(3)}\hat{B}, M_0^{(3)}\hat{B}] = \tau^2[M_2^{(3)}B,\ M_1^{(3)}B, M_0^{(3)}B] = \tau^2[B,\ \hat{A}_1 B,\ (\hat{A}_1^2 + \hat{A}_0)B], \tag{3.10}$$

here,

$$\hat{A}_1 B = \begin{bmatrix} 1.98 \\ 0.05 \\ 6.1 \end{bmatrix}, \quad (\hat{A}_1^2 + \hat{A}_0)B = \begin{bmatrix} 2.9188 \\ 0.2021 \\ 9.3987 \end{bmatrix},$$

and

$$det\, \mathbb{C} = -0.000221 \neq 0, \quad rank\, \mathbb{C} = 3 = n. \tag{3.11}$$

Therefore, discrete system (3.9) is controllable.

Example.2 The equations of controlling the motion of a spacecraft between the earth and the moon have the form [25]

$$\ddot{x}(t) + 2D\dot{x}(t) + Kx(t) = B(t)u(t) \tag{3.12}$$

Here,

$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}, \quad B(t) = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}, \quad \alpha_1, \alpha_2 = const.$$

For forward Euler method, the discrete form of system is as follows:

$$(I + 2\tau D)x_{k+1} + (\tau^2 K - 2I - 2\tau D)x_k + x_{k-1} = \tau^2 B_k u_k. \tag{3.13}$$

Then, choosing $\tau = 0.1s$, we have

$$I + 2\tau D = \begin{bmatrix} 1 & -2\tau \\ 2\tau & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.2 \\ 0.2 & 1 \end{bmatrix}, \quad det(I + 2\tau D) = 1 + 4\tau^2 = 1.04 \neq 0,$$

which means the inverse matrix of $I + 2\tau D$ exits and

$$(I + 2\tau D)^{-1} = \begin{bmatrix} \frac{1}{1+4\tau^2} & \frac{2\tau}{1+4\tau^2} \\ -\frac{2\tau}{1+4\tau^2} & \frac{1}{1+4\tau^2} \end{bmatrix} \approx \begin{bmatrix} 0.9615 & 0.1923 \\ -0.1923 & 0.9615 \end{bmatrix}.$$

The discrete system can be represented as

$$x_{k+1} = \hat{A}_0 x_{k-1} + \hat{A}_1 x_k + \hat{B}_k u_k \tag{3.14}$$

where

$$\hat{A}_0 = -(I + 2\tau D)^{-1}, \;\; \hat{A}_1 = (I + 2\tau D)^{-1}(2I + 2\tau D - \tau^2 K), \;\; \hat{B}_k = \tau^2 (I + 2\tau D)^{-1} B(k\tau).$$

According to Theorem 1, we have

$$n = 2, \;\; M_1^{(2)} = I_2, \;\; M_0^{(2)} = \hat{A}_1.$$

Then

$$\mathbb{C} = [M_1^{(2)} \hat{B}_1, \;\; M_0^{(2)} \hat{B}_0] = [\hat{B}_1, \; \hat{A}_1 \hat{B}_0], \tag{3.15}$$

here

$$\hat{B}_1 \approx \begin{bmatrix} 0.00937 \\ 0.00287 \end{bmatrix}, \quad \hat{B}_0 \approx \begin{bmatrix} -0.00962 \\ 0.00192 \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} \frac{2.04+0.01\times\alpha_1}{1.04} & \frac{0.2+0.002\times\alpha_2}{1.04} \\ \frac{-0.2-0.002\times\alpha_1}{1.04} & \frac{2.04+0.01\times\alpha_2}{1.04} \end{bmatrix}.$$

Because of $\alpha_1 = 1 + 2b, \;\; \alpha_2 = 1 - b, \;\; 0 < b << 1$, then

$$det\, \mathbb{C} \approx -0.000112 - 0.0000021\alpha_2 \neq 0, \quad rank\, \mathbb{C} = 2 = n. \tag{3.16}$$

Therefore, discrete system (3.13) is controllable. Similarly, we can prove the controllability of system (3.12) with forward Euler discretized form.

