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Research article

A new modified iterative scheme for finding common fixed points in Banach spaces: application in variational inequality problems

Shahram Rezapour[1,2,3], **Maryam Iqbal**[4], **Afshan Batool**[4], **Sina Etemad**[1] **and Thongchai Botmart**[5,*]

[1] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
[2] Department of Mathematics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul, Republic of Korea
[3] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
[4] Department of Mathematical Sciences, Fatima Jinnah Women University, Rawalpindi, Pakistan
[5] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

* **Correspondence:** Email: thongbo@kku.ac.th.

Abstract: This paper reports a modified F-iterative process for finding the fixed points of three generalized α-nonexpansive mappings. We assume certain assumptions to establish the weak and strong convergence of the scheme in the context of a Banach space. We suggest a numerical example of generalized α-nonexpansive mappings which exceeds, properly, the category of functions furnished with a condition (C). After that, we show that our modified F-iterative scheme of this example converges to a common fixed point of three generalized α-nonexpansive mappings. As an application of our main findings, we suggest a new projection-type iterative scheme to solve variational inequality problems in the setting of generalized α-nonexpansive mappings. The main finding of the paper is new and extends many known results of the literature.

Keywords: F-iteration; common fixed point; generalized α-nonexpansive mapping; convergence; condition (I)
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1. Introduction

Suppose that we have a Banach space $\mathcal{B}$ with the norm $\|.\|$. If M denotes any subset of $\mathcal{B}$, then the self-map Ω is called a contraction on M (sometimes called a θ–contraction to indicate the constant θ

involved in the definition) provided that, for all $\breve{w}, \hat{w} \in M$,

$$\|\Omega\breve{w} - \Omega\hat{w}\| \leq \theta\|\breve{w} - \hat{w}\| \tag{1.1}$$

holds, where $\theta \in [0, 1)$.

Once the inequality (1.1) is to be held for $\theta = 1$, then Ω is known as nonexpansive. If $w^* \in M$ exists such that $w^* = \Omega w^*$, then w^* is named as a fixed point for Ω, and for the set of all fixed points of Ω, we shall denote it by F_Ω. In 1922, Banach introduced his famous result in relation to contractions known as the Banach contraction principle (BCP), which suggests the existence of a unique fixed point for a given contraction. In 1965, Kirk [1], Browder [2] and Gohde [3] studied the fixed point's existence for nonexpansive mappings and finally obtained, independently, a very interesting fixed-point theorem for these mappings in the context of a uniformly convex Banach space (UCBS). Soon, Goebel [4] obtained an elementary proof for the Kirk-Browder-Gohde theorem. It has been shown by many authors that nonexpansive mappings appear naturally in the study of nonlinear problems in different structures of distance spaces [5–7]. Thus, it is very natural to study extensions of these mappings. Among the other things, Suzuki [8] came with a condition for mappings which he named as a condition (C). Notice that the self-map Ω is said to be equipped with the condition (C) provided that $\forall\, \breve{w}, \hat{w} \in M$; we have

$$\frac{1}{2}\|\breve{w} - \Omega\breve{w}\| \leq \|\breve{w} - \hat{w}\| \Rightarrow \|\Omega\breve{w} - \Omega\hat{w}\| \leq \|\breve{w} - \hat{w}\|.$$

It is obvious that every nonexpansive mapping satisfies this condition. However, in general, by a simple example, we see that the converse is not to be held.

Example 1.1. *[8] Consider Ω as a self-map on a subset $M = [0, 3]$ of $\mathcal{B} = \mathbb{R}$ defined by:*

$$\Omega\breve{w} = \begin{cases} 0 & \text{if } \breve{w} \neq 3, \\ 1 & \text{if } \breve{w} = 3. \end{cases}$$

Here, Ω admits the condition (C) but is not nonexpansive.

In 2011, Aoyama and Kohsaka [9] introduced a new type of generalization of nonexpansive mappings, as follows: The self-map Ω is said to be α-nonexpansive if one can find some $\alpha \in [0, 1)$ such that, for each $\breve{w}, \hat{w} \in M$, we have

$$\|\Omega\breve{w} - \Omega\hat{w}\|^2 \leq \alpha\|\breve{w} - \Omega\hat{w}\|^2 + \alpha\|\hat{w} - \Omega\breve{w}\|^2 + (1 - 2\alpha)\|\breve{w} - \hat{w}\|^2.$$

On the other hand, Pant and Shukla [10] introduced a larger class of nonlinear self-maps that includes properly nonexpansive mappings with the condition (C), and it extends the notion of being α-nonexpansive. Notice that the self-map Ω is said to be generalized α-nonexpansive if one can find some $\alpha \in [0, 1)$ such that $\forall\, \breve{w}, \hat{w} \in M$,

$$\frac{1}{2}\|\breve{w} - \Omega\breve{w}\| \leq \|\breve{w} - \hat{w}\| \Rightarrow \|\Omega\breve{w} - \Omega\hat{w}\| \leq \alpha\|\breve{w} - \Omega\hat{w}\| + \alpha\|\hat{w} - \Omega\breve{w}\| + (1 - 2\alpha)\|\breve{w} - \hat{w}\|.$$

The next example shows that the notion of generalized α-nonexpansive mappings is more general than the notions of mappings furnished with the condition (C), as well as α-nonexpansive mappings.

Example 1.2. *[10] Let $\mathcal{B} = \mathbb{R}^2$ with $\|(\breve{w}_1, \breve{w}_2)\| = |\breve{w}_1| + |\breve{w}_2|$. Take*

$$M = \{(0,0),(2,0),(0,4),(4,0),(4,5),(5,4)\},$$

and set $\Omega : M \to M$ by

$$\Omega\breve{w} = \begin{cases} (0,0) & \text{if } \breve{w} = (0,0), \\ (0,0) & \text{if } \breve{w} = (2,0), \\ (0,0) & \text{if } \breve{w} = (0,4), \\ (2,0) & \text{if } \breve{w} = (4,0), \\ (4,0) & \text{if } \breve{w} = (4,5), \\ (0,4) & \text{if } \breve{w} = (5,4). \end{cases}$$

Here, Ω is neither equipped with the condition (C), nor it is α-nonexpansive. However, Ω is generalized α-nonexpansive.

Now, we give an example of generalized α-nonexpansive maps on infinite dimensional space.

