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Research article

Existence and stability results of pantograph equation with three sequential fractional derivatives

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Abstract: The subject of this work is the existence and Mittag-Leffler-Ulam (MLU) stability of solutions for fractional pantograph equations with three sequential fractional derivatives. Sufficient conditions for the existence and uniqueness of solutions are constructed by utilizing well-known classical fixed point theorems such as the Banach contraction principle, and Leray-Schauder nonlinear alternative. The generalized singular Gronwall's inequality is used to show the MLU stability results. An illustrated example is provided to support the main findings.

Keywords: fractional derivative; fixed point; existence; fractional pantograph equation; Mittag-Leffler-Ulam stability
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1. Introduction

Differential equations (DEs) involving fractional operators of different orders have recently been studied by a number of scientific researchers because of the fact that they are valuable tools in the modeling of numerous problems in sciences and engineering such as biology, chemistry, physics, economics, signal and control theory, etc. The readers might refer to [13, 14, 18] for more information and the references therein. By utilizing diverse nonlinear analysis techniques, many researchers have identified the uniqueness and existence of solutions for numerous classes of DEs of fractional order. For more details, see [3, 6, 8, 9, 12, 16, 17, 19]. Stability analysis, on the other hand, is usually one of the most popular essential concerns in the concept and utilization of fractional differential equations (FDEs). Several authors have recently become interested in the Ulam and MLU-stability concerns, we refer to the papers [1, 2, 7, 8, 21–24] and the references therein. Following the concerns raised in this work, we will focus on a significant area of differential problems widely used in engineering and other scientific fields. The equation is referred to as the pantograph equation. For additional details and uses of the pantograph equation, we recommend reading works [10, 11, 20, 25–33]. It's worth mentioning that the standard expression of the pantograph equation is

$$\begin{cases} w'(\tau) = Aw(\tau) + Bw(\omega\tau), \\ w(0) = w_0, \\ 0 \le \tau \le T,\ 0 < \omega < 1. \end{cases}$$

Some scholars have studied several versions of the above pantograph equation. For example, the researchers examined the following multi-pantograph equation in [20, 25]

$$w'(\tau) = Aw(\tau) + \sum_{i=1}^{n} v_i(\tau)\, w(\omega_i\tau) + e(\tau),\ \tau \ge 0.$$

In [20], the authors investigated the non-linear neutral pantograph equation

$$\begin{cases} w'(\tau) = \psi\left(\tau, w(\tau), w(\omega\tau), w'(\omega\tau)\right), \\ w(0) = w_0, \\ 0 \le \tau \le T,\ 0 < \omega < 1. \end{cases}$$

Recently, in [4] the author evaluated the pantograph type's following difficulty

$$\begin{cases} {}^{C}D^{q} w(\tau) = \psi(\tau, w(\tau), z(\omega\tau)), \\ w(0) = w_0, \\ 0 \le \tau \le T,\ 0 < q < 1, 0 < \omega < 1, \end{cases}$$

where ${}^{C}D^{q}$ denote Caputo fractional derivative. The following sequential fractional pantograph equation is the subject of the current work, which examines the uniqueness, existence, and MLU-stability of solutions.

$$\begin{cases} {}^{RL}D^{\delta}\left({}^{C}D^{\vartheta}\left({}^{C}D^{\theta} w(t)\right)\right) = A\psi(\tau, w(\tau), w(\omega\tau)) + BI^{\alpha}\left[\phi(\tau, w(\tau), w(\varpi\tau))\right], \\ w(0) = 0, \lambda_1 w(1) - \lambda_2 w(\eta) = \varphi(w), {}^{C}D^{\theta} w(0) = 0, 0 < \eta < 1, \beta, \lambda_1, \lambda_2 \in \mathbb{R}, \\ \tau \in [0,1], 0 < \delta, \vartheta, \theta \le 1, \alpha \ge 0, 0 < \omega, \varpi < 1, A, B \in \mathbb{R}, \lambda_1 \ne \lambda_2 \eta^{\delta+\vartheta+\theta-1}, \end{cases} \tag{1.1}$$

where ${}^{RL}D^{\delta}, {}^{C}D^{q}, q \in \{\vartheta, \theta\}$ stand for the fractional derivatives of Riemann-Liouville (RL) and Caputo, $\phi, \psi : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\varphi : C([0,1], \mathbb{R}) \to \mathbb{R}$ are continuous functions. ${}^{RL}D^{\delta}$ [15, 18], is defined by

