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Research article

On the Caputo-Hadamard fractional IVP with variable order using the upper-lower solutions technique

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Abstract: This paper studies the existence of solutions for Caputo-Hadamard fractional nonlinear differential equations of variable order (CHFDEVO). We obtain some needed conditions for this purpose by providing an auxiliary constant order system of the given CHFDEVO. In other words, with the help of piece-wise constant order functions on some continuous subintervals of a partition, we convert the main variable order initial value problem (IVP) to a constant order IVP of the Caputo-Hadamard differential equations. By calculating and obtaining equivalent solutions in the form of a Hadamard integral equation, our results are established with the help of the upper-lower-solutions method. Finally, a numerical example is presented to express the validity of our results.

Keywords: fractional variable order; Caputo-Hadamard fractional derivative; upper-lower solutions; existence results
Mathematics Subject Classification: 34A12, 34A40, 47A10

1. Introduction

By comparing integer differential equations to fractional differential equations of a constant order, fractional calculus has been the subject of extensive studies for more than three centuries. The main and initial difference of fractional calculus is to replace the natural numbers in the order of derivative by arbitrary real ones. Although such a description of this widely used theory seems very superficial, it has a high power in describing physical phenomena. While numerous number of studies have been implemented for analyzing the existence theory in relation to fractional constant-order boundary value problems (BVPs) [1–15], this theory is rarely investigated for variable-order BVPs in other research studies [16–20]. Hence, at the same time, the technique we propose in this paper is new and valuable for such variable order structures. About the investigation of the existence theory for variable order BVPs, we mention some of them. Jiahui et al. [21] addressed unique solutions in relation to an IVP of Riemann-Liouville fractional differential equations in the case of variable order. In [22], Bouazza et al. discussed a new structure of variable-order Riemann-Liouville BVPs, and after that in [23], Benkerrouche et al. performed an analysis about Ulam-Hyers stable solutions for a Caputo nonlinear implicit fractional boundary value problem (FBVPs) of variable order. Simultaneously in 2021, Refice et al. [24] and Hristova et al. [25] focused on some research studies in relation to existence theory for BVPs of Hadamard FDEs with the help of complicated method of the Kuratowski measure of noncompactness in the case of variable order. For more information, we mention [26–29]. Of course, the stability analysis is one of the important aspects of fractional calculus, and some researchers have extended this area for constant-order systems [30–33], and it can be a motivational factor for other studies in variable-order systems.

Many real phenomena exist that expect the concept of Hadamard fractional derivative permitting the useful of physically initial conditions, which contain $\phi(p), \phi'(p)$, etc. The Caputo–Hadamard fractional derivative provides these conditions. Under this property, the basic notions of the Caputo–Hadamard fractional derivative are studied by Almeida [34]. After that, some researchers such as Ben Makhlouf and Mchiri [35] discussed some other properties of these operators. Moreover, Abuasbeh et al. [36,37], Khan et al. [38], Niazi et al. [39] and Shafqat et al. [40, 41] similarly investigated the existence and uniqueness of solution for the fuzzy fractional evolution equations. For other applications, see [42–44].

In particular, Bai et al. [45] studied the existence of solution for the following initial value problem

$$\begin{cases} {}^cD^{w}_{p^+}\phi(t) = \psi(t, \phi(t), I^{w}_{p^+}\phi(t)), \;\; t \in \Lambda := [p, T], \\ \phi(p) = \phi_p, \end{cases} \tag{1.1}$$

where ${}^cD^{w}_{p^+}$ and $I^{w}_{p^+}$ denote the Caputo derivative and Hadamard integral, respectively, $\psi : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\phi_p \in \mathbb{R}$, and $0 < p < T < \infty$.

In this paper, we study the existence of solutions for the following fractional nonlinear differential equation involving the Caputo-Hadamard fractional derivative of variable order

$$\begin{cases} {}^cD^{w(t)}_{p^+}\phi(t) = \psi(t, \phi(t)), \;\; t \in \Lambda := [p, T], \\ \phi(p) = \phi_p, \end{cases} \tag{1.2}$$

where $0 < p < T < \infty$, $\phi_p \in \mathbb{R}$ and $0 < w(t) \leq 1$ is a variable order, $\psi : \Lambda \times \mathbb{R} \to \mathbb{R}$ is a given function and ${}^cD^{w(t)}_{p^+}$ denotes the Caputo-Hadamard fractional derivative of order $w(t)$.

The organization of the rest of this paper is as follows. Some definitions and auxiliary results are given in Section 2. In Section 3, we try to obtain an equivalent system of constant order IVP by deriving Hadamard integral equations on some continuous subintervals and partitions. With the help of piecewise constant functions, we implement the technique of upper-lower solutions for such an equivalent system and generalize our results to the given Caputo-Hadamard variable order problem. One example is presented in Section 4, to show the efficiency and validity of the proposed results. Finally, some conclusion notes are given in Section 5. Note that there is no published work in which the technique of upper-lower solutions is used on a variable order system. This shows the originality of our research.

2. Auxiliary notions

In this section, we list some of definitions and propositions that are used in the following sections.

The space $E := C(\Lambda := [p, T], \mathbb{R})$ denotes the Banach space of continuous functions $\phi : \Lambda \to \mathbb{R}$, and by the function space $AC(p, q; \mathbb{R})$, we determines absolutely continuous $\mathbb{R}$-valued functions on $[p, q]$.

Definition 2.1. *([46, 47]) Let* $0 < p < q < \infty$ *and* $\phi : [p, q] \to \mathbb{R}$. *The Hadamard fractional integral of order* $w > 0$ *of the function* ϕ *is defined by*

$$I^{w}_{p^+}\phi(t) = \frac{1}{\Gamma(w)}\int_p^t (ln\frac{t}{s})^{w-1}\frac{\phi(s)}{s}ds \quad for \;\; t \in [p, q],$$

where the well-known Gamma function is denoted by

$$\Gamma(w) = \int_0^\infty t^{w-1}e^{-t}dt.$$

Definition 2.2. *([46,47]) Let* $0 < p < q < \infty$ *and* $\phi : [p, q] \to \mathbb{R}$. *The Hadamard fractional derivative of the order* $w \in (0, 1]$ *of the function* ϕ *is defined by*

$$D^{w}_{p^+}\phi(t) = \frac{1}{\Gamma(1-w)}t\frac{d}{dt}\int_p^t (ln\frac{t}{s})^{-w}\frac{\phi(s)}{s}ds \quad for \;\; t \in [p, q].$$

Clearly, we have

$$I^{w}_{p^+}\Big(ln\frac{t}{p}\Big)^{v-1} = \frac{\Gamma(v)}{\Gamma(v+w)}\Big(ln\frac{t}{p}\Big)^{v+w-1}, \;\; D^{w}_{p^+}\Big(ln\frac{t}{p}\Big)^{v-1} = \frac{\Gamma(v)}{\Gamma(v-w)}\Big(ln\frac{t}{p}\Big)^{v-w-1},$$

for each $t \in [p, q]$.

