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Research article

Local Lie derivations of generalized matrix algebras

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Abstract: In this paper, we investigate local Lie derivations of a certain class of generalized matrix algebras and show that, under certain conditions, every local Lie derivation of a generalized matrix algebra is a sum of a derivation and a linear central-valued map vanishing on each commutator. The main result is then applied to full matrix algebras and unital simple algebras with nontrivial idempotents.

Keywords: local Lie derivation; Lie derivation; generalized matrix algebra; full matrix algebra
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1. Introduction and main results

A well-known and active direction in the study of derivations is the local derivations problem, which was initiated by Kadison [8] and Larson and Sourour [9]. Recall that a linear map φ of an algebra A is called a local derivation if for each $x \in A$, there exists a derivation φ_x of A, depending on x, such that $\varphi(x) = \varphi_x(x)$. The question of determining under what conditions every local derivation must be a derivation has been studied by many authors (see [4, 6, 7, 13, 15]). Recently, Brešar [2] proved that each local derivation of algebras generated by all their idempotents is a derivation.

A linear map φ of an algebra A is called a Lie derivation if $\varphi([x, y]) = [\varphi(x), y] + [x, \varphi(y)]$ for all $x, y \in A$, where $[x, y] = xy - yx$ is the usual Lie product, also called a commutator. A Lie derivation φ of A is standard if it can be decomposed as $\varphi = d + \tau$, where d is a derivation from A into itself and τ is a linear map from A into its center vanishing on each commutator. The classical problem, which has been studied for many years, is to find conditions on A under which each Lie derivation is standard or standard-like. We say that a linear map φ from A into itself is a local Lie derivation if for each $x \in A$, there exists a Lie derivation φ_x of A such that $\varphi(x) = \varphi_x(x)$. In [3], Chen et al. studied local Lie derivations of operator algebras on Banach spaces. We remark that the methods in [3] depend heavily on rank one operators in $B(X)$. Later, Liu and Zhang [10] proved that each local Lie derivation of factor von Neumann algebras is a Lie derivation. Liu and Zhang [11] investigated local Lie derivations of a certain class of operator algebras. An et al. [1] proved that every local Lie derivation on von Neumann algebras is a Lie derivation.

It is quite common to study local derivations in algebras that contain many idempotents, in the sense that the linear span of all idempotents is 'large'. The main novelty of this paper is that we shall deal with the subalgebra generated by all idempotents instead of their span. Let $\mathcal{M}_2$ be the algebra of 2×2 matrices over $L^\infty[0,1]$. By [6], $\mathcal{M}_2$ is generated by, but not spanned by, its idempotents. In what follows, we denote by $\mathcal{J}(A)$ the subalgebra of A generated by all idempotents in A. The purpose of the present paper is to study local Lie derivations of a certain class of generalized matrix algebras. Finally we apply the main result to full matrix algebras and unital simple algebras with nontrivial idempotents.

Let A and B be two unital algebras with unit elements 1_A and 1_B, respectively. A Morita context consists of A, B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \otimes_B N \to A$ and $\Psi_{NM} : N \otimes_A M \to B$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} M \otimes_B N \otimes_A M & \xrightarrow{\Phi_{MN}\otimes I_M} & A \otimes_A M \\ {\scriptstyle I_M\otimes\Psi_{NM}}\big\downarrow & & \cong\big\downarrow \\ M \otimes_B B & \xrightarrow{\cong} & M \end{array}$$

and

$$\begin{array}{ccc} N \otimes_A M \otimes_B N & \xrightarrow{\Psi_{NM}\otimes I_N} & B \otimes_B N \\ {\scriptstyle I_N\otimes\Phi_{MN}}\big\downarrow & & \cong\big\downarrow \\ N \otimes_A A & \xrightarrow{\cong} & N \end{array}$$

If $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\mathcal{G} = \begin{pmatrix} A & M \\ N & B \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mid a \in A, m \in M, n \in N, b \in B \right\}$$

forms an algebra under matrix-like addition and multiplication. Such an algebra is called a *generalized matrix algebras*. We further assume that M is faithful as an (A, B)-bimodule. The most common examples of generalized matrix algebras are full matrix algebras and triangular algebras.

