Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 6026–6035.
DOI: 10.3934/math.2023304
Received: 05 August 2022
Revised: 11 December 2022
Accepted: 12 December 2022
Published: 29 December 2022

Research article

Jordan semi-triple derivations and Jordan centralizers on generalized quaternion algebras

Ai-qun Ma[1,2], **Lin Chen**[2,*] **and Zijie Qin**[2]

[1] Department of Mathematics and Statistics, Guizhou University, Guiyang 550025, China
[2] Department of Mathematics and Statistics, Changshu Institute of Technology, Changshu 215500, China

* **Correspondence:** Email: linchen198112@163.com.

Abstract: In this paper, we investigate Jordan semi-triple derivations and Jordan centralizers on generalized quaternion algebras over the field of real numbers. We prove that every Jordan semi-triple derivation on generalized quaternion algebras over the field of real numbers is a derivation. Also, we show that every left (resp, right) Jordan centralizer on generalized quaternion algebras over the field of real numbers is a left (resp, right) centralizer.

Keywords: generalized quaternion algebra; Jordan semi-triple derivation; Jordan centralizer
Mathematics Subject Classification: 46K15, 47B49

1. Introduction

Let $\mathcal{A}$ be an algebra. Recall that a linear map δ from $\mathcal{A}$ into $\mathcal{A}$ is called a *left (resp, right) centralizer* if $\delta(xy) = \delta(x)y$ (resp, $\delta(xy) = x\delta(y)$) for all $x, y \in \mathcal{A}$, and it is called a *centralizer* if δ is both a left centralizer and a right centralizer. Also, δ is called a *left (resp, right) Jordan centralizer* if $\delta(x^2) = \delta(x)x$ (resp, $\delta(x^2) = x\delta(x)$) for all $x \in \mathcal{A}$. We say that δ is a *Jordan centralizer* if $\delta(xy + yx) = x\delta(y) + \delta(y)x = y\delta(x) + \delta(x)y$ for all $x, y \in \mathcal{A}$. Clearly, each (right, left) centralizer is a (right, left) Jordan centralizer. The converse is not true in general. In [1] Zalar showed that any left (resp, right) Jordan centralizer on a 2-torsion free semi-prime ring is a left (resp, right) centralizer. We refer the reader to [1,2] for results concerning centralizers on rings and algebras.

Recall that a linear map δ from $\mathcal{A}$ into $\mathcal{A}$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$, and it is called a *Jordan semi-triple derivation* if $\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x)$ for all $x, y \in \mathcal{A}$. Without linearity, a Jordan semi-triple derivation is called a Jordan semi-triple derivable map. Du and Zhang in [3] gave a characterization of a Jordan semi-triple derivable map on matrix algebra over a 2-torsion free commutative ring with unity. The second author and Zhang in [4] gave a full characterization of a $*$-Jordan semi-triple derivable map (i.e., a map ϕ satisfying $\phi(xy^*x) = \phi(x)y^*x + x\phi(y)^*x + xy^*\phi(x)$ if the algebra is a $*$ algebra) on matrix algebra over a 2-torsion free commutative real ring with unity and on operator algebra $B(H)$. The derivations, centralizers and Jordan semi-triple derivations of an algebra give interesting insights for studying its algebraic structure.

The quaternion was discovered by Hamilton [5] in 1835. Up to now, quaternions and quaternion matrices have become increasingly useful for practitioners in theory and application. For example, a number of research papers related to quaternions appear in mathematical or physical journals, and quantum mechanics based on quaternion analysis is mainstream in physics. We refer the reader to [6–12] for further information regarding the important roles of quaternion algebra in other branches of mathematics. For a detailed account of quaternions, quaternion matrices and their applications, the reader can consult [13–16].