Example 1.3. *[11] Consider the Banach space $X = L^\infty(\mathbb{R})$ of all essentially bounded Lebesgue measurable functions endowed with the essential supremum norm*

$$\| f \|_\infty = ess\, sup_{\mathbb{R}} \mid f \mid = inf\{M :\mid f(x) \mid \leq M \ \ a.e. \ \ on\, \mathbb{R}\}.$$

Define $C = \{f : \mathbb{R} \to [0,7] : f(x) = f(0), \forall x \leq 0\}$ and $T : C \to C$ by

$$Tf(x) = \begin{cases} f(x), & x > 0, \\ \frac{2}{7}f(0), & x \leq 0, f(0) \neq 7, \\ 3, & x \leq 0, f(0) = 7. \end{cases}$$

The mapping T is generalized α-nonexpansive with $\alpha = 0$.

Fixed-point theory suggests very fruitful and alternative techniques for existence, as well as an iterative approximation of desired solutions for those problems of the applied sciences for which one is unable to find the analytical value of the requested solution. To do this, first, we need to express the requested desired solution in the form of a fixed point of an operator defined possibly on an appropriate subset of a Banach space (or a complete metric space, if possible). The proof of the BCP, however, suggests a Picard [12] iteration for finding the approximate value of the unique fixed point of contractions. Moreover, the applicability of different versions of the Banach principle can be found in different applied articles in which fixed-point theorems play a fundamental role in establishing the existence of solutions for a variety of mathematical models and boundary value problems [13–21]. A very important branch is the involvement of fixed points in approximation by algorithms. Numerous problems, such as equilibrium problems, optimization problems, feasibility problems and monotone variational inequalities, can be thought of as fixed-point problems [22–28]. Besides, if the operator is nonexpansive on its domain, then it possesses a fixed point if we impose some extra conditions on its domain, but the limit value of the Picard iteration may not give a fixed point for the given nonexpansive mapping. Hence, to overcome such a case and get relatively high accurate convergence, some authors proposed other iterative schemes, which are essentially generalized and faster than the Picard iterative scheme (for instance, see Mann [29], Ishikawa [30], three-step Noor [31], Agarwal et al. (also called the S-iterative scheme) [32], Abbas and Nazir [33], Thakur et al. [34], M-iteration of Ullah and Arshad [35] and others). On the other hand, Ali and Ali [36] suggested a new iteration, namely, the F-iteration, and proved its convergence and stablity for generalized contractions in the context of a Banach space. Later, Ahmad et al. [37] improved their results for the general class of nonlinear functions of generalized α-nonexpansive mappings.

Motivated by the above, we here implement a generalized F-iterative scheme as follows: Let us consider this hypothsis that we have three generalized α-nonexpansive mappings, namely, $\Omega_1, \Omega_2, \Omega_3 : M \to M$. Then, the modified F-iterative process acts as follows:

$$\begin{cases} \breve{w}_1 \in M, \\ \breve{v}_k = \Omega_1[(1-a_k)\breve{w}_k + a_k\Omega_1\breve{w}_k], \\ \breve{u}_k = \Omega_2\breve{v}_k, \\ \breve{w}_{k+1} = \Omega_3\breve{u}_k, k \geq 1, \end{cases} \tag{1.2}$$

where $0 < a_k < 1$.

Although Ahmad et al. [37] proved several convergence results of the F-iterative scheme for generalized α-nonexpansive mappings, in this paper, we extend their main outcome to the general structures of common fixed points, that is, we connect the modified F-iterative scheme (1.2) for finding common fixed points of three generalized α-non expansive mappings because there is limited literature on iterative schemes for more than one map. First, we will prove some basic weak and strong convergence theorems using the process described by (1.2). After this, we will give an appropriate numerical example. Using this example, we will prove that our scheme converges to a common fixed-point under the conditions of various parameters and starting points. Graphical representation of the convergence is also presented. Eventually, as an application of our main findings, we will introduce a projection-type scheme based on our scheme (1.2) and prove that it converges to a solution of a broad class of problems.

2. Preliminaries

In this section, to recall some preliminaries, and for our main results, we need some basic concepts and notions, as follows.

Definition 2.1. *[38, 39] Assume that $\{\breve{w}_k\}$ is bounded in a closed convex subset M of a UCBS $\mathcal{B}$. Notice that the asymptotic radius for the sequence $\{\breve{w}_k\}$ in relation to the set M is $r(M, \{\breve{w}_k\}) = \inf\{\limsup_{k\to\infty} \|\breve{w}_k - \breve{w} : \breve{w} \in M\}$, while the asymptotic center for the sequence $\{\breve{w}_k\}$ in relation to the set M is $A(M, \{\breve{w}_k\}) = \{\breve{w} \in M : \limsup_{k\to\infty} \|\breve{w}_k - \breve{w}\| = r(M, \breve{w}_k)\}$. In this case, the set $A(M, \{\breve{w}_k\})$ admits one and only one point.*

Definition 2.2. *[40] We say that a self-map Ω on the subset M is said to be equipped with the condition (I) whenever one has a function f s.t. $f(s) > 0\ \forall\, r > 0$ and $f(s) = 0 \Leftrightarrow s = 0$, and, furthermore, for all $\breve{w} \in M$, $\|\breve{w} - \Omega\breve{w}\| \geq f(d(\breve{w}, F_\Omega))$. Here, the notation $d(\breve{w}, F_\Omega)$ represents a norm distance between $\breve{w}$ and the set F_Ω.*

The following condition was taken from Opial [41].

Definition 2.3. *If $\mathcal{B}$ denotes a Banach space, then $\mathcal{B}$ is said to be furnished with Opial's condition if for every sequence $\{\breve{w}_k\} \subseteq \mathcal{B}$ weakly converging to some $\breve{w} \in \mathcal{B}$, one has*

$$\limsup_{k\to\infty} \|\breve{w}_k - \breve{w}\| < \limsup_{k\to\infty} \|\breve{w}_k - \hat{w}\|$$

for all other $\hat{w}$, which are different from $\breve{w}$ in the space $\mathcal{B}$.

In [10], the authors proved some basic characterizations of generalized α-nonexpansive mappings, as follows.

Proposition 2.4. *[10] We assume a self-map Ω on a subset M of a Banach space $\mathcal{B}$. In this case, we have the following*

(i) Whenever Ω admits the condition (C), then Ω forms a generalized α-nonexpansive mapping.
(ii) Whenever Ω forms a generalized α-nonexpansive mapping and admits at least one fixed point, namely, $\breve{p}$, then $\|\Omega\breve{w} - \Omega\breve{p}\| \leq \|\breve{w} - \breve{p}\|$ for all $\breve{w} \in M$.
(iii) Whenever Ω forms a generalized α-nonexpansive mapping, then its fixed point set F_Ω is closed.
(iv) When Ω forms a generalized α-nonexpansive mapping, the inequality

$$\|\breve{w} - \Omega\hat{w}\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\|\breve{w} - \Omega\breve{w}\| + \|\breve{w} - \hat{w}\|$$

holds, where $\breve{w}, \hat{w} \in M$ are any elements.
(v) When Ω forms a generalized α-nonexpansive mapping and $\mathcal{B}$ admits Opial's condition, in this case, if $\{\breve{w}_k\}$ admits a weak limit, namely, $\breve{p}$ and $\lim_{k\to\infty} \|\breve{w}_k - \Omega\breve{w}_k\| = 0$, then $\breve{p} \in F_\Omega$.