Consider algebra $\mathcal{G}$. Any element of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{G}$$

will be denoted by $a \oplus b$. Let us define two natural projections $\pi_A : \mathcal{G} \to A$ and $\pi_B : \mathcal{G} \to B$ by

$$\pi_A : \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mapsto b.$$

The center of $\mathcal{G}$ is

$$Z(\mathcal{G}) = \{a \oplus b \mid am = mb, na = bn \text{ for all } m \in M, n \in N\}.$$

Furthermore, $\pi_A(Z(\mathcal{G})) \subseteq Z(A)$ and $\pi_B(Z(\mathcal{G})) \subseteq Z(B)$, and there exists a unique algebra isomorphism η from $\pi_B(Z(\mathcal{G}))$ to $\pi_A(Z(\mathcal{G}))$ such that $\eta(b)m = mb$ and $n\eta(b) = bn$ for all $m \in M, n \in N$ (see [14]). Set

$$e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}.$$

We immediately notice that e and f are orthogonal idempotents of $\mathcal{G}$ and so $\mathcal{G}$ may be represented as $\mathcal{G} = (e+f)\mathcal{G}(e+f) = e\mathcal{G}e + e\mathcal{G}f + f\mathcal{G}e + f\mathcal{G}f$. Then each element $x = exe + exf + fxe + fxf \in \mathcal{G}$ can be represented in the form $x = eae + emf + fne + fbf = a + m + n + b$, where $a \in A, b \in B, m \in M, n \in N$.

We close this section with a well known result concerning Lie derivations.

Proposition 1.1. *(See* [5],Theorem 1 *) Let $\mathcal{G}$ be a generalized matrix algebra. Suppose that*
(1) $Z(A) = \pi_A(Z(\mathcal{G}))$ and $Z(B) = \pi_B(Z(\mathcal{G}))$;
(2) either A or B does not contain nonzero central ideals.
Then every Lie derivation $\varphi : \mathcal{G} \to \mathcal{G}$ is standard, that is, φ is the sum of a derivation d and a linear central-valued map τ vanishing on each commutator.

2. Main results

Our main result reads as follows.

Theorem 2.1. *Let $\mathcal{G}$ be a generalized matrix algebra. Suppose that*
(1) $A = \mathcal{J}(A)$ and $B = \mathcal{J}(B)$;
(2) $Z(A) = \pi_A(Z(\mathcal{G}))$ and $Z(B) = \pi_B(Z(\mathcal{G}))$;
(3) either A or B does not contain nonzero central ideals.
Then every local Lie derivation φ from $\mathcal{G}$ into itself is a sum of a derivation δ and a linear central-valued map h vanishing on each commutator.

To prove Theorem 2.1, we need some lemmas. In the following, φ is a local Lie derivation and, for any $x \in \mathcal{G}$, the symbol φ_x stands for a Lie derivation from $\mathcal{G}$ into itself such that $\varphi(x) = \varphi_x(x)$. It follows from $A = \mathcal{J}(A)$ that every a in A can be written as a linear combination of some elements $p_1 p_2 \cdots p_i$ $(i = 1, 2, \ldots, k)$, where $p_1, p_2, \ldots, p_i$ are idempotents in A.

Lemma 2.2. *Let $p, q \in \mathcal{G}$ be idempotents, then for every $x \in \mathcal{G}$, there exist linear maps $\tau_1, \tau_2, \tau_3, \tau_4 : \mathcal{G} \to Z(\mathcal{G})$ vanishing on each commutator such that*

$$\begin{aligned} \varphi(pxq) = {} & \varphi(px)q + p\varphi(xq) - p\varphi(x)q + p^{\perp}\tau_1(pxq)q^{\perp} \\ & - p\tau_2(p^{\perp}xq)q^{\perp} + p\tau_3(p^{\perp}xq^{\perp})q - p^{\perp}\tau_4(pxq^{\perp})q, \end{aligned}$$

where $p^{\perp} = 1 - p$ and $q^{\perp} = 1 - q$.

Proof. Proposition 1.1 implies that for every idempotents $p, q \in \mathcal{G}$ and $x \in \mathcal{G}$, there exist derivations $d_1, d_2, d_3, d_4 : \mathcal{G} \to \mathcal{G}$ and linear maps $\tau_1, \tau_2, \tau_3, \tau_4 : \mathcal{G} \to Z(\mathcal{G})$ vanishing on each commutator such that

$$\varphi(pxq) = \varphi_{pxq}(pxq) = d_1(pxq) + \tau_1(pxq), \tag{2.1}$$

$$\varphi(p^{\perp}xq) = \varphi_{p^{\perp}xq}(p^{\perp}xq) = d_2(p^{\perp}xq) + \tau_2(p^{\perp}xq), \tag{2.2}$$

$$\varphi(p^\perp xq^\perp) = \varphi_{p^\perp xq^\perp}(p^\perp xq^\perp) = d_3(p^\perp xq^\perp) + \tau_3(p^\perp xq^\perp), \tag{2.3}$$

$$\varphi(pxq^\perp) = \varphi_{pxq^\perp}(pxq^\perp) = d_4(pxq^\perp) + \tau_4(pxq^\perp). \tag{2.4}$$

