AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 7766–7786.
DOI: 10.3934/math.2023390
Received: 11 September 2022
Revised: 30 December 2022
Accepted: 16 January 2023
Published: 28 January 2023

Research article

Existence and compatibility of positive solutions for boundary value fractional differential equation with modified analytic kernel

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Abstract: In this article, a Green's function for a fractional boundary value problem in connection with modified analytic kernel has been constructed to study the existence of multiple solutions of a type of characteristic fractional boundary value problems. It is done here by using a well-known result: Krasnoselskii fixed point theorem. Moreover, a practical example is created to understand the importance of main results regarding the existence of solution of a boundary value fractional differential problem with homogeneous conditions. This example analytically and graphically, explains circumstances under which the Green's functions with different types of differential operator are compatible.

Keywords: fractional differential equation; boundary value problem; positive solution; fractional Green's function; fixed-point theorem
Mathematics Subject Classification: 34B18, 45B05

1. Introduction

Fractional calculus is natural generalization of ordinary calculus [1, 2], since, it took derivative and integral of a function with arbitrary order. The concept of fractional calculus is not new. The objective of fractional calculus concurs with that of classical calculus. In 1695, when Leibniz introduced the notation $d^n/(dx)^n$ for ordinary differential operator, L'Hopital raised the concept of fractional power differential operators by asking Leibniz about the arbitrary power of differential operator. In 1730, Euler devoted his efforts to it, followed by Lagrange in 1772 and Laplace in 1812. Lacroix, first time introduced the notion of differentiation with arbitrary powers and later the idea was modified by the other mathematicians of that era. Various fractional derivatives were thus introduced and Grünwald and Krug unified the results of Riemann and Liouville and introduced new integral and differential operators called Riemann-Liouville (RL) fractional differential operators [1]. After that time, the fractional calculus begins to assemble complications of dynamics of complex real world and new concepts are starting to be practice and tested on real data.

In 1996, Delbosco [3] established the results for the existence and uniqueness of nonlinear fractional differential problems for RL operator, after him many other mathematician [4–6] extended his results for the nonlinear fractional differential problem of order different orders for nonlinear boundary value problems. In 2010, Li [7] extended the results for differential operator of order $1 < \xi \leq 2$ for RL operator. After that, number of articles [8–24] were published on the existence and uniqueness of periodic, non-periodic, implicit, explicit and singular boundary value problems, coupled systems and integrodifferential problems with RL operator of different orders.

In recent decades, fractional calculus is used to fill the voids in models for better results. For instance, in rheology [25–27], dynamic problems [28, 29], continuum mechanics [30–32], and in other areas of science and engineering [33–39]. In [40], Xu *et al.* established existence results for the multiple positive solutions for the nonlinear boundary value fractional differential problem with RL operator.

In this article, we will construct the Green's function and some of its important properties for the boundary value fractional differential Problem 1.1 (stated below) defined in the space of continuous real valued functions with modified analytic kernel and prove existence results using fixed point techniques. We also check the compatibility of Green's function for boundary value Problem 1.1 with Green's function of ordinary differential problem and RL differential problem. The boundary value fractional differential problem

$$\left.\begin{array}{l}\mathcal{D}^{\xi,\eta}_{0+}\varphi(t) = \psi(t,\varphi(t)), \quad 0 < t < 1 \\ \varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0\end{array}\right\}, \tag{1.1}$$

where $3 < \xi \leq 4$, and $\eta > 0$ is fixed complex parameter.

This article consists of three section. In first section, introductory material and preliminaries of our main work is given. In second section, Green's function is constructed for the boundary value Problem 1.1 and its properties are provided, and existence results for the boundary value problem are established, an example is also provided in support of results and for the compatibility of solutions, we constructed the graph of Green's function of ordinary differential equation of order four, for the Green's function of fractional order differential problem of order $3 < \xi \leq 4$ with RL differential operator, analytic kernel and modified analytic kernel. In third section, article is concluded.

Fractional integral with RL differential operator [41] of order ξ with a as the constant of integration is stated as

$${}^{RL}\mathcal{I}^{\xi}_{a+}\varphi(t) = \int_a^t (t-\omega)^{\xi-1}\varphi(\omega)d\omega, \qquad Re(\xi) > 0, \tag{1.2}$$

on the other hand RL fractional derivative of order ξ depending on the constant a is stated as

$${}^{RL}\mathcal{D}^{\xi}_{a+}\varphi(t) = \frac{d^n}{dt^n}\left({}^{RL}\mathcal{I}^{n-\xi}_{a+} f(t)\right), \quad Re(\xi) \geq 0, \quad n = Re(\xi) + 1. \tag{1.3}$$

This model contributed in resolving multiple ordinary and partial differential problems through transformation methods [41, 42]. An important method to solve differential problems is Green's function method. Xu [40], in 2009, constructed the Green's function for the RL boundary value problems of fractional order $3 < \xi \leq 4$ to prove the existence results for the boundary value problems.

In 2019, Fernandez et al. [43] presented a new model of fractional integral and differential operators involving modified analytic kernel. Some important properties for the well-defined operator are also described.

Definition 1.1. Let $[a, b] \in \mathbb{R}, \xi, \eta$ be complex parameters with positive real parts and $R \in \mathbb{R}^+$ satisfying the inequality $R > (b-a)^{Re(\eta)}$. Let A be the complex function defined on the analytic disc $D(0, R)$ by locally uniformly convergent power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{1.4}$$

where the coefficients $a_n = a_n(\xi, \eta)$ may depend on complex parameters if required.

Using analytic kernel (1.4), a modified form of analytic kernel is also defined by Fernendez et al. [43] that is stated below:

Definition 1.2. For any analytic function as in Definition 1.1, modified analytic function A_Γ is defined as

$$A_\Gamma(x) = \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) x^n. \tag{1.5}$$

The series (1.5) has radius of convergence given by

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} (n\eta + \eta + \xi)^{-\eta} \right|.$$

Definition 1.3. Fractional integral operator with analytic kernel, acting on the function $g : [a, b] \to \mathbb{R}$ as

$${}^{A}\mathcal{I}_{a+}^{\xi,\eta}\varphi(t) = \int_a^t (t-\omega)^{\xi-1} A\left((t-\omega)^{\eta}\right) \varphi(\omega) d\omega. \tag{1.6}$$

The integral operator defined in (1.6) provides the generalization of RL [1, 2], Prabhakar [44], AB [45] and GPF model [46].

Definition 1.4. Fractional differential operator with analytical kernel is defined in both Caputo and RL forms. The operator is defined as follows:

$$\begin{aligned} {}^{A}_{RL}\mathcal{D}_{a+}^{\xi,\eta}\varphi(t) &= \frac{d^m}{dt^m}\left({}^{A}\mathcal{I}_{a+}^{\xi',\eta'} \varphi(t)\right) && (1.7) \\ {}^{A}_{C}\mathcal{D}_{a+}^{\xi,\eta}\varphi(t) &= {}^{A}\mathcal{I}_{a+}^{\xi',\eta'}\left(\frac{d^m}{dt^m}\varphi(t)\right), && (1.8) \end{aligned}$$

where $m \in \mathbb{N}$, ξ' and η' depends on ξ and η, respectively.