3.2. Numerical calculations of controllability of discrete magnetic attitude control system

To the specific discrete system (2.19) with backward difference, we have

$$M_2^{(3)} = I_3,$$

$$M_1^{(3)} = \hat{A}_1 = \widetilde{A}_1 = \begin{bmatrix} k_1\omega_0^2\tau^2 + 2 & 0 & d_1\tau\omega_0 \\ 0 & k_2\omega_0^2\tau^2 + 2 & 0 \\ -d_3\tau\omega_0 & 0 & k_3\omega_0^2\tau^2 + 2 \end{bmatrix},$$

$$M_0^{(3)} = M_1^{(3)}\hat{A}_1 + M_2^{(3)}\hat{A}_0 = \widetilde{A}_1^2 + \widetilde{A}_0 =$$
$$\begin{bmatrix} (k_1\omega_0^2\tau^2 + 2)^2 - d_1 d_3 \tau^2\omega_0^2 - 1 & 0 & d_1\tau\omega_0(k_1\omega_0^2\tau^2 + k_3\omega_0^2\tau^2 + 3) \\ 0 & (k_2\omega_0^2\tau^2 + 2)^2 - 1 & 0 \\ -d_3\tau\omega_0(k_1\omega_0^2\tau^2 + k_3\omega_0^2\tau^2 + 3) & 0 & -d_1 d_3\tau^2\omega_0^2 + (k_3\omega_0^2\tau^2 + 2)^2 - 1 \end{bmatrix},$$

$$\hat{B}_2 = \widetilde{\widetilde{B}}_2 = \tau^2\omega_0^2\epsilon\begin{bmatrix} 0 & 2b_1\sin 2\omega_0\tau & b_4 \\ -2b_2\sin 2\omega_0\tau & 0 & b_2\cos 2\omega_0\tau \\ -b_5 & -b_3\cos 2\omega_0\tau & 0 \end{bmatrix},$$

$$\hat{B}_1 = \widetilde{\widetilde{B}}_1 = \tau^2\omega_0^2\epsilon\begin{bmatrix} 0 & 2b_1\sin \omega_0\tau & b_4 \\ -2b_2\sin \omega_0\tau & 0 & b_2\cos \omega_0\tau \\ -b_5 & -b_3\cos \omega_0\tau & 0 \end{bmatrix}.$$

$$\hat{B}_0 = \widetilde{\widetilde{B}}_0 = \tau^2\omega_0^2\epsilon\begin{bmatrix} 0 & 0 & b_4 \\ 0 & 0 & b_2 \\ -b_5 & -b_3 & 0 \end{bmatrix}.$$

$$M_1^{(3)}\hat{B}_1 = \widetilde{A}_1\widetilde{\widetilde{B}}_1 = \tau^2\omega_0^2\epsilon\begin{bmatrix} -b_5d_1\omega_0\tau & l_{12} & b_4(k_1\omega_0^2\tau^2+2) \\ -2b_2\sin\omega_0\tau(k_2\omega_0^2\tau^2+2) & 0 & b_2\cos\omega_0\tau(k_2\omega_0^2\tau^2+2) \\ -b_5(k_3\omega_0^2\tau^2+2) & l_{32} & -b_4d_3\omega_0\tau \end{bmatrix}$$

where

$$\begin{cases} l_{12} = 2b_1\sin\omega_0\tau(k_1\omega_0^2\tau^2+2) - b_3d_1\omega_0\tau\cos\omega_0\tau, \\ l_{32} = -2b_1d_3\omega_0\tau\sin\omega_0\tau - b_3\cos\omega_0\tau(k_3\omega_0^2\tau^2+2). \end{cases}$$