It follows from (2.1)–(2.4) that

$$p^\perp\varphi(pxq)q^\perp = p^\perp\tau_1(pxq)q^\perp,\ p\varphi(p^\perp xq)q^\perp = p\tau_2(p^\perp xq)q^\perp,$$

$$p\varphi(p^\perp xq^\perp)q = p\tau_3(p^\perp xq^\perp)q,\ p^\perp\varphi(pxq^\perp)q = p^\perp\tau_4(pxq^\perp)q.$$

Hence

$$\begin{aligned}\varphi(pxq)q^\perp &= p\varphi(pxq)q^\perp + p^\perp\varphi(pxq)q^\perp \\ &= p\varphi(xq)q^\perp - p\varphi(p^\perp xq)q^\perp + p^\perp\varphi(pxq)q^\perp \\ &= p\varphi(xq)q^\perp + p^\perp\tau_1(pxq)q^\perp - p\tau_2(p^\perp xq)q^\perp \\ &= p\varphi(xq) - p\varphi(xq)q + p^\perp\tau_1(pxq)q^\perp - p\tau_2(p^\perp xq)q^\perp,\end{aligned}$$

$$\begin{aligned}\varphi(pxq^\perp)q &= p\varphi(pxq^\perp)q + p^\perp\varphi(pxq^\perp)q \\ &= p\varphi(xq^\perp)q - p\varphi(p^\perp xq^\perp)q + p^\perp\varphi(pxq^\perp)q \\ &= p\varphi(xq^\perp)q - p\tau_3(p^\perp xq^\perp)q + p^\perp\tau_4(pxq^\perp)q.\end{aligned}$$

Thus,

$$\begin{aligned}\varphi(pxq) &= \varphi(pxq)q^\perp + \varphi(pxq)q \\ &= \varphi(pxq)q^\perp + \varphi(px)q - \varphi(pxq^\perp)q \\ &= \varphi(px)q + p\varphi(xq) - p\varphi(x)q + p^\perp\tau_1(pxq)q^\perp \\ &\quad - p\tau_2(p^\perp xq)q^\perp + p\tau_3(p^\perp xq^\perp)q - p^\perp\tau_4(pxq^\perp)q.\end{aligned}$$

□

It is easy to verify that for each derivation $d : \mathcal{G} \to \mathcal{G}$, we have

$$d(e) = -d(f) \in M \oplus N,\ d(A) \subseteq A \oplus M \oplus N,\ d(M) \subseteq A \oplus M \oplus B. \tag{2.5}$$

Lemma 2.3. *$e\varphi(e)e + f\varphi(e)f \in Z(\mathcal{G})$.*

Proof. For any $m \in M$, there exists a Lie derivation φ_e of $\mathcal{G}$ such that

$$\begin{aligned}\varphi_e(m) &= \varphi_e([e, m]) \\ &= [\varphi(e), m] + [e, \varphi_e(m)] \\ &= \varphi(e)m - m\varphi(e) + e\varphi_e(m)f - f\varphi_e(m)e.\end{aligned}$$

Multiplying the above equality from the left by e and from the right by f, we arrive at

$$e\varphi(e)m = m\varphi(e)f.$$

Similarly, for any $n \in N$, we have from $\varphi_e(n) = \varphi_e([n, e]) = [\varphi_e(n), e] + [n, \varphi(e)]$ that

$$f\varphi(e)n = n\varphi(e)e.$$

Hence

$$e\varphi(e)e + f\varphi(e)f \in Z(\mathcal{G}).$$

□

In the sequel, we define $\phi : \mathcal{G} \to \mathcal{G}$ by $\phi(x) = \varphi(x) - [x, e\varphi(e)f - f\varphi(e)e]$. One can verify that ϕ is also a local Lie derivation. Moreover, by Lemma 2.3, we have $\phi(e) = e\varphi(e)e + f\varphi(e)f \in Z(\mathcal{G})$.