Beta function does not have direct interaction with fractional derivative and integrals but plays a vital role in proofs of fractional theories. It is the combination of multiple gamma functions, formally, beta function is defined as follow:

Definition 1.5. Beta function is usually defined in integral form

$$B(\xi,\eta)=\int_0^1 \omega^{\xi-1}(1-\omega)^{\eta-1}d\omega, \quad Re(\xi)>0,\ Re(\eta)>0. \tag{1.9}$$

Beta function can also be represented in the form

$$B(\xi,\eta)=\frac{\Gamma(\xi)\Gamma(\eta)}{\Gamma(\xi+\eta)}, \quad Re(\xi)>0,\ Re(\eta)>0.$$

The following Krasonel'skii fixed point theorem is fundamental for the proof of our existence results.

Lemma 1.1. [40] Let $\mathcal{W}$ be a complete norm space, and let $\gimel \subseteq \mathcal{W}$ be a cone in $\mathcal{W}$. Suppose ϖ_1, ϖ_2 are open subsets of $\mathcal{W}$ with $0 \in \varpi_1 \subset \overline{\varpi_1} \subset \varpi_2$, and let completely continuous operator $F : \gimel \to \gimel$ such that, either

(i) $\|F\mathfrak{r}\| \leq \|\mathfrak{r}\|,\ \mathfrak{r} \in \gimel \cap \partial\varpi_1,\ \|F\mathfrak{r}\| \geq \|\mathfrak{r}\|,\ \mathfrak{r} \in \gimel \cap \partial\varpi_2$ or;

(ii) $\|F\mathfrak{r}\| \leq \|\mathfrak{r}\|,\ \mathfrak{r} \in \gimel \cap \partial\varpi_2,\ \|F\mathfrak{r}\| \geq \|\mathfrak{r}\|,\ \mathfrak{r} \in \gimel \cap \partial\varpi_1$.

Then F has fixed point in $\gimel \cap \overline{\varpi_2} \backslash \varpi_1$.

2. Results

In this section, we will construct Green's function for fractional differential equations with modified analytic kernel and also provide solution of the problem. Some important properties of the Green's function are also verified. For the construction of Green's function, we first prove several important results for differential operators with modified analytic kernel, by utilizing Definition 1.4. Also, we will prove the existence of positive solutions for Problem 1.1 and gave the compatibility of solution with the help of graph.

Lemma 2.1. Consider $\varphi(t)=t^{\nu}$, where $\nu>-1$, then

$${}^{A}\mathcal{D}_{0+}^{\xi,\eta}t^{\nu}=\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)B(n\eta-\xi,\nu+1)t^{n\eta-\xi+\nu},$$

holds and in particular ${}^{A}\mathcal{D}_{0+}^{\xi,\eta}\left(t^{\xi-n\eta-p}\right)=0,\ p=0,1,2,\cdots,N$, where N is the smallest integer greater than or equal to $(\xi-n\eta)$.

Proof. For $\nu>-1$,

$$\begin{aligned}
{}^{A}\mathcal{D}_{0+}^{\xi,\eta}t^{\nu} &= \frac{d^m}{dt^m}\int_0^t \sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)(t-\omega)^{n\eta+m-\xi-1}\omega^{\nu}d\omega \\
&= \sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)\frac{d^m}{dt^m}t^{n\eta+m+\nu-\xi}\int_0^1 \zeta^{\nu}(1-\zeta)^{n\eta+m-\xi-1}d\zeta \\
&= \sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)B(n\eta+m-\xi,\nu+1)\frac{d^m}{dt^m}t^{n\eta+m+\nu-\xi}.
\end{aligned}$$

So, for $t^{\xi-n\eta-p}$, where $p=0,1,2,\cdots,N$,

$$\begin{aligned}{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\left(\sum_{n=0}^{\infty}t^{\xi-n\eta-p}\right) &= \sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)B(n\eta+m-\xi,\nu+1)\frac{d^m}{dt^m}t^{m-p}\\ &= 0.\end{aligned}$$

Lemma 2.2. Let $\xi>0$ and η be any fixed complex parameters, assume $\varphi\in C(0,1)\cap L(0,1)$, then the fractional differential equation

$${}^{A}\mathcal{D}_{0+}^{\xi,\eta}\varphi(t)=0,$$

has

$$\begin{aligned}\varphi(t)=c_1\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1}+c_2\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}\\ +\cdots+c_N\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-N};\quad c_i\in\mathbb{R},\ i=1,2,\cdots,N,\end{aligned}$$

as the unique solution.

Proof. Consider

$$\begin{aligned}\varphi(t)=c_1\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1}+c_2\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}\\ +\cdots+c_N\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-N}.\end{aligned}$$

By Definition 1.4, we have

$$\begin{aligned}{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\varphi(t)=c_1\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)\,{}^{A}\mathcal{D}_{0+}^{\xi,\eta}t^{\xi+n\eta-1}+c_2\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)\,{}^{A}\mathcal{D}_{0+}^{\alpha,\beta}t^{\alpha+n\beta-2}\\ +\cdots+c_N\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)\,{}^{A}\mathcal{D}_{0+}^{\xi,\eta}t^{\xi+n\eta-N}.\end{aligned}$$

By using remark with $p=c_i,\ i=1,2,\cdots,N$, we have

$${}^{A}\mathcal{D}_{0+}^{\xi,\eta}\varphi(t)=0.$$

From Lemma 2.2 and ${}^{A}\mathcal{D}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{I}_{0+}^{\xi,\eta}\varphi=\varphi$ for all $\varphi\in C(0,1)\cap L(0,1)$, we deduce the following:

Lemma 2.3. Let $\varphi\in L(0,1)\cap C(0,1)$ with a fractional differential operator involving modified analytic kernel that belonging $L(0,1)\cap C(0,1)$. Then

$${}^{A}\mathcal{I}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\varphi(t)=\varphi(t)+c_1\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1}+c_2\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}$$

$$+\cdots+c_N\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-N},$$

for some $c_i\in\mathbb{R},\ i=1,2,\cdots,N$.

Proof. Since

$$
\begin{aligned}
{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{I}_{0+}^{\xi,\eta}\varphi(t)&=\varphi(t),\\
{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\left({}^{A}\mathcal{D}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{I}_{0+}^{\xi,\eta}\varphi(t)\right)&={}^{A}\mathcal{D}_{0+}^{\xi,\eta}\varphi(t).
\end{aligned}
$$

By using Lemma 2.2, we have

$$
\begin{aligned}
{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\left({}^{A}\mathcal{D}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{I}_{0+}^{\xi,\eta}\varphi(t)\right)&=0,\\
{}^{A}\mathcal{I}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\left({}^{A}\mathcal{D}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{I}_{0+}^{\xi,\eta}\varphi(t)\right)&={}^{A}\mathcal{I}_{0+}^{\xi,\eta}(0),
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
{}^{A}\mathcal{I}_{0+}^{\xi,\eta}\,{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\varphi(t)=\varphi(t)&+c_1\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1}+c_2\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}\\
&+\cdots+c_N\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-N}.
\end{aligned}
$$

Now, we will construct the Green's function for the boundary value fractional differential problem with modified analytic kernel.