Lemma 2.5. *[42] Let $\mathcal{B}$ be a UCBS, $a_k \in (0,1)$ and $\{\breve{w}_k\}$ and $\{\hat{w}_k\}$ be any sequences in $\mathcal{B}$. If $\limsup_{k\to\infty} \|\breve{w}_k\| \leq \epsilon$, $\limsup_{k\to\infty} \|\hat{w}_k\| \leq \epsilon$ and $\lim_{k\to\infty} \|a_k\breve{w}_k + (1-a_k)\hat{w}_k\| = \epsilon$ for some $\epsilon \geq 0$, then*

$$\lim_{k\to\infty} \|\breve{w}_k - \hat{w}_k\| = 0.$$

3. Main results

Now, we use the scheme (1.2) to obtain certain weak and strong convergence results for three generalized α-nonexpansive mappings Ω_1, Ω_2 and Ω_3. From now on, in this section, we specify by $\mathcal{F}$, the set $F_{\Omega_1} \cap F_{\Omega_1} \cap F_{\Omega_1}$. We also write simply $\mathcal{B}$ for a given UCBS. We begin this section with a fundamental lemma.

Lemma 3.1. *Suppose that the set M is convex closed in $\mathcal{B}$ and Ω_1, Ω_2 and Ω_3 are three generalized α-nonexpansive mappings with $\mathcal{F} \neq \emptyset$. If the sequence $\{\breve{w}_k\}$ is generated by the modified F-iterates given by (1.2), then $\lim_{k\to\infty} \|\breve{w}_k - \breve{p}\|$ exists $\forall\, \breve{p} \in \mathcal{F}$.*

Proof. Set $\breve{p} \in \mathcal{F}$. By Proposition 2.4(ii), one has

$$\begin{aligned}\|\breve{v}_k - \breve{p}\| &= \|\Omega_1[(1-a_k)\breve{w}_k + a_k\Omega_1\breve{w}_k - \breve{p}]\| \\ &= \|(1-a_k)\breve{w}_k + a_k\Omega_1\breve{w}_k - \breve{p}\|\end{aligned}$$

$$\begin{aligned} &\leq (1-a_k)\|\breve{w}_k-\breve{p}\|+a_k\|\Omega_1\breve{w}_k-\breve{p}\| \\ &\leq (1-a_k)\|\breve{w}_k-\breve{p}\|+a_k\|\breve{w}_k-\breve{p}\| \\ &\leq \|\breve{w}_k-\breve{p}\|, \end{aligned}$$

and

$$\begin{aligned} \|\breve{u}_k-\breve{p}\| &= \|\Omega_2\breve{v}_k-\breve{p}\| \\ &\leq \|\breve{v}_k-\breve{p}\|. \end{aligned}$$

These relations imply that

$$\begin{aligned} \|\breve{w}_{k+1}-p\| &= \|\Omega_3\breve{u}_k-\breve{p}\| \\ &\leq \|\breve{u}_k-\breve{p}\| \\ &\leq \|\breve{v}_k-\breve{p}\| \\ &\leq \|\breve{w}_k-\breve{p}\|. \end{aligned}$$

Thus, we have seen that $\{\|\breve{w}_k-\breve{p}\|\}$ is bounded and also non-increasing. Accordingly, we can write that $\lim_{k\to\infty}\|\breve{w}_k-\breve{p}\|$ exists for each $\breve{p}\in\mathcal{F}$. □

Theorem 3.2. *Suppose that the set M is convex closed in $\mathcal{B}$, Ω_1, Ω_2 and Ω_3 are three generalized α-nonexpansive mappings and the sequence $\{\breve{w}_k\}$ is generated by the modified F-iterates given by (1.2). Then, $\mathcal{F}\neq\emptyset$ iff $\{\breve{w}_k\}$ is bounded in M and* $\lim_{k\to\infty}\|\Omega_1\breve{w}_k-\breve{w}_k\| = \lim_{k\to\infty}\|\Omega_2\breve{w}_k-\breve{w}_k\| = \lim_{k\to\infty}\|\Omega_3\breve{w}_k-\breve{w}_k\| = 0$.

Proof. Let $F\neq\emptyset$. Given a fixed $\breve{p}\in M$, thanks to Lemma 3.1, $\lim_{k\to\infty}\|\breve{w}_k-\breve{p}\|$ exists and $\{\breve{w}\}$ is bounded. For some $\epsilon>0$, put

$$\lim_{k\to\infty}\|\breve{w}_k-\breve{p}\| = \epsilon. \tag{3.1}$$

Also, considering the proof of Lemma 3.1 and keeping (3.1), give

$$\limsup_{k\to\infty}\|\breve{v}_k-\breve{p}\| \leq \limsup_{k\to\infty}\|\breve{w}_k-\breve{p}\| = \epsilon.$$

On the other hand, using Proposition 2.4(ii), one has

$$\limsup_{k\to\infty}\|\Omega_1\breve{w}_k-\breve{p}\| \leq \limsup_{k\to\infty}\|\breve{w}_k-\breve{p}\| = \epsilon,$$

$$\limsup_{k\to\infty}\|\Omega_2\breve{w}_k-\breve{p}\| \leq \limsup_{k\to\infty}\|\breve{w}_k-\breve{p}\| = \epsilon,$$

$$\limsup_{k\to\infty}\|\Omega_3\breve{w}_k-\breve{p}\| \leq \limsup_{k\to\infty}\|\breve{w}_k-\breve{p}\| = \epsilon.$$

Again, in the proof of Lemma 3.1, and by using (3.1), we get

$$\epsilon = \liminf_{k\to\infty}\|\breve{u}_{k+1}-\breve{p}\| \leq \liminf_{k\to\infty}\|\breve{v}_k-\breve{p}\|. \tag{3.2}$$