Lemma 2.4. *$\phi(M) \subseteq M$ and $\phi(N) \subseteq N$.*

Proof. Let $a \in A, m \in M$ and p_1 be any idempotent in A. Taking $p = p_1$, $x = a$ and $q = e + m$ in Lemma 2.2, it follows from the facts $p^\perp x q^\perp$ and $pxq^\perp$ can be written as commutators that $\tau_3(p^\perp x q^\perp) = \tau_4(pxq^\perp) = 0$, hence

$$\begin{aligned}
\phi(p_1a + p_1am) &= \phi(p_1a)(e+m) + p_1\phi(a+am) - p_1\phi(a)(e+m) \\
&\quad + (1-p_1)\tau_1(p_1a + p_1am)(f-m) \\
&\quad - p_1\tau_2(a + am - p_1a - p_1am)(f-m) \\
&= \phi(p_1a)e + \phi(p_1a)m + p_1\phi(a)f + p_1\phi(am) \\
&\quad - p_1\phi(a)m + \tau_1(p_1a)f - \tau_1(p_1a)m + p_1\tau_1(p_1a)m \\
&\quad + p_1\tau_2(a - p_1a)m. \qquad (2.6)
\end{aligned}$$

Multiplying (2.6) from the right by e, we arrive at

$$\phi(p_1am)e = p_1\phi(am)e.$$

In particular,

$$\phi(p_1m)e = p_1\phi(m)e.$$

By the above two equations, then

$$\begin{aligned}
\phi(p_1p_2\cdots p_nm)e &= p_1\phi(p_2\cdots p_nm)e \\
&= p_1p_2\cdots p_{n-1}\phi(p_nm)e \\
&= p_1p_2\cdots p_n\phi(m)e
\end{aligned}$$

for any idempotents $p_1, \ldots, p_n \in A$. It follows from $A = \mathcal{J}(A)$ that

$$\phi(am)e = a\phi(m)e \qquad (2.7)$$

for all $a \in A, m \in M$. This implies that $f\phi(M)e = 0$.

The hypothesis (2), (3) and Proposition 1.1 imply that there exist a derivation $d : \mathcal{G} \to \mathcal{G}$ and a linear map $\tau : \mathcal{G} \to Z(\mathcal{G})$ vanishing on each commutator such that

$$\begin{aligned}
\phi(e+m) &= d(e+m) + \tau(e+m) \\
&= d(e+m) + \tau(e). \qquad (2.8)
\end{aligned}$$

It follows from (2.5), (2.8) and the fact $f\phi(M)e = 0$ that

$$0 = f\phi(e+m)e = fd(e)e$$

and hence by (2.5) and (2.8) again,

$$e\phi(e)e + e\phi(m)e = ed(m)e + e\tau(e)e = ed(mf)e + e\tau(e)e$$

$$= md(f)e + e\tau(e)e = -md(e)e + e\tau(e)e$$
$$= e\tau(e)e$$

and

$$f\phi(e)f + f\phi(m)f = fd(m)f + f\tau(e)f = fd(e)m + f\tau(e)f$$
$$= f\tau(e)f.$$

Then we have from the fact $\phi(e) = e\phi(e)e + f\phi(e)f \in Z(\mathcal{G})$ that

$$e\phi(m)e + f\phi(m)f = \tau(e) - \phi(e) \in Z(\mathcal{G}). \tag{2.9}$$

We assume without loss of generality that A does not contain nonzero central ideals. By (2.7) and (2.9) that $e\phi(m)e$ in the central ideal of A. Thus $e\phi(M)e = 0$. So, by (2.9), we get $f\phi(M)f = 0$. Hence, $\phi(M) \subseteq M$.

With the same argument, we can obtain that $\phi(N) \subseteq N$. □

Lemma 2.5. *There exist a linear map h_1 from A into $Z(\mathcal{G})$ such that $\phi(a) - h_1(a) \in A$ for all $a \in A$ and a linear map h_2 from B into $Z(\mathcal{G})$ such that $\phi(b) - h_2(b) \in B$ for all $b \in B$.*

Proof. Taking $m = 0$ in (2.6), we have

$$e\phi(p_1a)f = p_1\phi(a)f \text{ and } f\phi(p_1a)f = \tau_{p_1a}(p_1a)f \in \pi_B(Z(\mathcal{G})). \tag{2.10}$$

In particular,

$$e\phi(p_1)f = p_1\phi(e)f = 0.$$

By the two equations above, we obtain

$$e\phi(p_1p_2\cdots p_n)f = p_1\phi(p_2\cdots p_n)f$$
$$= p_1p_2\cdots p_{n-1}\phi(p_n)f$$
$$= 0$$

for all idempotents p_i in A. It follows from $A = \mathcal{J}(A)$ that $e\phi(a)f = 0$. Similarly, by taking $p = e$, $x = a$ and $q = p_1$ in Lemma 2.2, we get

$$f\phi(ap_1)e = f\phi(a)p_1.$$

This implies that $f\phi(a)e = 0$. So $\phi(a) \in A \oplus B$.