Lemma 2.4. Given $\psi\in C[0,1],\ 3<\xi\leq 4$ and $\eta>0$ be fixed, the unique solution of

$$
\begin{aligned}
{}^{A}\mathcal{D}_{0+}^{\xi,\eta}\varphi(t)&=\psi(t),\quad 0<t<1, &(2.1)\\
\varphi(0)=\varphi'(0)&=\varphi(1)=\varphi'(1)=0, &(2.2)
\end{aligned}
$$

is

$$\varphi(t)=\int_0^1 G(t,\omega)\psi(\omega)d\omega,$$

where

$$
G(t,\omega)=\begin{cases}\left[\begin{array}{c}\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)(t-\omega)^{\xi+n\eta-1}+\\ \sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}(1-\omega)^{\xi+n\eta-2}\\ \times\{(\omega-t)+\omega(1-t)(\xi+n\eta-2)\}\end{array}\right], & 0\leq\omega\leq t;\\ \left[\begin{array}{c}\sum_{n=0}^{\infty}a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}(1-\omega)^{\xi+n\eta-2}\\ \times\{(\omega-t)+\omega(1-t)(\xi+n\eta-2)\}\end{array}\right], & t\leq\omega\leq 1.\end{cases}\tag{2.3}
$$

Proof. We apply Lemma 2.3 to reduce (2.1) to the following integral equation

$$\varphi(t) = {}^{A}\mathcal{I}_{0+}^{\xi,\eta}\psi(t) - c_1\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1} - c_2\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}$$
$$- c_3\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-3} - c_4\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-4},$$

for some $c_i \in \mathbb{R}$, where $i = 1,2,3,4$. By (2.2), we have $c_3, c_4 = 0$ and

$$c_1 = -\frac{1}{\mathfrak{R}}\int_0^1 \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(1-\omega)^{\xi+n\eta-2}\left\{\omega(2-\xi-n\eta)-1\right\}\psi(\omega)d\omega$$
$$c_2 = -\frac{1}{\mathfrak{R}}\int_0^1 \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-1)(1-\omega)^{\xi+n\eta-2}\omega\psi(\omega)d\omega,$$

where

$$\begin{aligned}\mathfrak{R} &= \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-1) - \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)\\ &= \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta).\end{aligned}$$

Therefore, unique solution of problem (2.1) with (2.2) is

$$\begin{aligned}\varphi(t) &= \int_0^t \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-\omega)^{\xi+n\eta-1}\psi(\omega)d\omega\\
&\quad+ \int_0^1 \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1}(1-\omega)^{\xi+n\eta-2}\left\{\omega(2-\xi-n\eta)-1\right\}\psi(\omega)d\omega\\
&\quad+ \int_0^1 \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-1)t^{\xi+n\eta-2}(1-\omega)^{\xi+n\eta-2}\omega\psi(\omega)d\omega\\
&= \int_0^t \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-\omega)^{\xi+n\eta-1}\psi(\omega)d\omega\\
&\quad+ \int_0^t \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1}(1-\omega)^{\xi+n\eta-2}\left\{\omega(2-\xi-n\eta)-1\right\}\psi(\omega)d\omega\\
&\quad+ \int_t^1 \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-1}(1-\omega)^{\xi+n\eta-2}\left\{\omega(2-\xi-n\eta)-1\right\}\psi(\omega)d\omega\\
&\quad+ \int_0^t \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-1)t^{\xi+n\eta-2}(1-\omega)^{\xi+n\eta-2}\omega\varphi(\omega)d\omega\\
&\quad+ \int_t^1 \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-1)t^{\xi+n\eta-2}(1-\omega)^{\xi+n\eta-2}\omega\psi(\omega)d\omega\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \left[\sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(t-\omega)^{\xi+n\eta-1} + \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) t^{\xi+n\eta-2} \right. \\
&\quad \times (1-\omega)^{\xi+n\eta-2} \left. \left(t \sum_{n=0}^{\infty} (\omega(2-\xi-n\eta)-1) + \sum_{n=0}^{\infty} \omega(\xi+n\eta-1) \right) \right] \\
&\quad \times \psi(\omega) d\omega + \int_t^1 \left[\sum_{n=0}^{\infty} a_n \Gamma(\xi+n\eta) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} \right. \\
&\quad \times \left. \left(t \sum_{n=0}^{\infty} (\omega(2-\xi-n\eta)-1) + \sum_{n=0}^{\infty} \omega(\xi+n\eta-1) \right) \right] \psi(\omega) d\omega \\
&= \int_0^t \left[\sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(t-\omega)^{\xi+n\eta-1} + \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) t^{\xi+n\eta-2} \ (1-\omega)^{\xi+n\eta-2} \right. \\
&\quad \times \left. \{(\omega - t) + \omega(1-t)(\xi+n\eta-2)\} \right] \psi(\omega) d\omega \\
&\quad + \int_t^1 \left[\sum_{n=0}^{\infty} a_n \Gamma(\xi+n\eta) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} \{(\omega-t) + \omega(1-t)(\xi+n\eta-2)\} \right] \\
&\quad \times \psi(\omega) d\omega \\
&= \int_0^1 G(t,\omega) \psi(\omega) d\omega.
\end{aligned}
$$

If we take $\eta = 0$ and $a_n = 1/(\Gamma(\xi))^2$ in Lemma 2.4, then we have the unique solution of RL fractional differential boundary value problem [40].

Few important properties of Green's function of differential Problem 2.1 are crucial in the proof of our main result.

Lemma 2.5. The function $G(t,\omega)$ defined in (2.3), holds the following properties:

(i) $G(t,\omega) > 0$;

(ii) $G(t,\omega) = G(1-\omega, 1-t)$;

(iii) $\sum_{n=0}^{\infty} a_n \Gamma(\xi+n\eta)(\xi+n\eta-2)(t-t\omega)^{\xi+n\eta-2}\omega^2(1-t)^2 \leq G(t,\omega) \leq M_0\omega^2(1-\omega)^{\xi+n\eta-2}$;

(iv) $\sum_{n=0}^{\infty} a_n \Gamma(\xi+n\eta)(\xi+n\eta-2)(t-t\omega)^{\xi+n\eta-2}\omega^2(1-t)^2 \leq G(t,\omega) \leq M_0 t^2(1-t)^{\xi+n\eta-2}$,

where $t,\omega \in (0,1)$ and $M_0 = \max\left\{\sum_{n=0}^{\infty}(\xi+n\eta-2)^2, \sum_{n=0}^{\infty}(\xi+n\eta-1)\right\}$.

Proof.

(i) and (ii) Observing the expression of $G(t,\omega)$, it is clear that $G(t,\omega) > 0$ and $G(t,\omega) = G(1-\omega, 1-t)$.

