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Research article

On fundamental algebraic characterizations of complex intuitionistic Q-fuzzy subfield

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Abstract: The main objective of this study is to propose a new notion of a complex intuitionistic Q-fuzzy subfield of a field F that is developed from the concept of a complex fuzzy subfield of a field F by adding the notion of intuitionistic Q-fuzzy into a complex fuzzy subfield. We establish a new structure of complex fuzzy subfields which is called complex intuitionistic Q-fuzzy subfield. The most significant advantage of this addition appears to be that it broadens the scope of the investigation from membership function values to membership and non-membership function values. The range of complex fuzzy subfields is expanded to the unit disc in the complex plane for both membership and non-membership functions. Some fundamental operations, especially the intersection, union, and complement of complex intuitionistic Q-fuzzy subfields are studied. We define the necessity and possibility operators on a complex intuitionistic Q-fuzzy subfield. Moreover, we show that each complex intuitionistic Q-fuzzy subfield generates two intuitionistic Q-fuzzy subfields. Subsequently, several related theorems are proven.

Keywords: fuzzy subfield; Q-fuzzy set; Q-fuzzy subfield; complex Q-fuzzy set; complex intuitionistic Q-fuzzy set; complex intuitionistic Q-fuzzy subfield
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1. Introduction

A fuzzy set theory plays a significant role in the field of mathematics. Zadeh [1] is the first to propose the fuzzy set, which is a generalization of the ordinary set. The fuzzy set is a function where its domain is the universal set X, and its co-domain is the closed interval between zero and one. The facts from the obscure, imprecise, and occasionally biased computational presupposition were overcome using fuzzy logic. The use of ambiguous human judgment in problem calculation is made possible by fuzzy logic. This idea also encourages us to look for a better approach to making decisions to handle daily problems. Similarly, Goguen [2] continued the work of Zadeh [1] and explored the concept of an L-fuzzy set. Afterwards, De Luca and Termini [3] established some algebraic properties of fuzzy sets. Recent studies have discussed the applications of the fuzzy set and complex fuzzy set to solve one of the famous real-life problems, which is the decision-making problem (see [4,5]). In these studies, a new algorithm was introduced and applied to the interval-valued complex neutrosophic soft-set model to solve a hypothetical decision-making problem.

After that, Rosenfeld [6] applied the idea of a fuzzy set in group theory and determined the concept of a fuzzy subgroup. Consequently, many mathematicians generalized fuzzy subgroups [7–9]. Solairaju and Nagarajan [10] established the Q-fuzzy group, where Q is a non-empty set, and presented a definition of a Q-fuzzy set as a mapping $\mu: G \times Q \longrightarrow [0,1]$ in G such that Q is any set and G is a group. Moreover, Selvam et al. [11] introduced certain properties of anti-Q-fuzzy normal subgroups. Rasuli [12] introduced the notion of anti-Q-fuzzy subgroups with respect to t-conorm and studied their important properties.

Later on, Rasuli [13] presented the notions of Q-fuzzy subrings and anti-Q-fuzzy subrings and proved some related properties by using two types of norms, which are t-conorm and t-norm. Emniyet and Sahin [14] introduced the concept of fuzzy normed rings and demonstrated a few algebraic characteristics of normed ring theory on a fuzzy set. Furthermore, Al Tahan [15] proposed certain findings on fuzzy multi-Hv-ideals of Hv-rings and established the notion of generalized fuzzy multi-Hv-ideals as a generalization of fuzzy Hv-ideals. Al-Masarwah and Ahmad [16] introduced structures on doubt neutrosophic ideals of BCK/BCI-Algebras under (S,T) -norms. Addis et al. [17] defined the notion of a fuzzy kernel of a fuzzy homomorphism on rings and showed that it is a fuzzy ideal of the domain ring. Kausar et al. [18] gave the characterizations of fuzzy bi-ideals in LA-rings and characterized left weakly regular LA-rings in terms of fuzzy right ideals. Moreover, Al-Masarwah et al. [19] presented the idea of m-polar fuzzy positive implicative ideals of BCK algebras. Alolaiyan et al. [20] published a research paper about a certain structure of bipolar fuzzy subrings and defined bipolar fuzzy homomorphism by using the notion of a natural ring homomorphism. Malik and Mordeson [21] identified several properties of fuzzy subfields. On top of that, basic structure theorems were given for fuzzy subfields of finite fields by Mordeson [22]. Feng and Yao [23] introduced the notions of (λ,μ) anti-fuzzy subfields and studied their properties. After that, Hussain [24] proposed the Q-fuzzy field and described the Q-fuzzy set as mapping $\mu: X \times Q \longrightarrow [0,1]$ in a field X. Later, Muthuraman et al. [25] defined the notion of fuzzy HX field of an HX field and some of their related properties were investigated and introduced the concept of an image and pre-image of a fuzzy set.

Moreover, the idea of defining an intuitionistic fuzzy set (IFS) was first published by Atanassov [26,27]. Then Szmidt and Kacprzyk [28] proposed new definitions of distances between IFSs. These new definitions were introduced and compared with the approach used for fuzzy sets. Additionally, G. Beliakov et al. [29] determined averaging operators for Atanassov's IFSs. After that, Broumi [30] presented the concept of Q-intuitionistic fuzzy soft sets, which combines Q-IFSs and soft sets. This concept is a generalization of Q-fuzzy soft sets. Furthermore, Atanassov [31] established new results on IFSs. Later, Hur et al. [32] examined the properties of intuitionistic fuzzy subrings and intuitionistic fuzzy subgroups. For more development about the intuitionistic fuzzy subgroup, one can refer to [33,34]. In addition, Yamin [35] studied the theory of intuitionistic fuzzy rings and provided some new concepts, such as an intuitionistic fuzzy ring with operators. Correspondingly, Abed Alhaleem and Ahmad [36,37] presented intuitionistic normal fuzzy subrings over normed rings. Intuitionistic fuzzy normed prime and maximal ideals were also presented by Abed Alhaleem and Ahmad [38].

Ramot et al. in [39] characterized a complex fuzzy set. This set is described by a unique membership function, which consists of two terms called amplitude term and phase term. In their work, the range of the fuzzy set was expanded to the unit disc in the plane of complex instead of the interval $[0,1]$ to present their innovative notion of complex fuzzy sets. Moreover, there are some applications of complex fuzzy sets in decision-making. Al-Sharqi et al. [40] analyzed a real-life economic problem utilizing the major aspects of interval complex neutrosophic soft relations. After that, Yang et al. [41] proposed bipolar complex fuzzy subgroups. Alolaiyan et al. [42] introduced a novel algebraic structure of (α,β) -complex fuzzy subgroups. Alsarahead and Ahmad [43] presented the complex fuzzy subring and established some new notions. Subsequently, Alsarahead and Ahmad [44] suggested the complex intuitionistic fuzzy subring and presented some new definitions. Gulzar et al. [45] introduced some characterization of Q-complex fuzzy subrings and discussed its various algebraic aspects. In 2021, Gulzar et al. [46] studied a complex fuzzy subfield and proved some properties.

The contributions of this paper lie in presenting the concept of a complex intuitionistic Q-fuzzy subfield as a new structure. Some definitions related to complex intuitionistic Q-fuzzy subfields are proposed. Furthermore, basic operations on complex intuitionistic Q-fuzzy subfields are introduced, and some related theorems are proven. The necessity and possibility operators on a complex intuitionistic Q-fuzzy subfield are defined, and some of its properties are given.

The novelty of this work can be viewed as:

- In this work, we have combined the following concepts complex fuzzy subfield and intuitionistic Q-fuzzy. This was done by adding the idea of intuitionistic Q-fuzzy into a complex fuzzy subfield. Thus, we got a new structure called a complex intuitionistic Q-fuzzy subfield.
- We have utilized this new concept to create and prove many theorems and results, which will be a new addition to knowledge in mathematics.

The paper is divided into the following sections. In Section 2, we provide several definitions and preliminary results. In Section 3, we characterize some algebraic properties of the complex intuitionistic Q-fuzzy subfield and describe the notions of the necessity and possibility operators on a complex intuitionistic Q-fuzzy subfield. Some related theorems are also discussed in this section. Finally, the conclusion is summarized in Section 5.

2. Preliminaries

We provide some basic definitions of Q-fuzzy set and complex intuitionistic Q-fuzzy set in this part that will be utilized throughout the study.

Definition 2.1. [1] *A function $\psi : X \longrightarrow [0,1]$ that receives values from X to the range $[0,1]$ is referred to as a fuzzy set in a universal set X. This function is referred to as a membership function and is indicated by the symbol by $\mu_\psi(d)$. A fuzzy set ψ is represented by the formula $\psi = \{\langle d, \mu_\psi(d)\rangle : d \in X\}$, where $0 \le \mu_\psi(d) \le 1$, for every $d \in X$.*

Definition 2.2. [21] *A fuzzy subset $\psi = \{\langle d, \mu_\psi(d)\rangle : d \in X\}$ of a field $(F, +, \cdot)$ is said to be a fuzzy subfield for all $d, k \in X$ if:*

(1) $\mu_\psi(d-k) \ge min\{\mu_\psi(d), \mu_\psi(k)\}$,

(2) $\mu_\psi(dk) \geq min\{\mu_\psi(d), \mu_\psi(k)\}$,
(3) $\mu_\psi(d^{-1}) \geq \mu_\psi(d)$.

Definition 2.3. [46] *Let* $(F, +, \cdot)$ *be a field. A complex fuzzy set* $\psi = \{\langle d, \mu_\psi(d)\rangle : d \in F\}$ *of a field* F *is said to be a complex fuzzy subfield of a field* F *if it satisfies the following for all* $d, k \in F$:
(1) $\mu_\psi(d - k) \geq min\{\mu_\psi(d), \mu_\psi(k)\}$,
(2) $\mu_\psi(dk) \geq min\{\mu_\psi(d), \mu_\psi(k)\}$,
(3) $\mu_\psi(d^{-1}) \geq \mu_\psi(d)$.

Definition 2.4. [10] *Let* X *and* Q *represent any two sets. In a set* X, *a* Q*-fuzzy set is a mapping with the notation* $\psi: X \times Q \longrightarrow [0,1]$.

Definition 2.5. [45] *A* Q*-complex fuzzy set (*Q*-CFS) of universe of discourse* X *is represented by the membership function* $\mu_\psi(d,q) = r_\psi(d,q)e^{i\omega_\psi(d,q)}$ *and described as* $\mu_\psi: X \times Q \longrightarrow \{z \in \mathbb{C}: |z| \leq 1\}$. *This membership function receives all membership values from the unit disc on a plane, where* $i = \sqrt{-1}$, *both* $r_\psi(d,q)$ *and* $\omega_\psi(d,q)$ *are real valued such that* $r_\psi(d,q) \in [0,1]$ *and* $\omega_\psi(d,q) \in [0, 2\pi]$.

Definition 2.6. [45] *Let* ψ_1 *and* ψ_2 *be* Q*-CFS of* X. *Then*
(1) *A* Q*-complex fuzzy set* ψ_1 *is a homogeneous* Q*-complex fuzzy set, if for all* $d, k \in X$, $q \in Q$, *we have* $r_{\psi_1}(d,q) \leq r_{\psi_1}(k,q)$ *if and only if* $\omega_{\psi_1}(d,q) \leq \omega_{\psi_1}(k,q)$.
(2) *A* Q*-complex fuzzy* ψ_1 *is homogeneous* Q*-complex fuzzy set with* ψ_2, *if for all* $d, k \in X$, $q \in Q$, *we have* $r_{\psi_1}(d,q) \leq r_{\psi_2}(k,q)$ *if and only if* $\omega_{\psi_1}(d,q) \leq \omega_{\psi_2}(k,q)$.

Definition 2.7. [47] *Let* X *and* Q *represent any two sets. The definition of an intuitionistic* Q*-fuzzy set* ψ *in a set* X *is* $\psi = \{\langle (d,q), \mu_\psi(d,q), \gamma_\psi(d,q)\rangle : d \in X, q \in Q\}$, *where* $\mu_\psi(d): X \times Q \longrightarrow [0,1]$ *describes the degree of membership and* $\gamma_\psi(d): X \times Q \longrightarrow [0,1]$ *describes the degree of non-membership of the element* $(d,q) \in X \times Q$, *such that* $0 \leq \mu_\psi(d,q) + \gamma_\psi(d,q) \leq 1$, *for every* $(d,q) \in X \times Q$.

Definition 2.8. [47] *In a set* X, *let* ψ_1 *and* ψ_2 *represent two intuitionistic* Q*-fuzzy sets where* $\psi_1 = \{\langle (d,q), \mu_{\psi_1}(d,q), \gamma_{\psi_1}(d,q)\rangle : d \in X, q \in Q\}$ *and* $\psi_2 = \{\langle (d,q), \mu_{\psi_2}(d,q), \gamma_{\psi_2}(d,q)\rangle : d \in X, q \in Q\}$. *For all* $d \in X, q \in Q$, *the next relations and operations are described as:*
(1) $\psi_1 = \psi_2 \Leftrightarrow \mu_{\psi_1}(d,q) = \mu_{\psi_2}(d,q)$ and $\gamma_{\psi_1}(d,q) = \gamma_{\psi_2}(d,q)$,
(2) ${\psi_1}^c = \{\langle (d,q), \gamma_{\psi_1}(d,q), \mu_{\psi_1}(d,q)\rangle : d \in X, q \in Q\}$,
(3) $\psi_1 \cap \psi_2 = \{\langle (d,q), \mu_{\psi_1 \cap \psi_2}(d,q), \gamma_{\psi_1 \cap \psi_2}(d,q)\rangle : d \in X, q \in Q\}$,
where
$\mu_{\psi_1 \cap \psi_2}(d,q) = min\{\mu_{\psi_1}(d,q), \mu_{\psi_2}(d,q)\}$,
$\gamma_{\psi_1 \cap \psi_2}(d,q) = max\{\gamma_{\psi_1}(d,q), \gamma_{\psi_2}(d,q)\}$,
(4) $\psi_1 \cup \psi_2 = \{\langle (d,q), \mu_{\psi_1 \cup \psi_2}(d,q), \gamma_{\psi_1 \cup \psi_2}(d,q)\rangle : d \in X, q \in Q\}$,
where
$\mu_{\psi_1 \cup \psi_2}(d,q) = max\{\mu_A(d,q), \mu_B(d,q)\}$,
$\gamma_{\psi_1 \cup \psi_2}(d,q) = min\{\gamma_{\psi_1}(d,q), \gamma_{\psi_2}(d,q)\}$,
(5) $\Box\, \psi = \{\langle (d,q), \mu_\psi(d,q), 1 - \mu_\psi(d,q)\rangle : d \in X, q \in Q\}$,
(6) $\Diamond\, \psi = \{\langle (d,q), 1 - \gamma_\psi(d,q), \gamma_\psi(d,q)\rangle : d \in X, q \in Q\}$.