By the hypothesis (2) of Theorem 2.1, there exists a algebra isomorphism $\eta : Z(B) \to Z(A)$ such that $\eta(b) \oplus b \in Z(\mathcal{G})$ for any $b \in Z(B)$.

It follows from (2.10) that $f\phi(a)f \in \pi_B(Z(\mathcal{G})) = Z(B)$. We define $h_1 : A \to Z(\mathcal{G})$ by $h_1(a) = \eta(f\phi(a)f) \oplus f\phi(a)f$. It is clear that h_1 is linear and

$$\phi(a) - h_1(a) = e\phi(a)e + f\phi(a)f - \eta(f\phi(a)f) - f\phi(a)f$$
$$= e\phi(a)e - \eta(f\phi(a)f) \in A.$$

With the similar argument, we can define a linear map $h_2 : B \to Z(\mathcal{G})$ such that $\phi(b) - h_2(b) \in B$ for all $b \in B$. □

Now for any $x \in \mathcal{G}$, we define two linear maps $h : \mathcal{G} \to Z(\mathcal{G})$ and $\delta : \mathcal{G} \to \mathcal{G}$ by

$$h(x) = h_1(exe) + h_2(fxf) \text{ and } \delta(x) = \phi(x) - h(x).$$

It is easy to verify that $\delta(e) = 0$. Moreover, we have

$$\delta(A) \subseteq A,\ \delta(B) \subseteq B,\ \delta(M) = \phi(M) \subseteq M,\ \delta(N) = \phi(N) \subseteq N.$$

Lemma 2.6. *δ is a derivation.*

Proof. We divide the proof into the following three steps.

Step 1. We first prove that

$$\delta(p_1p_2 \dots p_nm) = \delta(p_1p_2 \dots p_n)m + p_1p_2 \dots p_n\delta(m) \tag{2.11}$$

for all idempotents p_i in A and $m \in M$.

Let $a \in A$, $m \in M$ and p_1 be any idempotent in A. Taking $p = p_1$, $x = a$ and $q = e + m$ in (2.2), we have

$$\begin{aligned}\phi(a + am - p_1a - p_1am) &= d_2(a + am - p_1a - p_1am)\\ &\quad + \tau_2(a + am - p_1a - p_1am)\\ &= d_2(a + am - p_1a - p_1am)\\ &\quad + \tau_2(a - p_1a).\end{aligned} \tag{2.12}$$

It follows from (2.5) and (2.12) that

$$0 = fd_2(a - p_1a)e = fd_2(e(a - p_1a))e = fd_2(e)(a - p_1a)$$

and hence by (2.5) and (2.12) again,

$$\begin{aligned}f\phi(a - p_1a)f &= fd_2(am - p_1am)f + f\tau_2(a - p_1a)f\\ &= fd_2(e)(a - p_1a)m + f\tau_2(a - p_1a)f\\ &= f\tau_2(a - p_1a)f.\end{aligned} \tag{2.13}$$

Multiplying (2.6) by f from both sides, we arrive at

$$f\phi(p_1a)f = f\tau_1(p_1a)f. \tag{2.14}$$

By (2.13) and (2.14), then $m\tau_1(p_1a) = m\phi(p_1a)$ and

$$\begin{aligned}p_1m\tau_2(a - p_1a) &= p_1m\phi(a - p_1a)\\ &= p_1m\phi(a) - p_1m\phi(p_1a)\\ &= p_1m\phi(a) - p_1m\tau_1(p_1a).\end{aligned}$$

Hence (2.6) implies that

$$\delta(p_1am) = \phi(p_1am)$$

$$\begin{aligned}
&= \phi(p_1a)m + p_1\phi(am) - p_1\phi(a)m - m\phi(p_1a) + p_1m\phi(a) \\
&= (\delta(p_1a) + h(p_1a))m + p_1\delta(am) - p_1(\delta(a) + h(a))m \\
&\quad - m(\delta(p_1a) + h(p_1a)) + p_1m(\delta(a) + h(a)) \\
&= \delta(p_1a)m + p_1\delta(am) - p_1\delta(a)m. \qquad (2.15)
\end{aligned}$$

Taking $a = e$ in (2.15), we have from $\delta(e) = 0$ that

$$\delta(p_1m) = \delta(p_1)m + p_1\delta(m).$$

This shows that (2.11) is true for $n = 1$. One can verify that Eq (2.11) follows easily by induction based on (2.15). It follows from $A = \mathcal{J}(A)$ that $\delta(am) = \delta(a)m + a\delta(m)$.