A generalized quaternion is a generalization of a quaternion, and it can be found in [17]. A generalized quaternion x is of the form $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$ where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and the quaternionic units e_0, e_1, e_2 and e_3 obey the following equations:

$$\begin{aligned} &e_1^2 = -\alpha, e_2^2 = -\beta, e_3^2 = -\alpha\beta,\\ &e_1 \cdot e_2 = e_3 = -e_2 \cdot e_1,\\ &e_2 \cdot e_3 = \beta e_1 = -e_3 \cdot e_2,\\ &e_3 \cdot e_1 = \alpha e_2 = -e_1 \cdot e_3, \end{aligned}$$

for some $\alpha, \beta \in \mathbb{R}$. We denote by $H_{\alpha,\beta}$ the set of generalized quaternions over $\mathbb{R}$ with the basis $\mathcal{B}(H_{\alpha,\beta}) = \{e_0, e_1, e_2, e_3\}$ corresponding to the familiar $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$. Note that e_0 acts as identity, which means $e_0 \cdot e_i = e_i \cdot e_0$ for any i, and hence the center of $H_{\alpha,\beta}$ is $Z(H_{\alpha,\beta}) = \mathbb{R} \cdot e_0 = \mathbb{R}$.

In [18, 19] the authors study generalized Jordan derivations and generalized Lie derivations of quaternion ring. Recently, Kizil and Alagöz determine derivations of the algebra $H_{\alpha,\beta}$ of generalized quaternions over $\mathbb{R}$ in [20]. Motivated by [20], in this paper, we consider Jordan semi-triple derivations and Jordan centralizers on generalized quaternion algebras over $\mathbb{R}$, and we prove that every Jordan semi-triple derivation on generalized quaternion algebras over the field of real numbers is a derivation. Also, we show that every left (resp, right) Jordan centralizer on generalized quaternion algebras over the field of real numbers is a left (resp, right) centralizer.

2. Jordan semi-triple derivations on generalized quaternion algebras

In this section, we consider Jordan semi-triple derivations on generalized quaternion algebras over $\mathbb{R}$. We denote by $JDer(H_{\alpha,\beta})$ the set of Jordan semi-triple derivations on generalized quaternion algebras over $\mathbb{R}$, and let $Der(H_{\alpha,\beta})$ denote the derivation algebra of $H_{\alpha,\beta}$ over $\mathbb{R}$. The following lemma provides $Der(H_{\alpha,\beta})$ in its matrix form.

Lemma 2.1. The algebra $Der(H_{\alpha,\beta})$ of derivations for $H_{\alpha,\beta}$ is generated by the following matrices:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta}{\alpha}a & -\beta b \\ 0 & a & d & -\alpha c \\ 0 & b & c & d \end{bmatrix} \in Der(H_{\alpha,\beta}),$$

where $a, b, c, d \in \mathbb{R}$ such that $d = d(\beta) \neq 0$ if $\beta = 0$, and $d = 0$ otherwise.

The algebra $ad(H_{\alpha,\beta})$ of inner derivation for $H_{\alpha,\beta}$ is generated by the following matrices:

$$ad(e_1)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-2\alpha\\0&0&2&0\end{bmatrix}, ad(e_2)=\begin{bmatrix}0&0&0&0\\0&0&0&2\beta\\0&0&0&0\\0&-2&0&0\end{bmatrix}, ad(e_3)=\begin{bmatrix}0&0&0&0\\0&0&-2\beta&0\\0&2\alpha&0&0\\0&0&0&0\end{bmatrix}.$$

We are now in a position to state the main result of this section.

Theorem 2.2. The linear space $JDer(H_{\alpha,\beta})$ over $\mathbb{R}$ is generated by the following matrices:

$$\begin{bmatrix}0&0&0&0\\0&0&-\frac{\beta}{\alpha}a&-\beta b\\0&a&d_1&c_1\\0&b&c_2&d_2\end{bmatrix},$$

where $a,b,c_1,c_2,d_1,d_2\in\mathbb{R}$ such that $d_i=d_i(\beta)\neq 0$ $(i=1,2)$ if $\beta=0$, $d_i=0$ if $\beta\neq 0$, $c_1=-\alpha c_2$ if $\beta\neq 0$, and $c_i=c_i(\beta)\neq 0$ $(i=1,2)$ if $\beta=0$.