Proposition 2.1. [1] *Let* $\psi_1 = \{\langle d, \mu_{\psi_1}(d)\rangle : d \in X\}$, $\psi_2 = \{\langle d, \mu_{\psi_2}(d)\rangle : d \in X\}$ *two fuzzy sets. Then for all* $d \in X$:
(1) $1 - max\{\mu_{\psi_1}(d), \mu_{\psi_2}(d)\} = min\{1 - \mu_{\psi_1}(d), 1 - \mu_{\psi_2}(d)\}$,
(2) $1 - min\{\mu_{\psi_1}(d), \mu_{\psi_2}(d)\} = max\{1 - \mu_{\psi_1}(d), 1 - \mu_{\psi_2}(d)\}$.

Definition 2.9. [21] *A fuzzy subset* $\psi = \{\langle d, \mu_\psi(d)\rangle : d \in X\}$ *of a field* $(F, +, \cdot)$ *is said to be a fuzzy subfield for all* $d, k \in X$ *if:*
(3) $\mu_\psi(d-k) \geq min\{\mu_\psi(d), \mu_\psi(k)\}$,
(4) $\mu_\psi(dk) \geq min\{\mu_\psi(d), \mu_\psi(k)\}$,
(5) $\mu_\psi(d^{-1}) \geq \mu_\psi(d)$.
Definition 2.10. [48] *Let* X *be a universe of discourse. A complex intuitionistic fuzzy set is defined on* X *and characterized by membership* $\mu_\psi(d) = r_\psi(d)e^{i\omega_\psi(d)}$ *and non-membership functions* $\gamma_\psi(d) = \hat{r}_\psi(d)e^{i\widehat{\omega}_\psi(d)}$, *that determines for any element* $d \in X$ *a complex-valued grade of both membership and non-membership in* ψ. *According to the definition,* $\psi = \{\langle d, \mu_\psi(d), \gamma_\psi(d)\rangle : d \in X\}$ *where* $r_\psi(d) + \hat{r}_\psi(d) \leq 1$.
Definition 2.11. [48] *Let* ψ_1 *and* ψ_2 *be two complex intuitionistic fuzzy subsets of* X, *with membership functions* $\mu_{\psi_1}(d) = r_{\psi_1}(d)e^{i\omega_{\psi_1}(d)}$ *and* $\mu_{\psi_2}(d) = r_{\psi_2}(d)e^{i\omega_{\psi_2}(d)}$, *respectively, while the non-membership functions are* $\gamma_{\psi_1}(d) = \hat{r}_{\psi_1}(d)e^{i\widehat{\omega}_{\psi_1}(d)}$ *and* $\gamma_{\psi_2}(d) = \hat{r}_{\psi_2}(d)e^{i\widehat{\omega}_{\psi_2}(d)}$, *respectively. Then* $\psi_1 \cap \psi_2$ *is given by:*
$\psi_1 \cap \psi_2 = \{\langle d, \mu_{\psi_1\cap\psi_2}(d), \gamma_{\psi_1\cap\psi_2}(d)\rangle : d \in X\}$,
where
$\mu_{\psi_1\cap\psi_2}(d) = min\{ r_{\psi_1}(d), r_{\psi_2}(d)\}e^{imin\{\omega_{\psi_1}(d), \omega_{\psi_2}(d)\}}$,
$\gamma_{\psi_1\cap\psi_2}(d) = max\{ \hat{r}_{\psi_1}(d), \hat{r}_{\psi_2}(d)\}e^{imax\{\widehat{\omega}_{\psi_1}(d), \widehat{\omega}_{\psi_2}(d)\}}$.
Definition 2.12. [48] *Let* $\psi = \{\langle d, \mu_\psi(d), \gamma_\psi(d)\rangle : d \in X\}$ *be a complex intuitionistic fuzzy set. The complement of* ψ *is denoted by* ψ^c *and defined as:*
$\psi^c = \{\langle d, \gamma_\psi(d), \mu_\psi(d)\rangle : d \in X\} = \{\langle d, \hat{r}_\psi(d)e^{i\widehat{\omega}_{\psi^c}(d)}, r_\psi(d)e^{i\omega_{\psi^c}(d)}\rangle : d \in X\}$,
where
$\omega_{\psi^c}(d) = \omega_\psi(d), 2\pi - \omega_\psi(d)$, or $\omega_\psi(d) + \pi$.
Definition 2.13. [44] *Let* ψ_1 *and* ψ_2 *be two complex intuitionistic fuzzy subsets of* X *with membership functions* $\mu_{\psi_1}(d) = r_{\psi_1}(d)e^{i\omega_{\psi_1}(d)}$ *and* $\mu_{\psi_2}(d) = r_{\psi_2}(d)e^{i\omega_{\psi_2}(d)}$, *respectively. While the non-membership functions are* $\gamma_{\psi_1}(d) = \hat{r}_{\psi_1}(d)e^{i\widehat{\omega}_{\psi_1}(d)}$ *and* $\gamma_{\psi_2}(d) = \hat{r}_{\psi_2}(d)e^{i\widehat{\omega}_{\psi_2}(d)}$, *respectively. Then we have*
(1) *A complex intuitionistic fuzzy subset* ψ_1 *is said to be a homogeneous complex intuitionistic fuzzy set if for all* $d, k \in X$ *the following hold:* $r_{\psi_1}(d) \leq r_{\psi_1}(k)$ *if and only if* $\omega_{\psi_1}(d) \leq \omega_{\psi_1}(k)$, *and* $\hat{r}_{\psi_1}(d) \leq \hat{r}_{\psi_1}(k)$ *if and only if* $\widehat{\omega}_{\psi_1}(d) \leq \widehat{\omega}_{\psi_1}(k)$.
(2) *A complex intuitionistic fuzzy subset* ψ_1 *is said to be homogeneous with* ψ_2, *if for all* $d, k \in X$ *the following hold:* $r_{\psi_1}(d) \leq r_{\psi_2}(k)$ *if and only if* $\omega_{\psi_1}(d) \leq \omega_{\psi_2}(k)$, *and* $\hat{r}_{\psi_1}(d) \leq \hat{r}_{\psi_2}(k)$ *if and only if* $\widehat{\omega}_{\psi_1}(d) \leq \widehat{\omega}_{\psi_2}(k)$.
Definition 2.14. [44] *Let* $\psi = \{\langle d, \mu_\psi(d), \gamma_\psi(d)\rangle : d \in X\}$ *be an intuitionistic fuzzy set. Then the set* $\psi_\pi(d) = \{ d, \nu_{\psi_\pi}(d), \rho_{\psi_\pi}(d) : d \in X\}$ *is said to be an intuitionistic* π*-fuzzy set where* $\nu_{\psi_\pi}(d) = 2\pi\mu_\psi(d)$ *and* $\rho_{\psi_\pi}(d) = 2\pi\gamma_\psi(d)$. *Note that the condition* $\nu_{\psi_\pi}(d) + \rho_{\psi_\pi}(d) \leq 2\pi$ *is already satisfied.*

3. Characterizations of complex intuitionistic Q-fuzzy subfield

In this part, we start by providing an overview of the main definition of a complex intuitionistic Q-fuzzy subfield. We examine a few fundamental characterizations of complex intuitionistic Q-fuzzy subfields and establish some associated theorems.

Definition 3.1. *Let* $(F,+,\cdot)$ *be a field. If a homogeneous complex intuitionistic* Q*-fuzzy subset* $\psi=\{\langle(d,q),\mu_\psi(d,q),\gamma_\psi(d,q)\rangle: d\in F,\ q\in Q\}$ *of a field* F *holds the following conditions for any* $d,k\in F,\ q\in Q$, *it is said to be a complex intuitionistic* Q*-fuzzy subfield of that field:*

(1) $\mu_\psi(d-k,q)\geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(2) $\mu_\psi(dk,q)\geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(3) $\mu_\psi(d^{-1},q)\geq\mu_\psi(d,q)$,

(4) $\gamma_\psi(d-k,q)\leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,

(5) $\gamma_\psi(dk,q)\leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,

(6) $\gamma_\psi(d^{-1},q)\leq\gamma_\psi(d,q)$.

Definition 3.2. *Let* $(F,+,\cdot)$ *be a field. If a homogeneous complex intuitionistic* Q*-fuzzy subset* $\psi=\{\langle(d,q),\mu_\psi(d,q),\gamma_\psi(d,q)\rangle: d\in F,\ q\in Q\}$ *of a field* F *holds the following conditions for any* $d,k\in F,\ q\in Q$, *it is said to be a complex intuitionistic anti-*Q*-fuzzy subfield of that field:*

(1) $\mu_\psi(d-k,q)\leq max\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(2) $\mu_\psi(dk,q)\leq max\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(3) $\mu_\psi(d^{-1},q)\leq\mu_\psi(d,q)$,

(4) $\gamma_\psi(d-k,q)\geq min\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,

(5) $\gamma_\psi(dk,q)\geq min\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,

(6) $\gamma_\psi(d^{-1},q)\geq\gamma_\psi(d,q)$.

Definition 3.3. *An intuitionistic* π*-*Q*-fuzzy set of a field* F *is defined as* $\psi_\pi=\{\langle(d,q),\nu_{\psi_\pi}(d,q),\rho_{\psi_\pi}(d,q)\rangle: d\in F,\ q\in Q\}$. *If the following holds for any* $d,k\in F$, $q\in Q$, *then* ψ_π *is said to be an intuitionistic* π*-*Q*-fuzzy subfield of a field* F*:*

(1) $\nu_{\psi_\pi}(d-k,q)\geq min\{\nu_{\psi_\pi}(d,q),\nu_{\psi_\pi}(k,q)\}$,

(2) $\nu_{\psi_\pi}(dk,q)\geq min\{\nu_{\psi_\pi}(d,q),\nu_{\psi_\pi}(k,q)\}$,

(3) $\nu_{\psi_\pi}(d^{-1},q)\geq\nu_{\psi_\pi}(d,q)$,

(4) $\rho_{\psi_\pi}(d-k,q)\leq max\{\rho_{\psi_\pi}(d,q),\rho_{\psi_\pi}(k,q)\}$,

(5) $\rho_{\psi_\pi}(dk,q)\leq max\{\rho_{\psi_\pi}(d,q),\rho_{\psi_\pi}(k,q)\}$,

(6) $\rho_{\psi_\pi}(d^{-1},q)\leq\rho_{\psi_\pi}(d,q)$.

Theorem 3.1. *Let* $\psi_\pi=\{\langle(d,q),\nu_{\psi_\pi}(d,q),\rho_{\psi_\pi}(d,q)\rangle: d\in F,\ q\in Q\}$ *be an intuitionistic* π*-*Q*-fuzzy set a field* F. *Then* ψ_π *is said to be an intuitionistic* π*-*Q*-fuzzy subfield if and only if* ψ *is an intuitionistic* Q*-fuzzy subfield.*

Theorem 3.2. *Let* F *be a field and* $\psi=\{\langle(d,q),\mu_\psi(d,q),\gamma_\psi(d,q)\rangle: d\in F,\ q\in Q\}$ *be a homogeneous complex intuitionistic* Q*-fuzzy set with membership function* $\mu_\psi(d,q)=r_\psi(d,q)e^{i\omega_\psi(d,q)}$ *and non-membership function* $\gamma_\psi(d,q)=\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}$. *Then* ψ *is a complex intuitionistic* Q*-fuzzy subfield of a field* F *if and only if:*

(1) *The intuitionistic* Q*-fuzzy set* $\bar{\psi}=\{\langle(d,q),r_\psi(d,q),\hat{r}_\psi(d,q)\rangle: d\in F,\ q\in Q, r_\psi(d,q),\hat{r}_\psi(d,q)\in[0,1]\}$ *is an intuitionistic* Q*-fuzzy subfield.*

(2) *The intuitionistic* π*-*Q*-fuzzy set* $\underline{\psi}=\{\langle(d,q),\omega_\psi(d,q),\widehat{\omega}_\psi(d,q)\rangle: d\in F,\ q\in Q,\omega_\psi(d,q),\widehat{\omega}_\psi(d,q)\in[0,2\pi]\}$ *is an intuitionistic* π*-* Q*-fuzzy subfield.*

Proof. Let ψ be a complex intuitionistic Q-fuzzy subfield and $d,k\in F, q\in Q$. Then we have:

$$\begin{aligned} r_\psi(d-k,q)e^{i\omega_\psi(d-k,q)} &= \mu_\psi(d-k,q)\\ &\geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\} \end{aligned}$$

$$= min\{r_\psi(d,q)e^{i\omega_\psi(d,q)}, r_\psi(k,q)e^{i\omega_\psi(k,q)}\}$$
$$= min\{r_\psi(d,q), r_\psi(k,q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(k,q)\}}.$$

(Since ψ is homogeneous)
$\Rightarrow r_\psi(d-k,q) \geq min\{r_\psi(d,q), r_\psi(k,q)\}$ and $\omega_\psi(d-k,q) \geq min\{\omega_\psi(d,q), \omega_\psi(k,q)\}$.
Also, we have

$$r_\psi(dk,q)e^{i\omega_\psi(dk,q)} = \mu_\psi(dk,q)$$
$$\geq min\{\mu_\psi(d,q), \mu_\psi(k,q)\}$$
$$= min\{r_\psi(d,q)e^{i\omega_\psi(d,q)}, r_\psi(k,q)e^{i\omega_\psi(k,q)}\}$$
$$= min\{r_\psi(d,q), r_\psi(k,q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(k,q)\}}.$$