(iii) Consider $G(t,\omega)$ when $\omega \leq t$, we have

$$
G(t,\omega) = \sum_{n=0}^{\infty} a_n \Gamma(\xi+n\eta)(t-\omega)^{\xi+n\eta-1} + \sum_{n=0}^{\infty} a_n \Gamma(\xi+n\eta) t^{\xi+n\eta-2}
$$

$$
\begin{aligned}
&\times (1-\omega)^{\xi+n\eta-2}\left\{(\omega-t)+\omega(1-t)(\xi+n\eta-2)\right\}\\
&=\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-\omega)\left[(t-\omega)^{\xi+n\eta-2}-(t-t\omega)^{\xi+n\eta-2}\right]\\
&\quad+\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-t\omega)^{\xi+n\eta-2}\omega(1-\omega)(\xi+n\eta-2)\\
&\geq\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-\omega)(t-t\omega)^{\xi+n\eta-3}\omega(t-1)(\xi+n\eta-2)\\
&\quad+\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-t\omega)^{\xi+n\eta-2}\omega(1-\omega)(\xi+n\eta-2)\\
&=\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)(t-t\omega)^{\xi+n\eta-3}\omega^2(1-t)^2\\
&\geq\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)(t-t\omega)^{\xi+n\eta-2}\omega^2(1-t)^2,
\end{aligned}
\tag{2.4}
$$

where $\xi > 3$ is used.

On the other hand $1-t \leq 1-\omega$ and for $\xi > 3$, we have

$$
\begin{aligned}
G(t,\omega)&=\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-\omega)^{\xi+n\eta-1}+\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)t^{\xi+n\eta-2}\\
&\quad\times (1-\omega)^{\xi+n\eta-2}\left\{(\omega-t)+\omega(1-t)(\xi+n\eta-2)\right\}\\
&=\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-\omega)\left[(t-\omega)^{\xi+n\eta-2}-(t-t\omega)^{\xi+n\eta-2}\right]\\
&\quad+\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-t\omega)^{\xi+n\eta-2}\omega(1-t)(\xi+n\eta-2)\\
&\leq\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-\omega)^{\xi+n\eta-2}\omega(t-1)(\xi+n\eta-2)\\
&\quad+\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(t-t\omega)^{\xi+n\eta-2}\omega(1-t)(\xi+n\eta-2)\\
&=\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)\omega(1-t)[(t-t\omega)^{\xi+n\eta-2}\\
&\quad-(t-\omega)^{\xi+n\eta-2}]\\
&\leq\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)\omega^2(1-t)^2(t-t\omega)^{\xi+n\eta-3}\\
&\leq\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)\omega^2(1-\omega)^{\xi+n\eta-2}
\end{aligned}
$$

$$\leq \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) M_0 \omega^2 (1-\omega)^{\xi+n\eta-2}. \tag{2.5}$$

Now for $t \leq \omega$, we have

$$G(t,\omega) = \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} [(\omega - t) + \omega(1-t)(\xi + n\eta - 2)]. \tag{2.6}$$

Since $\omega - t \geq 0$, we have (2.6) as

$$\begin{aligned} G(t,\omega) &= \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} \omega(1-t)(\xi + n\eta - 2) \\ &= \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} \omega^2 (1-t)^2. \end{aligned} \tag{2.7}$$

On the other hand for $\xi > 3$ and $t \leq \omega$, we have

$$\begin{aligned} G(t,\omega) &= \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} [(\omega - t) + \omega(1-t)(\xi + n\eta - 2)] \\ &= \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} [\omega + \omega(\xi + n\eta - 2) - t\omega(\xi + n\eta - 2) - t]. \end{aligned}$$

Since $[-t\omega(\xi + n\eta - 2) - t] \leq 0$ and $t \leq \omega$, by using these values in above, we have

$$\begin{aligned} G(t,\omega) &\leq \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) t^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} [\omega + \omega(\xi + n\eta - 2)] \\ &\leq \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta) \omega^{\xi+n\eta-2} (1-\omega)^{\xi+n\eta-2} \omega [\xi + n\eta - 1] \\ &\leq \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 1) \omega^{\xi+n\eta-1} (1-\omega)^{\xi+n\eta-2} \\ &\leq M_0 \omega^{\xi+n\eta-1} (1-\omega)^{\xi+n\eta-2}, \end{aligned} \tag{2.8}$$

So, from (2.4)–(2.8), we get the result.

(iv) Since properties from (i)–(iii) holds, (iv) comes from (i) and (iii), the proof is complete.

Now onward, we assume $q(t) = t^2(1-t)^{\xi+n\eta-2}$ and $k(\tau) = \tau^2(1-\tau)^{\xi+n\eta-2}$, so we have $\sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2) k(\tau) q(t) \leq G(t,\tau) \leq M_0 k(\tau)$.

Now, we will prove the existence of positive solutions for Problem 1.1 and gave the compatibility of solution with the help of graph. For ease, we first symbolize some expressions as follow:

$$\psi_0 = \lim_{\mu \to 0+} \inf \min_{x \in [0,1]} \frac{\psi(x,\mu)}{\mu}, \quad \psi^0 = \lim_{\mu \to 0+} \sup \max_{x \in [0,1]} \frac{\psi(x,\mu)}{\mu},$$

$$\psi_\infty = \lim\inf_{\mu\to+\infty} \min_{x\in[0,1]} \frac{\psi(x,\mu)}{\mu}, \quad \psi^\infty = \lim\sup_{\mu\to+\infty} \max_{x\in[0,1]} \frac{\psi(x,\mu)}{\mu},$$

$$\overline{N} = \left(\sum_{n=0}^{\infty}\left(a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q\left(1-\frac{2}{\xi}\right)\right)^2 \int_0^1 k(\omega)d\omega\right)^{-1},$$

$$\widetilde{N} = \left(\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q\left(1-\frac{2}{\xi}\right)\sigma \int_0^1 k(\omega)d\omega\right)^{-1},$$

$$\sigma = \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q(t)}{M_0},$$

$$M = \left(M_0 \int_0^1 k(\omega)d\omega\right)^{-1}.$$

For the proof of main results, following hypothesis holds:

(P) $\psi : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous.

(H_1) $\psi_0 > \overline{N}$, $\psi_\infty > \widetilde{N}$;

(H_2) $\psi^0 < M$, $\psi^\infty < M$;

(H_3) There exists $p \in \mathbb{R}^+$ in such a way that $0 \le \mu \le p$ and $0 \le t \le 1$ imply $\psi(t,\mu) < Mp$;

(H_4) There is a $p > 0$ such that $\sigma p \le \mu \le p$ and $\frac{1}{4} \le t \le \frac{3}{4}$ imply $\psi(t,\mu) > \widetilde{N}\mu$.

Let φ is the solution of Problem 1.1. Then

$$\varphi(t) = \int_0^1 G(t,\omega)\psi(\omega,\varphi(\omega))d\omega, \quad 0 \le t \le 1. \tag{2.9}$$

Let K be the cone in $E = C[0,1]$ defined by

$$K = \left\{\varphi \in E : \varphi(t) \ge \frac{\sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q(t)}{M_0}\|\varphi\|, t\in[0,1]\right\}.$$

where $\|\varphi\| = \sup_{0\le t\le 1} |\varphi(t)|$. Define an operator $A : K \to E$ as stated below:

$$(A\varphi)(t) = \int_0^1 G(t,\omega)\psi(\omega,\varphi(\omega))d\omega. \tag{2.10}$$

Then we have the following lemma.

Proposition 2.1. Let hypothesis (P) holds. Then $A(K) \subset K$.

Proof. Using (2.10) and Lemma 2.5, we have

$$\|A\varphi\| \le \int_0^1 M_0 k(\omega)\psi(\omega,\varphi(\omega))d\omega.$$

Also, we have

$$
\begin{aligned}
(A\varphi)(t) &\geq \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2)q(t) \int_0^1 k(\omega)\psi(\omega, \varphi(\omega))d\omega \\
&\geq \sum_{n=0}^{\infty} \frac{a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2)q(t)}{M_0} \|A\varphi\|.
\end{aligned}
$$

Thus we have $A(K) \subset K$.