Proof. Let $\phi\in JDer(H_{\alpha,\beta})$, since ϕ admits a matrix representation with respect to the basis $\mathcal{B}(H_{\alpha,\beta})$, which is the 4×4 matrix $[\phi]=(d_{ij})^{\mathrm{T}}$ whose entries are defined by the following equations:

$$\phi(e_{i-1})=\sum_{j=1}^{4}d_{ij}e_{j-1},\qquad 1\leq i\leq 4.$$

Each column of $[\phi]$ is an element of $H_{\alpha,\beta}$. In order to obtain $[\phi]$, we apply ϕ to the products $e_ie_je_i$ with $0\leq i,j\leq 3$. e_0 is a central idempotent and

$$\begin{aligned}\phi(e_0e_ie_0)=&\phi(e_0)e_ie_0+e_0\phi(e_i)e_0+e_0e_i\phi(e_0),\forall i=1,2,3,\\ &\Updownarrow\\ \phi(e_i)=&\phi(e_0)e_i+\phi(e_i)+e_i\phi(e_0),\end{aligned}$$

which occurs if and only if

$$\phi(e_0)e_i+e_i\phi(e_0)=0,\forall i=1,2,3.$$

Hence, $\phi(e_0)=0$ for every $\phi\in JDer(H_{\alpha,\beta})$. Moreover, we obtain $d_{11}=0,d_{12}=0,d_{13}=0,d_{14}=0$ only by evaluating.

Let us apply ϕ to the quaternionic units:

$$\begin{aligned}0=&-\alpha\phi(e_0)=\phi(e_1e_0e_1)=\phi(e_1)e_0e_1+e_1e_0\phi(e_1)\\ =&-2\alpha d_{22}e_0+2d_{21}e_1,\end{aligned}$$

which implies $d_{21}=0,d_{22}=0$. We are now going to check the same procedure for $e_1e_2e_1=\alpha e_2$, $e_1e_3e_1=\alpha e_3$, $e_2e_0e_2=-\beta e_0$, $e_2e_3e_2=\beta e_3$ and $e_3e_0e_3=-\alpha\beta e_0$.

From $\phi(e_1e_2e_1)=\alpha\phi(e_2)$ we obtain $d_{31}=0,\ d_{32}=-\frac{\beta}{\alpha}d_{23}$. Similarly, since $\phi(e_1e_3e_1)=\alpha\phi(e_3)$, we get $d_{41}=0,\ d_{42}=-\beta d_{24}$. Continuing this way, from $-\beta\phi(e_0)=\phi(e_2e_0e_2)$, we have

$$0=-\beta\phi(e_0)=\phi(e_2e_0e_2)=\phi(e_2)e_0e_2+e_2e_0\phi(e_2)$$

$$= -2\beta d_{33}e_0 + 2d_{31}e_2,$$

which gives $-2\beta d_{33} = 0, 2d_{31} = 0$. Hence, $d_{31} = 0$ and

$$d_{33} = \begin{cases} 0, & \text{if } \beta \neq 0 \\ 0 \neq d_{33}, & \text{if } \beta = 0 \end{cases}.$$

Also, from $\phi(e_2e_3e_2) = \beta\phi(e_3)$, we have $-2\alpha\beta d_{34} - 2\beta d_{43} = 0$, that is, if $\beta = 0$, d_{34} and d_{43} are any real numbers, if $\beta \neq 0$, $d_{43} = -\alpha d_{34}$. Since $\phi(e_3e_0e_3) = -\alpha\beta\phi(e_0)$, from which we obtain $d_{41} = 0$, and

$$d_{44} = \begin{cases} 0, & \text{if } \beta \neq 0 \\ 0 \neq d_{44}, & \text{if } \beta = 0 \end{cases},$$

combining all these together, we obtain

$$\begin{aligned} &d_{11} = d_{21} = d_{31} = d_{41} = d_{12} = d_{13} = d_{14} = d_{22} = 0, \\ &\beta d_{33} = 0, \quad \beta d_{44} = 0, \quad d_{32} = -\frac{\beta}{\alpha} d_{23}, \quad d_{42} = -\beta d_{24}, \quad \beta(\alpha d_{34} + d_{43}) = 0. \end{aligned}$$

Let $d_{23} = a$, $d_{24} = b$, $d_{34} = c_2$, $d_{43} = c_1$, $d_{33} = d_1$ and $d_{44} = d_2$. Thus, we obtain ϕ in its matrix form. □

The following corollary is a direct consequence of Theorem 2.2.