Now, using (3.1) and (3.2), we have

$$\lim_{k\to\infty} \|\breve{v}_k - \breve{p}\| = \epsilon. \tag{3.3}$$

Hence, using (3.3), we write

$$\begin{aligned}
\epsilon = \lim_{k\to\infty} \|\breve{v}_k - \breve{p}\| &= \lim_{k\to\infty} \|\Omega_1[(1-a_k)\breve{w}_k + a_k\Omega_1\breve{w}_k] - \breve{p}\| \\
&\leq \lim_{k\to\infty} \|(1-a_k)\breve{w}_k + a_k\Omega_1\breve{w}_k - \breve{p}\| \\
&= \lim_{k\to\infty} \|(1-a_k)(\breve{w}_k - \breve{p}) + \alpha_k(\Omega_1\breve{w}_k - \breve{p})\| \\
&\leq \lim_{k\to\infty} \|(1-a_k)(\breve{w}_k - \breve{p})\| + \lim_{k\to\infty} \|a_k(\Omega_1\breve{w}_k - \breve{p})\| \\
&\leq \lim_{k\to\infty}(1-a_k)\|\breve{w}_k - \breve{p}\| + \lim_{k\to\infty} a_k\|\breve{w}_k - \breve{p}\| \\
&= \lim_{k\to\infty} \|\breve{w}_k - \breve{p}\| \\
&= \epsilon,
\end{aligned}$$

if and only if

$$\epsilon = \lim_{k\to\infty} \|(1-a_k)(\breve{w}_k - \breve{p}) + a_k(\Omega_1\breve{w}_k - \breve{p})\|.$$

Applying Lemma 2.5, we obtain

$$\lim_{k\to\infty} \|\Omega_1\breve{w}_k - \breve{w}_k\| = 0.$$

Similarly, one can show that $\lim_{k\to\infty} \|\Omega_2\breve{w}_k - \breve{w}_k\| = 0$ and $\lim_{k\to\infty} \|\Omega_3\breve{w}_k - \breve{w}_k\| = 0$.

In contrast, we consider the bounded sequence $\{\breve{w}_k\}$ in M and $\lim_{k\to\infty} \|\Omega_1\breve{w}_k - \breve{w}_k\| = \lim_{k\to\infty} \|\Omega_2\breve{w}_k - \breve{w}_k\| = \lim_{k\to\infty} \|\Omega_3\breve{w}_k - \breve{w}_k\| = 0$. It should be proved that $\mathcal{F} \neq \emptyset$. For this, let $\breve{p} \in A(M, \{\breve{w}_k\})$. If we apply Proposition 2.4(iv), then one can observe the following inequalities:

$$\begin{aligned}
r(\Omega_1\breve{p}, \{\breve{w}_k\}) &= \limsup_{k\to\infty} \|\breve{w}_k - \Omega_1\breve{p}\| \\
&\leq \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{k\to\infty} \|\Omega_1\breve{w}_k - \breve{w}_k\| + \limsup_{k\to\infty} \|\breve{w}_k - \breve{p}\| \\
&= \limsup_{k\to\infty} \|\breve{w}_k - \breve{p}\| \\
&= r(\breve{p}, \{\breve{w}_k\}),
\end{aligned}$$

and

$$\begin{aligned}
r(\Omega_2\breve{p}, \{\breve{w}_k\}) &= \limsup_{k\to\infty} \|\breve{w}_k - \Omega_2\breve{p}\| \\
&\leq \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{k\to\infty} \|\Omega_2\breve{w}_k - \breve{w}_k\| + \limsup_{k\to\infty} \|\breve{w}_k - \breve{p}\| \\
&= \limsup_{k\to\infty} \|\breve{w}_k - \breve{p}\|
\end{aligned}$$

$$= r(\breve{p}, \{\breve{w}_k\}),$$

and

$$\begin{aligned}
r(\Omega_3\breve{p}, \{\breve{w}_k\}) &= \limsup_{k\to\infty} \|\breve{w}_k - \Omega_3\breve{p}\| \\
&\leq \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{k\to\infty} \|\Omega_3\breve{w}_k - \breve{w}_k\| + \limsup_{k\to\infty} \|\breve{w}_k - \breve{p}\| \\
&= \limsup_{k\to\infty} \|\breve{w}_k - \breve{p}\| \\
&= r(\breve{p}, \{\breve{w}_k\}).
\end{aligned}$$

In the above cases, we obtained $\Omega_1\breve{p} \in A(M, \{\breve{w}_k\})$, $\Omega_2\breve{p} \in A(M, \{\breve{w}_k\})$ and $\Omega_3\breve{p} \in A(M, \breve{w}_k\})$. Since $A(M, \{\breve{w}_k\})$ possesses only one element, we get $\Omega_1\breve{p} = \Omega_2\breve{p} = \Omega_3\breve{p} = \breve{p}$. Thus, $\breve{p} \in \mathcal{F}$ and, hence, $\mathcal{F} \neq \emptyset$. □

First, we obtain the weak convergence for our scheme.

Theorem 3.3. *Suppose that the set M is convex closed in $\mathcal{B}$ and Ω_1, Ω_2 and Ω_3 are three generalized α-nonexpansive mappings with $\mathcal{F} \neq \emptyset$. If $\mathcal{B}$ fulfills Opial's criterion, then the sequence $\{\breve{w}_k\}$ generated by the modified F-iterates given by (1.2) is weakly convergent to a common fixed point of Ω_1, Ω_2 and Ω_3.*

Proof. By Theorem 3.2, $\{\breve{w}_k\}$ is bounded in M. Since $\mathcal{B}$ is a UCBS, $\mathcal{B}$ will be reflexive; hence, the bounded sequence $\{\breve{w}_k\}$ admits a weakly convergent subsequence $\{\breve{w}_{k_r}\}$ with a weak limit, namely, $\breve{w}_1 \in M$. If we apply Theorem 3.2 to this subsequence, we obtain $\lim_{r\to\infty} \|\breve{w}_{k_r} - \Omega_i\breve{w}_{k_r}\| = 0$, where $i = 1, 2, 3$. Thus, by Proposition 2.4(v), one has $\breve{w}_1 \in F_{\Omega_i}$ where $i = 1, 2, 3$. If we prove that $\breve{w}_1$ is a weak limit for $\{\breve{w}_k\}$, then the proof will be finished. We prove this by contradiction. Suppose that $\breve{w}_1$ is not a weak limit for $\{\breve{w}_k\}$, that is, another subsequence $\{\breve{w}_{k_s}\}$ of $\{\breve{w}_k\}$ exists which admits a weak limit $\breve{w}_2 \in M$. The same calculations give $\breve{w}_2 \in F_{\Omega_i}$. Now, we know that $\mathcal{B}$ admits Opial's criterion, so one has

$$\begin{aligned}
\lim_{k\to\infty} \|\breve{w}_k - \breve{w}_1\| &= \lim_{r\to\infty} \|\breve{w}_{k_r} - \breve{w}_1\| \\
&< \lim_{r\to\infty} \|\breve{w}_{k_r} - \breve{w}_2\| \\
&= \lim_{k\to\infty} \|\breve{w}_k - \breve{w}_2\| \\
&= \lim_{s\to\infty} \|\breve{w}_{k_s} - \breve{w}_2\| \\
&< \lim_{s\to\infty} \|\breve{w}_{k_s} - \breve{w}_1\| \\
&= \lim_{k\to\infty} \|\breve{w}_k - \breve{w}_1\|.
\end{aligned}$$