The following theorems are our main results.

Theorem 2.1. Let (P), (H_1) and (H_3) holds. Then Problem 1.1 has at least two positive solutions φ_1 and φ_2 with

$$
0 < \|\varphi_1\| < p < \|\varphi_2\|,
$$

where $\|\varphi\| = \sup_{t\in[0,1]} |\varphi(t)|$.

Proof. Let (H_1) is satisfied, then $\psi_0 > \overline{N}$. Let $\epsilon > 0$ and $0 < r_0 < p$ so that

$$
\frac{\psi(t, \varphi)}{\varphi} - \epsilon \geq \overline{N} \;\Rightarrow\; \psi(t, \varphi) \geq (\overline{N} + \epsilon)\varphi, \quad \forall\, t \in [0, 1],\; 0 \leq \varphi \leq r_0. \tag{2.11}
$$

Let $r \in (0, r_0), \Omega_r = \{\varphi \in K : \|\varphi\| < r\}$. Then for $\varphi \in \partial\Omega_r$ we have

$$
\frac{\sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2)q(t)}{M_0} r \leq \varphi(t) < r, \tag{2.12}
$$

for $t \in [0, 1]$, and so

$$
\begin{aligned}
(A\varphi)\left(1 - \frac{2}{\xi}\right) &= \int_0^1 G\left(1 - \frac{2}{\xi}, \omega\right)\psi(\omega, \varphi(\omega))d\omega \\
&\geq \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2)q\left(1 - \frac{2}{\xi}\right)\int_0^1 k(\omega)\psi(\omega, \varphi(\omega))d\omega \\
&\geq \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2)q\left(1 - \frac{2}{\xi}\right)\int_0^1 k(\omega)(\overline{N} + \epsilon)\varphi(\omega)d\omega \\
&\geq \sum_{n=0}^{\infty} \frac{\left(a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2)q\left(1 - \frac{2}{\xi}\right)\right)^2}{M_0}(\overline{N} + \epsilon)r \int_0^1 k(\omega)d\omega \\
&> \sum_{n=0}^{\infty} \frac{\left(a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2)q\left(1 - \frac{2}{\xi}\right)\right)^2}{M_0}(\overline{N})r \int_0^1 k(\omega)d\omega \\
&= r = \|\varphi\|,
\end{aligned}
$$

thus, we have $\|A\varphi\| > r$, for $\varphi \in \partial\Omega_r$. On the other hand, since $\psi_\infty > \widetilde{N}$ (from (H_1)), there exist $\epsilon > 0$ and $H > 0$ in such a way that

$$
\frac{\psi(t, \varphi)}{\varphi} - \epsilon \geq \widetilde{N} \;\Rightarrow\; \psi(t, \varphi) \geq (\widetilde{N} + \epsilon)\varphi, \quad \forall\, t \in [0, 1],\; u \geq H. \tag{2.13}
$$

Choose $R > R_0 = \max\left\{\frac{H}{\sigma}, p\right\}$. Let $\Omega_R = \{\varphi \in K \big| \, \|\varphi\| < R\}$, $\varphi \in \partial\Omega_R$. Since $\varphi(t) \geq \sigma\|\varphi\| > H$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, from (2.13) we see that

$$\begin{aligned}(A\varphi)\left(1-\frac{2}{\xi}\right) &= \int_0^1 G\left(1-\frac{2}{\xi},\omega\right)\psi(\omega,\varphi(\omega))d\omega \\ &\geq \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q\left(1-\frac{2}{\xi}\right)\int_0^1 k(\omega)\psi(\omega,\varphi(\omega))d\omega \\ &\geq \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q\left(1-\frac{2}{\xi}\right)\int_0^1 k(\omega)(\widetilde{N}+\epsilon)\varphi(\omega)d\omega \\ &\geq \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q\left(1-\frac{2}{\xi}\right)(\widetilde{N}+\epsilon)\sigma R\int_{\frac{1}{4}}^{\frac{3}{4}} k(\omega)d\omega \\ &> \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q\left(1-\frac{2}{\xi}\right)(\widetilde{N})R\sigma\int_{\frac{1}{4}}^{\frac{3}{4}} k(\omega)d\omega \\ &= R = \|\varphi\|.\end{aligned}$$

which implies that $\|A\varphi\| > R$, for $\varphi \in \partial\Omega_R$. Let (H_3) holds. Let $\Omega_p = \{\varphi \in K \big| \, \|\varphi\| < p\}$. For $\varphi \in \partial\Omega_p$, we have

$$\psi(t,\varphi(t)) < Mp, \quad \forall\, t \in [0,1]. \tag{2.14}$$

so we have

$$\begin{aligned}\|A\varphi\| &\leq \int_0^1 M_0 k(\omega)\psi(\omega,\varphi(\omega))d\omega, \\ &< Mp\int_0^1 M_0 k(\omega)d\omega, \\ &= p = \|\varphi\|.\end{aligned}$$

Therefore, $0 < r < p < R$, by Lemma 1.1, we have $0 \leq \|\varphi_1\| \leq p \leq \|\varphi_2\|$.

Corollary 2.1. If (H_1) is replaced by following inequality (H_1^*), conclusion of Theorem 2.1 holds.

$$(H_1^*) \qquad \psi_0 = \infty, \quad \psi_\infty = \infty.$$

Proof. Assume $\psi_0 = \infty$, i.e.,

$$\liminf_{\varphi\to 0+}\min_{t\in[0,1]}\frac{\psi(t,\varphi)}{\varphi} = \infty.$$

Then for every $0 < r_0 < p$, there exist $\overline{N} > 0$, in such a way that for all $\varphi \in K$ with $\varphi(t) \geq \sigma\|\varphi\| > \overline{N}$ and for any $\epsilon > 0$, we have

$$\psi(t,\varphi) \geq (\overline{N}+\epsilon)\varphi, \quad \forall\, t \in [0,1], \quad \varphi \geq \overline{N}.$$

Assume $r \in (0, r_0)$, $\Omega_r = \{\varphi \in K \big| \|\varphi\| < r\}$. Then for $\varphi \in \partial\Omega_r$, we have $\frac{\sum_{n=0}^{\infty} a_n \Gamma(\xi+n\eta)(\xi+n\eta-2)q(t)}{M_0} r \le \varphi(t) \le r$ for $t \in [0, 1]$. So, for

$$(A\varphi)\left(1 - \frac{2}{\xi}\right) = \int_0^1 G\left(1 - \frac{2}{\xi}, \omega\right)\psi(\omega, u(\omega))d\omega > r.$$

Similarly, for $\psi_\infty = \infty$, we must have $\epsilon > 0$ and $\widetilde{N} > 0$, we have

$$\psi(t, \varphi(t)) \ge (\widetilde{N} + \epsilon)u, \quad \forall t \in [0, 1], \ \varphi \ge \widetilde{N}.$$

Choosing $R > max\left\{\frac{\widetilde{N}}{\sigma}, p\right\}$. Assume $\Omega_R = \left\{\varphi \in K \big| \|\varphi\| < R\right\}$. Also by definition of cone and limit of the function we have, $\varphi(t) \ge \sigma\|\varphi\| > \widetilde{N}$ for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$. So we have

$$(A\varphi)\left(1 - \frac{2}{\xi}\right) = \int_0^1 G\left(1 - \frac{2}{\xi}, \omega\right)\psi(\omega, \varphi(\omega))d\omega > R.$$

which completes the proof.