Corollary 2.3. If we pick $\alpha \neq 0$ and $\beta \neq 0$, then Jordan semi-triple derivations on generalized quaternion algebras over $\mathbb{R}$ are derivations, that is,

$$\phi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta}{\alpha}a & -\beta b \\ 0 & a & 0 & -\alpha c \\ 0 & b & c & 0 \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$.

3. Centralizers on generalized quaternion algebras

In this section, we investigate centralizers on generalized quaternion algebras over $\mathbb{R}$. The following lemma provides an algebra isomorphism from a generalized quaternion algebra to a subalgebra of the 4×4 matrix algebra.

Lemma 3.1. The generalized quaternion algebra $H_{\alpha,\beta}$ is isomorphic to

$$G = \left\{ \begin{bmatrix} x_0 & -\alpha x_1 & -\beta x_2 & -\alpha\beta x_3 \\ x_1 & x_0 & -\beta x_3 & \beta x_2 \\ x_2 & \alpha x_3 & x_0 & -\alpha x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix} \mid x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Proof. For $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 \in H_{\alpha,\beta}$, let $\sigma : H_{\alpha,\beta} \longrightarrow G$, $x \mapsto \begin{bmatrix} x_0 & -\alpha x_1 & -\beta x_2 & -\alpha\beta x_3 \\ x_1 & x_0 & -\beta x_3 & \beta x_2 \\ x_2 & \alpha x_3 & x_0 & -\alpha x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix}$.

Obviously, σ is bijective, and σ preserves addition and scalar multiplication. Since

$$\sigma(e_0) = E,\ \sigma(e_1) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},\ \sigma(e_2) = \begin{bmatrix} 0 & -\beta J \\ J & 0 \end{bmatrix},\ \sigma(e_3) = \begin{bmatrix} 0 & -\beta K \\ K & 0 \end{bmatrix},$$

where $I = \begin{bmatrix} 0 & -\alpha \\ 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, K = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$, E is the 4×4 identity matrix. One can verify that

$$\begin{aligned}
&\sigma(e_1)^2 = -\alpha\sigma(e_0), \sigma(e_2)^2 = -\beta\sigma(e_0), \sigma(e_3)^2 = -\alpha\beta\sigma(e_0),\\
&\sigma(e_1)\cdot\sigma(e_2) = \sigma(e_3) = -\sigma(e_2)\cdot\sigma(e_1),\\
&\sigma(e_2)\cdot\sigma(e_3) = \beta\sigma(e_1) = -\sigma(e_3)\cdot\sigma(e_2),\\
&\sigma(e_3)\cdot\sigma(e_1) = \alpha\sigma(e_2) = -\sigma(e_1)\cdot\sigma(e_3).
\end{aligned}$$

This shows that σ preserves basis of $H_{\alpha,\beta}$. Let $P, Q \in H_{\alpha,\beta}$, since σ is a linear map and preserves basis of $H_{\alpha,\beta}$, from which we obtain $\sigma(PQ) = \sigma(P)\sigma(Q)$. Thus, σ preserves the multiplication of all elements of $H_{\alpha,\beta}$. Therefore, σ is an isomorphism, and the conclusion is established. □

Similarly, we have the following Lemma.

Lemma 3.2. The generalized quaternion algebra $H_{\alpha,\beta}$ is anti-isomorphic to

$$G^* = \left\{ \begin{bmatrix} x_0 & -\alpha x_1 & -\beta x_2 & -\alpha\beta x_3 \\ x_1 & x_0 & \beta x_3 & -\beta x_2 \\ x_2 & -\alpha x_3 & x_0 & \alpha x_1 \\ x_3 & x_2 & -x_1 & x_0 \end{bmatrix} \mid x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Proof. For $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 \in H_{\alpha,\beta}$, let $\psi : H_{\alpha,\beta} \longrightarrow G^*$, $x \mapsto \begin{bmatrix} x_0 & -\alpha x_1 & -\beta x_2 & -\alpha\beta x_3 \\ x_1 & x_0 & \beta x_3 & -\beta x_2 \\ x_2 & -\alpha x_3 & x_0 & \alpha x_1 \\ x_3 & x_2 & -x_1 & x_0 \end{bmatrix}$.