The above estimate suggests a contradiction; hence, we must accept that $\breve{w}_1 = \breve{w}_2$. Accordingly, $\{\breve{w}_k\}$ converges weakly to $\breve{w}_1 \in F_{\Omega_i}$, where $i = 1, 2, 3$. Thus, $\{\breve{w}_k\}$ converges weakly to a common fixed point of Ω_1, Ω_2 and Ω_3. □

We now obtain a strong convergence for our scheme.

Theorem 3.4. *Suppose that the set M is convex closed in $\mathcal{B}$ and Ω_1, Ω_2 and Ω_3 are three generalized α-nonexpansive mappings with $\mathcal{F} \neq \emptyset$. If M is compact, then the sequence $\{\breve{w}_k\}$ generated by the modified F-iterates given by (1.2) is weakly convergent to a common fixed point of Ω_1, Ω_2 and Ω_3.*

Proof. Due to the convexity of M, we have that $\{\breve{w}_k\} \subseteq M$. Accordingly, we have a subsequence $\{\breve{w}_{k_r}\}$ of $\{\breve{w}_k\}$ with $\lim_{r\to\infty} \|\breve{w}_{k_r} - \breve{q}\| = 0$ for some $\breve{q} \in M$. On the other hand, using Theorem 3.2, $\lim_{r\to\infty} \|\Omega_i \breve{w}_{k_r} - \breve{w}_{k_r}\| = 0$, where $i = 1, 2, 3$. Thus, by Proposition 2.4(iv), one has

$$\|\breve{w}_{k_r} - \Omega_i \breve{q}\| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \|\breve{w}_{k_r} - \Omega_i \breve{w}_{k_r}\| + \|\breve{w}_{k_r} - \breve{q}\|.$$

Hence, if we let $r \to \infty$, then $\Omega_i \breve{q} = \breve{q}$, where $i = 1, 2, 3$. Hence, $\breve{q}$ is a common fixed point of Ω_1, Ω_2 and Ω_3, and, by Lemma 3.1, $\lim_{k\to\infty} \|\breve{w}_k - \breve{q}\|$ exists. Accordingly, $\breve{q}$ is also a strong limit of $\{\breve{w}_k\}$. □

Note that the strong convergence of our scheme on a non-compact domain is valid by following Theorem 3.5.

Theorem 3.5. *Suppose that the set M is closed and convex in $\mathcal{B}$ and Ω_1, Ω_2 and Ω_3 are three generalized α-nonexpansive mappings with $\mathcal{F} \neq \emptyset$. If*

$$\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_1}) = \liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_2}) = \liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_3}) = 0,$$

then the sequence $\{\breve{w}_k\}$ generated by the modified F-iterates given by (1.2) is strongly convergent to a common fixed point of Ω_1, Ω_2 and Ω_3.

Proof. Fix $\breve{p} \in F$. Thanks to Lemma 3.1, $\lim_{k\to\infty} \|\breve{w}_k - \breve{p}\|$ exists. Accordingly, $\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_1})$, $\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_2})$ and $\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_3})$ exist. Applying our assumptions, one has

$$\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_1}) = 0, \tag{3.4}$$

$$\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_2}) = 0, \tag{3.5}$$

and

$$\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_3}) = 0. \tag{3.6}$$

Now, (3.4) gives us subsequences $\{\breve{w}_{k_i}\}$ of $\{\breve{w}_k\}$ and $\{\breve{p}_i\}$ in F_{Ω_1} s.t.

$$\|\breve{w}_{k_{i+1}} - \breve{p}_i\| \leq \|\breve{w}_{k_i} - \breve{p}_i\| \leq \frac{1}{2^i}.$$

Therefore,

$$\begin{aligned} \|\breve{p}_{i+1} - \breve{p}_i\| &\leq \|\breve{p}_{i+1} - \breve{w}_{k_{i+1}}\| + \|\breve{w}_{k_{i+1}} - \breve{p}_i\| \\ &\leq \frac{1}{2^{i+1}} + \frac{1}{2^i} \\ &\leq \frac{1}{2^{i-1}} \to 0 \text{ as } i \to \infty. \end{aligned}$$

Consequently, we obtained that $\{\breve{p}_i\}$ is Cauchy in F_{Ω_1} and thus converges to some $\breve{p}_0$. By Proposition 2.4(iii) and (iv), F_{Ω_1} is closed; so, $\breve{p}_0 \in F_{\Omega_1}$. Similarly, using (3.5) and (3.6), we can prove that $\breve{p}_0 \in F_{\Omega_2}$ and $\breve{p}_0 \in F_{\Omega_3}$. Consequently, $\breve{p}_0 \in F$; thus, by Lemma 3.1, $\lim_{k\to\infty} \|\breve{w}_k - \breve{p}_0\|$ exists. In consequence, $\breve{p}_0$ is a common fixed point of Ω_1, Ω_2 and Ω_3, and it is also the strong limit of $\{\breve{w}_k\}$. □

Eventually, we discuss the strong convergence for our scheme by using the condition (I) given in Definition 2.2.

Theorem 3.6. *Suppose that the set M is closed and convex in $\mathcal{B}$ and Ω_1, Ω_2 and Ω_3 are three generalized α-nonexpansive mappings with $\mathcal{F} \neq \emptyset$. If Ω_1, Ω_2 and Ω_3 satisfy the condition (I) given in Definition 2.2, the sequence $\{\breve{w}_k\}$ generated by the modified F-iterates given by (1.2) is strongly convergent to a common fixed point of Ω_1, Ω_2 and Ω_3.*