We now state some important characteristics for Hadamard fractional integral and derivative operators. The proofs of them can be found in [47].

Lemma 2.3. *([47]). Let* $w > 0$ *and* $v > 0$.

(i) *For* $\phi \in L^r(p, q; \mathbb{R})$*, if* $1 \leq r < \infty$*, then we have*

$$I^{v}_{p^+}I^{w}_{p^+}\phi(t) = I^{w+v}_{p^+}\phi(t) \quad for \;\; t \in [p, q].$$

(ii) *For $\phi \in L^r(p,q;\mathbb{R})$, if $1 \le r < \infty$ and $w > v$, then we have*

$$D^{v}_{p^+} I^{w}_{p^+} \phi(t) = I^{w-v}_{p^+} \phi(t) \quad for \quad t \in [p,q].$$

Definition 2.4. *([46, 47]). Let $0 < p < q < \infty$ and $\phi : [p,q] \to \mathbb{R}$. The Caputo-Hadamard fractional derivative of order $w \in (0,1]$ of the function ϕ is defined by*

$${}^cD^{w}_{p^+} \phi(t) \quad = D^{w}_{p^+}[\phi(t) - \phi(p)] \quad for \quad t \in [p,q].$$

Remark 2.5. *It should be obvious that the Caputo-Hadamard fractional derivative, i.e., Definition 2.4, is equivalent to the following expression that if $\phi \in AC(p,q;\mathbb{R})$, then*

$${}^cD^{w}_{p^+} \phi(t) = \frac{1}{\Gamma(1-w)} \int_p^t (\ln \frac{t}{s})^{-w} \phi'(s) ds, \quad for \quad t \in [p,q].$$

Definition 2.6. *[48] The left variable-order Caputo-Hadamard fractional derivative of the functional order $w(t)$ is defined by*

$$\begin{aligned} {}^cD^{w(t)}_{p^+} \phi(t) = & \frac{t w'(t)}{\Gamma(2-w(t))} \int_p^t (ln\frac{t}{s})^{1-w(t)} \phi'(s) \Big[\frac{1}{1-w(t)} - \ln\Big(\ln \frac{t}{s}\Big) \Big] ds \\ & + \frac{1}{\Gamma(1-w(t))} \int_p^t (ln\frac{t}{s})^{-w(t)} \phi'(s) ds. \end{aligned}$$

Remark 2.7. *If $w(t) \equiv w$, (w is constant), then Definition 2.6 is transformed into the Caputo-Hadamard derivative given in [46] as*

$${}^cD^{w}_{p^+} \phi(t) \quad = \frac{1}{\Gamma(1-w)} \int_p^t (ln\frac{t}{s})^{-w} \phi'(s) ds.$$

The component characteristics for the Caputo-Hadamard fractional operators are listed below, and this section is concluded by mentioning them.

Lemma 2.8. *([47]) Let $n = [w] + 1$ be the case for $w > 0$.*

(i) *If $\phi \in C(p,q;\mathbb{R})$, then*

$${}^cD^{w}_{p^+} (I^{w}_{p^+} \phi(t)) = \phi(t) \quad for \quad t \in [p,q].$$

(ii) *If $\phi \in AC(p,q;\mathbb{R})$, then*

$$I^{w}_{p^+} ({}^cD^{w}_{p^+} \phi(t)) = \phi(t) - \phi(p) \quad for \quad t \in [p,q].$$

3. Main results

Let's state the underlying assumptions. It will be the basic step in proving the results of this section.

(H1) For $n \in \mathbb{N}$, the finite sequence of points $\{T_k\}_{k=0}^n$ such that $p = T_0 < T_k < T_n = T,\ k = 1, \ldots, n-1$ is given. Denote $\Lambda_k := (T_{k-1}, T_k],\ k = 1, 2, \ldots, n$. Consequently, $\mathcal{P} = \bigcup_{k=1}^n \Lambda_k$ is a partition of Λ.

The symbol $E_m = C(\Lambda_m, \mathbb{R}), m = 1, 2, \ldots, n$ denotes the Banach space of continuous functions $\phi : \Lambda_m \to \mathbb{R}$ endowed with $\|\phi\|_{E_m} = \sup_{t\in\Lambda_m} |\phi(t)|$.

Suppose that $w(t) : \Lambda \to (0, 1]$ is defined by $w(t) = \sum_{m=1}^{n} w_m I_m(t)$, where $0 < w_m \leq 1$ are constants and I_m is the indicator of Λ_m be a piecewise constant function with respect to $\mathcal{P}$, where

$$I_m(t) = \begin{cases} 1, & for\ t \in \Lambda_m, \\ 0, & elsewhere. \end{cases}$$

The left Caputo-Hadamard derivative for the function $\phi \in C(\Lambda, \mathbb{R})$ with variable order $w(t)$, given by Definition 2.6, might then be stated as the sum of the left Caputo-Hadamard derivatives of the constant orders $w_k,\ k = 1, 2, \ldots, n$, i.e.,

$$\begin{aligned} D_{p^+}^{w(t)}\phi(t) &= \frac{tw'(t)}{\Gamma(2-w(t))}\int_p^t (ln\frac{t}{s})^{1-w(t)}\phi'(s)\Big[\frac{1}{1-w(t)} - \ln\Big(\ln\frac{t}{s}\Big)\Big]ds \\ &\quad + \frac{1}{\Gamma(1-w(t))}\int_p^t (ln\frac{t}{s})^{-w(t)}\phi'(s)ds \\ &= \frac{1}{\Gamma(1-w(t))}\Big(\sum_{k=1}^{m-1}\int_{T_{k-1}}^{T_k}(\ln\frac{t}{s})^{-w_k}\phi'(s)ds + \int_{T_{m-1}}^{t}(\ln\frac{t}{s})^{-w_m}\phi'(s)ds\Big). \end{aligned}$$