Similarly, we can get $\delta(mb) = \delta(m)b + m\delta(b)$, $\delta(mb) = \delta(m)b + m\delta(b)$ and $\delta(na) = \delta(n)a + n\delta(a)$.

Step 2. Let $a, a' \in A$. For any $m \in M$, on one hand, by Step 1, we have

$$\begin{aligned}
\delta(aa'm) &= \delta(a)a'm + a\delta(a'm) \\
&= \delta(a)a'm + a\delta(a')m + aa'\delta(m).
\end{aligned}$$

On the other hand,

$$\delta(aa'm) = \delta(aa')m + aa'\delta(m).$$

Comparing these two equalities, we have

$$(\delta(aa') - \delta(a)a' - a\delta(a'))m = 0$$

for any $m \in M$. Since M is a faithful left A-module, we get

$$\delta(aa') = \delta(a)a' + a\delta(a').$$

Similarly, by considering $\delta(mbb')$, we can get

$$\delta(bb') = \delta(b)b' + b\delta(b').$$

Step 3. Let $m, m' \in M$ and $n \in N$. Taking $p = e - m'$, $x = n + m'n$ and $q = e - m'$ in Lemma 2.2, we have from $pxq = pxq^{\perp} = 0$ that

$$\begin{aligned}
0 &= (e - m')\phi(m'n - m'nm' + n - nm') - (e - m')\phi(m'n + n)(e - m') \\
&\quad - (e - m')\tau_2(m'n - nm')(f + m') + (e - m')\tau_3(nm')(e - m') \\
&= -\phi(m'nm') - e\phi(nm') - m'\phi(m'n) + m'\phi(nm') \\
&\quad + \phi(m'n)m' - m'\phi(n)m' + e\tau_3(nm')e - \tau_3(nm')m'. \qquad (2.16)
\end{aligned}$$

This implies that

$$e\phi(nm') = e\tau_3(nm')e.$$

Then $e\phi(nm')m' = \tau_3(nm')m'$ and hence by (2.16),

$$\begin{aligned}
\delta(m'nm') &= \phi(m'nm') \\
&= -m'\phi(m'n) + m'\phi(nm') + \phi(m'n)m' - m'\phi(n)m' - \phi(nm')m'
\end{aligned}$$

$$\begin{aligned}
&= -m'h(m'n) + m'\delta(nm') + m'h(nm') + \delta(m'n)m' \\
&\quad + h(m'n)m' - m'\delta(n)m' - h(nm')m' \\
&= m'\delta(nm') + \delta(m'n)m' - m'\delta(n)m'.
\end{aligned}$$

Replacing m' with $m + m'$, we arrive at

$$\begin{aligned}
\delta(m'nm + mnm') = \delta(m'n)m + m'\delta(nm) - m'\delta(n)m \\
+ \delta(mn)m' + m\delta(nm') - m\delta(n)m'.
\end{aligned}$$

On the other hand, by Steps 1 and 2, we have

$$\delta(m'nm + mnm') = \delta(m'n)m + m'n\delta(m) + \delta(m)nm' + m\delta(nm').$$

Comparing these two equalities, we have

$$(\delta(mn) - \delta(m)n - m\delta(n))m' = -m'(\delta(nm) - n\delta(m) - \delta(n)m). \tag{2.17}$$

Set

$$f(m, n) := \delta(mn) - \delta(m)n - m\delta(n)$$

and

$$g(m, n) := \delta(nm) - n\delta(m) - \delta(n)m.$$

We assume without loss of generality that A does not contain nonzero central ideals. For any $a \in A$, by (2.17),

$$f(m, n)am' = -am'g(m, n) = af(m, n)m'.$$

which is equivalent to $(f(m, n)a - af(m, n))m' = 0$. Since M is a faithful left A-module, we get $f(m, n)a = af(m, n)$. Then

$$f(m, n) \in Z(A).$$

By Steps 1 and 2, we have

$$\begin{aligned}
f(am, n) &= \delta(amn) - \delta(am)n - am\delta(n) \\
&= \delta(a)mn + a\delta(mn) - \delta(a)mn - a\delta(m)n - am\delta(n) \\
&= af(m, n).
\end{aligned}$$