Proof. First, looking into Theorem 3.2, we derive the following estimates:

$$\liminf_{k\to\infty} \|\Omega_1 \breve{w}_k - \breve{w}_k\| = 0, \tag{3.7}$$

$$\liminf_{k\to\infty} \|\Omega_2 \breve{w}_k - \breve{w}_k\| = 0, \tag{3.8}$$

and

$$\liminf_{k\to\infty} \|\Omega_3 \breve{w}_k - \breve{w}_k\| = 0. \tag{3.9}$$

Also, from the condition (I) given in Definition 2.2, one has

$$\|\breve{w}_k - \Omega_1 \breve{w}_k\| \geq f(d(\breve{w}_k, F_{\Omega_1})), \tag{3.10}$$

$$\|\breve{w}_k - \Omega_2 x_k\| \geq f(d(\breve{w}_k, F_{\Omega_2})), \tag{3.11}$$

$$\|\breve{w}_k - \Omega_3 \breve{w}_k\| \geq f(d(\breve{w}_k, F_{\Omega_3})). \tag{3.12}$$

Applying (3.7) to (3.10), (3.8) to (3.11) and (3.9) to (3.12), we have

$$\liminf_{k\to\infty} f(d(\breve{w}_k, F_{\Omega_1})) = 0,$$

$$\liminf_{k\to\infty} f(d(\breve{w}_k, F_{\Omega_2})) = 0,$$

and

$$\liminf_{k\to\infty} f(d(\breve{w}_k, F_{\Omega_3})) = 0.$$

Therefore,

$$\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_1}) = 0,$$

$$\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_2}) = 0,$$

and

$$\liminf_{k\to\infty} d(\breve{w}_k, F_{\Omega_3}) = 0.$$

Subsequently, all of the required conditions for Theorem 3.5 are valid; hence, $\{\breve{w}_k\}$ is strongly convergent to a common fixed point of Ω_1, Ω_2 and Ω_3. □

4. Numerical example

We now offer an example and prove numerically and graphically that our iterative scheme given in (1.2) converges to a common fixed-point for various cases.

Example 4.1. *Let $M = [5, 15]$. For the sake of simplicity, take $A = [5, 15)$ and $B = \{15\}$. Then, consider the self-maps $\Omega_i (i = 1, 2, 3)$ on M according to the following rules:*

$$\Omega_1 \breve{w} = \begin{cases} \frac{\breve{w}+5}{2} & \text{for } \breve{w} \in A, \\ 5 & \text{for } \breve{w} \in B, \end{cases}$$

$$\Omega_2 \breve{w} = \begin{cases} \frac{\breve{w}+10}{3} & \text{for } \breve{w} \in A, \\ 5 & \text{for } \breve{w} \in B, \end{cases}$$

$$\Omega_3 \breve{w} = \begin{cases} \frac{\breve{w}+15}{4} & \text{for } \breve{w} \in A, \\ 5 & \text{for } \breve{w} \in B. \end{cases}$$

Now, for $\breve{w} = 12.5$ and $\hat{w} = 15$, we have

(i) $\frac{1}{2}|\breve{w} - \Omega_1 \breve{w}| \leq |\breve{w} - \hat{w}| \Rightarrow |\Omega_1 \breve{w} - \Omega_1 \hat{w}| > |\breve{w} - \hat{w}|$,
(ii) $\frac{1}{2}|\breve{w} - \Omega_2 \breve{w}| \leq |\breve{w} - \hat{w}| \Rightarrow |\Omega_2 \breve{w} - \Omega_2 \hat{w}| > |\breve{w} - \hat{w}|$,
(iii) $\frac{1}{2}|\breve{w} - \Omega_3 \breve{w}| \leq |\breve{w} - \hat{w}| \Rightarrow |\Omega_3 \breve{w} - \Omega_3 \hat{w}| > |\breve{w} - \hat{w}|$.

Hence, all $\Omega_i, i = 1, 2, 3$, do not admit the condition (C). Now, for $\alpha = \frac{1}{2}$, all $\Omega_i (i = 1, 2, 3)$ are generalized α-nonexpansive. We prove this fact only for Ω_1, and for Ω_2 and Ω_3, one can use the same techniques.

***Case I**: $\forall\, \breve{w}, \hat{w} \in A$; we have*

$$\frac{1}{2}|\breve{w} - \Omega_1 \hat{w}| + \frac{1}{2}|\hat{w} - \Omega_1 \breve{w}| + (1 - 2(\frac{1}{2}))|\breve{w} - \hat{w}| = \frac{1}{2}|\breve{w} - (\frac{\hat{w}+5}{2})| + \frac{1}{2}|\hat{w} - (\frac{\breve{w}+5}{2})|$$
$$\geq \frac{1}{2}|\frac{3\breve{w}}{2} - \frac{3\hat{w}}{2}| \geq \frac{1}{2}|\breve{w} - \hat{w}| = |\Omega_1 \breve{w} - \Omega_1 \hat{w}|.$$

***Case II**: $\forall\, \breve{w}, \hat{w} \in B$; we get*

$$\frac{1}{2}|\breve{w} - \Omega_1 \hat{w}| + \frac{1}{2}|\hat{w} - \Omega_1 \breve{w}| + (1 - 2(\frac{1}{2}))|\breve{w} - \hat{w}| \geq 0 = |\Omega_1 \breve{w} - \Omega_1 \hat{w}|.$$

***Case III**: When $\breve{w} \in A$ and $\hat{w} \in B$, we have*

$$\frac{1}{2}|\breve{w} - \Omega_1 \hat{w}| + \frac{1}{2}|\hat{w} - \Omega_1 \breve{w}| + (1 - 2(\frac{1}{2}))|\breve{w} - \hat{w}| = \frac{1}{2}|\breve{w} - 5| + \frac{1}{2}|\hat{w} - (\frac{\breve{w}+5}{2})|$$
$$\geq \frac{1}{2}|\breve{w} - 5| = |\Omega_1 \breve{w} - \Omega_1 \hat{w}|.$$

Now, in our Example 4.1, we see that $F = \{5\}$. Moreover, the domain $[5, 15]$ of Ω_1, Ω_2 and Ω_3 is closed convex in a UCBS $\mathcal{B} = \mathbb{R}$. Hence, all conditions for our main results are available. Thus, the sequence of the modified F-iterates given by (1.2) converges to 5. This fact is confirmed in Tables 1–3 and Figures 1 and 2.