It can be shown

$$\det(\mathrm{M}_2^{(3)}\hat{\mathrm{B}}_2) = \det(\hat{\mathrm{B}}_2) = \mathrm{b}_2(\mathrm{b}_3\mathrm{b}_4 - \mathrm{b}_1\mathrm{b}_5)\sin 4\omega_0\tau = 0 \tag{3.17}$$

where

$$b_3b_4 - b_1b_5 = \frac{\sin\beta}{J_3}\frac{\cos\beta}{J_1} - \frac{\sin\beta}{J_1}\frac{\cos\beta}{J_3} = 0. \tag{3.18}$$

Based on the results in equation (3.17), the condition of full rank of $\mathbb{C}$ is not satisfied when the determinant of matrix $\mathbb{C}$ composed of the first three columns. However, if the other three independent columns (such as,1,2,5) are selected, for convenience, we choose $\tau = 0.1$ as the period of the discrete system without loss of generality, it is easy to see that $\det(\mathbb{C}) \neq 0$ on the basic of satellite and its orbit parameters in [27]. Then

$$\operatorname{rank}\mathbb{C} = \operatorname{rank}\,[\mathrm{M}_2^{(3)}\hat{\mathrm{B}}_2,\ \ \mathrm{M}_1^{(3)}\hat{\mathrm{B}}_1,\ \ \mathrm{M}_0^{(3)}\hat{\mathrm{B}}_0] = 3. \tag{3.19}$$

According to Theorem 1 , the system (2.19) by backward Euler method is controllable.

Moreover, we make $\tau = 0.2,\ 0.05$ respectivly to compute the rank of $\mathbb{C}$ and the results show different τ have no effect on the controllability of system (2.19). In the same way, we can also calculate the rank of $\mathbb{C}$ for the system (2.14) with Matlab and then prove the discretized system (2.14) by forward Euler method is controllable.

4. Observability of the discrete system

4.1. Observability analysis

Based on the definition of observability of discrete linear time-varying systems in [28], we can also present the following definition of observability of system (2.20).

Definition 2. The second order discrete linear time-varying system (2.20) is observable if for any unknown initial state $(x_0,\ x_1)$, there exists a finite $k_\beta \in N(k_\beta > 0)$ such that $(x_0,\ x_1)$ can be determined uniquely from the knowledge of output y_k and input $u_k,\ k \in [0, k_\beta]$. Otherwise the system is said to be unobservable.

Theorem 2. The second-order linear discrete-time system (2.20) is observable if and only if the observability matrix $\mathbb{S}$ has rank equal to $2n$ and

$$\mathbb{S} = \begin{bmatrix} C_0 & O \\ O & C_1 \\ C_2\hat{A}_0 & C_2\hat{A}_1 \\ C_3\hat{A}_1\hat{A}_0 & C_3(\hat{A}_0 + \hat{A}_1^2) \\ ... & ... \\ C_{2n-1}Q(2n-1) & C_{2n-1}P(2n-1) \end{bmatrix}$$

where the definitions of $M(k), P(k)$ are same as above, and O denotes the relative dimensions of zero matrix.

Proof Taking $k = 0, 1, ...$ in system (2.20) and equations (3.3), we generate the following sequence

$$\begin{aligned} y_0 &= C_0x_0, \quad y_1 = C_1x_1, \\ y_2 &= C_2x_2 = C_2(\hat{A}_0x_0 + \hat{A}_1x_1 + \hat{B}_1u_1) = C_2\hat{A}_0x_0 + C_2\hat{A}_1x_1 + C_2\hat{B}_1u_1, \\ y_3 &= C_3x_3 = C_3\hat{A}_1\hat{A}_0x_0 + C_3(\hat{A}_0 + \hat{A}_1^2)x_1 + C_3\hat{A}_1\hat{B}_1u_1 + C_3\hat{B}_2u_2. \end{aligned}$$