Theorem 2.2. Suppose that the hypothesis (P), (H_2) and (H_4) holds. Then Problem 1.1 has at least two positive solutions φ_1 and φ_2 satisfying the following inequality

$$0 < \|\varphi_1\| < p < \|\varphi_2\|.$$

Proof. Let (H_2) is satisfied. Since $\psi^0 < M$, there exists $\epsilon > 0$ and $0 < r_0 < p$, thus we have

$$\psi(t, \varphi) \le (M - \epsilon)\varphi, \quad \forall \ t \in [0, 1], \ 0 \le \varphi \le r_0.$$

Let $r \in (0, r_0)$, $\Omega_r = \{\varphi \in K \big| \ \|\varphi\| < r\}$. Then for $\varphi \in \partial\Omega_r$, we have $\sum_{n=0}^{\infty} \frac{a_n\Gamma(\xi+n\eta)}{M_0} q(t)r \le \varphi(t) \le r$, for $t \in [0, 1]$ and so

$$\begin{aligned}
(A\varphi)(t) &= \int_0^1 G(t, \omega)\psi(\omega, \varphi(\omega))d\omega \\
&\le \int_0^1 M_0 k(\omega)\psi(\omega, \varphi(\omega))d\omega \\
&\le \int_0^1 M_0 k(\omega)(M - \epsilon)\varphi(\omega)d\omega \\
&\le \int_0^1 M_0 k(\omega)(M - \epsilon) r d\omega \\
&< rM\left(\int_0^1 M_0 k(\omega)d\omega\right) \\
&= r = \|\omega\|.
\end{aligned}$$

From which we see that $\|A\varphi\| < r$ for $\varphi \in \partial\Omega_r$.

Also from (H_2), $\psi^\infty < M$, there exists $\epsilon > 0$ and $H > 0$ in such a way that

$$\psi(t, \varphi) \le (M - \epsilon)\varphi, \quad \forall \ t \in [0, 1], \varphi \ge H. \tag{2.15}$$

If $\max_{t\in[0,1]}\psi(t,\varphi)$ is unbounded for $\varphi \geq H$, then we choose $R > r + p$, which implies

$$\psi(t,\varphi) \leq \max_{t\in[0,1]}\psi(t,R), \quad \forall\, u \in (0,R], \quad t \in [0,1]. \tag{2.16}$$

For $\varphi \in K$ with norm $\|\varphi\| = R$, from (2.15) and (2.16), we have

$$\begin{aligned}
(A\varphi)(t) &= \int_0^1 G(t,\omega)\psi(\omega,\varphi(\omega))d\omega \\
&\leq \int_0^1 M_0 k(\omega)\psi(\omega,\varphi(\omega))d\omega \\
&\leq \int_0^1 M_0 k(\omega)(M-\epsilon)\varphi(\omega)d\omega \\
&\leq \int_0^1 M_0 k(\omega)(M-\epsilon)R d\omega \\
&< RM\left(\int_0^1 M_0 k(\omega)d\omega\right) \\
&= R = \|\varphi\|.
\end{aligned}$$

Also if $\max_{t\in[0,1]}\psi(t,\varphi)$ be bounded on nonnegative real interval, say

$$\psi(t,\varphi) \leq L, \quad \forall\, \varphi \geq 0,\ t \in [0,1]. \tag{2.17}$$

On the other hand, we assume $R > p + \frac{L}{M}$, for $\varphi \in K$ with $\|\varphi\| = R$, from (2.17), we have

$$\begin{aligned}
(A\varphi)(t) &= \int_0^1 G(t,\omega)\psi(\omega,\varphi(\omega))d\omega \\
&\leq \int_0^1 M_0 k(\omega)\psi(\omega,\varphi(\omega))d\omega \\
&\leq \int_0^1 M_0 k(\omega)L d\omega \\
&\leq \frac{L}{M} < R = \|\varphi\|.
\end{aligned}$$

Thus, in either case, we may put $\Omega_R = \{\varphi \in K \big| \, \|\varphi\| < R\}$ and we have $\|A\varphi\| < \|\varphi\|$ for all $\varphi \in \partial\Omega_R$. Now suppose that (H_4) holds. For any $\varphi \in \partial\Omega_p$. Since $\sigma p \leq \varphi(t) \leq p$ for $t \in \left[\frac{1}{4},\frac{3}{4}\right]$, we have

$$\psi(t,\varphi(t)) > \tilde{N}\varphi, \quad \forall\, t \in \left[\frac{1}{4},\frac{3}{4}\right],$$

and so

$$\begin{aligned}
(A\varphi)\left(1-\frac{2}{\xi}\right) &= \int_0^1 G\left(1-\frac{2}{\xi},\omega\right)\psi(\omega,\varphi(\omega))d\omega \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n\Gamma(\xi+n\eta)(\xi+n\eta-2)q\left(1-\frac{2}{\xi}\right)k(\omega)\psi(\omega,\varphi(\omega))d\omega
\end{aligned}$$

$$
\begin{aligned}
&> \int_{\frac{1}{4}}^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2) q \left(1 - \frac{2}{\xi}\right) k(\omega) \tilde{N} \varphi(\omega) d\omega \\
&> \int_{\frac{1}{4}}^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2) q \left(1 - \frac{2}{\xi}\right) k(\omega) \tilde{N} \sigma p d\omega \\
&= p\tilde{N} \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \sum_{n=0}^{\infty} a_n \Gamma(\xi + n\eta)(\xi + n\eta - 2) q \left(1 - \frac{2}{\xi}\right) k(\omega) \sigma d\omega \right) \\
&= p = \|\varphi\|,
\end{aligned}
$$

which implies that $\|A\varphi\| > p$ for $\varphi \in \partial\Omega_p$. Thus, by (i) and (ii) of Lemma 1.1, we complete the proof.

Corollary 2.2. If (H_2^*) is used in place of (H_2), the conclusion of Theorem 2.2 still holds.

$$\psi^0 = 0, \quad \psi^\infty = 0. \tag{H_2^*}$$

Theorem 2.3. Let hypothesis (P) and the following condition holds:

$$\psi_0 > \overline{N}, \qquad \psi^\infty < M.$$

Then there exists at least one positive solution for fractional differential Problem 1.1.

Corollary 2.3. Let hypothesis (P) and the equalities stated below holds:

$$\psi_0 = \infty, \qquad \psi^\infty = 0.$$

Then there exists at least one positive solution for fractional differential Problem 1.1.

Theorem 2.4. Let hypothesis (P) and the following condition holds:

$$\psi^0 < M, \qquad \psi_\infty < \widetilde{N}.$$

Then there exists at least one positive solution for fractional differential Problem 1.1.

Corollary 2.4. Assume that (P) and the following condition holds:

$$\psi^0 = 0, \qquad \psi_\infty = \infty.$$

Then Problem 1.1 has at least one positive solution.