For each $t \in \Lambda_m$, where $m = 1, 2, \ldots, n$, the Caputo-Hadamard derivative for the system of CHFDEVO (1.2) can be stated in the following form

$$\frac{1}{\Gamma(1-w(t))}\Big(\sum_{k=1}^{m-1}\int_{T_{k-1}}^{T_k}(\ln\frac{t}{s})^{-w_k}\phi'(s)ds + \int_{T_{m-1}}^{t}(\ln\frac{t}{s})^{-w_m}\phi'(s)ds\Big) = \psi(t, \phi(t)). \tag{3.1}$$

To solve the integral equation (3.1), let the function $\tilde{\phi} \in C(\Lambda_m, \mathbb{R})$ be such that $\tilde{\phi}(t) \equiv 0$ on $t \in [p, T_{m-1}]$. Then (3.1) is transformed into

$$D_{T_{m-1}^+}^{w_m}\tilde{\phi}(t) = \psi(t, \tilde{\phi}(t)), \quad t \in \Lambda_m.$$

For obtained Caputo-Hadamard constant order fractional differential equations, we consider the following auxiliary Caputo-Hadamard fractional differential equations (CHFDE) of constant order

$$\begin{cases} {}^cD_{T_{m-1}^+}^{w_m}\phi_m(t) = \psi(t, \phi_m(t)),\ t \in \Lambda_m, \\ \phi_m(T_{m-1}) = \phi_{T_{m-1}}, \end{cases} \tag{3.2}$$

for each $m = 1, 2, \ldots, n$.

The main basic theorem can be stated now.

Theorem 3.1. *Assume that $\psi : \Lambda_m \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The solution to the integral equation (i.e., $\phi_m \in C(T_{m-1}, T_m; \mathbb{R})$) given by*

$$\phi_m(t) = \phi_{T_{m-1}} + \frac{1}{\Gamma(w_m)}\int_{T_{m-1}}^{t}(\ln\frac{t}{s})^{w_m-1}\frac{\psi(s, \phi_m(s))}{s}ds \quad for\ \ t \in \Lambda_m, \tag{3.3}$$

solves the auxiliary CHFDE of constant order (3.2).

Proof. Assume that $\phi_m \in C(T_{m-1}, T_m; \mathbb{R})$ is a solution of (3.3). Naturally, we take $\phi(T_{m-1}) = \phi_{T_{m-1}}$ and $t \to I^{w_m}_{T^+_{m-1}} \phi_m(t) \in C(T_{m-1}, T_m; \mathbb{R})$. The definition of the Hadamard integral $I^{w_m}_{T^+_{m-1}}$ and the continuity of ψ guarantee that $t \to \psi(t, \phi_m(t))$ is continuous as well and

$$I^{w_m}_{T^+_{m-1}} \psi(t, \phi_m(t))|_{t=T_{m-1}} = 0.$$

Since $t \to I^{w_m}_{T^+_{m-1}} \psi(t, \phi_m(t))$ is continuous, we can conclude that ϕ_m is differentiable for a.e. $t \in (T_{m-1}, T_m)$, (see (3.3)), i.e., $\phi_m \in AC(T_{m-1}, T_m; \mathbb{R})$. From Lemma 2.8, we have

$${}^cD^{w_m}_{T^+_{m-1}} I^{w_m}_{T^+_{m-1}} \psi(t, \phi_m(t)) = \psi(t, \phi_m(t)) \quad \text{for} \ \ t \in \Lambda_m.$$

On the other hand, Remark 2.5 gives

$$\begin{aligned} {}^cD^{w_m}_{T^+_{m-1}} [\phi_m(t) - \phi_{T_{m-1}}] &= \frac{1}{\Gamma(1-w_m)} \int_{T_{m-1}}^t (\ln \frac{t}{s})^{-w_m} [\phi_m(s) - \phi_{T_{m-1}}]' ds \\ &= \frac{1}{\Gamma(1-w_m)} \int_{T_{m-1}}^t (\ln \frac{t}{s})^{-w_m} \phi'_m(s) ds \\ &= {}^cD^{w_m}{}_{T^+_{m-1}} \phi_m(t), \end{aligned}$$

for each $t \in \Lambda_m$. By all above, we conclude that $\phi_m \in C(T_{m-1}, T_m; \mathbb{R})$ is a solution of the auxiliary CHFDE of constant order (3.2). □

Definition 3.2. *Let* $(\underline{\phi_m}, \overline{\phi_m}) \in C(T_{m-1}, T_m; \mathbb{R}) \times C(T_{m-1}, T_m; \mathbb{R})$*. A pair of functions* $(\underline{\phi_m}, \overline{\phi_m})$ *is called an upper-lower solutions of the auxiliary CHFDE of constant order (3.2), respectively, if*

$$\underline{\phi_m}(t) \le \phi_{T_{m-1}} + \frac{1}{\Gamma(w_m)} \int_{T_{m-1}}^t (ln\frac{t}{s})^{w_m - 1} \frac{\psi(s, \underline{\phi_m}(s))}{s} ds \quad \textit{for all} \ \ t \in \Lambda_m,$$

and

$$\overline{\phi_m}(t) \ge \phi_{T_{m-1}} + \frac{1}{\Gamma(w_m)} \int_{T_{m-1}}^t (ln\frac{t}{s})^{w_m - 1} \frac{\psi(s, \overline{\phi_m}(s))}{s} ds \quad \textit{for all} \ \ t \in \Lambda_m.$$

Assume that the upper-lower solution to the the auxiliary CHFDE of constant order (3.2) is $(\underline{\phi_m}, \overline{\phi_m})$. In the following, we define an acceptable set of solutions for the auxiliary CHFDE of constant order (3.2) which is controlled by two upper-lower solutions $(\underline{\phi_m}, \overline{\phi_m})$ as follows

$$S_{(\underline{\phi_m}, \overline{\phi_m})} := \Big\{\phi_m \in C(T_{m-1}, T_m; \mathbb{R}) : \underline{\phi_m}(t) \le \phi_m(t) \le \overline{\phi_m}(t), t \in \Lambda_m \text{ and } \phi_m \text{ is a solution of (3.2)}\Big\}.$$