Then the measurement y_k according to the equation (3.4) of state vector x_k is

$$y_k = C_kQ(k)x_0 + C_kP(k)x_1 + \sum_{i=1}^{k-1} C_kM_i^{(k)}\hat{B}_iu_i, \tag{4.1}$$

In general, we have

$$y_{2n-1} = C_{2n-1}Q(2n-1)x_0 + C_{2n-1}P(2n-1)x_1 + \sum_{i=1}^{2n-2} C_{2n-1}M_i^{(2n-1)}\hat{B}_iu_i, \tag{4.2}$$

As consequence, equation (4.2) can be rewritten in the following relevant matrix form

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 - C_2\hat{B}_1u_1 \\ y_3 - C_3\hat{A}_1\hat{B}_1u_1 - C_3\hat{B}_2u_2 \\ ...... \\ y_{2n-1} - \sum_{i=1}^{2n-2} C_{2n-1}M_i^{(2n-1)}\hat{B}_iu_i \end{bmatrix} = \begin{bmatrix} C_0 & O \\ O & C_1 \\ C_2\hat{A}_0 & C_2\hat{A}_1 \\ C_3\hat{A}_1\hat{A}_0 & C_3(\hat{A}_0 + \hat{A}_1^2) \\ ... & ... \\ C_{2n-1}Q(2n-1) & C_{2n-1}P(2n-1) \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

We know from linear algebra that the system of linear algebra equations with 2n unknowns, equation (4.1) has a unique solution (x_0, x_1) if and only if the system matrix has rank $2n$:

$$\text{rank} \begin{bmatrix} C_0 & O \\ O & C_1 \\ C_2\hat{A}_0 & C_2\hat{A}_1 \\ C_3\hat{A}_1\hat{A}_0 & C_3(\hat{A}_0 + \hat{A}_1^2) \\ ... & ... \\ C_{2n-1}Q(2n-1) & C_{2n-1}P(2n-1) \end{bmatrix} = 2\text{n}. \tag{4.3}$$

The matrix in (4.3) we denote by $\mathbb{S}$. Then the initial values x_0, x_1 are determined uniquely, if and only if rank $\mathbb{S} = 2n$. □

Analogously, a numerical example is given as follows to verify the validity of observability analysis.

Example.3 Considering the following system with discrete method [20]

$$\begin{aligned} \ddot{x}(t) &= \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t), \\ y(t) &= [1 \quad 3] x(t). \end{aligned} \tag{4.4}$$

The discrete system has the form

$$\begin{aligned} x_{k+1} &= \hat{A}_0 x_{k-1} + \hat{A}_1 x_k + \hat{B} u_k, \\ y_k &= C x_k \end{aligned} \tag{4.5}$$

where

$$\hat{A}_0 = -I_2, \ \ \hat{A}_1 = 2I_2 + \tau^2 A, \ \ \hat{B} = \tau^2 B, \quad A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \ \ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \ C = [1 \quad 3]. \tag{4.6}$$

Letting $\tau = 0.1s$, according to Theorem 2, we have

$$n = 2, \ \hat{A}_1 = \begin{bmatrix} 2.02 & 0.01 \\ 0.03 & 2.04 \end{bmatrix}, \ \hat{A}_1^2 = \begin{bmatrix} 4.0807 & 0.0406 \\ 0.1218 & 4.1619 \end{bmatrix}, \tag{4.7}$$

$$\mathbb{S} = \begin{bmatrix} C & O \\ O & C \\ C\hat{A}_0 & C\hat{A}_1 \\ C\hat{A}_1\hat{A}_0 & C(\hat{A}_0 + \hat{A}_1^2) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ -1 & -3 & 2.11 & 6.13 \\ -2.11 & -6.13 & 4.4461 & 12.5263 \end{bmatrix}, \tag{4.8}$$

$$det\mathbb{S} = -0.04 \neq 0, \ \ rank\mathbb{S} = 4 = 2n. \tag{4.9}$$

Therefore, the discrete system (4.5) is observable.