The above two equalities show that $f(m, n)$ in the central ideal of A and hence

$$f(m, n) = 0, \tag{2.18}$$

that is

$$\delta(mn) = \delta(m)n + m\delta(n)$$

for all $m \in M, n \in N$. Since M is a faithful right B-module, it follows from (2.17) that

$$\delta(nm) = n\delta(m) + \delta(n)m$$

for all $m \in M, n \in N$. □

Lemma 2.7. *The map $h : \mathcal{G} \to Z(\mathcal{G})$ vanishes on each commutator.*

Proof. **Step 1.** Let $a \in A$, $m \in M$, $n \in N$ and $b \in B$, by the definition of h, we have $h([a, m]) = h([m, b]) = h([n, a]) = h([b, n]) = 0$.

Step 2. Let $a, a' \in A$, we have $\phi([a, a']) = e\phi([a, a'])e + f\phi([a, a'])f \in A \oplus B$. On the other hand, Proposition 1.1 implies that $\phi([a, a']) = d([a, a']) \in A \oplus M \oplus N$, where d is a derivation. Thus, $f\phi([a, a'])f = 0$. This implies that $h([a, a']) = h_1([a, a']) = \eta(f\phi([a, a'])f) + f\phi([a, a'])f = 0$.

Similarly, we can get $h([b, b']) = 0$, for all $b, b' \in B$.

Step 3. It follows from (2.18) that

$$\begin{aligned}&(\phi(mn) - \eta(f\phi(mn)f) - \phi(m)n - m\phi(n))m' \\ &= -m'(\phi(nm) - \eta^{-1}(e\phi(nm)e) - n\phi(m) - \phi(n)m).\end{aligned} \tag{2.19}$$

Since $f\phi(a)f \in \pi_B(Z(\mathcal{G}))$, $e\phi(b)e \in \pi_A(Z(\mathcal{G}))$, we get that

$$m'f\phi(mn)f = \eta(f\phi(mn)f)m', e\phi(nm)em' = m'\eta^{-1}(e\phi(nm)e).$$

It further follows from (2.19) that

$$\begin{aligned}&\phi(mn)m' - m'f\phi(mn)f - \phi(m)nm' - m\phi(n)m' \\ &= -m'\phi(nm) + e\phi(nm)m' + m'n\phi(m) + m'\phi(n)m.\end{aligned}$$

Hence

$$\begin{aligned}&(\phi(mn) - e\phi(nm) - \phi(m)n - m\phi(n))m' \\ &= m'(-\phi(nm) + f\phi(mn)f + n\phi(m) + \phi(n)m).\end{aligned}$$

Using an argument similar to that in the proof of (2.18), we arrive that

$$e\phi(mn)e - e\phi(nm) - \phi(m)n - m\phi(n) = 0, \tag{2.20}$$

and

$$-f\phi(nm)f + f\phi(mn)f + n\phi(m) + \phi(n)m = 0.$$

By (2.19) and (2.20), we get that $e\phi(nm)e = \eta(f\phi(mn)f)$. Note that $h([m, n]) = h_1(mn) - h_2(nm) = \eta(f\phi(mn)f) + f\phi(mn)f - e\phi(nm)e - \eta^{-1}(e\phi(nm)e)$, thus $h([m, n]) = 0$.

Therefore it is easily verify that h vanishing on each commutator. □

Proof of Theorem 1.1 By the definition of δ, we have $\varphi(x) = \delta(x) + [x, e\varphi(e)f - f\varphi(e)e] + h(x)$ for all $x \in A$, where δ is a derivation and h is a linear map from A into its center vanishing on each commutator. The proof is complete. □

Let A be a unital algebra and $M_{k\times m}(A)$ be the set of all $k \times m$ matrices over A. For $n \geq 2$ and each $2 \leq l < n - 1$, the full matrix algebra $M_n(A)$ can be represented as a generalized matrix algebra of the form

$$\begin{pmatrix} M_{l\times l}(A) & M_{l\times(n-l)}(A) \\ M_{(n-l)\times l}(A) & M_{(n-l)\times(n-l)}(A) \end{pmatrix}.$$

Corollary 2.8. *Let $M_n(A)$ be a full matrix algebra with $n \geq 4$. Then each local Lie derivation φ on $M_n(A)$ is of the form $\varphi = d + \tau$, where d is a derivation of $M_n(A)$ and τ is a linear map from $M_n(A)$ into its center $Z(A) \cdot I_n$ vanishing on each commutator.*