Obviously, ψ is bijective, and ψ preserves addition and scalar multiplication. Since

$$\psi(e_0) = E,\ \psi(e_1) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix},\ \psi(e_2) = \begin{bmatrix} 0 & -\beta J \\ J & 0 \end{bmatrix},\ \psi(e_3) = \begin{bmatrix} 0 & \beta K \\ K & 0 \end{bmatrix},$$

where $I = \begin{bmatrix} 0 & \alpha \\ -1 & 0 \end{bmatrix}, J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, K = \begin{bmatrix} 0 & -\alpha \\ 1 & 0 \end{bmatrix}$, E is the 4×4 identity matrix. Similar to the proof of Lemma 3.1, one can verify that ψ preserves basis of $H_{\alpha,\beta}$ and preserves the reversed multiplication of all elements of $H_{\alpha,\beta}$. Therefore, ψ is an anti-isomorphism, and the conclusion is established. □

The main result of this section is stated as follows.

Theorem 3.3. Every left centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$ is an algebra isomorphism, and every right centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$ is an algebra anti-isomorphism.

Proof. Let ω be a left centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$, since ω admits a matrix representation with respect to the basis $\mathcal{B}(H_{\alpha,\beta})$, which is the 4×4 matrix $[\omega]=(d_{ij})^{\mathrm{T}}$ whose entries are defined by the following equations:

$$\omega(e_{i-1})=\sum_{j=1}^{4} d_{ij}e_{j-1},\qquad 1\le i\le 4.$$

Each column of $[\omega]$ is an element of $H_{\alpha,\beta}$. In order to obtain $[\omega]$, we apply ω to the products e_ie_j with $0\le i,j\le 3$. Since

$$\begin{aligned} d_{21}e_0+d_{22}e_1+d_{23}e_2+d_{24}e_3 &= \omega(e_1)=\omega(e_0e_1)\\ &= \omega(e_0)e_1=(d_{11}e_0+d_{12}e_1+d_{13}e_2+d_{14}e_3)e_1\\ &= -\alpha d_{12}e_0+d_{11}e_1+\alpha d_{14}e_2-d_{13}e_3, \end{aligned}$$

we have $d_{21}=-\alpha d_{12}$, $d_{22}=d_{11}$, $d_{23}=\alpha d_{14}$, $d_{24}=-d_{13}$. Since

$$\begin{aligned} d_{41}e_0+d_{42}e_1+d_{43}e_2+d_{44}e_3 &= \omega(e_3)=\omega(e_1e_2)\\ &= \omega(e_1)e_2=(d_{21}e_0+d_{22}e_1+d_{23}e_2+d_{24}e_3)e_2\\ &= -\beta d_{23}e_0-\beta d_{24}e_1+d_{21}e_2+d_{22}e_3, \end{aligned}$$

we have $d_{41}=-\beta d_{23}$, $d_{42}=-\beta d_{24}$, $d_{43}=d_{21}$, $d_{44}=d_{22}$.

We are now going to check the same procedure for $e_2e_1=-e_3$. We get

$$-\alpha d_{32}e_0+d_{31}e_1+\alpha d_{34}e_2-d_{33}e_3=-d_{41}e_0-d_{42}e_1-d_{43}e_2-d_{44}e_3,$$

which implies $d_{41}=\alpha d_{32}$, $d_{42}=-d_{31}$, $d_{43}=-\alpha d_{34}$, $d_{44}=d_{33}$. Let $d_{11}=a$, $d_{12}=b$, $d_{13}=c$, $d_{14}=d$, and combining all these together, we obtain

$$\begin{aligned} &d_{22}=d_{33}=d_{44}=a,\\ &d_{21}=-\alpha b,\quad d_{34}=b,\quad d_{43}=-\alpha b,\\ &d_{31}=-\beta c,\quad d_{24}=-c,\quad d_{42}=\beta c,\\ &d_{23}=\alpha d,\quad d_{32}=-\beta d,\quad d_{41}=-\alpha\beta d. \end{aligned}$$