Theorem 3.3. *Let* $\psi \in C(\Lambda_m \times \mathbb{R}; \mathbb{R})$ *and* $(\underline{\phi_m}, \overline{\phi_m}) \in C(T_{m-1}, T_m; \mathbb{R}) \times C(T_{m-1}, T_m; \mathbb{R})$*. The auxiliary CHFDE of constant order* (3.2) *has the pair of upper-lower solutions with* $\underline{\phi_m}(t) \le \overline{\phi_m}(t)$ *and* $t \in \Lambda_m$*. If* $\phi_m \to \psi(t, \phi_m)$ *is nondecreasing, that is*

$$\psi(t, \phi_1) \le \psi(t, \phi_2) \quad \textit{for} \ \ \phi_1 \le \phi_2,$$

then, there are minimum and maximum solutions $\phi_{M,m}, \phi_{L,m} \in S_{(\underline{\phi_m}, \overline{\phi_m})}$ *in* $S_{(\underline{\phi_m}, \overline{\phi_m})}$*; i.e., for each* $\phi_m \in S_{(\underline{\phi_m}, \overline{\phi_m})}$,

$$\phi_{L,m}(t) \le \phi_m(t) \le \phi_{M,m}(t) \quad \textit{for} \ \ t \in \Lambda_m.$$

Proof. We provide two sequences $\{\vartheta_{n,m}\}$ and $\{\beta_{n,m}\}$ as

$$\begin{cases} \vartheta_{0,m} = \underline{\phi_m}, \\ \vartheta_{n+1,m}(t) = \phi_{T_{m-1}} + \dfrac{1}{\Gamma(w_m)} \displaystyle\int_{T_{m-1}}^{t} (ln\frac{t}{s})^{w_m-1} \frac{\psi(s, \vartheta_{n,m}(s))}{s} ds, t \in \Lambda_m \text{ and } n = 0, 1, ..., \end{cases} \tag{3.4}$$

and

$$\begin{cases} \beta_{0,m} = \overline{\phi_m}, \\ \beta_{n+1,m}(t) = \phi_{T_{m-1}} + \dfrac{1}{\Gamma(w_m)} \displaystyle\int_{T_{m-1}}^{t} (ln\frac{t}{s})^{w_m-1} \frac{\psi(s, \beta_{n,m}(s))}{s} ds, t \in \Lambda_m \text{ and } n = 0, 1, ... \end{cases} \tag{3.5}$$

The proof is now divided into three steps.

Step1. Sequences $\{\vartheta_{n,m}\}$ and $\{\beta_{n,m}\}$ satisfy the following relation:

$$\underline{\phi_m}(t) = \vartheta_{0,m}(t) \leq \vartheta_{1,m}(t) \leq \vartheta_{2,m}(t) \leq ... \leq \vartheta_{n,m}(t) \leq ... \leq \beta_{n,m}(t) \leq ... \leq \beta_{1,m}(t) \leq \beta_{0,m}(t) = \overline{\phi_m}(t) \tag{3.6}$$

for each $t \in \Lambda_m$.

We will first demonstrate that the sequence $\{\vartheta_{n,m}\}$ is nondecreasing and

$$\vartheta_{n,m}(t) \leq \beta_{0,m}(t), t \in \Lambda_m \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by a recurrence relation, we prove

$$\vartheta_{n-1,m}(t) \leq \vartheta_{n,m}(t), \quad \forall t \in \Lambda_m. \tag{3.7}$$

By the definition of $\vartheta_{0,m}(t)$, we have $\vartheta_{0,m}(t) \leq \vartheta_{1,m}(t)$ for each $t \in \Lambda_m$. We suppose that (3.7) is true for n and we prove for $n + 1 : \vartheta_{n,m}(t) \leq \vartheta_{n+1,m}(t), \ \ \forall t \in \Lambda_m$.

We have

$$\vartheta_{n,m}(t) = \phi_{T_{m-1}} + \frac{1}{\Gamma(w_m)} \int_{T_{m-1}}^{t} (ln\frac{t}{s})^{w_m-1} \frac{\psi(s, \vartheta_{n-1,m}(s))}{s} ds.$$

$$\vartheta_{n+1,m}(t) = \phi_{T_{m-1}} + \frac{1}{\Gamma(w_m)} \int_{T_{m-1}}^{t} (ln\frac{t}{s})^{w_m-1} \frac{\psi(s, \vartheta_{n,m}(s))}{s} ds.$$

Using the monotonicity of ψ, we obtain

$$\vartheta_{n,m}(t) \leq \vartheta_{n+1,m}(t).$$

As $\vartheta_{n,m}(t)$ is noncreasing, by the definition of $\beta_{0,m}(t)$, we have

$$\vartheta_{n,m}(t) \leq \vartheta_{n+1,m}(t) \leq \beta_{0,m}(t).$$

Further, we will show that

$$\vartheta_{n,m}(t) \leq \beta_{n,m}(t) \quad \text{for } t \in \Lambda_m \quad \text{and} \quad n \in \mathbb{N}.$$

Since $n = 0$, it is evident that $\underline{\phi_m}(t) = \vartheta_{0,m}(t) \leq \beta_{0,m}(t) = \overline{\phi_m}(t)$ for each $t \in \Lambda_m$. Now, we make an inductive assumption

$$\vartheta_{n,m}(t) \leq \beta_{n,m}(t), t \in \Lambda_m.$$

Accordingly, given that ψ is monotonic with respect to the second variable, it is simple to conclude that

$$\vartheta_{n+1,m}(t) \leq \beta_{n+1,m}(t), t \in \Lambda_m.$$

Also, we have that the sequence $\{\beta_{n,m}\}$ is nonincreasing.

Step2. Both sequences $\{\vartheta_{\mathbf{n,m}}\}$ and $\{\beta_{\mathbf{n,m}}\}$ are relatively compact in $\mathbf{C}(\mathbf{T_{m-1}}, \mathbf{T_m}; \mathbb{R})$.