4.2. Numerical calculations of observability of discrete magnetic attitude control system

To system (2.19), according to equations (3.5), we also have

$$\begin{aligned} Q(2) &= \hat{A}_0 Q(0) + \hat{A}_1 Q(1) = \hat{A}_0, \\ Q(3) &= \hat{A}_0 Q(1) + \hat{A}_1 Q(2) = \hat{A}_1 \hat{A}_0, \\ Q(4) &= \hat{A}_0 Q(2) + \hat{A}_1 Q(3) = \hat{A}_0^2 + \hat{A}_1^2 \hat{A}_0, \\ Q(5) &= \hat{A}_0 Q(3) + \hat{A}_1 Q(4) = \hat{A}_0 \hat{A}_1 \hat{A}_0 + \hat{A}_1 (\hat{A}_0^2 + \hat{A}_1^2 \hat{A}_0), \\ P(2) &= \hat{A}_0 P(0) + \hat{A}_1 P(1) = \hat{A}_1, \\ P(3) &= \hat{A}_0 P(1) + \hat{A}_1 P(2) = \hat{A}_0 + \hat{A}_1^2, \\ P(4) &= \hat{A}_0 P(2) + \hat{A}_1 P(3) = \hat{A}_0 \hat{A}_1 + \hat{A}_1 \hat{A}_0 + \hat{A}_1^3, \\ P(5) &= \hat{A}_0 P(3) + \hat{A}_1 P(4) = \hat{A}_0 (\hat{A}_0 + \hat{A}_1 \hat{A}_0) + \hat{A}_1 (\hat{A}_0 \hat{A}_1 + \hat{A}_1 \hat{A}_0 + \hat{A}_1^3), \end{aligned} \tag{4.10}$$

and

$$\hat{A}_0 = \widetilde{A}_0 = \begin{bmatrix} -1 & 0 & -d_1\omega_0\tau \\ 0 & -1 & 0 \\ d_3\omega_0\tau & 0 & -1 \end{bmatrix},$$

$$\hat{A}_1 = \widetilde{A}_1 = \begin{bmatrix} k_1\omega_0^2\tau^2 + 2 & 0 & d_1\omega_0\tau \\ 0 & k_2\omega_0^2\tau^2 + 2 & 0 \\ -d_3\omega_0\tau & 0 & k_3\omega_0^2\tau^2 + 2 \end{bmatrix},$$

and we can rewrite the measurement matrix C_k as the following form

$$C_k = \begin{bmatrix} \alpha_2 & 0 & -\alpha_1 \sin\omega_0 k\tau - \alpha_3 \cos\omega_0 k\tau \\ 0 & -\alpha_2 & \alpha_1 \cos\omega_0 k\tau - \alpha_3 \sin\omega_0 k\tau \end{bmatrix}, \ \ k = 0, 1, ..., 5$$

letting

$$-\alpha_1 \sin\omega_0 k\tau - \alpha_3 \cos\omega_0 k\tau = \sqrt{\alpha_1^2 + \alpha_3^2}\sin(\omega_0 k\tau + \phi) \triangleq \eta_k(\phi),$$
$$\alpha_1 \cos\omega_0 k\tau - \alpha_3 \sin\omega_0 k\tau = \sqrt{\alpha_1^2 + \alpha_3^2}\sin(\omega_0 k\tau + \varphi) \triangleq \eta_k(\varphi),$$

where

$$\phi = \arctan\frac{\alpha_3}{\alpha_1},$$
$$\varphi = \arctan(-\frac{\alpha_1}{\alpha_3}),$$

we get

$$C_0 = \begin{bmatrix} \alpha_2 & 0 & -\alpha_3 \\ 0 & -\alpha_2 & \alpha_1 \end{bmatrix}, \ C_1 = \begin{bmatrix} \alpha_2 & 0 & -\eta_1(\phi) \\ 0 & -\alpha_2 & \eta_1(\varphi) \end{bmatrix}, \ C_2 = \begin{bmatrix} \alpha_2 & 0 & -\eta_2(\phi) \\ 0 & -\alpha_2 & \eta_2(\varphi) \end{bmatrix},$$