Proof. It follows from the example (C) of [2] that the matrix algebras $M_l(A)$ and $M_{n-l}(A)$ are generated by their idempotents for $2 \leq l < n-1$. Since $Z(M_n(A)) = Z(A) \cdot I_n$, $Z(M_l(A)) = Z(A) \cdot I_l$ and $Z(M_{n-l}(A)) = Z(A) \cdot I_{n-l}$, the condition (2) of Theorem 2.1 is satisfied. By [5, Lemma 1], $M_k(A)$ does not contain nonzero central ideals for $k \geq 2$. Hence by Theorem 2.1, every local Lie derivation of $M_n(A)$ is a sum of a derivation and a linear central-valued map vanishing on each commutator. □

Corollary 2.9. *Let R be an unital simple algebra with a nontrivial idempotent. If $\varphi : R \to R$ is a local Lie derivation, then there exit a derivation d and a linear central map τ vanishing on each commutator, such that $\varphi = d + \tau$.*

Proof. Let R be an unital simple algebra with a nontrivial idempotent e_0 and let f_0 denote the idempotent $1 - e_0$. Then R can be represented in the so-called Peirce decomposition form

$$R = e_0Re_0 + e_0Rf_0 + f_0Re_0 + f_0Rf_0,$$

where e_0Re_0 and f_0Rf_0 are subalgebras with unitary element e_0 and f_0, respectively, e_0Rf_0 is an (e_0Re_0, f_0Rf_0)-bimodule.

Next, we will show that

$$e_0xe_0 \cdot e_0Rf_0 = \{0\} \text{ implies } e_0xe_0 = 0$$

and

$$e_0Rf_0 \cdot f_0xf_0 = \{0\} \text{ implies } f_0xf_0 = 0.$$

That is e_0Rf_0 is faithful as an (e_0Re_0, f_0Rf_0)-bimodule. Let $e = f_0 + e_0Rf_0$, then $e^2 = e$ and $[e, R] \subseteq eR(1-e) + (1-e)Re$. Note that

$$(1-e)Re = (e_0 - e_0Rf_0)R(f_0 + e_0Rf_0) \subseteq e_0Rf_0.$$

Furthermore, the assumption $e_0xe_0 \cdot e_0Rf_0 = \{0\}$ implies

$$e_0xe_0eR(1-e) = e_0xe_0(f_0 + e_0Rf_0)R(e_0 + e_0Rf_0) = \{0\}$$

and then

$$e_0xe_0[e, R] = \{0\}.$$

Let $r = [e, y]$ and $z, w \in R$. It follows from

$$zrw = [e, z[e, r]w] - [e, z][e, rw] - [e, zr][e, w] + 2[e, z]r[e, w]$$

that $e_0xe_0zrw = 0$. Then

$$e_0xe_0R[e, R]R = 0. \tag{2.21}$$

It is clear that $I = R[e, R]R$ is a nonzero ideal of R. R is a simple algebra, which implies $I = R$. By (2.21), $e_0xe_0R = 0$. Since $1 \in R$, we get $e_0xe_0 = 0$. Similarly, we can show that $e_0Rf_0 \cdot f_0xf_0 = \{0\}$ implies $f_0xf_0 = 0$. Now, we can conclude that R can be represented as a generalized matrix algebra of the form $R = e_0Re_0 + e_0Rf_0 + f_0Re_0 + f_0Rf_0$.

It follows from the example (A) of [2] that the unital simple algebra with a nontrivial idempotent is generated by its idempotents, the condition (1) of Theorem 2.1 is satisfied. It is clear that e_0Re_0 and f_0Rf_0 satisfy the conditions (2) and (3) of Theorem 2.1. Hence by Theorem 2.1, every local Lie derivation of R is the sum of a derivation and a linear central-valued map vanishing on each commutator. □

Let $B(H)$ be the set of bounded linear operators acting on a complex Hilbert space H, and let $K(H)$ be the ideal of compact operators on H. If H is an infinite-dimensional separable Hilbert space, by [12, Theorem 4.1.16], the Calkin algebra $B(H)/K(H)$ is a simple C^*-algebra.

Corollary 2.10. *If H is an infinite-dimensional separable Hilbert space, then every local Lie derivation of the Calkin algebra $B(H)/K(H)$ is the sum of a derivation and a linear central map vanishing on each commutator.*

3. Conclusions

In this paper, we investigate local Lie derivations of a certain class of generalized matrix algebras and show that, under certain conditions every local Lie derivation of a generalized matrix algebra is a sum of a derivation and a linear central-valued map vanishing on each commutator. The main result is then applied to full matrix algebras and unital simple algebras with nontrivial idempotents.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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