Thus, we obtain the matrix $[\omega]$ as the following:

$$\begin{bmatrix} a & -\alpha b & -\beta c & -\alpha\beta d\\ b & a & -\beta d & \beta c\\ c & \alpha d & a & -\alpha b\\ d & -c & b & a \end{bmatrix}(a,b,c,d\in\mathbb{R}).$$

Moreover, by Lemma 3.1 we know that $[\omega]$ is the same as G. Hence, ω is an algebra isomorphism.

Similarly, let δ be a right centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$, and we obtain the matrix form of δ as the following:

$$\begin{bmatrix} a & -\alpha b & -\beta c & -\alpha\beta d\\ b & a & \beta d & -\beta c\\ c & -\alpha d & a & \alpha b\\ d & c & -b & a \end{bmatrix}(a,b,c,d\in\mathbb{R}).$$

By Lemma 3.2 we know that $[\delta]$ is the same as G^*. Hence, δ is an algebra anti-isomorphism. □

Remark For $q \in H_{\alpha,\beta}$, define right multiplication operator $q^{(r)} : H_{\alpha,\beta} \to H_{\alpha,\beta}, x \mapsto xq$. Next, we can verify that $q^{(r)}$ is a right centralizer. Let $H^{(r)} = \{q^{(r)}, q \in H_{\alpha,\beta}\}$. For $q_1^{(r)}, q_2^{(r)} \in H^{(r)}$ and $x \in H_{\alpha,\beta}$, we define

$$(q_1^{(r)} + q_2^{(r)})(x) = q_1^{(r)}(x) + q_2^{(r)}(x),$$
$$(q_1^{(r)} q_2^{(r)})(x) = q_1^{(r)}(q_2^{(r)}(x)) = (q_1^{(r)} \circ q_2^{(r)})(x).$$

Let $\Phi : H_{\alpha,\beta} \to H^{(r)}, q \mapsto q^{(r)}$. For $q_1, q_2 \in H_{\alpha,\beta}$, we have $\Phi(q_1 q_2) = \Phi(q_2)\Phi(q_1)$. If we regard $H_{\alpha,\beta}$ as a vector space M of dimension 4 over $\mathbb{R}$, and let $Aut(M)$ be anti-automorphism ring of M, then we have $H^{(r)} \subseteq Aut(M)$.

4. Jordan centralizers on generalized quaternion algebras

In this last section, we study Jordan centralizers on generalized quaternion algebras over $\mathbb{R}$, and we show that every left (resp, right) Jordan centralizer is a left (resp, right) centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$.

Theorem 4.1. Every left Jordan centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$ is a left centralizer, and every right Jordan centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$ is a right centralizer.

Proof. Let τ be a left Jordan centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$, since τ admits a matrix representation with respect to the basis $\mathcal{B}(H_{\alpha,\beta})$, which is the 4×4 matrix $[\tau] = (d_{ij})^{\mathrm{T}}$ whose entries are defined by the following equations:

$$\tau(e_{i-1}) = \sum_{j=1}^{4} d_{ij} e_{j-1}, \qquad 1 \le i \le 4.$$

Each column of $[\tau]$ is an element of $H_{\alpha,\beta}$. In order to obtain $[\tau]$, we apply ω to the products $e_i e_j$ with $0 \le i, j \le 3$. From

$$\begin{aligned} -\alpha d_{11} e_0 - \alpha d_{12} e_1 - \alpha d_{13} e_2 - \alpha d_{14} e_3 &= -\alpha\tau(e_0) = \tau(e_1^2) \\ &= \tau(e_1)e_1 = (d_{21}e_0 + d_{22}e_1 + d_{23}e_2 + d_{24}e_3)e_1 \\ &= -\alpha d_{22}e_0 + d_{21}e_1 + \alpha d_{24}e_2 - d_{23}e_3, \end{aligned}$$