Because ψ is continuous and $(\underline{\phi_m}, \overline{\phi_m}) \in C(T_{m-1}, T_m; \mathbb{R})$, from Step 1, we find out that $\{\vartheta_{n,m}\}$ and $\{\beta_{n,m}\}$ belong to $C(T_{m-1}, T_m; \mathbb{R})$ as well. It follows from (3.6) that $\{\vartheta_{n,m}\}$ and $\{\beta_{n,m}\}$ are uniformlly bounded. On the other hand, for any $t_1, t_2 \in \Lambda_m$, without loss of generality, let $t_1 \leq t_2$. We have

$$\begin{aligned}
|\vartheta_{n+1,m}(t_1) - \vartheta_{n+1,m}(t_2)| &= \frac{1}{\Gamma(w_m)}\Big| \int_{T_{m-1}}^{t_2} (ln\frac{t_2}{s})^{w_m-1} \frac{\psi(s, \vartheta_{n,m}(s))}{s} ds \\
&- \int_{T_{m-1}}^{t_1} (ln\frac{t_1}{s})^{w_m-1} \frac{\psi(s, \vartheta_{n,m}(s))}{s} ds\Big| \\
&= \frac{1}{\Gamma(w_m)}\Big| \int_{T_{m-1}}^{t_1} \Big[(ln\frac{t_2}{s})^{w_m-1} - (ln\frac{t_1}{s})^{w_m-1}\Big] \frac{\psi(s, \vartheta_{n,m}(s))}{s} ds \\
&+ \int_{t_1}^{t_2} (ln\frac{t_2}{s})^{w_m-1} \frac{\psi(s, \vartheta_{n,m}(s))}{s} ds\Big| \\
&\leq \frac{M}{\Gamma(w_m)}\Big| \int_{T_{m-1}}^{t_1} \frac{1}{s}\Big[(ln\frac{t_2}{s})^{w_m-1} - (ln\frac{t_1}{s})^{w_m-1}\Big] ds + \int_{t_1}^{t_2} \frac{1}{s}(ln\frac{t_2}{s})^{w_m-1} ds\Big| \\
&= \frac{M}{\Gamma(w_m)}\Big| - \frac{1}{w_m}\Big[\Big(ln\frac{t_2}{s}\Big)^{w_m}\Big]_{T_{m-1}}^{t_1} + \frac{1}{w_m}\Big[\Big(ln\frac{t_1}{s}\Big)^{w_m}\Big]_{T_{m-1}}^{t_1} - \frac{1}{w_m}\Big[\Big(ln\frac{t_2}{s}\Big)^{w_m}\Big]_{t_1}^{t_2}\Big| \\
&= \frac{M}{\Gamma(w_m)}\Big|\frac{1}{w_m}\Big(\Big(ln\frac{t_2}{T_{m-1}}\Big)^{w_m} - \Big(ln\frac{t_2}{t_1}\Big)^{w_m}\Big) + \Big(\frac{1}{w_m}\Big(- \Big(ln\frac{t_1}{T_{m-1}}\Big)^{w_m} + \Big(ln\frac{t_1}{t_1}\Big)^{w_m}\Big)\Big) \\
&+ \Big(\frac{1}{w_m}\Big(\Big(ln\frac{t_2}{t_1}\Big)^{w_m} - \Big(ln\frac{t_2}{t_2}\Big)^{w_m}\Big)\Big| \\
&= \frac{M}{\Gamma(w_m)}\Big|\frac{1}{w_m}\Big(\Big(ln\frac{t_2}{T_{m-1}}\Big)^{w_m} - \Big(ln\frac{t_1}{T_{m-1}}\Big)^{w_m}\Big)\Big| \\
&= \frac{M}{\Gamma(w_m + 1)}\Big|\Big(ln\frac{t_2}{T_{m-1}}\Big)^{w_m} - \Big(ln\frac{t_1}{T_{m-1}}\Big)^{w_m}\Big| \\
&\to 0, \quad as\; t_1 \to t_2,
\end{aligned}$$

where $M > 0$ is a constant independent of n, t_1, and t_2. It gives this fact that $\{\vartheta_{n,m}\}$ is equicontinuous in $C(T_{m-1}, T_m; \mathbb{R})$.

We conclude that $\{\vartheta_{n,m}\}$ is relatively compact in $C(T_{m-1}, T_m; \mathbb{R})$ based on the Arzela-Ascoli Theorem. Similar to this, we find that $\{\beta_{n,m}\}$ is also relatively compact in $C(\Lambda_m; \mathbb{R})$.

Step3. In $\mathbf{S}_{(\underline{\phi_m}, \overline{\phi_m})}$, there are minimum and maximum solutions.

The sequences $\{\vartheta_{n,m}\}$ and $\{\beta_{n,m}\}$ are monotone and relatively compact in $C(T_{m-1}, T_m; \mathbb{R})$, as shown in Steps 1 and 2. Evidently, continuous functions ϑ_m and β_m exist with $\vartheta_{n,m}(t) \leq \vartheta_m(t) \leq \beta_m(t) \leq \beta_{n,m}(t)$ for all $t \in \Lambda_m$ and $n \in \mathbb{N}$, such that $\{\vartheta_{n,m}\}$ and $\{\beta_{n,m}\}$ converge uniformly to ϑ_m and β_m, respectively, in $C(T_{m-1}, T_m; \mathbb{R})$. Therefore, the solutions to the auxiliary CHFDE of constant order (3.2) are ϑ_m and β_m; i.e.,

$$\begin{aligned} \vartheta_m(t) &= \phi_{T_{m-1}} + \frac{1}{\Gamma(w_m)} \int_{T_{m-1}}^{t} (ln\frac{t}{s})^{w_m - 1} \frac{\psi(s, \vartheta_m(s))}{s} ds, \\ \beta_m(t) &= \phi_{T_{m-1}} + \frac{1}{\Gamma(w_m)} \int_{T_{m-1}}^{t} (ln\frac{t}{s})^{w_m - 1} \frac{\psi(s, \beta_m(s))}{s} ds, \end{aligned}$$

for each $t \in \Lambda_m$. Therefore,

$$\underline{\phi_m}(t) \leq \vartheta_m(t) \leq \beta_m(t) \leq \overline{\phi_m}(t) \quad for\ t \in \Lambda_m.$$

Finally, we will prove that ϑ_m and β_m are the minimum and maximum solutions in $S_{(\underline{\phi_m}, \overline{\phi_m})}$. If $\phi_m \in S_{(\underline{\phi_m}, \overline{\phi_m})}$, then

$$\underline{\phi_m}(t) \leq \phi_m(t) \leq \overline{\phi_m}(t), \quad t \in \Lambda_m.$$

Remembering that the second and third arguments do not cause ψ to decrease, we introduce

$$\underline{\phi_m}(t) \leq \vartheta_{n,m}(t) \leq \phi_m(t) \leq \beta_{n,m}(t) \leq \overline{\phi_m}(t) \quad \text{for } t \in \Lambda_m \quad \text{and } n \in \mathbb{N}.$$

As $n \to \infty$ in the above inequality, it implies that

$$\underline{\phi_m}(t) \leq \vartheta_m(t) \leq \phi_m(t) \leq \beta_m(t) \leq \overline{\phi_m}(t) \quad \text{for } t \in \Lambda_m.$$

This concludes the proof of theorem by considering $\phi_{L,m} = \vartheta_m$ and $\phi_{M,m} = \beta_m$, respectively, which are the minimum and maximum solutions in $S_{(\underline{\phi_m}, \overline{\phi_m})}$. □

Theorem 3.4. *Assume that the hypotheses of Theorem 3.3 to be satisfied. The auxiliary CHFDE of constant order* (3.2) *has at least one solution in* $C(\Lambda_m; \mathbb{R})$.