(Since ψ is homogeneous)
$\Rightarrow r_\psi(dk,q) \geq min\{r_\psi(d,q), r_\psi(k,q)\}$ and $\omega_\psi(dk,q) \geq min\{\omega_\psi(d,q), \omega_\psi(k,q)\}$.
Moreover,

$$r_\psi(d^{-1},q)e^{i\omega_\psi(d^{-1},q)} = \mu_\psi(d^{-1},q) \geq \mu_\psi(d,q) = r_\psi(d,q)e^{i\omega_\psi(d,q)}.$$

So,

$$r_\psi(d^{-1},q) \geq r_\psi(d,q) \text{ and } \omega_\psi(d^{-1},q) \geq \omega_\psi(d,q).$$

On the other hand,

$$\hat{r}_\psi(d-k,q)e^{i\widehat{\omega}_\psi(d-k,q)} = \gamma_\psi(d-k,q)$$
$$\leq max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}$$
$$= max\{\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}, \hat{r}_\psi(k,q)e^{i\widehat{\omega}_\psi(k,q)}\}$$
$$= max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}e^{imax\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}}.$$

(Since ψ is homogeneous)
$\Rightarrow \hat{r}_\psi(d-k,q) \leq max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}$ and $\widehat{\omega}_\psi(d-k,q) \leq max\{\widehat{\omega}_\psi(d,q), \widehat{\omega}_\psi(k,q)\}$.
Also, we have

$$\hat{r}_\psi(dk,q)e^{i\widehat{\omega}_\psi(dk,q)} = \gamma_\psi(dk,q)$$
$$\leq max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}$$
$$= max\{\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}, \hat{r}_\psi(k,q)e^{i\widehat{\omega}_\psi(k,q)}\}$$
$$= max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}e^{imax\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}}.$$

(Since ψ is homogeneous)
$\Rightarrow \hat{r}_\psi(dk,q) \leq max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}$ and $\widehat{\omega}_\psi(dk,q) \leq max\{\widehat{\omega}_\psi(d,q), \widehat{\omega}_\psi(k,q)\}$.
Moreover,

$$\hat{r}_\psi(d^{-1},q)e^{i\widehat{\omega}_\psi(d^{-1},q)} = \gamma_\psi(d^{-1},q) \leq \gamma_\psi(d,q) = \hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}.$$

So,

$$\hat{r}_\psi(d^{-1}, q) \leq \hat{r}_\psi(d, q) \text{ and } \widehat{\omega}_\psi(d^{-1}, q) \leq \widehat{\omega}_\psi(d, q).$$

Therefore, $\bar{\psi}$ is an intuitionistic Q-fuzzy subfield and $\underline{\psi}$ is an intuitionistic π-Q-fuzzy subfield. Conversely, assume that $\bar{\psi}$ is an intuitionistic Q-fuzzy subfield and $\underline{\psi}$ is an intuitionistic π-Q-fuzzy subfield. Thus,

$$r_\psi(d-k, q) \geq min\{r_\psi(d, q), r_\psi(k, q)\}, \omega_\psi(d-k, q) \geq min\{\omega_\psi(d, q), \omega_\psi(k, q)\},$$

$$r_\psi(dk, q) \geq min\{r_\psi(d, q), r_\psi(k, q)\}, \omega_\psi(dk, q) \geq min\{\omega_\psi(d, q), \omega_\psi(k, q)\},$$

$$r_\psi(d^{-1}, q) \geq r_\psi(d, q),\ \omega_\psi(d^{-1}, q) \geq \omega_\psi(d, q),$$

$$\hat{r}_\psi(d-k, q) \leq max\{\hat{r}_\psi(d, q), \hat{r}_\psi(k, q)\}, \widehat{\omega}_\psi(d-k, q) \leq max\{\widehat{\omega}_\psi(d, q), \widehat{\omega}_\psi(k, q)\},$$

$$\hat{r}_\psi(dk, q) \leq max\{\hat{r}_\psi(d, q), \hat{r}_\psi(k, q)\}, \widehat{\omega}_\psi(dk, q) \leq max\{\widehat{\omega}_\psi(d, q), \widehat{\omega}_\psi(k, q)\},$$

$$\hat{r}_\psi(d^{-1}, q) \leq \hat{r}_\psi(d, q),\ \widehat{\omega}_\psi(d^{-1}, q) \leq \widehat{\omega}_\psi(d, q).$$

Then,

$$\begin{aligned}\mu_\psi(d-k, q) &= r_\psi(d-k, q)e^{i\omega_\psi(d-k,q)} \\ &\geq min\{r_\psi(d, q), r_\psi(k, q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(k,q)\}} \\ &= min\{r_\psi(d, q)e^{i\omega_\psi(d,q)}, r_\psi(k, q)e^{i\omega_\psi(k,q)}\} \\ &= min\{\mu_\psi(d, q), \mu_\psi(k, q)\}.\end{aligned}$$

Also, we have:

$$\begin{aligned}\mu_\psi(dk, q) &= r_\psi(dk, q)e^{i\omega_\psi(dk,q)} \\ &\geq min\{r_\psi(d, q), r_\psi(k, q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(k,q)\}} \\ &= min\{r_\psi(d, q)e^{i\omega_\psi(d,q)}, r_\psi(k, q)e^{i\omega_\psi(k,q)}\} \\ &= min\{\mu_\psi(d, q), \mu_\psi(k, q)\}.\end{aligned}$$

Further,

$$\mu_\psi(d^{-1}, q) = r_\psi(d^{-1}, q)e^{i\omega_\psi(d^{-1},q)} \geq r_\psi(d, q)e^{i\omega_\psi(d,q)} = \mu_\psi(d, q).$$

On the other hand,

$$\begin{aligned}\gamma_\psi(d-k, q) &= \hat{r}_\psi(d-k, q)e^{i\widehat{\omega}_\psi(d-k,q)} \\ &\leq max\{\hat{r}_\psi(d, q), \hat{r}_\psi(k, q)\}e^{imax\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}} \\ &= max\{\hat{r}_\psi(d, q)e^{i\widehat{\omega}_\psi(d,q)}, \hat{r}_\psi(k, q)e^{i\widehat{\omega}_\psi(k,q)}\} \\ &= max\{\gamma_\psi(d, q), \gamma_\psi(k, q)\}.\end{aligned}$$

Also, we have

$$\gamma_\psi(dk,q) = \hat{r}_\psi(dk,q)e^{i\widehat{\omega}_\psi(dk,q)}$$
$$\leq max\{\hat{r}_\psi(d,q),\hat{r}_\psi(k,q)\}e^{imax\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}}$$
$$= max\{\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)},\hat{r}_\psi(k,q)e^{i\widehat{\omega}_\psi(k,q)}\}$$
$$= max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}.$$

Moreover,

$$\gamma_\psi(d^{-1},q) = \hat{r}_\psi(d^{-1},q)e^{i\widehat{\omega}_\psi(d^{-1},q)} \leq \hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)} = \gamma_\psi(d,q).$$

So, ψ is a complex intuitionistic Q-fuzzy subfield.

Theorem 3.3. *Let* $\psi = \{\langle(d,q),\mu_\psi(d,q),\gamma_\psi(d,q)\rangle : d \in F,\ q \in Q\}$ *be a complex intuitionistic* Q-*fuzzy subfield of a field* F, *for* $d \in F,\ q \in Q$. *Then*

(1) $r_\psi(0,q) \geq r_\psi(d,q),\ \omega_\psi(0,q) \geq \omega_\psi(d,q)$,

(2) $r_\psi(1,q) \geq r_\psi(d,q),\ \omega_\psi(1,q) \geq \omega_\psi(d,q)$,

(3) $\hat{r}_\psi(0,q) \leq \hat{r}_\psi(d,q),\ \widehat{\omega}_\psi(0,q) \leq \widehat{\omega}_\psi(d,q)$,

(4) $\hat{r}_\psi(1,q) \leq \hat{r}_\psi(d,q),\ \widehat{\omega}_\psi(1,q) \leq \widehat{\omega}_\psi(d,q)$,

where the identity elements of F *are* 0 *and* 1.

Proof. (1) In the case when the identity element of F is 0 and d in F:

$$r_\psi(0,q)e^{i\omega_\psi(0,q)} = r_\psi(d-d,q)e^{i\omega_\psi(d-d,q)}$$
$$= \mu_\psi(d-d,q)$$
$$\geq min\{\mu_\psi(d,q),\mu_\psi(d,q)\}$$
$$= min\{r_\psi(d,q)e^{i\omega_\psi(d,q)},r_\psi(d,q)e^{i\omega_\psi(d,q)}\}$$
$$= min\{r_\psi(d,q),r_\psi(d,q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(d,q)\}}$$
$$= r_\psi(d,q)e^{i\omega_\psi(d,q)}.$$

(As ψ is homogeneous)
$\Rightarrow r_\psi(0,q) \geq r_\psi(d,q)$, and $\omega_\psi(0,q) \geq \omega_\psi(d,q)$.

(2) For $d \neq 0$ in F and 1 is the identity element of F:

$$r_\psi(1,q)e^{i\omega_\psi(1,q)} = r_\psi(dd^{-1},q)e^{i\omega_\psi(dd^{-1},q)}$$
$$= \mu_\psi(dd^{-1},q)$$
$$\geq min\{\mu_\psi(d,q),\mu_\psi(d,q)\}$$
$$= min\{r_\psi(d,q)e^{i\omega_\psi(d,q)},r_\psi(d,q)e^{i\omega_\psi(d,q)}\}$$
$$= min\{r_\psi(d,q),r_\psi(d,q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(d,q)\}}$$
$$= r_\psi(d,q)e^{i\omega_\psi(d,q)}.$$

(As ψ is homogeneous)
$\Rightarrow r_\psi(1,q) \geq r_\psi(d,q)$, and $\omega_\psi(1,q) \geq \omega_\psi(d,q)$.

(3) For d in F and 0 is the identity element of F:

$$\begin{aligned}\hat{r}_{\psi}(0,q)e^{i\omega_{\hat{\psi}}(0,q)} &= \hat{r}_{\psi}(d-d,q)e^{i\widehat{\omega}_{\psi}(d-d,q)} \\ &= \gamma_{\psi}(d-d,q) \\ &\leq max\{\gamma_{\psi}(d,q),\gamma_{\psi}(d,q)\} \\ &= max\{\hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)},\hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)}\} \\ &= max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(d,q)\}e^{imax\{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(d,q)\}} \\ &= \hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)}.\end{aligned}$$

(As ψ is homogeneous)
$\Rightarrow \hat{r}_{\psi}(0,q) \leq \hat{r}_{\psi}(d,q)$, and $\widehat{\omega}_{\psi}(0,q) \leq \widehat{\omega}_{\psi}(d,q)$.

(4) In the case when the identity element of F is 1 and $d \neq 0$ in F:

$$\begin{aligned}\hat{r}_{\psi}(1,q)e^{i\widehat{\omega}_{\psi}(1,q)} &= \hat{r}_{\psi}(dd^{-1},q)e^{i\widehat{\omega}_{\psi}(dd^{-1},q)} \\ &= \gamma_{\psi}(dd^{-1},q) \\ &\leq max\{\gamma_{\psi}(d,q),\gamma_{\psi}(d,q)\} \\ &= max\{\hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)}\hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)}\} \\ &= max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(,q)\}e^{imax\{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(d,q)\}} \\ &= \hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)}.\end{aligned}$$

(As ψ is homogeneous)
$\Rightarrow \hat{r}_{\psi}(1,q) \leq \hat{r}_{\psi}(d,q)$, and $\widehat{\omega}_{\psi}(1,q) \leq \widehat{\omega}_{\psi}(d,q)$.