we have $d_{21} = -\alpha d_{12}, d_{22} = d_{11}, d_{23} = \alpha d_{14}, d_{24} = -d_{13}$. Since

$$\begin{aligned} -\beta d_{11} e_0 - \beta d_{12} e_1 - \beta d_{13} e_2 - \beta d_{14} e_3 &= -\beta\tau(e_0) = \tau(e_2^2) \\ &= \tau(e_2)e_2 = (d_{31}e_0 + d_{32}e_1 + d_{33}e_2 + d_{34}e_3)e_2 \\ &= -\beta d_{33}e_0 - \beta d_{34}e_1 + d_{31}e_2 + d_{32}e_3, \end{aligned}$$

we have $d_{31} = -\beta d_{13}, d_{32} = -\beta d_{14}, d_{33} = d_{11}, d_{34} = d_{12}$.

We are now going to check the same procedure for $e_3^2 = -\alpha\beta e_0$. We get

$$-\alpha\beta d_{11}e_0 - \alpha\beta d_{12}e_1 - \alpha\beta d_{13}e_2 - \alpha\beta d_{14}e_3 = -\alpha\beta d_{44}e_0 + \beta d_{43}e_1 - \alpha d_{42}e_2 + d_{41}e_3,$$

which implies $d_{41} = -\alpha\beta d_{14}, d_{42} = \beta d_{13}, \beta d_{43} = -\alpha\beta d_{12}, d_{44} = d_{11}$. Let $d_{11} = a, d_{12} = b, d_{13} = c$, $d_{14} = d$, and combining all these together, we obtain

$$d_{22} = d_{33} = d_{44} = a,$$

$$
\begin{aligned}
&d_{21} = -\alpha b, \quad d_{34} = b, \quad d_{43} = -\alpha b, \\
&d_{31} = -\beta c, \quad d_{24} = -c, \quad d_{42} = \beta c, \\
&d_{23} = \alpha d, \quad d_{32} = -\beta d, \quad d_{41} = -\alpha\beta d.
\end{aligned}
$$

Thus, we obtain $[\tau]$ as the following:

$$
\begin{bmatrix} a & -\alpha b & -\beta c & -\alpha\beta d \\ b & a & -\beta d & \beta c \\ c & \alpha d & a & -\alpha b \\ d & -c & b & a \end{bmatrix} (a, b, c, d \in \mathbb{R}).
$$

Let ρ be a Jordan right centralizer on $H_{\alpha,\beta}$ over $\mathbb{R}$, and similarly, one has $[\rho]$ as the following:

$$
\begin{bmatrix} a & -\alpha b & -\beta c & -\alpha\beta d \\ b & a & \beta d & -\beta c \\ c & -\alpha d & a & \alpha b \\ d & c & -b & a \end{bmatrix} (a, b, c, d \in \mathbb{R}).
$$

By Theorem 3.3, τ is a left centralizer, and ρ is a right centralizer. □

5. Conclusions

In this paper, we first obtain the matrix representation of the Jordan semi-triple derivation on generalized quaternion algebras over the field of real numbers and have given the matrix representation of the derivation on generalized quaternion algebras over the real number field in [20]. Thus, we obtain the condition that a Jordan semi-triple derivation on generalized quaternion algebras over the field of real numbers is a derivation. Second, we show that the left centralizer on the generalized quaternion algebra over the real number field is an algebra isomorphism, and the right centralizer is an algebra anti isomorphism. We further obtain the equivalent relationship between the left (resp, right) Jordan centralizer and the left (resp, right) centralizer on generalized quaternion algebras over the field of real numbers. In future work, we will further study other mappings on generalized quaternion algebra and their relationships.

Acknowledgments

The authors thank the referee for his constructive suggestion and careful reading of the manuscript.

Conflict of interest

We declare that we have no conflicts of interest.