and

$$C_2Q(2) = \begin{bmatrix} u_{51} & 0 & u_{53} \\ u_{61} & \alpha_2 & u_{63} \end{bmatrix}, \ \ C_2P(2) = \begin{bmatrix} u_{54} & 0 & u_{56} \\ u_{64} & u_{65} & u_{66} \end{bmatrix},$$

where

$$\begin{aligned} u_{51} &= -\alpha_2 + d_3\omega_0\tau\eta_2(\phi), \\ u_{53} &= -\alpha_2 d_1\omega_0\tau - \eta_2(\phi), \\ u_{61} &= d_3\omega_0\tau\eta_2(\varphi), \\ u_{63} &= -\eta_2(\varphi), \\ u_{54} &= \alpha_2(k_1\omega_0^2\tau^2 + 2) - d_3\omega_0\tau\eta_2(\phi), \\ u_{56} &= \alpha_2 d_1\omega_0\tau + (k_3\omega_0^2\tau^2 + 2)\eta_2(\phi), \\ u_{64} &= -d_3\omega_0\tau\eta_2(\varphi), \\ u_{65} &= -\alpha_2(k_2\omega_0^2\tau^2 + 2), \\ u_{66} &= (k_3\omega_0^2\tau^2 + 2)\eta_2(\varphi). \end{aligned}$$

The following matrix U is chosen as the first six columns of the observability matrix $\mathbb{S}$

$$U = \begin{bmatrix} \alpha_2 & 0 & -\alpha_3 & 0 & 0 & 0 \\ 0 & -\alpha_2 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & \eta_1(\phi) \\ 0 & 0 & 0 & 0 & -\alpha_2 & \eta_1(\varphi) \\ u_{51} & 0 & u_{53} & u_{54} & 0 & u_{56} \\ u_{61} & \alpha_2 & u_{63} & u_{64} & u_{65} & u_{66} \end{bmatrix}. \tag{4.11}$$

The determination of matrix U is

$$\begin{aligned}(-\alpha_2^2)(\alpha_2(u_{53}(u_{66} + (k_2\omega_0^2\tau^2 + 2)\eta_1(\varphi)) - u_{56}(\alpha_1 + u_{63})) + \eta_1(\phi)(u_{53}u_{64} - u_{54}(\alpha_1 + u_{63}))\\ +\alpha_3((-\alpha_2)(u_{51}(u_{66} + (k_2\omega_0^2\tau^2)\eta_1(\varphi) - u_{56}u_{61}) + \eta_1(\phi)(u_{51}u_{64} - u_{61}u_{54})))).\end{aligned}$$

Generally speaking, the determination of matrix U is not equal to 0 if and only if α_j $(j = 1, 2, 3) \neq 0$, which is valid according to the definition of α_j. In addition, using Matlab, choosing $\tau = 0.1$, we also have

$$\text{rank U} = \text{ rank } \mathbb{S} = 6. \tag{4.12}$$

Based on Theorem 2, the system (2.19) with backward Euler method is observable.

Moreover, we make $\tau = 0.2$, 0.05 respectivly to compute the rank of $\mathbb{S}$ and the results show different τ have no effect on the observability of system (2.19). The observability of system (2.14) with forward Euler method can be proven similarly.

5. Conclusions

In this paper, two different discrete schemes of the second-order linear time-varying system represented by the linearized satellite magnetic attitude control motion equation are obtained by Euler method. Subsequently, the controllability and observability conditions of a new discrete second-order linear time-varying system are proposed, which are applied to investigate the controllability and observability of the discretized satellite magnetic control system. Some numerical examples are given to further verify the correctness of theoretical results. Research results show that, generally speaking, different periods τ and parameters in coefficient matrices have no effect on the controllability and observability of the resulting discrete system.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (12171196).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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