Theorem 3.4. *Let* $(F,+,\cdot)$ *be a field. Let* $\psi = \{\langle (d,q),\mu_{\psi}(d,q),\gamma_{\psi}(d,q)\rangle : d \in F,\ q \in Q\}$ *be a complex intuitionistic* Q*-fuzzy set. Then the statements below are equivalent:*
(1) *The fuzzy set* ψ *is a complex intuitionistic* Q*-fuzzy subfield of a field* F.
(2) *The fuzzy set* ψ^c *is a complex intuitionistic anti-*Q*-fuzzy subfield of a field* F.
Proof. $(1) \Rightarrow (2)$ Suppose ψ is a complex intuitionistic Q-fuzzy subfield. So,

$$\begin{aligned}\mu_{\psi}(d-k,q) &\geq min\{\mu_{\psi}(d,q),\mu_{\psi}(k,q)\} \\ &= min\{r_{\psi}(d,q)e^{i\omega_{\psi}(d,q)},r_{\psi}(k,q)e^{i\omega_{\psi}(k,q)}\} \\ &= min\{r_{\psi}(d,q),r_{\psi}(k,q)\}e^{imin\{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}}\end{aligned}$$

$$\begin{aligned}\Rightarrow\ \mu_{\psi^c}(d-k,q) &\leq \left(1-min\{r_{\psi}(d,q),r_{\psi}(k,q)\}\right)e^{i\left(2\pi-mi\ \{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}\right)} \\ &= max\{1-r_{\psi}(d,q),1-r_{\psi}(k,q)\}e^{imax\{2\pi-\omega_{\psi}(d,q),2\pi-\omega_{\psi}(k,q)\}} \\ &= max\{r_{\psi^c}(d,q),r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c}(d,q),\omega_{\psi^c}(k,q)\}} \\ &= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)},r_{\psi^c}(k,q)e^{i\omega_{\psi^c}(k,q)}\} \\ &= max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}.\end{aligned}$$

Also,

$$\mu_\psi(dk,q) \ \geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$$
$$= min\{r_\psi(d,q)e^{i\omega_\psi(d,q)}, r_\psi(k,q)e^{i\omega_\psi(k,q)}\}$$
$$= min\{r_\psi(d,q), r_\psi(k,q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(k,q)\}}$$

$$\Rightarrow \ \mu_{\psi^c}(dk,q) \leq \left(1-min\{r_\psi(d,q),r_\psi(k,q)\}\right)e^{i\left(2\pi-min\{\omega_\psi(d,q),\omega_\psi(k,q)\}\right)}$$
$$= max\{1-r_\psi(d,q),1-r_\psi(k,q)\}e^{imax\{2\pi-\omega_\psi(d,q),2\pi-\omega_\psi(k,q)\}}$$
$$= max\{r_{\psi^c}(d,q), r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c}(d,q),\omega_{\psi^c}(k,q)\}}$$
$$= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)}, r_{\psi^c}(k,q)e^{i\omega_{\psi^c}(k,q)}\}$$
$$= max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}.$$

Further,

$$\mu_\psi(d^{-1},q) \geq \mu_\psi(d,q) = r_\psi(d,q)e^{i\omega_\psi(d,q)}$$

$$\Rightarrow \ \mu_{\psi^c}(d^{-1},q) \leq (1-r_\psi(d,q))e^{i\left(2\pi-\omega_\psi(d,q)\right)} = r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)} = \mu_{\psi^c}(d,q).$$

On the other hand,

$$\gamma_\psi(d-k,q) \ \leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$$
$$= max\{\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}, \hat{r}_\psi(k,q)e^{i\widehat{\omega}_\psi(k,q)}\}$$
$$= max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}e^{imax\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}}$$

$$\Rightarrow \ \gamma_{\psi^c}(d-k,q) \geq \left(1-max\{\hat{r}_\psi(d,q),\hat{r}_\psi(k,q)\}\right)e^{i\left(2\pi-max\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}\right)}$$
$$= min\{1-\hat{r}_\psi(d,q),1-\hat{r}_\psi(k,q)\}e^{imin\{2\pi-\widehat{\omega}_\psi(d,q),2\pi-\widehat{\omega}_\psi(k,q)\}}$$
$$= min\{\hat{r}_{\psi^c}(d,q), \hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c}(d,q),\widehat{\omega}_{\psi^c}(k,q)\}}$$
$$= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)}, \hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c}(k,q)}\}.$$
$$= min\{\gamma_{\psi^c}(d,q),\gamma_{\psi^c}(k,q)\}.$$

Moreover,

$$\gamma_\psi(dk,q) \ \leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$$
$$= max\{\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}, \hat{r}_\psi(k,q)e^{i\widehat{\omega}_\psi(k,q)}\}$$
$$= max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}e^{imax\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}}$$

$$\Rightarrow \ \gamma_{\psi^c}(dk,q) \geq \left(1-max\{\hat{r}_\psi(d,q),\hat{r}_\psi(k,q)\}\right)e^{i\left(2\pi-ma \ \{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}\right)}$$
$$= min\{1-\hat{r}_\psi(d,q),1-\hat{r}_\psi(k,q)\}e^{imin\{2\pi-\widehat{\omega}_\psi(d,q),2\pi-\widehat{\omega}_\psi(k,q)\}}$$
$$= min\{\hat{r}_{\psi^c}(d,q), \hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c}(d,q),\widehat{\omega}_{\psi^c}(k,q)\}}$$

$$= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c(d,q)}}, \hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c(k,q)}}\}$$

$$= min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}.$$

Father,

$$(d^{-1},q)\,\gamma_{\psi} \leq \gamma_{\psi}(d,q) = \hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)}$$

$$\Rightarrow \ \gamma_{\psi^c}(d^{-1},q) \geq (1-\hat{r}_{\psi}(d,q)) = \hat{r}_{\psi^c}(d,q) = \gamma_{\psi^c}(d,q).$$

Therefore, ψ^c is a complex intuitionistic anti-Q-fuzzy subfield.
Conversely, assume that ψ^c is a complex anti-Q-fuzzy subfield. So,

$$\mu_{\psi^c}(d-k,q) \leq max\{\mu_{\psi^c}(d,q), \mu_{\psi^c}(k,q)\}$$

$$= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c(d,q)}}, r_{\psi^c}(k,q)e^{i\omega_{\psi^c(k,q)}}\}$$

$$= max\{r_{\psi^c}(d,q), r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c(d,q)},\omega_{\psi^c(k,q)}\}}$$

$$= max\{1-r_{\psi}(d,q), 1-r_{\psi}(k,q)\}e^{imax\{2\pi-\omega_{\psi}(d,q),2\pi-\omega_{\psi}(k,q)\}}$$

$$= \left(1-min\{r_{\psi}(d,q), r_{\psi}(k,q)\}\right)e^{i\left(2\pi-min\{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}\right)}$$

$$\Rightarrow 1-\mu_{\psi}(d-k,q) \leq \left(1-min\{r_{\psi}(d,q), r_{\psi}(k,q)\}\right)e^{i\left(2\pi-mi\ \{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}\right)}$$

$$\Rightarrow \mu_{\psi}(d-k,q) \geq min\{r_{\psi}(d,q), r_{\psi}(k,q)\}e^{imin\{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}}$$

$$= min\{r_{\psi}(d,q)e^{i\omega_{\psi}(d,q)}, r_{\psi}(k,q)e^{i\omega_{\psi}(k,q)}\}$$

$$= min\{\mu_{\psi}(d,q), \mu_{\psi}(k,q)\}.$$

Also, we have

$$\mu_{\psi^c}(dk,q) \ \leq max\{\mu_{\psi^c}(d,q), \mu_{\psi^c}(k,q)\}$$

$$= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c(d,q)}}, r_{\psi^c}(k,q)e^{i\omega_{\psi^c(k,q)}}\}$$

$$= max\{r_{\psi^c}(d,q), r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c(d,q)},\omega_{\psi^c(k,q)}\}}$$

$$= max\{1-r_{\psi}(d,q), 1-r_{\psi}(k,q)\}e^{imax\{2\pi-\omega_{\psi}(d,q),2\pi-\omega_{\psi}(k,q)\}}$$

$$= \left(1-min\{r_{\psi}(d,q), r_{\psi}(k,q)\}\right)e^{i\left(2\pi-min\{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}\right)}$$

$$\Rightarrow 1-\mu_{\psi}(dk,q) \leq \left(1-min\{r_{\psi}(d,q), r_{\psi}(k,q)\}\right)e^{i\left(2\pi-min\{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}\right)}$$

$$\Rightarrow \mu_{\psi}(dk,q) \geq min\{r_{\psi}(d,q), r_{\psi}(k,q)\}e^{imin\{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\}}$$

$$= min\{r_{\psi}(d,q)e^{i\omega_{\psi}(d,q)}, r_{\psi}(k,q)e^{i\omega_{\psi}(k,q)}\}$$

$$= min\{\mu_{\psi}(d,q), \mu_{\psi}(k,q)\}.$$

Further,

$$\mu_{\psi^c}(d^{-1},q) \leq \mu_{\psi^c}(d,q) = r_{\psi^c}(d,q)e^{i\omega_{\psi^c(d,q)}} = (1-r_{\psi}(d,q))e^{i\left(2\pi-\omega_{\psi}(d,q)\right)}$$

$$\Rightarrow \mu_{\psi}(d^{-1},q) \geq r_{\psi}(d,q)e^{i\omega_{\psi}(d,q)} = \mu_{\psi}(d,q).$$

On other hand,

$$\gamma_{\psi^c}(d-k,q) \geq min\{\gamma_{\psi^c}(d,q),\gamma_{\psi^c}(k,q)\}$$
$$= min\,\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)},\hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c}(k,q)}\}$$
$$= min\{\hat{r}_{\psi^c}(d,q),\hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c}(d,q),\widehat{\omega}_{\psi^c}(k,q)\}}$$
$$= min\{1-\hat{r}_{\psi}(d,q),1-\hat{r}_{\psi}(k,q)\}e^{imin\{2\pi-\widehat{\omega}_{\psi}(d,q),2\pi-\widehat{\omega}_{\psi}(k,q)\}}$$
$$= \left(1-max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}\right)e^{i\left(2\pi-max\{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\}\right)}$$
$$\Rightarrow (1-\gamma_{\psi}(d-k,q)) \geq \left(1-max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}\right)e^{i\left(2\pi-ma\ \{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\}\right)}$$
$$\Rightarrow \gamma_{\psi}(d-k,q) \leq max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}e^{imax\{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\}}$$
$$= max\{\hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)},\hat{r}_{\psi}(k,q)e^{i\widehat{\omega}_{\psi}(k,q)}\}$$
$$= max\{\gamma_{\psi}(d,q),\gamma_{\psi}(k,q)\}.$$

Moreover,

$$\gamma_{\psi^c}(dk,q) \geq min\{\gamma_{\psi^c}(d,q),\gamma_{\psi^c}(k,q)\}$$
$$= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)},\hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c}(k,q)}\}$$
$$= min\{\hat{r}_{\psi^c}(d,q),\hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c}(d,q),\widehat{\omega}_{\psi^c}(k,q)\}}$$
$$= min\{1-\hat{r}_{\psi}(d,q),1-\hat{r}_{\psi}(k,q)\}e^{imin\{2\pi-\widehat{\omega}_{\psi}(d,q),2\pi-\widehat{\omega}_{\psi}(k,q)\}}$$
$$= \left(1-max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}\right)e^{i\left(2\pi-max\{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\}\right)}$$
$$\Rightarrow 1-\gamma_{\psi}(dk,q) \geq \left(1-max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}\right)e^{i\left(2\pi-ma\ \{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\}\right)}$$
$$\Rightarrow \gamma_{\psi}(dk,q) \leq max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}e^{imax\{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\}}$$
$$= max\{\hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)},\hat{r}_{\psi}(k,q)e^{i\widehat{\omega}_{\psi}(k,q)}\}$$
$$= max\{\gamma_{\psi}(d,q),\gamma_{\psi}(k,q)\}.$$

Also, we have

$$\gamma_{\psi^c}(d^{-1},q) \geq \gamma_{\psi^c}(d,q) = \hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)} = (1-\hat{r}_{\psi}(d,q))e^{i\left(2\pi-\widehat{\omega}_{\psi}(d,q)\right)}$$
$$\Rightarrow \gamma_{\psi}(d^{-1},q) \leq \hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)} = \gamma_{\psi}(d,q).$$

Thus, the theorem is proven.

Definition 3.4. *Let* $\psi = \{\langle (d,q),\mu_{\psi}(d,q),\gamma_{\psi}(d,q)\rangle : d \in F,\ q \in Q\}$ *be a complex intuitionistic* Q-*fuzzy subfield of a field* F. *Then*

(1) *The necessity operator* $\Box\,\psi = \{\langle (d,q),\mu_{\psi}(d,q),1-\mu_{\psi}(d,q)\rangle : d \in F,\ q \in Q\}$,

(2) *The possibility operator* $\Diamond\ \psi = \{\langle (d,q),1-\gamma_{\psi}(d,q),\gamma_{\psi}(d,q)\rangle : d \in F,\ q \in Q\}$.

Theorem 3.5. *Let* $(F,+,\cdot)$ *be a field. If* ψ *is a complex intuitionistic* Q*-fuzzy subfield of* $(F,+,\cdot)$*, then the necessity operator* $\square\psi$ *is a complex intuitionistic* Q*-fuzzy subfield of* $(F,+,\cdot)$.

Proof. Let $\square\psi=\{\langle(d,q),\mu_\psi(d,q),\mu_{\psi^c}(d,q)\rangle: d\in F,\ q\in Q\}$. To show that $\square\psi$ is a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$, let $\psi=\{\langle(d,q),\mu_\psi(d,q),\gamma_\psi(d,q)\rangle: d\in F,\ q\in Q\}$ be a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$. So, for all $d,k\in F,\ q\in Q$

(i) $\mu_\psi(d-k,q)\geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(ii) $\mu_\psi(dk,q)\geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(iii) $\mu_\psi(d^{-1},q)\geq\mu_\psi(d,q)$,

(iv)$\gamma_\psi(d-k,q)\leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,

(v) $\gamma_\psi(dk,q)\leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,

(vi)$\gamma_\psi(d^{-1},q)\leq\gamma_\psi(d,q)$.

To show that $\square\psi$ is a complex intuitionistic Q-fuzzy subfield, we must prove the following:

(1) $\mu_\psi(d-k,q)\geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(2) $\mu_\psi(dk,q)\geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,

(3) $\mu_\psi(d^{-1},q)\geq\mu_\psi(d,q)$,

(4) $\mu_{\psi^c}(d-k,q)\leq max\{\mu_{\psi^c}(d,q),\mu_{(\psi)^c}(k,q)\}$,

(5) $\mu_{\psi^c}(dk,q)\leq max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}$,

(6) $\mu_{\psi^c}(d^{-1},q)\leq\mu_{\psi^c}(d,q)$.