Example 2.1. Consider boundary value fractional differential problem with homogeneous conditions as

$$
\begin{aligned}
&{}^{A}\mathcal{D}_{0+}^{3.5,1.5}\varphi(t) = \varphi^a(t) + \varphi^b(t), \quad 0 < a < 1 < b,\ 0 < t < 1, \\
&\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0.
\end{aligned} \tag{2.18}
$$

Then there exists at least two positive solutions φ_1 and φ_2 with

$$0 < \|\varphi_1\| < 1 < \|\varphi_2\|.$$

Proof. Set $\psi(t,\varphi) = \varphi^a + \varphi^b$ and note that

$$\liminf_{\varphi\to 0+} \min_{t\in[0,1]} \frac{\psi(t,\varphi)}{\varphi} = \infty, \qquad \liminf_{\varphi\to +\infty} \min_{t\in[0,1]} \frac{\psi(t,\varphi)}{\varphi} = \infty.$$

So (H_1^*) holds, (P) holds. Also note for $n = 1$, we have

$$M = \left(\int_0^1 M_0 k(\omega)d\omega\right)^{-1} = \left(\frac{11.25}{60}\right)^{-1} \approx 5.33.$$

Since $p = 1$ such that $0 \le \varphi \le 1$, which implies

$$\psi(t,\varphi) \le 2 < M = Mp.$$

Thus, (H_3) holds. So, by using Corollary 2.1, the proof is completed.

Example 2.2. Consider the differential problem

$$\begin{aligned} &{}^{A}D_{0+}^{3.5,1.3}\varphi(t) = \left(te^{-2t} + 16\right)\varphi^a(t), \quad a > 1,\ 0 < t < 1, \\ &\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0. \end{aligned} \tag{2.19}$$

Then there exists at least two positive solutions φ_1 and φ_2 with

$$0 < \|\varphi_1\| < 1 < \|\varphi_2\|.$$

Proof. Consider $\psi(t,\varphi) = \left(te^{-2t} + 16\right)\varphi^a(t)$. Then we have

$$\limsup_{\varphi\to 0+} \max_{t\in[0,1]} \frac{\psi(t,\varphi)}{\varphi} = 0, \qquad \limsup_{\varphi\to +\infty} \max_{t\in[0,1]} \frac{\psi(t,\varphi)}{\varphi} = 0.$$

Therefore, (H_2^*) holds and for $n = 2$, we have

$$\widetilde{N} = \left(\sum_{n=0}^{\infty} a_n\Gamma(\xi + n\eta)(\xi + n\eta - 2)q\left(1 - \frac{2}{\xi}\right)\sigma \int_0^1 k(\omega)d\omega\right)^{-1} \approx \left(7\times 10^{-6}\right)^{-1}$$

$$\sigma = \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{\sum_{n=0}^{\infty} a_n\Gamma(\xi + n\eta)(\xi + n\eta - 2)q(t)}{M_0} \approx 0.0015.$$

Since $p = 1$ such that $0.0015 \le \varphi^a \le 1$ and $\frac{1}{4} \le t \le \frac{3}{4}$ implies $\psi(t,\varphi) > \widetilde{N}\varphi$. Then by Corollary 2.2, the result holds.

The following Figure 1 represents the Green's function for Problem 2.19 with $t = 0.5$.

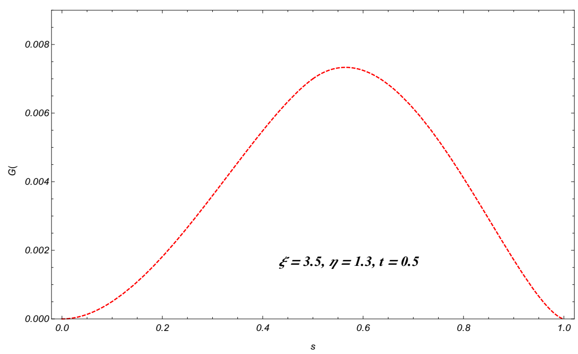

Figure 1. Green's function for $n = 2$.

In Figure 2, we take $t = 0.5, \xi = 4, \eta = 0$ for Green's function of ordinary differential equation, and $t = 0.5, \xi = 3.9, \eta = 0.5$ with radius of convergence $R > 1, a_n = (0.7)^{n+1}/\Gamma(3.9 + 0.5n)$ and n varies from 0 to 5, for RL fractional differential operator, fractional differential operator with analytic kernel and fractional differential operator with modified analytic kernel. All the Green's functions with different definition of differential operator are compatible.

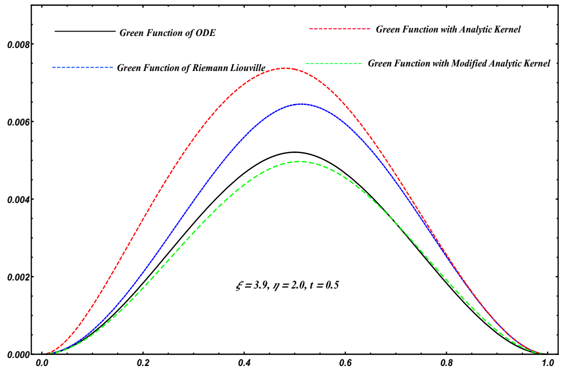

Figure 2. Green's function of ODE, fractional differential problem with RL operator, analytic kernel and modified analytic kernel.

3. Conclusions

The Green's function for a type of boundary value fractional problems with modified kernel and homogeneous conditions has been built along with interesting properties. Using Krasnoselskii fixed point theorem, existence result for fractional differential boundary value problem with modified analytic kernel is also established. A specific example is also provided to explain existence result and to show all the Green's function with different definitions of kernels are compatible.

Acknowledgement

The authors extended their appreciation to the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) for funding and supporting this work through Research Partnership Program No. RP-21-09-04.

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. K. S. Miller, B. Ross, *An introduction to fractional calculus and fractional diffrential equations*, New York: Wiley, 1993.
2. K. B. Oldham, J. Spanier, *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order*, Elsevier, 1974.

3. D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **204** (1996), 609–625. https://doi.org/10.1006/jmaa.1996.0456

4. S. Zhang, The existence of a positive solution for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **252** (2000), 804–812. https://doi.org/10.1006/jmaa.2000.7123

5. A. Babakhani, V. Daftardar-Gejji, Existence of positive solutions of nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **278** (2003), 434–442. https://doi.org/10.1016/S0022-247X(02)00716-3

6. Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311** (2005), 495–505. https://doi.org/10.1016/j.jmaa.2005.02.052

7. C. F. Li, X. N. Luo, Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, *Comput. Math. Appl.*, **59** (2010), 1363–1375. https://doi.org/10.1016/j.camwa.2009.06.029

8. Y. Wang, L. Liu, Y. Wu, Positive solutions for a nonlocal fractional differential equation, *Nonlinear Anal. Theor.*, **74** (2011), 3599–3605. https://doi.org/10.1016/j.na.2011.02.043

9. B. Ahmad, S. K. Ntouyas, A. Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Differ. Equ.*, **2011** (2011), 107384. https://doi.org/10.1155/2011/107384