Table 1. Convergence of our scheme (1.2) for $a_k = 0.1$ in Example 4.1.

k	Scheme (1.2)	Scheme (1.2)	Scheme (1.2)
1	5.4	10.4	14.4
2	5.01583333333333	5.21375000000000	5.37208333333333
3	5.00062673611111	5.00846093750000	5.01472829861111
4	5.00002480830440	5.00033491210938	5.00058299515336
5	5.00000098199538	5.00001325693766	5.00002307689149
6	5.00000003887065	5.00000052475378	5.00000091346029
7	5.00000000153865	5.00000002077150	5.00000003615780
8	5.00000000006090	5.00000000082221	5.00000000143125
9	5.00000000000241	5.00000000003255	5.00000000005665
10	5.00000000000010	5.00000000000129	5.00000000000224
11	5	5.00000000000005	5.00000000000009
12	5	5	5

Table 2. Convergence of our scheme (1.2) for $a_k = 0.5$ in Example 4.1.

k	Scheme (1.2)	Scheme (1.2)	Scheme (1.2)
1	5.4	10.4	14.4
2	5.01250000000000	5.16870000000000	5.29375000000000
3	5.00039062500000	5.00527343750000	5.00917968750000
4	5.00001220703125	5.00016479492188	5.00028686523438
5	5.00000038146973	5.00000514984131	5.00000896453858
6	5.00000001192093	5.00000016093254	5.00000028014183
7	5.00000000037253	5.00000000502914	5.00000000875443
8	5.00000000001164	5.00000000015716	5.00000000027358
9	5.00000000000036	5.00000000000491	5.00000000000855
10	5.00000000000001	5.00000000000015	5.00000000000027
11	5	5	5.00000000000001
12	5	5	5

Table 3. Convergence of our scheme (1.2) for $a_k = 0.9$ in Example 4.1.

k	Scheme (1.2)	Scheme (1.2)	Scheme (1.2)
1	5.4	10.4	14.4
2	5.00916666666667	5.12375000000000	5.21541666666667
3	5.00021006944444	5.00283593770000	5.00493663194444
4	5.00000481409144	5.00006499023438	5.00011313114873
5	5.00000011032293	5.00000148935954	5.00000259258883
6	5.00000000252823	5.00000003413116	5.00000005941349
7	5.00000000005794	5.00000000078217	5.00000000136156
8	5.00000000000133	5.00000000001793	5.00000000003120
9	5.00000000000003	5.00000000000041	5.00000000000072
10	5	5.00000000000001	5.00000000000002
11	5	5	5
12	5	5	5

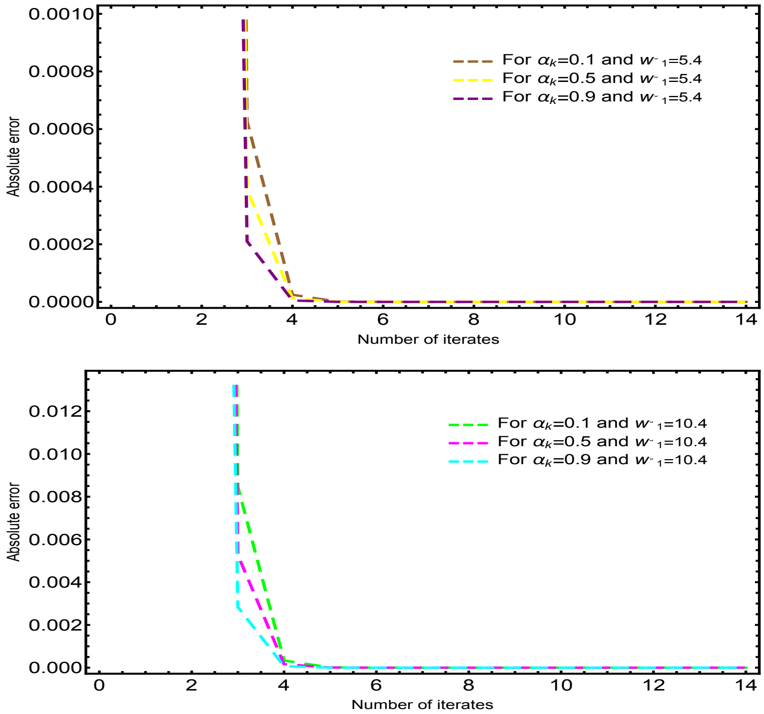

Figure 1. Graphical analysis of our scheme (1.2) for different sets of parameters and starting points.

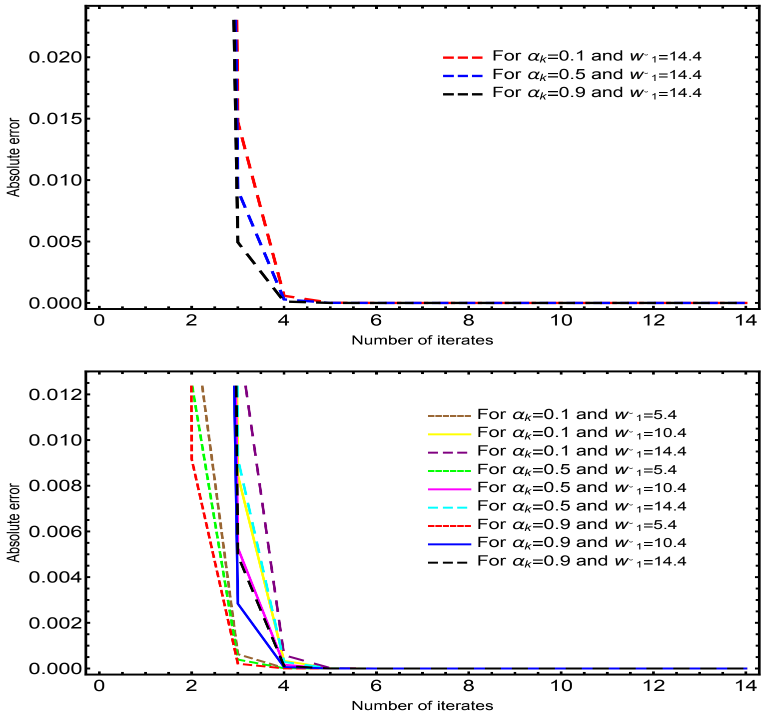

Figure 2. Graphical analysis of our scheme (1.2) for different sets of parameters and starting points.

5. Application

In some cases, and perhaps always, some problems admit a solution, but it is not easy (or it is impossible) to obtain the value of such a solution using analytical methods. For example, see the following equations:

$$\breve{w}^2 - \sin\breve{w} = 0 \text{ and } \breve{w}^3 \ln\breve{w} - e^{\breve{w}} = 0.$$

Here, analytical methods cannot be used to solve these equations. In such a situation, we need to calculate the approximate numerical value of such a solution. To calculate the approximate solutions, it is the same to find a solution for the operator equation $\breve{w} = \Omega\breve{w}$, where the operator Ω is an operator on a Hilbert or Banach space. In this case, the fixed-point set of Ω corresponds to the solution set of the operator equation $\breve{w} = \Omega\breve{w}$. The Banach principle [43] offers a unique fixed point if the operator Ω is a contraction and $\mathcal{B}$ is a Banach space. Moreover, the proof of the Banach principle suggests the Picard [12] iteration for finding this unique fixed point. But, for nonexpansive mappings, it is known that the Picard iteration does not behave well. Thus, in this section, we suggest a new projection-type iteration for solving a variational inequality problem which is different but better than the Picard and many other iterative schemes.