$${}^{RL}D^{\delta} y(\tau) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{d\tau}\right)^{n} \int_0^{\tau} (\tau - s)^{n-\delta-1}\, y(s)\, ds, n = [\delta] + 1,$$

where $\Gamma(.)$ represents Euler gamma function. The operator, ${}^{C}D^{q}$ [15, 18], is defined by

$${}^{C}D^{q} y(\tau) = \frac{1}{\Gamma(n-q)} \int_0^{\tau} (\tau - s)^{n-q-1}\, y^{(n)}(s)\, ds, n = [q] + 1,$$

and the RL fractional integral [15, 18] of order $\alpha > 0$, stated as

$$I^{\vartheta}y(\tau) = \frac{1}{\Gamma(\vartheta)} \int_0^{\tau} (\tau - s)^{\vartheta-1} y(s)\, ds, \ \tau > 0.$$

Following are several lemmas that we notice [13, 18].

Lemma 1. Let $q, p > 0$ and $y \in L^1([0,1])$. Then $I^q I^p y(\tau) = I^{q+p} y(\tau)$ and $D^q I^q y(\tau) = y(\tau)$.

Lemma 2. Let $q > p > 0$ and $y \in L^1([0,1])$. Then $D^p I^q y(\tau) = I^{q-p} y(\tau)$.

Also, we recall the observing lemmas.

Lemma 3. [13] Let $\delta > 0$. Then for $w \in C(0,1) \cap L^1(0,1)$ and ${}^{RL}D^{\delta} w \in C(0,1) \cap L^1(0,1)$, we have

$$I^{\delta}\left[{}^{RL}D^{\delta} w(\tau)\right] = w(\tau) + \sum_{i=1}^{n} c_i \tau^{\delta - i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n, n = [\delta] + 1$.

Lemma 4. [13] Let $q > 0$. Then

$$I^{q}\left[{}^{C}D^{q} w(\tau)\right] = w(\tau) + \sum_{i=0}^{n-1} c_i \tau^{i},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1, n = [q] + 1$.

In what follow, we require an important singular type Gronwall inequality.

Theorem 5. [15] For each $\tau \in [0,1)$. If

$$u(\tau) \leq p(\tau) + \sum_{i=1}^{n} k_i(\tau) \int_0^{\tau} (\tau - s)^{\varepsilon_i - 1} u(s)\, ds,$$

where all of the functions are continuous and non-negative. The costants $\varepsilon_i > 0$, k_i $(i = 1, 2, ..., n)$ are monotonic increasing and bounded functions on $[0, 1)$, then

$$u(\tau) \leq p(\tau) + \sum_{j=1}^{\infty} \left(\sum_{1', 2', ..., j' = 1}^{n} \frac{\prod_{i=1}^{j} [k_{i'}(\tau) \Gamma(\varepsilon_{i'})]}{\Gamma\left(\sum_{i=1}^{j} \varepsilon_{i'}\right)} \int_0^{\tau} (\tau - s)^{\sum_{i=1}^{j} \varepsilon_{i'} - 1} p(s)\, ds \right).$$

Remark 6. For $n = 2$, if $\varepsilon_1, \varepsilon_2 > 0, k_1, k_2 \geq 0, p(\tau)$ is locally integrable and non-negative on $[0, 1)$ and $u(\tau)$ is locally integrable and non-negative on $[0, 1)$ with

$$u(\tau) \leq p(\tau) + k_1 \int_0^{\tau} (\tau - s)^{\varepsilon_1 - 1} u(s)\, ds + k_2 \int_0^{\tau} (\tau - s)^{\varepsilon_2 - 1} u(s)\, ds,$$

then

$$u(\tau) \leq p(\tau) + \sum_{j=1}^{\infty} \left(\frac{(k_1 \Gamma(\varepsilon_1))^j}{\Gamma(j\varepsilon_1)} \int_0^{\tau} (\tau - s)^{j\varepsilon_1 - 1} p(s)\, ds \right.$$

$$+\frac{(k_2\Gamma(\varepsilon_2))^j}{\Gamma(j\varepsilon_2)}\int_0^\tau (\tau-s)^{j\varepsilon_2-1}p(s)\,ds\Bigg).$$

Remark 7. Let $p(\tau)$ be a non-decreasing function on $[0,1)$ under the criteria of Remark 6. Then we have

$$u(\tau)\le p(\tau)\left(E_{\varepsilon_1}[k_1\Gamma(\varepsilon_1)\tau^{\varepsilon_1}]+E_{\varepsilon_2}[k_2\Gamma(\varepsilon_2)\tau^{\varepsilon_2}]\right),$$

where the Mittag-Leffler function, E_ε [24] defined by: $E_\varepsilon[x]=\sum_{j=1}^{\infty}\frac{x^\varepsilon}{\Gamma(j\varepsilon+1)}, x\in\mathbb{C}$.