We note that conditions (1)–(3) are given. So,

$$\begin{aligned}\mu_{\psi^c}(d-k,q)&=r_{\psi^c}(d-k,q)e^{i\omega_{\psi^c}(d-k,q)}\\&=\{1-r_\psi(d-k,q)\}e^{i\{2\pi-\omega_\psi(d-k,q)\}}\\&\leq\left(1-min\{r_\psi(d,q),r_\psi(k,q)\}\right)e^{i\left(2\pi-min\{\omega_\psi(d,q),\omega_\psi(k,q)\}\right)}\\&=max\{1-r_\psi(d,q),1-r_\psi(k,q)\}e^{imax\{2\pi-\omega_\psi(d,q),2\pi-\omega_\psi(k,q)\}}\\&=max\{r_{\psi^c}(d,q),r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c}(d,q),\omega_{\psi^c}(k,q)\}}\\&=max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)},r_{\psi^c}(k,q)e^{i\omega_{\psi^c}(k,q)}\}\\&=max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\mu_{\psi^c}(dk,q)&=r_{\psi^c}(dk,q)e^{i\omega_{\psi^c}(dk,q)}\\&=\{1-r_\psi(dk,q)\}e^{i\{2\pi-\omega_\psi(dk,q)\}}\\&\leq\left(1-min\{r_\psi(d,q),r_\psi(k,q)\}\right)e^{i\left(2\pi-min\{\omega_\psi(d,q),\omega_\psi(k,q)\}\right)}\\&=max\{1-r_\psi(d,q),1-r_\psi(k,q)\}e^{imax\{2\pi-\omega_\psi(d,q),2\pi-\omega_\psi(k,q)\}}\\&=max\{r_{\psi^c}(d,q),r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c}(d,q),\omega_{\psi^c}(k,q)\}}\\&=max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)},r_{\psi^c}(k,q)e^{i\omega_{\psi^c}(k,q)}\}\\&=max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}.\end{aligned}$$

Also,

$$\mu_{(\psi)^c}(d^{-1},q) = r_{(\psi)^c}(d^{-1},q)e^{i\omega_{(\psi)^c}(d^{-1},q)}$$
$$= \{1 - r_\psi(d^{-1},q)\}e^{i\{2\pi-\omega_\psi(d^{-1},q)\}}$$
$$\leq \{1 - r_\psi(d,q)\}e^{i\{2\pi-\omega_\psi(d,q)\}}$$
$$= r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)}$$
$$= \mu_{\psi^c}(d,q).$$

Thus,
$\mu_\psi(d-k,q) \geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$, $\mu_\psi(dk,q) \geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$, $\mu_\psi(d^{-1},q) \geq \mu_\psi(d,q)$,
$\mu_{\psi^c}(d-k,q) \leq max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}$, $\mu_{\psi^c}(dk,q) \leq max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}$, and $\mu_{\psi^c}(d^{-1},q) \leq \mu_{\psi^c}(d,q)$.
Therefore, $\square\,\psi = \{\langle(d,q),\mu_\psi(d,q),\mu_{\psi^c}(d,q)\rangle: d \in F,\ q \in Q\}$ is a complex intuitionistic Q-fuzzy subfield of a field F.

Theorem 3.6. *Let* $(F,+,\cdot)$ *be a field. If* ψ *is a complex intuitionistic* Q*-fuzzy subfield of* $(F,+,\cdot)$*, then the possibility operator* $\Diamond\,\psi$ *is a complex intuitionistic* Q*-fuzzy subfield of* $(F,+,\cdot)$.

Proof. Let $\Diamond\,\psi = \{\langle(d,q),\gamma_{\psi^c}(d,q),\gamma_\psi(d,q)\rangle: d \in F, q \in Q\}$. To show that $\Diamond\,\psi$ is a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$, let $\psi = \{\langle(d,q),\mu_\psi(d,q),\gamma_\psi(d,q)\rangle: d \in F,\ q \in Q\}$ be a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$. Thus,
(i) $\mu_\psi(d-k,q) \geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,
(ii) $\mu_\psi(dk,q) \geq min\{\mu_\psi(d,q),\mu_\psi(k,q)\}$,
(iii) $\mu_\psi(d^{-1},q) \geq \mu_\psi(d,q)$,
(iv) $\gamma_\psi(d-k,q) \leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,
(v) $\gamma_\psi(dk,q) \leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,
(vi) $\gamma_\psi(d^{-1},q) \leq \gamma_\psi(d,q)$.
To show that $\Diamond\,\psi$ is a complex intuitionistic Q-fuzzy subfield, we must prove the following:
(1) $\gamma_{\psi^c}(d-k,q) \geq min\{\gamma_{\psi^c}(d,q),\gamma_{\psi^c}(k,q)\}$,
(2) $\gamma_{\psi^c}(dk,q) \geq min\{\gamma_{\psi^c}(d,q),\gamma_{\psi^c}(k,q)\}$,
(3) $\gamma_{\psi^c}(d^{-1},q) \geq \gamma_{\psi^c}(d,q)$,
(4) $\gamma_\psi(d-k,q) \leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,
(5) $\gamma_\psi(dk,q) \leq max\{\gamma_\psi(d,q),\gamma_\psi(k,q)\}$,
(6) $\gamma_\psi(d^{-1},q) \leq \gamma_\psi(d,q)$.
We note that conditions (4)–(6) are given. Now,

$$\gamma_{\psi^c}(d-k,q) = \hat{r}_{\psi^c}(d-k,q)e^{i\widehat{\omega}_{\psi^c}(d-k,q)}$$
$$= \{1 - \hat{r}_\psi(d-k,q)\}e^{i\{2\pi-\widehat{\omega}_\psi(d-k,q)\}}$$
$$\geq \left(1 - max\{\hat{r}_\psi(d,q),\hat{r}_\psi(k,q)\}\right)e^{i\left(2\pi-max\{\widehat{\omega}_\psi(d,q),\widehat{\omega}_\psi(k,q)\}\right)}$$
$$= min\{1-\hat{r}_\psi(d,q),1-\hat{r}_\psi(k,q)\}e^{i min\{2\pi-\widehat{\omega}_\psi(d,q),2\pi-\widehat{\omega}_\psi(k,q)\}}$$
$$= min\{\hat{r}_{\psi^c}(d,q),\hat{r}_{\psi^c}(k,q)\}e^{i min\{\widehat{\omega}_{\psi^c}(d,q),\widehat{\omega}_{\psi^c}(k,q)\}}$$

$$= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)}, \hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c}(k,q)}\}$$

$$= min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}.$$

Also,

$$\gamma_{\psi^c}(dk,q) = \hat{r}_{\psi^c}(dk,q)e^{i\widehat{\omega}_{\psi^c}(dk,q)}$$

$$= \{1 - \hat{r}_{\psi}(dk,q)\}e^{i\{2\pi - \widehat{\omega}_{\psi}(dk,q)\}}$$

$$\geq \left(1 - max\{\hat{r}_{\psi}(d,q), \hat{r}_{\psi}(k,q)\}\right)e^{i\left(2\pi - m \quad \{\widehat{\omega}_{\psi}(d,q), \widehat{\omega}_{\psi}(k,q)\}\right)}$$

$$= min\{1 - \hat{r}_{\psi}(d,q), 1 - \hat{r}_{\psi}(k,q)\}e^{imin\{2\pi - \widehat{\omega}_{\psi}(d,q), 2\pi - \widehat{\omega}_{\psi}(k,q)\}}$$

$$= min\{\hat{r}_{\psi^c}(d,q), \hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c}(d,q), \widehat{\omega}_{\psi^c}(k,q)\}}$$

$$= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)}, \hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c}(k,q)}\}$$

$$= min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}.$$

Moreover,

$$\gamma_{\psi^c}(d^{-1},q) = \hat{r}_{\psi^c}(d^{-1},q)e^{i\widehat{\omega}_{\psi^c}(d^{-1},q)}$$

$$= \{1 - \hat{r}_{\psi}(d^{-1},q)\}e^{i\{2\pi - \widehat{\omega}_{\psi}(d^{-1},q)\}}$$

$$\geq \{1 - \hat{r}_{\psi}(d,q)\}e^{i\{2\pi - \widehat{\omega}_{\psi}(d,q)\}}$$

$$= \hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)}$$

$$= \gamma_{\psi^c}(d,q).$$

So,
$\gamma_{\psi^c}(d-k,q) \geq min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}$, $\gamma_{\psi^c}(dk,q) \geq min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}$, $\gamma_{\psi^c}(d^{-1},q) \geq \gamma_{\psi^c}(d,q)$,
$\gamma_{\psi}(d-k,q) \leq max\{\gamma_{\psi}(d,q), \gamma_{\psi}(k,q)\}$, $\gamma_{\psi}(dk,q) \leq max\{\gamma_{\psi}(d,q), \gamma_{\psi}(k,q)\}$, and $\gamma_{\psi}(d^{-1},q) \leq \gamma_{\psi}(d,q)$.
Therefore, $\lozenge\ \psi = \{\langle (d,q), \gamma_{\psi^c}(d,q), \gamma_{\psi}(d,q)\rangle : d \in F, q \in Q\}$ is a complex intuitionistic Q-fuzzy subfield of a field F.

Theorem 3.7. *A complex intuitionistic Q-fuzzy subset $\psi = \{\langle (d,q), \mu_{\psi}(d,q), \gamma_{\psi}(d,q)\rangle : d \in F, q \in Q\}$ of a field F is a complex intuitionistic Q-fuzzy subfield of a field F if and only if the complex Q-fuzzy subsets $\mu_{\psi}(d,q)$, $\gamma_{\psi^c}(d,q)$ are complex Q-fuzzy subfields of a field F.*

Proof. Let $\psi = \{\langle (d,q), \mu_{\psi}(d,q), \gamma_{\psi}(d,q)\rangle : d \in F, q \in Q\}$ be a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$. Thus,
(1) $\mu_{\psi}(d-k,q) \geq min\{\mu_{\psi}(d,q), \mu_{\psi}(k,q)\}$,
(2) $\mu_{\psi}(dk,q) \geq min\{\mu_{\psi}(d,q), \mu_{\psi}(k,q)\}$,
(3) $\mu_{\psi}(d^{-1},q) \geq \mu_{\psi}(d,q)$,
(4) $\gamma_{\psi}(d-k,q) \leq max\{\gamma_{\psi}(d,q), \gamma_{\psi}(k,q)\}$,
(5) $\gamma_{\psi}(dk,q) \leq max\{\gamma_{\psi}(d,q), \gamma_{\psi}(k,q)\}$,
(6) $\gamma_{\psi}(d^{-1},q) \leq \gamma_{\psi}(d,q)$.

Clearly, $\mu_{\psi}(d,q)$ is a complex Q-fuzzy subfield of $(F,+,\cdot)$ by the given (1), (2), and (3). Now we must show that $\gamma_{\psi^c}(d,q)$ is a complex Q-fuzzy subfield of $(F,+,\cdot)$.
To show that $\gamma_{\psi^c}(d,q)$ is a complex Q-fuzzy subfield of a field F, we will prove the following:
(i) $\gamma_{\psi^c}(d-k,q) \geq min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}$,
(ii) $\gamma_{\psi^c}(dk,q) \geq min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}$,
(iii) $\gamma_{\psi^c}(d^{-1},q) \geq \gamma_{\psi^c}(d,q)$.
Now,

$$\begin{aligned}\gamma_{\psi^c}(d-k,q) &= \hat{r}_{\psi^c}(d-k,q)e^{i\widehat{\omega}_{\psi^c}(d-k,q)}\\ &= \{1-\hat{r}_{\psi}(d-k,q)\}e^{i\{2\pi-\widehat{\omega}_{\psi}(d-k,q)\}}\\ &\geq \left(1-max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}\right)e^{i(2\pi-ma\ \{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\})}\\ &= min\{1-\hat{r}_{\psi}(d,q),1-\hat{r}_{\psi}(k,q)\}e^{imin\{2\pi-\widehat{\omega}_{\psi}(d,q),2\pi-\widehat{\omega}_{\psi}(k,q)\}}\\ &= min\{\hat{r}_{\psi^c}(d,q),\hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c}(d,q),\widehat{\omega}_{\psi^c}(k,q)\}}\\ &= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)},\hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c}(k,q)}\}\\ &= min\{\gamma_{\psi^c}(d,q),\gamma_{\psi^c}(k,q)\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\gamma_{\psi^c}(dk,q) &= \hat{r}_{\psi^c}(dk,q)e^{i\widehat{\omega}_{\psi^c}(dk,q)}\\ &= (1-\hat{r}_{\psi}(dk,q))e^{i(2\pi-\widehat{\omega}_{\psi}(dk,q))}\\ &\geq \left(1-max\{\hat{r}_{\psi}(d,q),\hat{r}_{\psi}(k,q)\}\right)e^{i(2\pi-max\{\widehat{\omega}_{\psi}(d,q),\widehat{\omega}_{\psi}(k,q)\})}\\ &= min\{1-\hat{r}_{\psi}(d,q),1-\hat{r}_{\psi}(k,q)\}e^{imin\{2\pi-\widehat{\omega}_{\psi}(d,q),2\pi-\widehat{\omega}_{\psi}(k,q)\}}\\ &= min\{\hat{r}_{\psi^c}(d,q),\hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c}(d,q),\widehat{\omega}_{\psi^c}(k,q)\}}\\ &= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)},\hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c}(k,q)}\}\\ &= min\{\gamma_{\psi^c}(d,q),\gamma_{\psi^c}(k,q)\}.\end{aligned}$$

Further,

$$\begin{aligned}\gamma_{\psi^c}(d^{-1},q) &= \hat{r}_{\psi^c}(d^{-1},q)e^{i\widehat{\omega}_{\psi^c}(d^{-1},q)}\\ &= \{1-\hat{r}_{\psi}(d^{-1},q)\}e^{i\{2\pi-\widehat{\omega}_{\psi}(d^{-1},q)\}}\\ &\geq \{1-\hat{r}_{\psi}(d,q)\}e^{i\{2\pi-\widehat{\omega}_{\psi}(d,q)\}}\\ &= \hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)}\\ &= \gamma_{\psi^c}(d,q).\end{aligned}$$

Therefore, the complex Q-fuzzy subsets $\mu_{\psi}(d,q)$, $\gamma_{\psi^c}(d,q)$ are complex Q-fuzzy subfields of $(F,+,\cdot)$.

Conversely, let $\mu_\psi(d,q)$,$\gamma_{\psi^c}(d,q)$ are complex Q-fuzzy subfields of $(F,+,\cdot)$. To prove that $\psi = \{\langle (d,q), \mu_\psi(d,q), \gamma_\psi(d,q) \rangle : d \in F, q \in Q\}$ be a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$, we must prove that ψ satisfies all conditions of a complex intuitionistic Q-fuzzy subfield of a field F. So, we want to show that:

(1) $\mu_\psi(d-k,q) \geq min\{\mu_\psi(d,q), \mu_\psi(k,q)\}$,

(2) $\mu_\psi(dk,q) \geq min\{\mu_\psi(d,q), \mu_\psi(k,q)\}$,

(3) $\mu_\psi(d^{-1},q) \geq \mu_\psi(d,q)$,

(4) $\gamma_\psi(d-k,q) \leq max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}$,

(5) $\gamma_\psi(dk,q) \leq max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}$,

(6) $\gamma_\psi(d^{-1},q) \leq \gamma_\psi(d,q)$.