More precisely, we provide some application of our main findings. Let us assume a Hilbert space, namely, $\mathcal{W}$ with an inner product $\langle .,. \rangle$, and that $\emptyset \neq M \subset \mathcal{W}$ is convex. It is known that $\mathcal{U} : \mathcal{W} \to \mathcal{W}$ is said to be monotone whenever

$$\langle \mathcal{U}\breve{w} - \mathcal{U}\hat{w}, \breve{w} - \hat{w} \rangle \geq 0,$$

where $\breve{w}$ and $\hat{w}$ are any points in $\mathcal{W}$. In this step, we denote simply $\mathcal{V}_{M\mathcal{U}}$, a variational inequality problem on the set M and under the mapping $\mathcal{U}$ which is defined by the following way:

$$\text{Search } \breve{w}^* \in M \ : \ \langle \mathcal{U}\breve{w}^*, \breve{w} - \breve{w}^* \rangle \geq 0 \text{ for each } \breve{w} \in \mathcal{W}.$$

Now, we denote I as an identity function on $\mathcal{W}$, and P_M as the nearest point projection onto M. Then, thanks to Byrne [5], if $\lambda > 0$, then the element $\breve{w}^*$ solves our problem $\mathcal{V}_{M\mathcal{U}}$ if and only if $\breve{w}^*$ satisfies the equation $\breve{w} = P_M(I - \lambda\mathcal{U})\breve{w}$.

It should be noted that we will denote S as the solution set for $\mathcal{V}_{M\mathcal{U}}$. Under suitable assumptions, Byrne [5] has shown that, if S is nonempty and $I - \lambda\mathcal{U}$, $P_M(I - \lambda\mathcal{U})$ are averaged nonexpansive, the sequence $\{\breve{w}_k\}$ generated by the iterative scheme $\breve{w}_{k+1} = P_M(I - \lambda\mathcal{U})\breve{w}_k$ converges weakly to a solution of $\mathcal{V}_{M\mathcal{U}}$.

Note that we suggest an alternative approach to solve the problem $\mathcal{V}_{M\mathcal{U}}$ based on the generalized α-nonexpansive mappings that are discontinuous in general (as shown by an example in this paper), instead of nonexpansive operators, which are already well known to be uniformly continuous. In fact, we suggest a new projection-type scheme based on our scheme (1.2), which is better than many other iterative methods, as shown in this paper. It should be noted that, once a weak convergence for a certain problem is established, then the strong convergence is desirable. Therefore, in what follows, we discuss the weak convergence and also obtain the strong convergence for our scheme, which extends the weak convergence of Byrne [5] to the settings of strong convergence.

Before going to establish the strong convergence, first, we establish the weak convergence for a given variational inequality problem $\mathcal{V}_{M\mathcal{U}}$.

Theorem 5.1. *Assume that S admits at least one point and $P_M(I-\lambda\mathcal{U})$, where $\lambda > 0$, is generalized α–nonexpansive with a sequence $\{\breve{w}_k\}$ defined as follows*

$$\begin{cases} \breve{w}_1 \in M, \\ \breve{v}_k = P_M(I-\lambda\mathcal{U})[(1-a_k)\breve{w}_k + a_k P_M(I-\lambda\mathcal{U})\breve{w}_k], \\ \breve{u}_k = P_M(I-\lambda\mathcal{U})\breve{v}_k, \\ \breve{w}_{k+1} = P_M(I-\lambda\mathcal{U})\breve{u}_k, k \geq 1, \end{cases}$$

where $0 < \alpha_k < 1$. In this case, $\{\breve{w}_k\}$ is weakly convergent to a point $\breve{w}^*$ of S.

Proof. Put $\Omega = \Omega_1 = \Omega_2 = \Omega_3 = P_M(I-\lambda\mathcal{U})$. Then, since Ω is generalized α-nonexpansive, by Theorem 3.3, $\{\breve{w}_k\}$ converges to a point of $\mathcal{F}$, and, hence, to the point of S. This finishes the proof. □

After the weak convergence result, we now give a strong convergence result for a given variational inequality problem $\mathcal{V}_{M\mathcal{U}}$.

Theorem 5.2. *Let S admits at least one point and $P_M(I-\eta\mathcal{U})$, where $\eta > 0$, is generalized α–nonexpansive with a sequence $\{\breve{w}_k\}$ defined as follows*

$$\begin{cases} \breve{w}_1 \in M, \\ \breve{v}_k = P_M(I-\lambda\mathcal{U})[(1-a_k)\breve{w}_k + a_k P_M(I-\lambda\mathcal{U})\breve{w}_k], \\ \breve{u}_k = P_M(I-\lambda\mathcal{U})\breve{v}_k, \\ \breve{w}_{k+1} = P_M(I-\lambda\mathcal{U})\breve{u}_k, k \geq 1, \end{cases}$$

where $0 < \alpha_k < 1$. In this case, $\{\breve{w}_k\}$ is strongly convergent to a point $\breve{w}^*$ of S.

Proof. Put $\Omega = \Omega_1 = \Omega_2 = \Omega_3 = P_M(I-\lambda\mathcal{U})$. Then, since Ω is generalized α–nonexpansive, by Theorem 3.5, $\{\breve{w}_k\}$ converges to a point of $\mathcal{F}$, and, hence, to the point of S. This finishes the proof. □

6. Conclusions

This paper presented the following new findings.

(i) We applied the F-iterative scheme of Ali and Ali [36] to the setting of three generalized α-nonexpansive maps, namely, Ω_1, Ω_2 and Ω_3. We proved the weak and the strong convergence of the proposed modified iterative scheme to a common fixed point of these mappings.

(ii) The main findings of the paper have been supported by some numerical experiments. Our results improved the findings of Ali and Ali [36], as well as those of Ahmad et al. [37], from the ordinary case of one mapping to the case of three mappings.

(iii) As an application of our main outcome, we suggested a new projection-type iterative scheme to solve variational inequality problems in the context of generalized α-nonexpansive mappings.

(iv) Thus, our results unify the main outcome of Ahmad et al. [37] from the setting of an α-nonexpansive mapping to the setting of three generalized α-nonexpansive mappings. In a similar way, our results are an improvement and refinements of the results obtained by Ali and Ali [36] and Ahmad et al. [37] for contraction and nonexpansive mappings by extending them to the case of three generalized nonexpansive mappings and the high speed of convergence.

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Conflict of interest

The authors declare no conflicts of interest.

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