The subsequent auxiliary result is also necessary.

Lemma 8. Suppose that $y(\tau)\in C([0,1],\mathbb{R})$ and let's examine the fractional problem

$$^{RL}D^\delta\left({}^CD^\vartheta\left({}^CD^\theta\right)w(\tau)\right)=y(\tau),\ \tau\in[0,1],\ 0<\delta,\vartheta,\theta\le 1, \tag{1.2}$$

with the condition

$$w(0)=0,\lambda_1w(1)-\lambda_2w(\eta)=\varphi(w),{}^C D^\theta w(0)=0. \tag{1.3}$$

Then, we have

$$\begin{aligned} w(\tau) &= \frac{1}{\Gamma(\delta+\vartheta+\theta)}\int_0^\tau(\tau-s)^{\delta+\vartheta+\theta-1}y(s)\,ds \\ &+\frac{\tau^{\delta+\vartheta+\theta-1}}{\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}}\left[\frac{\lambda_2}{\Gamma(\delta+\vartheta+\theta)}\int_0^\eta(\eta-s)^{\delta+\vartheta+\theta-1}y(s)\,ds\right. \\ &\left.-\frac{\lambda_1}{\Gamma(\delta+\vartheta+\theta)}\int_0^1(1-s)^{\delta+\vartheta+\theta-1}y(s)\,ds\right]+\frac{t^{\delta+\vartheta+\theta-1}}{\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}}\varphi(w). \end{aligned} \tag{1.4}$$

Proof. Utilizing Lemma 3 and the RL fractional integral of order δ on both sides of Eq (1.2), we obtain

$$^CD^\vartheta\left({}^CD^\theta\right)w(\tau)=I^\delta y(\tau)+c_1\tau^{\delta-1} \tag{1.5}$$

where $c_1\in\mathbb{R}$. Next, using Lemma 4 and the RL fractional integral of order ϑ on both sides of Eq (1.5), we obtain

$$^CD^\theta w(\tau)=I^{\delta+\vartheta}y(\tau)+\frac{\Gamma(\delta)c_1}{\Gamma(\delta+\vartheta)}\tau^{\delta+\vartheta-1}+c_2, \tag{1.6}$$

where $c_2\in\mathbb{R}$. When both sides of Eq (1.6) are solved using the RL fractional integral of order θ, we obtain

$$w(\tau)=I^{\delta+\vartheta+\theta}y(\tau)+\frac{\Gamma(\delta)c_1}{\Gamma(\delta+\vartheta+\theta)}\tau^{\delta+\vartheta+\theta-1}+\frac{c_2}{\Gamma(\theta+1)}\tau^\theta+c_3, c_3\in\mathbb{R}. \tag{1.7}$$

Using (1.3), we obtain

$$c_1=\frac{\Gamma(\delta+\vartheta+\theta)}{(\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1})\Gamma(\delta)}\left[\varphi(w)+\lambda_2I^{\delta+\vartheta+\theta}y(\eta)-\lambda_1I^{\delta+\vartheta+\theta}y(1)\right],$$

and

$$c_2=c_3=0,$$

inserting the values of c_0, c_1 and c_2 in (1.7) provides the solution (1.4). □

2. Existence results for pantograph problem (1.1)

Under this section, let's look at the sequential fractional pantograph problem using fixed point theory.

So, we need to introduce the following space.

Let $Z = C([0,1],\mathbb{R})$, the space of continuous Banach functions from $[0,1]$ into $\mathbb{R}$ with the norm, $\|w\|$ where $\|w\| = \sup\{|w(\tau)| : \tau \in [0,1]\}$. We construct operator $O : Z \to Z$ in the context of Lemma 8 by