Given that $\mu_\psi(d,q)$ is a complex Q-fuzzy subfield of $(F,+,\cdot)$., we remark that (1)–(3) are satisfied. As a result, we need to demonstrate conditions (4)–(6). Given that $\gamma_{\psi^c}(d,q)$ is a complex Q-fuzzy subfield, thus:

$$\begin{aligned}
\gamma_{\psi^c}(d-k,q) &\geq min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\} \\
&= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c(d,q)}}, \hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c(k,q)}}\} \\
&= min\{\hat{r}_{\psi^c}(d,q), \hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c(d,q)}, \widehat{\omega}_{\psi^c(k,q)}\}} \\
&= min\{1-\hat{r}_\psi(d,q), 1-\hat{r}_\psi(k,q)\}e^{imin\{2\pi-\widehat{\omega}_\psi(d,q), 2\pi-\widehat{\omega}_\psi(k,q)\}} \\
&= \left(1-max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}\right)e^{i(2\pi-ma\ \{\widehat{\omega}_\psi(d,q), \widehat{\omega}_\psi(k,q)\})}.
\end{aligned}$$

So,

$$\begin{aligned}
\gamma_\psi(d-k,q) &\leq max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}e^{imax\{\widehat{\omega}_\psi(d,q), \widehat{\omega}_\psi(k,q)\}} \\
&= max\{\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}, \hat{r}_\psi(k,q)e^{i\widehat{\omega}_\psi(k,q)}\} \\
&= max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}.
\end{aligned}$$

Also, since $\gamma_{\psi^c}(d,q)$ is a complex Q-fuzzy subfield of $(F,+,\cdot)$, hence

$$\begin{aligned}
\gamma_{\psi^c}(dk,q) &\geq min\{\gamma_{\psi^c}(d,q), \gamma_{\psi^c}(k,q)\}, \\
&= min\{\hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c(d,q)}}, \hat{r}_{\psi^c}(k,q)e^{i\widehat{\omega}_{\psi^c(k,q)}}\} \\
&= min\{\hat{r}_{\psi^c}(d,q), \hat{r}_{\psi^c}(k,q)\}e^{imin\{\widehat{\omega}_{\psi^c(d,q)}, \widehat{\omega}_{\psi^c(k,q)}\}} \\
&= min\{1-\hat{r}_\psi(d,q), 1-\hat{r}_\psi(k,q)\}e^{imin\{2\pi-\widehat{\omega}_\psi(d,q), 2\pi-\widehat{\omega}_\psi(k,q)\}} \\
&= \left(1-max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}\right)e^{i(2\pi-max\{\widehat{\omega}_\psi(d,q), \widehat{\omega}_\psi(k,q)\})}.
\end{aligned}$$

So,

$$\begin{aligned}
\gamma_\psi(dk,q) &\leq max\{\hat{r}_\psi(d,q), \hat{r}_\psi(k,q)\}e^{imax\{\widehat{\omega}_\psi(d,q), \widehat{\omega}_\psi(k,q)\}} \\
&= max\{\hat{r}_\psi(d,q)e^{i\widehat{\omega}_\psi(d,q)}, \hat{r}_\psi(k,q)e^{i\widehat{\omega}_\psi(k,q)}\} \\
&= max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}.
\end{aligned}$$

Also, since $\gamma_{\psi^c}(d,q)$ is a complex Q-fuzzy subfield of $(F,+,\cdot)$, hence:

$$\gamma_{\psi^c}(d^{-1},q) \geq \gamma_{\psi^c}(d,q)$$
$$= \hat{r}_{\psi^c}(d,q)e^{i\widehat{\omega}_{\psi^c}(d,q)}$$
$$= \{1-\hat{r}_{\psi}(d,q)\}e^{i\{2\pi-\widehat{\omega}_{\psi}(d,q)\}}$$

Then,
$\gamma_{\psi}(d^{-1},q) \leq \hat{r}_{\psi}(d,q)e^{i\widehat{\omega}_{\psi}(d,q)} = \gamma_{\psi}(d,q)$.
Already we have
$\mu_{\psi}(d-k,q) \geq min\{\mu_{\psi}(d,q),\mu_{\psi}(k,q)\}$, $\mu_{\psi}(dk,q) \geq min\{\mu_{\psi}(d,q),\mu_{\psi}(k,q)\}$, and $\mu_{\psi}(d^{-1},q) \geq \mu_{\psi}(d,q)$.
Hence, $\psi = \{\langle(d,q),\mu_{\psi}(d,q),\gamma_{\psi}(d,q)\rangle : d \in F, q \in Q\}$ be a complex intuitionistic Q-fuzzy subfield of a field F.

Theorem 3.8. *Let $(F,+,\cdot)$ be a field. A complex intuitionistic Q-fuzzy subset $\psi = \{\langle(d,q),\mu_{\psi}(d,q),\gamma_{\psi}(d,q)\rangle : d \in F, q \in Q\}$ of $(F,+,\cdot)$ is a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$ if and only if the complex Q-fuzzy subsets $\mu_{\psi^c}(d,q)$, $\gamma_{\psi}(d,q)$ are complex anti-Q-fuzzy subfields of $(F,+,\cdot)$.*

Proof. Let $\psi = \{\langle(d,q),\mu_{\psi}(d,q),\gamma_{\psi}(d,q)\rangle : d \in F, q \in Q\}$ be a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$. Then
(1) $\mu_{\psi}(d-k,q) \geq min\{\mu_{\psi}(d,q),\mu_{\psi}(k,q)\}$,
(2) $\mu_{\psi}(dk,q) \geq min\{\mu_{\psi}(d,q),\mu_{\psi}(k,q)\}$,
(3) $\mu_{\psi}(d^{-1},q) \geq \mu_{\psi}(d,q)$,
(4) $\gamma_{\psi}(d-k,q) \leq max\{\gamma_{\psi}(d,q),\gamma_{\psi}(k,q)\}$,
(5) $\gamma_{\psi}(dk,q) \leq max\{\gamma_{\psi}(d,q),\gamma_{\psi}(k,q)\}$,
(6) $\gamma_{\psi}(d^{-1},q) \leq \gamma_{\psi}(d,q)$.
From (4), (5), and (6), it is clear that $\gamma_{\psi}(d,q)$ is a complex anti-Q-fuzzy subfield of $(F,+,\cdot)$. Now we must show that $\mu_{\psi^c}(d,q)$ is a complex anti-Q-fuzzy subfield of $(F,+,\cdot)$.
To show that $\mu_{\psi^c}(d,q)$ is a complex anti-Q-fuzzy subfield of $(F,+,\cdot)$, we will prove the following:
(i) $\mu_{\psi^c}(d-k,q) \leq max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}$,
(ii) $\mu_{\psi^c}(dk,q) \leq max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}$,
(iii) $\mu_{\psi^c}(d^{-1},q) \leq \mu_{\psi^c}(d,q)$.
Now,

$$\mu_{\psi^c}(d-k,q) = r_{\psi^c}(d-k,q)e^{i\omega_{\psi^c}(d-k,q)}$$
$$= \{1-r_{\psi}(d-k,q)\}e^{i\{2\pi-\omega_{\psi}(d-k,q)\}}$$
$$\leq \left(1-min\{r_{\psi}(d,q),r_{\psi}(k,q)\}\right)e^{i(2\pi-min\{\omega_{\psi}(d,q),\omega_{\psi}(k,q)\})}$$
$$= max\{1-r_{\psi}(d,q),1-r_{\psi}(k,q)\}e^{imax\{2\pi-\omega_{\psi}(d,q),2\pi-\omega_{\psi}(k,q)\}}$$
$$= max\{r_{\psi^c}(d,q),r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c}(d,q),\omega_{\psi^c}(k,q)\}}$$
$$= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)},r_{\psi^c}(k,q)e^{i\omega_{\psi^c}(k,q)}\}$$
$$= max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}.$$

Also,

$$\mu_{\psi^c}(dk,q) = r_{\psi^c}(dk,q)e^{i\omega_{\psi^c}(dk,q)}$$
$$= \{1 - r_\psi(dk,q)\}e^{i\{2\pi - \omega_\psi(dk,q)\}}$$
$$\leq (1 - min\{r_\psi(d,q), r_\psi(k,q)\})e^{i(2\pi - mi\ \{\omega_\psi(d,q),\omega_\psi(k,q)\})}$$
$$= max\{1 - r_\psi(d,q), 1 - r_\psi(k,q)\}e^{imax\{2\pi - \omega_\psi(d,q), 2\pi - \omega_\psi(k,q)\}}$$
$$= max\{r_{\psi^c}(d,q), r_{\psi^c}(k,q)\}e^{imax\{\omega_{\psi^c}(d,q),\omega_{\psi^c}(k,q)\}}$$
$$= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)}, r_{\psi^c}(k,q)e^{i\omega_{\psi^c}(k,q)}\}$$
$$= max\{\mu_{\psi^c}(d,q), \mu_{\psi^c}(k,q)\}.$$

Further,

$$\mu_{\psi^c}(d^{-1},q) = r_{\psi^c}(d^{-1},q)e^{i\omega_{\psi^c}(d^{-1},q)}$$
$$= \{1 - r_\psi(d^{-1},q)\}e^{i\{2\pi - \omega_\psi(d^{-1},q)\}}$$
$$\leq \{1 - r_\psi(d,q)\}e^{i\{2\pi - \omega_\psi(d,q)\}}$$
$$= r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)}$$
$$= \mu_{\psi^c}(d,q).$$

So, $\mu_{\psi^c}(d,q)$ and $\gamma_\psi(d,q)$ are a complex anti-Q-fuzzy subfield of $(F,+,\cdot)$.
Conversely, let $\mu_{\psi^c}(d,q)$ and $\gamma_\psi(d,q)$ are a complex anti-Q-fuzzy subfield of $(F,+,\cdot)$. To prove that $\psi = \{\langle (d,q), \mu_\psi(d,q), \gamma_\psi(d,q)\rangle : d \in F, q \in Q\}$ be a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$, we must prove that ψ satisfies all conditions of a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$. It is clear that $\gamma_\psi(d-k,q) \leq max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}$, $\gamma_\psi(dk,q) \leq max\{\gamma_\psi(d,q), \gamma_\psi(k,q)\}$, and $\gamma_\psi(d^{-1},q) \leq \gamma_\psi(d,q)$ are satisfied.
Now,

$$\mu_{\psi^c}(d-k,q) \leq max\{\mu_{\psi^c}(d,q), \mu_{\psi^c}(k,q)\}$$
$$= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c}(d,q)}, r_{\psi^c}(k,q)e^{i\omega_{\psi^c}(k,q)}\}$$
$$= max\{1 - r_\psi(d,q), 1 - r_\psi(k,q)\}e^{imax\{2\pi - \omega_\psi(d,q), 2\pi - \omega_\psi(k,q)\}}$$
$$= (1 - min\{r_\psi(d,q), r_\psi(k,q)\})e^{i(2\pi - min\{\omega_\psi(d,q),\omega_\psi(k,q)\})}.$$

Then,

$$\mu_\psi(d-k,q) \geq min\{r_\psi(d,q), r_\psi(k,q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(k,q)\}}$$
$$= min\{r_\psi(d,q)e^{i\omega_\psi(d,q)}, r_\psi(k,q)e^{i\omega_\psi(k,q)}\}$$
$$= min\{\mu_\psi(d,q), \mu_\psi(k,q)\}.$$

Also,

$$\mu_{\psi^c}(dk,q) \leq max\{\mu_{\psi^c}(d,q),\mu_{\psi^c}(k,q)\}$$
$$= max\{r_{\psi^c}(d,q)e^{i\omega_{\psi^c(d,q)}}, r_{\psi^c}(k,q)e^{i\omega_{\psi^c(k,q)}}\}$$
$$= max\{1-r_\psi(d,q),1-r_\psi(k,q)\}e^{imax\{2\pi-\omega_\psi(d,q),2\pi-\omega_\psi(k,q)\}}$$
$$= \left(1-min\{r_\psi(d,q),r_\psi(k,q)\}\right)e^{i(2\pi-min\{\omega_\psi(d,q),\omega_\psi(k,q)\})}.$$

Then,

$$\mu_\psi(dk,q) \geq min\{r_\psi(d,q),r_\psi(k,q)\}e^{imin\{\omega_\psi(d,q),\omega_\psi(k,q)\}}$$
$$= min\{r_\psi(d,q)e^{i\omega_\psi(d,q)}, r_\psi(k,q)e^{i\omega_\psi(k,q)}\}$$
$$= min\{\mu_\psi(d,q),\mu_\psi(k,q)\}.$$

Also,

$$\mu_{\psi^c}(d^{-1},q) \leq \mu_{\psi^c}(d,q)$$
$$= r_{\psi^c}(d,q)e^{i\omega_{\psi^c(d,q)}}$$
$$= (1-r_\psi(d,q))e^{i(2\pi-\omega_\psi(d,q))}.$$

Then,

$$\mu_\psi(d^{-1},q) \geq r_\psi(d,q) = \mu_\psi(d,q).$$

Therefore, $\psi(d) = \{\langle (d,q),\mu_\psi(d,q),\gamma_\psi(d,q)\rangle : d \in F, q \in Q\}$ is a complex intuitionistic Q-fuzzy subfield of $(F,+,\cdot)$.