10. J. Wang, Y. Zhou, M. Feckan, On recent developments in the theory of boundary value problems for impulsive fractional differential equations, *Comput. Math. Appl.*, **64** (2012), 3008–3020. https://doi.org/10.1016/j.camwa.2011.12.064

11. M. Feckan, Y. Zhou, J. Wang, On the concept and existence of solution for impulsive fractional differential equations, *Commun. Nonlinear Sci.*, **17** (2012), 3050–3060. https://doi.org/10.1016/j.cnsns.2011.11.017

12. J. Jiang, L. Liu, Y. Wu, Positive solutions to singular fractional differential system with coupled boundary conditions, *Commun. Nonlinear Sci.*, **18** (2013), 3061–3074. https://doi.org/10.1016/j.cnsns.2013.04.009

13. J. Henderson, R. Luca, Positive solutions for a system of nonlocal fractional boundary value problems, *Fract. Calc. Appl. Anal.*, **16** (2013), 985–1008. https://doi.org/10.2478/s13540-013-0061-4

14. K. Shah, R. A. Khan, Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions, *Differ. Equ. Appl.*, **7** (2015), 245–262. https://doi.org/10.7153/dea-07-14

15. Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, *Appl. Math. Lett.*, **51** (2016), 48–54. https://doi.org/10.1016/j.aml.2015.07.002

16. Y. Zou, G. He, On the uniqueness of solutions for a class of fractional differential equations, *Appl. Math. Lett.*, **74** (2017), 68–73. https://doi.org/10.1016/j.aml.2017.05.011

17. B. Ahmad, R. Luca, Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions, *Appl. Math. Comput.*, **339** (2018), 516–534. https://doi.org/10.1016/j.amc.2018.07.025

18. M. Benchohra, S. Bouriah, J. J. Nieto, Existence and Ulam stability for nonlinear implicit differential equations with Riemann-Liouville fractional derivative, *Demonstr. Math.*, **52** (2019), 437–450. https://doi.org/10.1515/dema-2019-0032

19. Z. Yue, Y. Zou, New uniqueness results for fractional differential equation with dependence on the first order derivative, *Adv. Differ. Equ.*, **2019** (2019), 38. https://doi.org/10.1186/s13662-018-1923-1

20. A. Amara, Existence results for hybrid fractional differential equations with three-point boundary conditions, *AIMS Math.*, **5** (2020), 1074–1088. https://doi.org/10.3934/math.2020075

21. H. Afshari, M. S. Abdo, J. Alzabut, Further results on existence of positive solutions of generalized fractional boundary value problems, *Adv. Differ. Equ.*, **2020** (2020), 600. https://doi.org/10.1186/s13662-020-03065-2

22. H. R. Marasi, H. Aydi, Existence and uniqueness results for two-term nonlinear fractional differential equations via a fixed point technique, *J. Math.*, **2021** (2021), 6670176. https://doi.org/10.1155/2021/6670176

23. A. Tudorache, R. Luca, Positive solutions for a system of fractional boundary value problems with r-Laplacian operators, uncoupled nonlocal conditions and positive parameters, *Axioms*, **11** (2022), 164. https://doi.org/10.3390/axioms11040164

24. X. Zhang, Y. Tian, Sharp conditions for the existence of positive solutions for a second-order singular impulsive differential equation, *Appl. Anal.*, **101** (2022), 1–13. https://doi.org/10.1080/00036811.2017.1370542

25. C. E. Wagner, A. C. Barbati, J. Engmann, A. S. Burbidge, G. H. McKinley, Quantifying the consistency and rheology of liquid foods using fractional calculus, *Food Hydrocolloid.*, **69** (2017), 242–254. https://doi.org/10.1016/j.foodhyd.2017.01.036

26. L. L. Ferras, N. J. Ford, M. L. Morgado, M. Rebelo, G. H. McKinley, J. M. Nobrega, Theoretical and numerical analysis of unsteady fractional viscoelastic flows in simple geometries, *Comput. Fluids*, **174** (2018), 14–33. https://doi.org/10.1016/j.compfluid.2018.07.004

27. A. Stankiewicz, Fractional Maxwell model of viscoelastic biological materials, *BIO Web of Conferences*, **10** (2018), 02032. https://doi.org/10.1051/bioconf/20181002032

28. Y. A. Rossikhin, M. V. Shitikova, Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results, *Appl. Mech. Rev.*, **63** (2010), 010801. https://doi.org/10.1115/1.4000563

29. H. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci.*, **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019

30. A. Carpinteri, F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Springer, 2014.

31. K. Lazopoulos, Non-local continuum mechanics and fractional calculus, *Mech. Res. Commun.*, **33** (2006), 753–757. https://doi.org/10.1016/j.mechrescom.2006.05.001

32. C. S. Drapaca, S. Sivaloganathan, A fractional model of continuum mechanics, *J. Elast.*, **107** (2012), 105–123. https://doi.org/10.1007/s10659-011-9346-1

33. F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics*, Vienna: Springer, 1997.

34. D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative, *Chaos Soliton. Fract.*, **134** (2020), 109705. https://doi.org/10.1016/j.chaos.2020.109705

35. V. E. Tarasov, Mathematical economics: Application of fractional calculus, *Mathematics*, **8** (2020), 660. https://doi.org/10.3390/math8050660

36. D. Kumar, D. Baleanu, Fractional calculus and its applications in physics, *Front. Phys.*, **7** (2019), 81. https://doi.org/10.3389/fphy.2019.00081

37. Y. Wei, Y. Kang, W. Yin, Y. Wang, Generalization of the gradient method with fractional order gradient direction, *J. Frankl. I.*, **357** (2020), 2514–2532. https://doi.org/10.1016/j.jfranklin.2020.01.008

38. K. A. Abro, M. H. Laghari, J. F. Gómez-Aguilar, Application of Atangana-Baleanu fractional derivative to carbon nanotubes based non-Newtonian nanofluid: applications in nanotechnology, *J. Appl. Comput. Mech.*, **6** (2020), 1260–1269. https://doi.org/10.22055/JACM.2020.33461.2229

39. S. Chávez-Vázquez, J. F. Gómez-Aguilar, J. E. Lavín-Delgado, R. F. Escobar-Jiménez, V. H. Olivares-Peregrino, Applications of fractional operators in robotics: a review, *J. Intell. Robot. Syst.*, **104** (2022), 63. https://doi.org/10.1007/s10846-022-01597-1

40. X. Xu, D. Jiang, C. Yuan, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation, *Nonlinear Anal.*, **71** (2009), 4676–4688. https://doi.org/10.1016/j.na.2009.03.030

41. I. Podlubny, *Fractional differential equations*, 1998.

42. L. Debnath, D. Bhatta, *Integral transforms and their applications*, Chapman & Hall, 2007.

43. A. Fernandez, M. A. Ozarslan, D. Baleanu, On fractional calculus with general analytical kernels, *Appl. Math. Comput.*, **354** (2019), 248–265. https://doi.org/10.1016/j.amc.2019.02.045

44. T. R. Prabhakar, A singular integral equation with a generalized Mittag Leffler function in the kernel, *Yokohama Math. J.*, **19** (1971), 7–15.

45. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A

46. F. Jarad, T. Abdeljawad, J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Spec. Top.*, **226** (2017), 3457–3471. https://doi.org/10.1140/epjst/e2018-00021-7