Proof. According to Theorem 3.3, we get $S_{(\underline{\phi_m}, \overline{\phi_m})} \neq \emptyset$, implying that the solution set associated with the auxiliary CHFDE of constant order (3.2) is not empty in $C(T_{m-1}, T_m; \mathbb{R})$. By proving that the auxiliary CHFDE of constant order (3.2) has at least one solution in $C(T_{m-1}, T_m; \mathbb{R})$, this completes the proof of theorem. □

We shall now investigate the existence result for the Caputo-Hadamard fractional nonlinear differential equation of variable order (CHFDEVO) (1.2).

Theorem 3.5. *Let all* $m \in \{1, 2, \ldots, n\}$ *satisfy the condition (H1). Then, there is at least one solution for the given nonlinear IVP of CHFDEVO* (1.2) *in E.*

Proof. Based on the above proofs, we know that the nonlinear IVP of constant order Caputo-Hadamard fractional differential equation (3.2) has at least one solution $\widetilde{\phi}_m \in E_m$, $m \in \{1, 2, \ldots, n\}$. This is in accordance with Theorems 3.3 and 3.4.

We define the solution function for each $m \in \{1, 2, \ldots, n\}$ as

$$\phi_m = \begin{cases} 0, & t \in [p, T_{m-1}], \\ \widetilde{\phi}_m, & t \in \Lambda_m. \end{cases} \tag{3.8}$$

Thus, $\phi_m \in C(T_{m-1}, T_m; \mathbb{R})$ solves the Hadamard integral equation (3.1) for each $t \in \Lambda_m$, which means that $\phi_m(p) = 0, \phi_m(T_m) = \widetilde{\phi}_m(T_m) = 0$. Then, the function

$$\phi(t) = \begin{cases} \phi_1(t), & t \in \Lambda_1, \\ \phi_2(t) = \begin{cases} 0, & t \in \Lambda_1, \\ \widetilde{\phi}_2, & t \in \Lambda_2, \end{cases} \\ \cdot \\ \cdot \\ \cdot \\ \phi_n(t) = \begin{cases} 0, & t \in [p, T_{n-1}], \\ \widetilde{\phi}_n, & t \in \Lambda_n. \end{cases} \end{cases}$$

is a solution of the given nonlinear IVP of CHFDEVO (1.2) in E. □

4. Numerical example

Let $\Lambda := [1, e^2]$, $T_0 = 1$, $T_1 = e$, $T_2 = e^2$. Consider the following nonlinear variable order IVP of CHFDE

$$\begin{cases} {}^cD_{1^+}^{w(t)}\phi(t) = \frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi(t), & t \in \Lambda, \\ \phi(1) = 0, \end{cases} \tag{4.1}$$

where

$$w(t) = \begin{cases} \frac{1}{2}, & t \in \Lambda_1 := [1, e], \\ \frac{2}{3}, & t \in \Lambda_2 :=]e, e^2]. \end{cases} \tag{4.2}$$

Denote

$$\psi(t, \phi) = \frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi(t),\ (t, \phi) \in [1, e^2] \times \mathbb{R}.$$

Using (4.2) and (3.2), we consider two auxiliary constant order IVPs of CHFDEs as

$$\begin{cases} {}^cD_{1^+}^{\frac{1}{2}}\phi(t) = \frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi(t), & t \in \Lambda_1, \\ \phi(1) = 0, \end{cases} \tag{4.3}$$

and

$$\begin{cases} {}^cD^{\frac{2}{3}}_{e^+}\phi(t) = \dfrac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi(t), \quad t \in \Lambda_2, \\ \phi(e) = 1. \end{cases} \tag{4.4}$$

For m = 1: By Theorem 3.1, the auxiliary IVP of constant order CHFDE (4.3) has at least one solution $\widetilde{\phi}_1 \in E_1$ as

$$\phi_1(t) = I^{\frac{1}{2}}_{1^+}\Big(\frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi_1(t)\Big) \text{ for } t \in \Lambda_1. \tag{4.5}$$

In fact, as one can see, $(\underline{\phi_1(t)}, \overline{\phi_1(t)}) = (0, \ln t + (\ln t)^5)$ denotes the upper-lower bounds of the solution to (4.5). We can calculate the sequences $\{\vartheta_{n,1}\}$ and $\{\beta_{n,1}\}$ by

$$\begin{cases} \vartheta_{0,1} = \underline{\phi_1} \\ \vartheta_{n+1,1}(t) = I^{\frac{1}{2}}_{1^+}\psi(t, \vartheta_{n,1}(t)), n = 0, 1, ..., \end{cases}$$

and

$$\begin{cases} \beta_{0,1} = \overline{\phi_1} \\ \beta_{n+1,1}(t) = I^{\frac{1}{2}}_{1^+}\psi(t, \beta_{n,1}(t)), n = 0, 1, ..., \end{cases}$$

for each $t \in \Lambda_1$. We can now use Theorem 3.3 to determine that $\vartheta_{n,1} \to \vartheta_1 \in E_1$ and $\beta_{n,1} \to \beta_1 \in E_1$ as $n \to \infty$. In the meanwhile, we may obtain $t \in \Lambda_1$ for $\beta_1(t) = \vartheta_1(t) = \dfrac{\pi(\ln t)^2}{3}$.

We use Maple to calculate the sequences $\{\vartheta_{n,2}\}$ and $\{\beta_{n,2}\}$ for each n which are defined as integrals with different initial values. Then, we take the values of these sequences at each instant t and plot them with Matlab. In Table 1, we present the error (which is the sup of the absolute value of the defference) between the sequences $\{\vartheta_{n,1}\}$, $\{\beta_{n,1}\}$ and the exact solution for $n = 5, 10, 15, 20$. In Figure 1, we plot the sequences $\{\vartheta_{n,1}\}$, $\{\beta_{n,1}\}$ and the exact solution for $n = 0, 1, 2, 10, 30$.