Definition 3.5. *Let $(F,+,\cdot)$.be a field. Let ψ_1 and ψ_2 be any two complex intuitionistic Q-fuzzy subfields of $(F,+,\cdot)$ where $\psi_1 = \{\langle (d,q),\mu_{\psi_1}(d,q),\gamma_{\psi_1}(d,q)\rangle : d \in F, q \in Q\}$ and $\psi_2 = \{\langle (d,q),\mu_{\psi_2}(d,q),\gamma_{\psi_2}(d,q)\rangle : d \in F, q \in Q\}$. Then their intersection is defined as:*

$$\psi_1 \cap \psi_2 = \{\langle (d,q),\mu_{\psi_1\cap\psi_2}(d,q),\gamma_{\psi_1\cap\psi_2}(d,q)\rangle : d \in F, q \in Q\}$$

where

$$\mu_{\psi_1\cap\psi_2}(d,q) = min\{r_{\psi_1}(d,q),r_{\psi_2}(d,q)\}e^{imin\{\omega_{\psi_1}(d,q),\omega_{\psi_2}(d,q)\}},$$
$$\gamma_{\psi_1\cap\psi_2}(d,q) = max\{\hat{r}_{\psi_1}(d,q),\hat{r}_{\psi_2}(d,q)\}e^{imax\{\widehat{\omega}_{\psi_1}(d,q),\widehat{\omega}_{\psi_2}(d,q)\}}.$$

Definition 3.6. *Let $(F,+,\cdot)$ be a field. Let ψ_1 and ψ_2 be any two complex intuitionistic Q-fuzzy subfields of $(F,+,\cdot)$ where $\psi_1 = \{\langle (d,q),\mu_{\psi_1}(d,q),\gamma_{\psi_1}(d,q)\rangle : d \in F, q \in Q\}$ and $\psi_2 = \{\langle (d,q),\mu_{\psi_2}(d,q),\gamma_{\psi_2}(d,q)\rangle : d \in F, q \in Q\}$. Then their union is defined as:*

$$\psi_1 \cup \psi_2 = \{\langle (d,q),\mu_{\psi_1\cup\psi_2}(d,q),\gamma_{\psi_1\cup\psi_2}(d,q)\rangle : d \in F, q \in Q\}$$

where

$$\mu_{\psi_1\cup\psi_2}(d,q) = max\{r_{\psi_1}(d,q),r_{\psi_2}(d,q)\}e^{imax\{\omega_{\psi_1}(d,q),\omega_{\psi_2}(d,q)\}},$$
$$\gamma_{\psi_1\cup\psi_2}(d,q) = min\{\hat{r}_{\psi_1}(d,q),\hat{r}_{\psi_2}(d,q)\}e^{imin\{\widehat{\omega}_{\psi_1}(d,q),\widehat{\omega}_{\psi_2}(d,q)\}}.$$

Theorem 3.9. *Let* $(F,+,\cdot)$ *be a field. If* ψ_1 *and* ψ_2 *be two complex intuitionistic* Q*-fuzzy subfields of* $(F,+,\cdot)$*, then* $\psi_1\cap\psi_2$ *is a complex intuitionistic* Q*-fuzzy subfield of* $(F,+,\cdot)$*.*

Proof. Let $\psi_1=\{\langle(d,q),\mu_{\psi_1}(d,q),\gamma_{\psi_1}(d,q)\rangle: d\in F, q\in Q\}$ and $\psi_2=\{\langle(d,q),\mu_{\psi_2}(d,q),\gamma_{\psi_2}(d,q)\rangle: d\in F, q\in Q\}$ be two complex intuitionistic Q-fuzzy subfields of $(F,+,\cdot)$. To prove that $\psi_1\cap\psi_2$ is also a complex intuitionistic Q-fuzzy subfield, we must show that $\psi_1\cap\psi_2$ satisfies all conditions of complex intuitionistic Q-fuzzy subfield for all $d,k\in F, q\in Q$. Note that $r_{\psi_1}(d,q), r_{\psi_2}(d,q), \hat{r}_{\psi_1}(d,q)$ and $\hat{r}_{\psi_2}(d,q)$ are intuitionistic Q-fuzzy subfields and $\omega_{\psi_1}(d,q), \omega_{\psi_2}(d,q), \widehat{\omega}_{\psi_1}(d,q)$, and $\widehat{\omega}_{\psi_2}(d,q)$ are intuitionistic π - Q -fuzzy subfields by Theorem 3.2. Then $r_{\psi_1\cap\psi_2}(d,q), r_{\psi_1\cap\psi_2}(d,q)$ are intuitionistic Q -fuzzy subfields and $\omega_{\psi_1\cap\psi_2}(d,q), \omega_{\psi_1\cap\psi_2}(d,q)$ are intuitionistic π-Q-fuzzy subfields.

Now, let $d,k\in F, q\in Q$. Then,

$$\begin{aligned}\mu_{\psi_1\cap\psi_2}(d-k,q) &= r_{\psi_1\cap\psi_2}(d-k,q)e^{i\omega_{\psi_1\cap\psi_2}(d-k,q)}\\ &\geq min\{r_{\psi_1\cap\psi_2}(d,q), r_{\psi_1\cap\psi_2}(k,q)\}e^{imin\{\omega_{\psi_1\cap\psi_2}(d,q),\omega_{\psi_1\cap\psi_2}(k,q)\}}\\ &= min\{r_{\psi_1\cap\psi_2}(d,q)e^{i\omega_{\psi_1\cap\psi_2}(d,q)}, r_{\psi_1\cap\psi_2}(k,q)e^{i\omega_{\psi_1\cap\psi_2}(k,q)}\}.\\ &= min\{\mu_{\psi_1\cap\psi_2}(d,q), \mu_{\psi_1\cap\psi_2}(k,q)\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\mu_{\psi_1\cap\psi_2}(dk,q) &= r_{\psi_1\cap\psi_2}(dk,q)e^{i\omega_{\psi_1\cap\psi_2}(dk,q)}\\ &\geq min\{r_{\psi_1\cap\psi_2}(d,q), r_{\psi_1\cap\psi_2}(k,q)\}e^{imin\{\omega_{\psi_1\cap\psi_2}(d,q),\omega_{\psi_1\cap\psi_2}(k,q)\}}\\ &= min\{r_{\psi_1\cap\psi_2}(d,q)e^{i\omega_{\psi_1\cap\psi_2}(d,q)}, r_{\psi_1\cap\psi_2}(k,q)e^{i\omega_{\psi_1\cap\psi_2}(k,q)}\}.\\ &= min\{\mu_{\psi_1\cap\psi_2}(d,q), \mu_{\psi_1\cap\psi_2}(k,q)\}.\end{aligned}$$

Also,

$$\begin{aligned}\mu_{\psi_1\cap\psi_2}(d^{-1},q) &= r_{\psi_1\cap\psi_2}(d^{-1},q)e^{i\omega_{\psi_1\cap\psi_2}(d^{-1},)}\\ &\geq r_{\psi_1\cap\psi_2}(d,q)e^{i\omega_{\psi_1\cap\psi_2}(d,q)}\\ &= \mu_{\psi_1\cap\psi_2}(d,q).\end{aligned}$$

Further,

$$\begin{aligned}\gamma_{\psi_1\cap\psi_2}(d-k,q) &= \hat{r}_{\psi_1\cap\psi_2}(d-k,q)e^{i\widehat{\omega}_{\psi_1\cap\psi_2}(d-k,q)}\\ &\leq max\{\hat{r}_{\psi_1\cap\psi_2}(d,q), \hat{r}_{\psi_1\cap\psi_2}(k,q)\}e^{imax\{\widehat{\omega}_{\psi_1\cap\psi_2}(d,q),\widehat{\omega}_{\psi_1\cap\psi_2}(k,q)\}}\\ &= max\{\hat{r}_{\psi_1\cap\psi_2}(d,q)e^{i\widehat{\omega}_{\psi_1\cap\psi_2}(d,q)}, \hat{r}_{\psi_1\cap\psi_2}(k,q)e^{i\widehat{\omega}_{\psi_1\cap\psi_2}(k,q)}\}.\\ &= max\{\gamma_{\psi_1\cap\psi_2}(d,q), \gamma_{\psi_1\cap\psi_2}(k,q)\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\gamma_{\psi_1\cap\psi_2}(dk,q) &= \hat{r}_{\psi_1\cap\psi_2}(dk,q)e^{i\widehat{\omega}_{\psi_1\cap\psi_2}(dk,q)}\\ &\leq max\{\hat{r}_{\psi_1\cap\psi_2}(d,q), \hat{r}_{\psi_1\cap\psi_2}(k,q)\}e^{imax\{\widehat{\omega}_{\psi_1\cap\psi_2}(d,q),\widehat{\omega}_{\psi_1\cap\psi_2}(k,q)\}}\end{aligned}$$

$$= max\{\hat{r}_{\psi_1\cap\psi_2}(d,q)e^{i\widehat{\omega}_{\psi_1\cap\psi_2}(d,q)}, \hat{r}_{\psi_1\cap\psi_2}(k,q)e^{i\widehat{\omega}_{\psi_1\cap\psi_2}(k,q)}\}$$

$$= max\{\gamma_{\psi_1\cap\psi_2}(d,q), \gamma_{\psi_1\cap\psi_2}(k,q)\}.$$

Also,

$$\gamma_{\psi_1\cap\psi_2}(d^{-1},q) = \hat{r}_{\psi_1\cap\psi_2}(d^{-1},q)e^{i\widehat{\omega}_{\psi_1\cap\psi_2}(d^{-1},q)}$$

$$\leq \hat{r}_{\psi_1\cap\psi_2}(d,q)e^{i\omega_{\psi_1\cap\psi_2}(d,q)}$$

$$= \gamma_{\psi_1\cap\psi_2}(d,q).$$

Hence, the intersection of two complex intuitionistic Q-fuzzy subfields is a complex intuitionistic Q-fuzzy subfield.

Theorem 3.10. *Let $\{\psi_i : i \in I\}$ represent a set of complex intuitionistic Q-fuzzy subfields of a field F. Then $\cap_{i\in I} \psi_i$ is a complex intuitionistic Q-fuzzy subfield.*

Proof. The proof is straightforward.

Remark 3.1. *The union of two complex intuitionistic Q-fuzzy subfields of a field F may not be a complex intuitionistic Q-fuzzy subfield of a field F.*

Example 3.1. *Let $F = \mathbb{Z}_{11}$ be a field under ordinary addition and multiplication of integers where $\mathbb{Z}_{11} = \{0,1,2,\ldots,10\}$ is the set of integers modulo 11.*

Suppose that ψ_1 and ψ_2 are two complex intuitionistic Q-fuzzy subfields of a field $\mathbb{Z}_{11}$ where $q \in Q$ and defined as:

$\mu_{\psi_1}(d,q) = \begin{cases} 0.2e^{i\frac{\pi}{2}} & , if\ d \in 3\mathbb{Z}_{11} \\ 0 & , otherwise \end{cases}$ and $\gamma_{\psi_1}(d,q) = \begin{cases} 0.1e^{i\frac{\pi}{9}} & , if\ d \in 3\mathbb{Z}_{11} \\ 0.5e^{i\frac{\pi}{3}} & , otherwise \end{cases}$

$\mu_{\psi_2}(d,q) = \begin{cases} 0.1e^{i\frac{\pi}{3}} & , if\ d \in 2\mathbb{Z}_{11} \\ 0.01e^{i\frac{\pi}{8}} & , otherwise \end{cases}$ and $\gamma_{\psi_2}(d,q) = \begin{cases} 0.3e^{i\frac{\pi}{8}} & , if\ d \in 2\mathbb{Z}_{11} \\ 0.4e^{i\frac{\pi}{4}} & , otherwise \end{cases}$.

It is possible to verify that ψ_1 and ψ_2 are complex intuitionistic Q-fuzzy subfields of a field $\mathbb{Z}_{11}$. From Definition 3.6 $\psi_1 \cup \psi_2 = \{\langle (d,q), \mu_{\psi_1\cup\psi_2}(d,q), \gamma_{\psi_1\cup\psi_2}(d,q)\rangle : d \in F, q \in Q\}$. Then,

$$\mu_{\psi_1\cup\psi_2}(d,q) = \begin{cases} 0.2e^{i\frac{\pi}{2}} & , if\ d \in 3\mathbb{Z}_{11} \\ 0.1e^{i\frac{\pi}{3}} & , if\ d \in 2\mathbb{Z}_{11} - 3\mathbb{Z}_{11} \\ 0.01e^{i\frac{\pi}{8}} & , otherwise \end{cases}$$

and

$$\gamma_{\psi_1\cup\psi_2}(d,q) = \begin{cases} 0.1e^{i\frac{\pi}{10}} & , if\ d \in 3\mathbb{Z}_{11} \\ 0.3e^{i\frac{\pi}{8}} & , if\ d \in 2\mathbb{Z}_{11} - 3\mathbb{Z}_{11} \\ 0.4e^{i\frac{\pi}{4}} & , otherwise \end{cases}.$$

Let $d = 3$ and $k = 2$. Then $\mu_{\psi_1\cup\psi_2}(3,q) = 0.2e^{i\frac{\pi}{2}}$, $\mu_{\psi_1\cup\psi_2}(2,q) = 0.1e^{i\frac{\pi}{3}}$, $\mu_{\psi_1\cup\psi_2}(3-2,q) = \mu_{\psi_1\cup\psi_2}(1,q) = 0.01e^{i\frac{\pi}{8}}$, and $min\{\mu_{\psi_1\cup\psi_2}(3,q), \mu_{\psi_1\cup\psi_2}(2,q)\} = min\left\{0.2e^{i\frac{\pi}{2}}, 0.1e^{i\frac{\pi}{3}},\right\} = 0.1e^{i\frac{\pi}{3}}$. We note that $\mu_{\psi_1\cup\psi_2}(3-2,q) < min\{\mu_{\psi_1\cup\psi_2}(3,q), \mu_{\psi_1\cup\psi_2}(2,q)\}$. This means that $\psi_1 \cup \psi_2$ does not satisfy one of the conditions of the complex intuitionistic Q-fuzzy subfield.

Therefore, the union of two complex intuitionistic Q-fuzzy subfields of a field F may not be a complex intuitionistic Q-fuzzy subfield of a field.