References

1. B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolin.*, **32** (1991), 609–614.

2. J. Vukman, I. Kosi-Ulbl, Centralisers on rings and algebras, *B. Aust. Math. Soc.*, **71** (2005), 225–234. https://doi.org/10.1017/S000497270003820X

3. W. Du, J. Zhang, Jordan semi-triple derivable maps of matrix algebras, *Acta. Math. Sinica.*, **51** (2008), 571–578. https://doi.org/10.3321/j.issn:0583-1431.2008.01.016

4. L. Chen L, J. Zhang, $*$-Jordan semi-triple derivable mappings, *Indian J. Pure Appl. Math.*, **51** (2020), 825–837. https://doi.org/10.1007/s13226-020-0434-4

5. W. R. Hamilton, Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time, *Trans. R. Irish Acad.*, **17** (1835), 293–422.

6. Z. Kurt, Ö. N. Gerek, A. Bilge, K. Özkan, A graph-based recommendation algorithm on quaternion algebra, *SN Comput. Sci.*, **3** (2022), 299. https://doi.org/10.1007/s42979-022-01171-4

7. A. M. Grigoryan, S. S. Agaian, Commutative quaternion algebra and DSP fundamental properties: Quaternion convolution and Fourier transform, *Signal Process.*, **196** (2022), 108533. https://doi.org/10.1016/j.sigpro.2022.108533

8. J. Voight, *The arithmetic of quaternion algebras*, 2014.

9. A. Bouhlal, N. Safouane, A. Achak, R. Daher, Wavelet transform of Dini Lipschitz functions on the quaternion algebra, *Adv. Appl. Clifford Algebras*, **31** (2021), 8. https://doi.org/10.1007/s00006-020-01112-5

10. S. Malev, The images of noncommutative polynomials evaluated on the quaternion algebra, *J. Algebra Appl.*, **20** (2021), 2150074. https://doi.org/10.1142/S0219498821500742

11. T. Csahók, P. Kutas, M. Montessinos, G. Zábrádi, Explicit isomorphisms of quaternion algebras over quadratic global fields, *Res. Number Theory*, **8** (2022), 77. https://doi.org/10.1007/s40993-022-00380-3

12. H. Boylan, N. P. Skoruppa, H. Zhou, Counting zeros in quaternion algebras using Jacobi forms, *Trans. Amer. Math. Soc.*, **371** (2019), 6487–6509. https://doi.org/10.1090/tran/7575

13. L. Rodman, Topics in quaternion linear algebra, In: *Topics in quaternion linear algebra*, Princeton: Princeton University Press, 2014. https://doi.org/10.23943/princeton/9780691161853.001.0001

14. Z. Jia, M. K. Ng, G. J. Song, Robust quaternion matrix completion with applications to image inpainting, *Numer. Linear Algebra Appl.*, **26** (2019), e2245. https://doi.org/10.1002/nla.2245

15. Z. H. He, M. Wang, X. Liu, On the general solutions to some systems of quaternion matrix equations, *RACSAM*, **114** (2020), 95. https://doi.org/10.1007/s13398-020-00826-2

16. L. S. Liu, Q. W. Wang, J. F. Chen, Y. Z. Xie, An exact solution to a quaternion matrix equation with an application, *Symmetry*, **14** (2022), 375. https://doi.org/10.3390/sym14020375

17. A. B. Mamagani, M. Jafari, On properties of generalized quaternion algebra, *J. Nov. Appl. Sci.*, **2** (2013), 683–689.

18. H. Ghahramani, M. N. Ghosseiriand, L. H. Zadeh, Generalized derivations and generalized Jordan derivations of quaternion rings, *Iran. J. Sci. Technol. Trans. Sci.*, **45** (2021), 305–308. https://doi.org/10.1007/s40995-020-01046-4

19. H. Ghahramani, M. N. Ghosseiriand, L. H. Zadeh, On the Lie derivations and generalized Lie derivations of quaternion rings, *Commun. Algebra*, **47** (2019), 1215–1221. https://doi.org/10.1080/00927872.2018.1501577

20. E. Kizil E, Y. Alagöz, Derivations of generalized quaternion algebra, *Turk. J. Math.*, **43** (2019), 2649–2657. https://doi.org/10.3906/mat-1905-86