Table 1. Error analysis for $m = 1$.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t\in[1,e]} \lvert\vartheta_{n,1}(t) - \vartheta_1(t)\rvert$	4.7692×10^{-2}	6.3900×10^{-4}	4×10^{-6}	10^{-15}
$\sup_{t\in[1,e]} \lvert\beta_{n,1} - \beta_1\rvert$	4.6818×10^{-2}	$7.8299 \times 10^{e-4}$	5×10^{-6}	9×10^{-8}

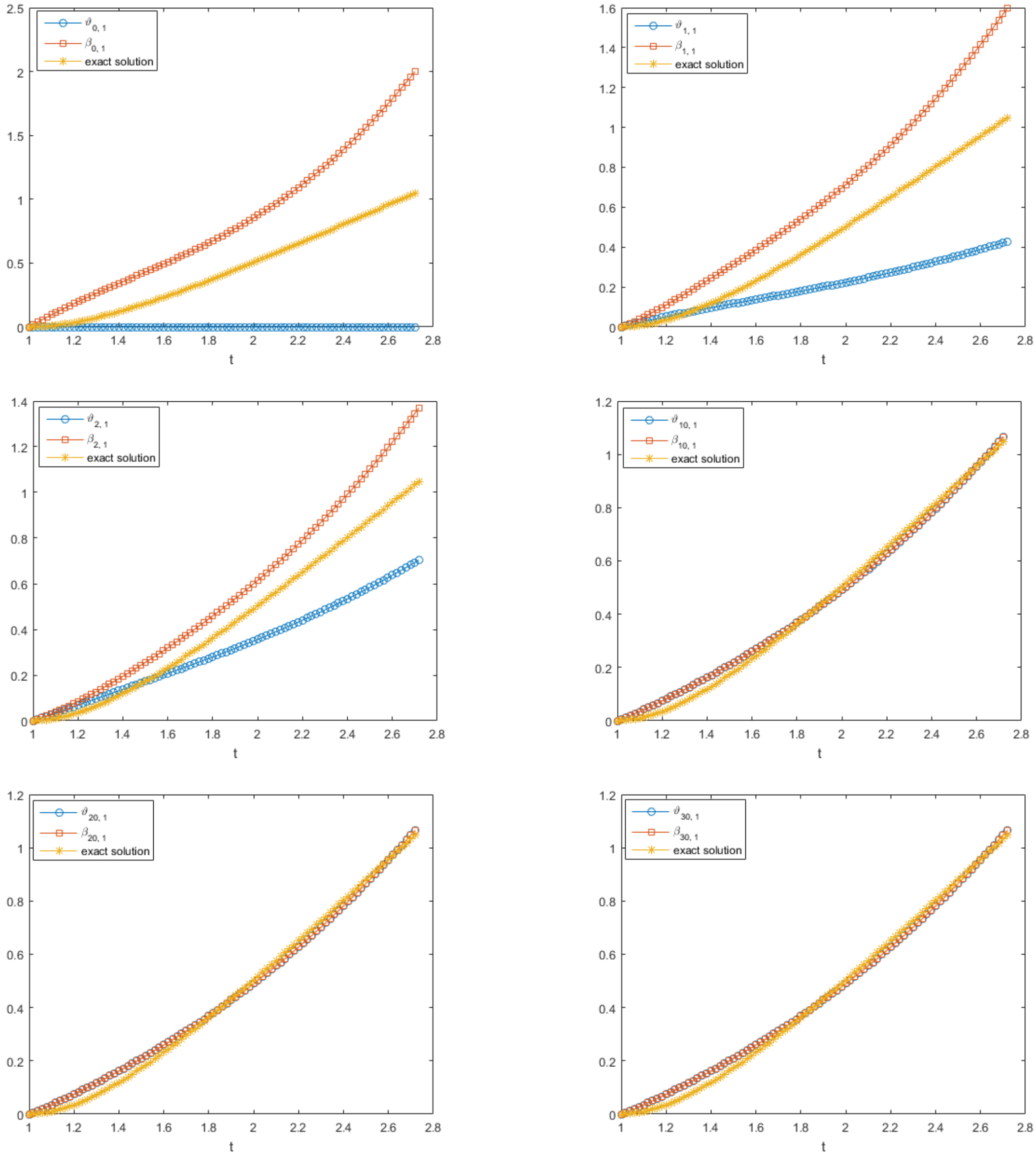

Figure 1. A plot of $\vartheta_{n,1}, \beta_{n,1}$ and exact solution for $n = 0, 1, 2, 10, 20, 30$.

In Figure 1, We notice that when n is larger, the sequences $\{\vartheta_{n,1}\}$ and $\{\beta_{n,1}\}$ are approximated to the exact solution $\dfrac{\pi(\ln t)^2}{3}$. Moreover, in Table 1, we confirm our previous remark, because the error approaches to 0 when n converges to $+\infty$.

For m = 2: By Theorem 3.1, the auxiliary IVP of constant order CHFDE (4.4) has at least one solution $\widetilde{\phi_2} \in E_2$ as

$$\phi_2(t) = I^{1}_{e+}\Big(\frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi_2(t)\Big) \quad \text{for} \quad t \in \Lambda_2. \tag{4.6}$$

In fact, we are able to observe that $(\underline{\phi_2(t)}, \overline{\phi_2(t)}) = (1, \ln t + (\ln t)^5)$ is upper-lower solution to (4.6).

We can calculate the sequences $\{\vartheta_{n,2}\}$ and $\{\beta_{n,2}\}$ by

$$\begin{cases} \vartheta_{0,2} = \underline{\phi_2} \\ \vartheta_{n+1,2}(t) = I_{e+}^{1} f(t, \vartheta_{n,2}(t)), n = 0, 1, ..., \end{cases}$$

and

$$\begin{cases} \beta_{0,2} = \overline{\phi_2} \\ \beta_{n+1,2}(t) = I_{e+}^{\frac{2}{3}} f(t, \beta_{n,2}(t)), n = 0, 1, ..., \end{cases}$$

for each $t \in \Lambda_2$. We can now use Theorem 3.3 to prove $\vartheta_{n,2} \to \vartheta_2 \in E_2$ and $\beta_{n,2} \to \beta_2 \in E_2$ as $n \to \infty$. In the meanwhile, we may obtain $t \in \Lambda_2$ for $\beta_2(t) = \vartheta_2(t) = \pi\frac{(\ln t)^5}{30}$.