Definition 3.7. *Let fields F_1 and F_2, have two complex intuitionistic Q-fuzzy subfields ψ_1 and ψ_2 such that $\psi_1 = \{ \langle (d,q), \mu_{\psi_1}(d,q), \gamma_{\psi_1}(d,q) \rangle : d \in F_1, q \in Q\}$ and $\psi_2 = \{ \langle (d,q), \mu_{\psi_2}(d,q), \gamma_{\psi_2}(d,q) \rangle : d \in F_2, q \in Q\}$. Then their direct product is denoted by $\psi_1 \times \psi_2$ and defined as:*

$$(\psi_1 \times \psi_2)\big((d,k),q\big) = \left\{ \langle \big((d,k),q\big), \mu_{\psi_1 \times \psi_2}\big((d,k),q\big), \gamma_{\psi_1 \times \psi_2}\big((d,k),q\big) \rangle : (d,k) \in F_1 \times F_2, q \in Q\right\}$$

where

$$\mu_{\psi_1 \times \psi_2}\big((d,k),q\big) = r_{\psi_1 \times \psi_2}\big((d,k),q\big) e^{i\omega_{\psi_1 \times \psi_2}((d,k),q)} = min\{r_{\psi_1}(d,q), r_{\psi_2}(k,q)\} e^{i\, min\{\omega_{\psi_1}(d,q), \omega_{\psi_2}(k,q)\}},$$

$$\gamma_{\psi_1 \times \psi_2}\big((d,k),q\big) = \hat{r}_{\psi_1 \times \psi_2}\big((d,k),q\big) e^{i\hat{\omega}_{\psi_1 \times \psi_2}((d,k),q)} = max\{\hat{r}_{\psi_1}(d,q), \hat{r}_{\psi_2}(k,q)\} e^{i\, max\{\hat{\omega}_{\psi_1}(d,q), \hat{\omega}_{\psi_2}(k,q)\}}.$$

Theorem 3.11. *Let fields F_1 and F_2, have two complex intuitionistic Q-fuzzy subfields ψ_1 and ψ_2. Then $\psi_1 \times \psi_2$ is a complex intuitionistic fuzzy subfield of a field $F_1 \times F_2$.*

Proof. Let $\psi_1 = \{ \langle (d,q), \mu_{\psi_1}(d,q), \gamma_{\psi_1}(d,q) \rangle : d \in F_1, q \in Q\}$ and $\psi_2 = \{ \langle (d,q), \mu_{\psi_2}(d,q), \gamma_{\psi_2}(d,q) \rangle : d \in F_2, , q \in Q\}$ be two complex intuitionistic Q-fuzzy subfields of fields F_1 and F_2, respectively. We must demonstrate that $\psi_1 \times \psi_2$ satisfies all conditions of a complex intuitionistic Q-fuzzy subfield to establish that $\psi_1 \times \psi_2$ is a complex intuitionistic Q-fuzzy subfield. For each $(d,k),(a,b) \in F_1 \times F_2$ and $q \in Q$, we have

$$\begin{aligned}
\mu_{\psi_1 \times \psi_2}\big((d,k)-(a,b),q\big) &= \mu_{\psi_1 \times \psi_2}\big((d-a,k-b),q\big) \\
&= r_{\psi_1 \times \psi_2}\big((d-a,k-b),q\big) e^{i\omega_{\psi_1 \times \psi_2}((d-a,k-b),q)} \\
&= min\{r_{\psi_1}(d-a,q), r_{\psi_2}(k-b,q)\} e^{i\, min\{\omega_{\psi_1}(d-a,q), \omega_{\psi_2}(k-b,q)\}} \\
&= min\{r_{\psi_1}(d-a,q) e^{i\omega_{\psi_1}(d-a,q)}, r_{\psi_2}(k-b,q) e^{i\omega_{\psi_2}(k-b,q)}\} \\
&= min\{\mu_{\psi_1}(d-a,q), \mu_{\psi_2}(k-b,q)\} \\
&\geq min\left\{min\{\mu_{\psi_1}(d,q), \mu_{\psi_1}(a,q)\}, min\{\mu_{\psi_2}(k,q), \mu_{\psi_2}(b,q)\}\right\} \\
&= min\left\{min\{\mu_{\psi_1}(d,q), \mu_{\psi_2}(k,q)\}, min\{\mu_{\psi_1}(a,q), \mu_{\psi_2}(b,q)\}\right\} \\
&= min\{\mu_{\psi_1 \times \psi_2}\big((d,k),q\big), \mu_{\psi_1 \times \psi_2}\big((a,b),q\big)\}.
\end{aligned}$$

Hence, $\mu_{\psi_1 \times \psi_2}\big((d,k)-(a,b),q\big) \geq min\{\mu_{\psi_1 \times \psi_2}\big((d,k),q\big), \mu_{\psi_1 \times \psi_2}\big((a,b),q\big)\}$.

Moreover,

$$\begin{aligned}
\mu_{\psi_1 \times \psi_2}\big((d,k)(a,b),q\big) &= \mu_{\psi_1 \times \psi_2}\big((da,kb),q\big) \\
&= r_{\psi_1 \times \psi_2}\big((da,kb),q\big) e^{i\omega_{\psi_1 \times \psi_2}((da,kb),q)}
\end{aligned}$$

$$= min\{r_{\psi_1}(da,q), r_{\psi_2}(kb,q)\}e^{i\, min\{\omega_{\psi_1}(da,q),\omega_{\psi_2}(kb,q)\}}$$

$$= min\{r_{\psi_1}(da,q)e^{i\omega_{\psi_1}(da,q)}, r_{\psi_2}(kb,q)e^{i\omega_{\psi_2}(kb,q)}\}$$

$$= min\{\mu_{\psi_1}(da,q), \mu_{\psi_2}(kb,q)\}$$

$$\geq min\left\{min\{\mu_{\psi_1}(d,q), \mu_{\psi_1}(a,q)\}, min\{\mu_{\psi_2}(k,q), \mu_{\psi_2}(b,q)\}\right\}$$

$$= min\left\{min\{\mu_{\psi_1}(d,q), \mu_{\psi_2}(k,q)\}, min\{\mu_{\psi_1}(a,q), \mu_{\psi_2}(b,q)\}\right\}$$

$$= min\{\mu_{\psi_1\times\psi_2}\big((d,k),q\big), \mu_{\psi_1\times\psi_2}\big((a,b),q\big)\}.$$

Hence, $\mu_{\psi_1\times\psi_2}\big((d,k)(a,b),q\big) \geq min\{\mu_{\psi_1\times\psi_2}\big((d,k),q\big), \mu_{\psi_1\times\psi_2}\big((a,b),q\big)\}$.
Furthermore,

$$\mu_{\psi_1\times\psi_2}\big((d^{-1},k^{-1}),q\big) = r_{\psi_1\times\psi_2}\big((d^{-1},k^{-1}),q\big)e^{i\omega_{\psi_1\times\psi_2}\left((d^{-1},k^{-1}),q\right)}$$

$$= min\{r_{\psi_1}(d^{-1},q), r_{\psi_2}(k^{-1},q)\}e^{i\, min\{\omega_{\psi_1}(d^{-1},q),\omega_{\psi_2}(k^{-1},q)\}}$$

$$= min\left\{r_{\psi_1}(d^{-1},q)e^{i\omega_{\psi_1}(d^{-1},q)}, r_{\psi_2}(k^{-1},q)e^{i\omega_{\psi_2}(k^{-1},q)}\right\}$$

$$= min\{\mu_{\psi_1}(d^{-1},q), \mu_{\psi_2}(k^{-1},q)\}$$

$$\geq min\{\mu_{\psi_1}(d,q), \mu_{\psi_2}(k,q)\}.$$

Hence, $\mu_{\psi_1\times\psi_2}\big((d^{-1},k^{-1}),q\big) \geq \mu_{\psi_1\times\psi_2}\big((d,k),q\big)$.
On the other hand,

$$\gamma_{\psi_1\times\psi_2}\big((d,k)-(a,b),q\big) = \gamma_{\psi_1\times\psi_2}\big((d-a,k-b),q\big)$$

$$= \hat{r}_{\psi_1\times\psi_2}\big((d-a,k-b),q\big)e^{i\widehat{\omega}_{\psi_1\times\psi_2}((d-a,k-b),q)}$$

$$= max\{\hat{r}_{\psi_1}(d-a,q), \hat{r}_{\psi_2}(k-b,q)\}e^{i\, max\{\widehat{\omega}_{\psi_1}(d-a,q),\widehat{\omega}_{\psi_2}(k-b,q)\}}$$

$$= max\{\hat{r}_{\psi_1}(d-a,q)e^{i\widehat{\omega}_{\psi_1}(d-a,q)}, \hat{r}_{\psi_2}(k-b,q)e^{i\widehat{\omega}_{\psi_2}(k-b,q)}\}$$

$$= max\{\gamma_{\psi_1}(d-a,q), \gamma_{\psi_2}(k-b,q)\}$$

$$\leq max\left\{max\{\gamma_{\psi_1}(d,q), \gamma_{\psi_1}(a,q)\}, max\{\gamma_{\psi_2}(k,q), \gamma_{\psi_2}(b,q)\}\right\}$$

$$= max\left\{max\{\gamma_{\psi_1}(d,q), \gamma_{\psi_2}(k,q)\}, max\{\gamma_{\psi_1}(a,q), \gamma_{\psi_2}(b,q)\}\right\}$$

$$= max\{\gamma_{\psi_1\times\psi_2}\big((d,k),q\big), \gamma_{\psi_1\times\psi_2}\big((a,b),q\big)\}.$$

Hence, $\gamma_{\psi_1\times\psi_2}\big((d,k)-(a,b),q\big) \leq max\{\gamma_{\psi_1\times\psi_2}\big((d,k),q\big), \gamma_{\psi_1\times\psi_2}\big((a,b),q\big)\}$.
Also,

$$\gamma_{\psi_1\times\psi_2}\big((d,k)(a,b),q\big) = \gamma_{\psi_1\times\psi_2}\big((da,kb),q\big)$$

$$= \hat{r}_{\psi_1\times\psi_2}\big((da,kb),q\big)e^{i\widehat{\omega}_{\psi_1\times\psi_2}((da,kb),q)}$$

$$= max\{\hat{r}_{\psi_1}(da,q), \hat{r}_{\psi_2}(kb,q)\}e^{i\, max\{\widehat{\omega}_{\psi_1}(da,q),\widehat{\omega}_{\psi_2}(kb,q)\}}$$

$$= max\{\hat{r}_{\psi_1}(da,q)e^{i\widehat{\omega}_{\psi_1}(da,q)}, \hat{r}_{\psi_2}(kb,q)e^{i\widehat{\omega}_{\psi_2}(kb,q)}\}$$

$$= max\{\gamma_{\psi_1}(da,q), \gamma_{\psi_2}(kb,q)\}$$

$$\leq max\left\{max\{\gamma_{\psi_1}(d,q), \gamma_{\psi_1}(a,q)\}, max\{\gamma_{\psi_2}(k,q), \gamma_{\psi_2}(b,q)\}\right\}$$

$$= max\left\{max\{\gamma_{\psi_1}(d,q), \gamma_{\psi_2}(k.q)\}, max\{\gamma_{\psi_1}(a,q), \gamma_{\psi_2}(b,q)\}\right\}$$

$$= max\{\gamma_{\psi_1\times\psi_2}((d,k),q), \gamma_{\psi_1\times\psi_2}((a,b),q)\}.$$

Hence, $\gamma_{\psi_1\times\psi_2}((d,k)(a,b),q) \leq max\{\gamma_{\psi_1\times\psi_2}((d,k),q), \gamma_{\psi_1\times\psi_2}((a,b),q)\}$.

Further,

$$\gamma_{\psi_1\times\psi_2}((d^{-1},k^{-1}),q) = \hat{r}_{\psi_1\times\psi_2}((d^{-1},k^{-1}),q)e^{i\widehat{\omega}_{\psi_1\times\psi_2}((d^{-1},k^{-1}),q)}$$

$$= max\{\hat{r}_{\psi_1}(d^{-1},q), \hat{r}_{\psi_2}(k^{-1},q)\}e^{i\, max\{\widehat{\omega}_{\psi_1}(d^{-1},q),\widehat{\omega}_{\psi_2}(k^{-1},q)\}}$$

$$= max\left\{\hat{r}_{\psi_1}(d^{-1},q)e^{i\widehat{\omega}_{\psi_1}(d^{-1},q)}, \hat{r}_{\psi_2}(k^{-1},q)e^{i\widehat{\omega}_{\psi_2}(k^{-1},q)}\right\}$$

$$= max\{\gamma_{\psi_1}(d^{-1},q), \gamma_{\psi_2}(k^{-1},q)\}$$

$$\leq max\{\gamma_{\psi_1}(d,q), \gamma_{\psi_2}(k,q)\}.$$

Hence, $\gamma_{\psi_1\times\psi_2}((d^{-1},k^{-1}),q) \leq \gamma_{\psi_1\times\psi_2}((d,k),q)$.

Therefore, $\psi_1 \times \psi_2$ is a complex intuitionistic Q-fuzzy subfield of a field $F_1 \times F_2$.

Corollary 3.1. Let $\psi_1, \psi_2, \psi_3, \dots, \psi_n$ be a complex intuitionistic Q-fuzzy subfield of fields $F_1, F_2, F_2, \dots, F_n$. Then $\psi_1 \times \psi_2 \times \psi_3 \times \dots \times \psi_n$ is a complex intuitionistic Q-fuzzy subfield of a field $F_1 \times F_2 \times F_2 \times \dots \times F_n$.

4. Conclusions

This study proposed a concept of a complex intuitionistic Q-fuzzy subfield as a new structure. We extend the concept of a complex fuzzy subfield to a complex intuitionistic Q-fuzzy subfield by adding the idea of the intuitionistic Q-fuzzy set to a complex fuzzy subfield. This valuable contribution expands the investigation from membership function values to include both membership and non-membership function values. In the complex plane, the range of complex fuzzy subfields is extended to the unit disc for membership and non-membership functions. Furthermore, we suggested basic operations such as intersection, union, and complement and investigated some properties of these operations. In addition, we introduce the necessity operator and possibility operator on a complex intuitionistic Q-fuzzy subfield. Some important and related theorems have been studied.

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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