In Table 2, we present the error (which is the sup of the absolute value of the defference) between the sequences $\{\vartheta_{n,2}\}$, $\{\beta_{n,2}\}$ and the exact solution for $n = 5, 10, 15, 20$. In Figure 2, we plot the sequences $\{\vartheta_{n,2}\}$, $\{\beta_{n,2}\}$ and the exact solution for $n = 0, 1, 2, 10, 30$. In this figure, we notice that when n is larger, the sequences $\{\vartheta_{n,2}\}$ and $\{\beta_{n,2}\}$ are approximated to the exact solution $\frac{\pi(\ln t)^5}{30}$. In Table 2, we confirm our previous remark, because the error approaches to 0 when n converges to $+\infty$.

Consequently, in accordance with Theorem 3.5, the given nonlinear IVP of CHFDEVO (4.1) has a solution

$$\phi(t) = \begin{cases} \widetilde{\phi}_1(t), & t \in \Lambda_1, \\ \\ \phi_2(t), & t \in \Lambda_2, \end{cases}$$

where

$$\phi_2(t) = \begin{cases} 0, & t \in \Lambda_1, \\ \\ \widetilde{\phi}_2(t), & t \in \Lambda_2. \end{cases}$$

Table 2. Error analysis for $m = 2$.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t\in]e,e^2]} \lvert\vartheta_{n,2}(t) - \vartheta_2(t)\rvert$	6.8509×10^{-3}	10^{-6}	6×10^{-10}	10^{-10}
$\sup_{t\in]e,e^2]} \lvert\beta_{n,2}(t) - \beta_2(t)\rvert$	3.1123×10^{-2}	10^{-6}	10^{-7}	10^{-12}

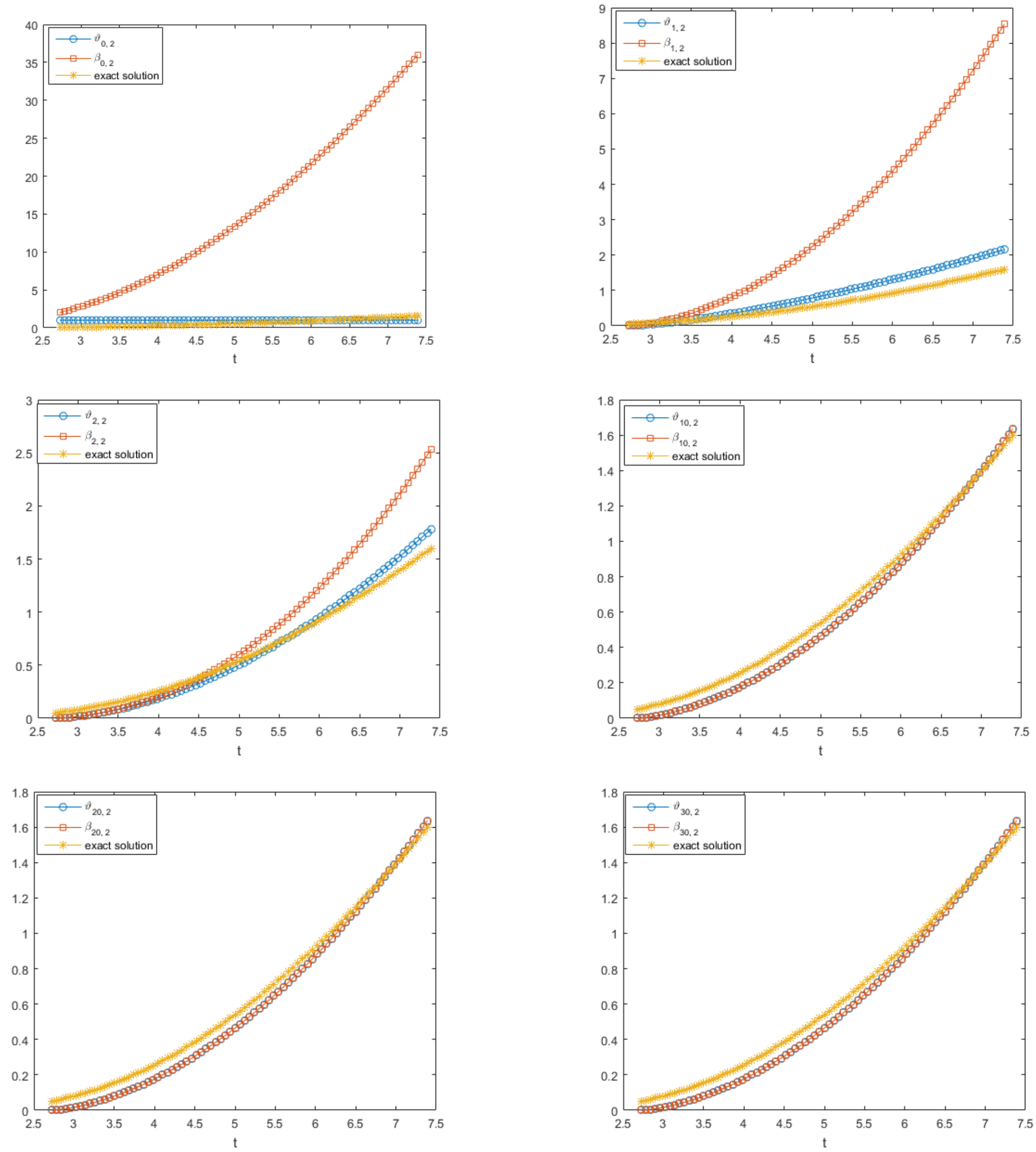

Figure 2. A plot of $\vartheta_{n,2}, \beta_{n,2}$ and exact solution, for $n = 0, 1, 2, 10, 20, 30$.

5. Conclusions

In this paper, a Caputo-Hadamard fractional nonlinear differential equation of variable order was considered and discussed. With the help of piece-wise constant order functions on some continuous subintervals of a partition, we converted the main variable order IVP to a constant order IVP of the Caputo-Hadamard differential equation. By calculating and obtaining equivalent solutions in the form of a Hadamard integral equation, we used the upper-lower solution technique to prove the relevant existence theorems. By plotting some graphs and providing some numerical tables, we presented an example of the variable order IVP to apply and demonstrate the results of our method. In the future, we will extend our studies on different IVPs and BVPs (implicit, resonance, thermostat model, etc.) with changing conditions (terminal, integral conditions, etc.) in the future. Also, if we can define variable order tempered fractional derivative, then it will be a new idea for this purpose [49, 50].

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Conflict of interest

The authors declare no conflicts of interest.

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