$$\begin{aligned} Ow(\tau) = & \frac{A}{\Gamma(\delta+\vartheta+\theta)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta-1} \psi(s, w(s), w(\omega s))\, ds \\ & + \frac{B}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta+\alpha-1} \phi(s, w(s), w(\varpi s))\, ds \\ & + \frac{\tau^{\delta+\vartheta+\theta-1}}{\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}} \left[\frac{\lambda_2 A}{\Gamma(\delta+\vartheta+\theta)} \int_0^\eta (\eta-s)^{\delta+\vartheta+\theta-1} \psi(s, w(s), w(\omega s))\, ds \right. \\ & + \frac{\lambda_2 B}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^\eta (\eta-s)^{\delta+\vartheta+\theta+\alpha-1} \phi(s, w(s), w(\varpi s))\, ds \\ & - \frac{\lambda_1 A}{\Gamma(\delta+\vartheta+\theta)} \int_0^1 (1-s)^{\delta+\vartheta+\theta-1} \psi(s, w(s), w(\omega s))\, ds \\ & \left. - \frac{\lambda_1 B}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^1 (1-s)^{\delta+\vartheta+\theta+\alpha-1} \phi(s, w(s), w(\varpi s))\, ds \right] \\ & + \frac{\tau^{\delta+\vartheta+\theta-1}}{\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}} \varphi(w). \end{aligned} \tag{2.1}$$

For convenience, we consider the following hypotheses:

(H_1) Continuous functions: $\psi, \phi : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\exists$ constants $\mu_i > 0, i = 1,2$ such that for each $\tau \in [0,1]$ and $z_j, w_j \in \mathbb{R}$, $j = 1,2$

$$|\psi(\tau, z_1, z_2) - \psi(\tau, w_1, w_2)| \leq \mu_1 (|z_1 - w_1| + |z_2 - w_2|),$$

$$|\phi(\tau, z_1, z_2) - \phi(\tau, w_1, w_2)| \leq \mu_2 (|z_1 - w_1| + |z_2 - w_2|).$$

(H_2) $\varphi : C([0,1],\mathbb{R}) \to \mathbb{R}$ is continuous function with $\varphi(0) = 0$ and $\exists$ a constant $\sigma > 0$ such that

$$|\varphi(z) - \varphi(w)| \leq \sigma |z - w|, \ z, w \in C([0,1],\mathbb{R}).$$

We also introduce the notations shown below:

$$\begin{aligned} \Delta_1 := & \frac{|A|}{\Gamma(\delta+\vartheta+\theta+1)} \left[1 + \frac{1}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|} \left(|\lambda_2| \eta^{\delta+\vartheta+\theta} + |\lambda_1| \right) \right], \\ \Delta_2 := & \frac{|B|}{\Gamma(\delta+\vartheta+\theta+\alpha+1)} \left[1 + \frac{1}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|} \left(|\lambda_2| \eta^{\delta+\vartheta+\theta+\alpha} + |\lambda_1| \right) \right]. \end{aligned} \tag{2.2}$$

The first result of our existence is based on Banach's principle of contraction.

Theorem 9. If (H_i), $i = 1, 2$, are satisfied and also

$$2\mu\left(\Delta_1 + \Delta_2\right) < 1 - \frac{\sigma}{\left|\lambda_1 - \lambda_2 \eta^{\vartheta+\theta}\right|}, \tag{2.3}$$

where $\mu = \max\{\mu_i, i = 1, 2\}$ and Δ_i, $i = 1, 2$, are given by (2.2), then problem (1.1) has a unique solution on $[0, 1]$.

Proof. Let's define $\Lambda = \max\{\Lambda_i : i = 1, 2\}$, where Λ_i are finite numbers given by

$$\Lambda_1 = \sup_{\tau\in[0,1]} |\psi(\tau, 0, 0)| \quad \text{and} \quad \Lambda_2 = \sup_{\tau\in[0,1]} |\phi(\tau, 0, 0)|.$$

Setting

$$r \geq \frac{\Lambda\Delta_1 + \Lambda\Delta_2}{1 - \left(2\mu\left(\Delta_1 + \Delta_2\right) + \frac{\sigma}{\left|\lambda_1 - \lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\right)},$$

we demonstrate that $OB_r \subset B_r$, where $B_r = \{w \in Z : \|w\| \leq r\}$.

For $w \in B_r$ and for each $\tau \in [0, 1]$, by using the hypothesis (H_i), $i = 1, 2$, we can write

$$\begin{aligned} |\psi(\tau, w(\tau), w(\lambda\tau))| &\leq |\psi(\tau, w(\tau), w(\omega\tau)) - \psi(\tau, 0, 0)| + |\psi(\tau, 0, 0)| \\ &\leq 2\mu_1 \|w\| + \Lambda_1 \leq 2\mu_1 r + \Lambda_1, \end{aligned}$$

$$\begin{aligned} |\phi(\tau, w(\tau), w(\varpi\tau))| &\leq |\phi(\tau, w(\tau), w(\mu\tau)) - \varphi(\tau, 0, 0)| + |\varphi(\tau, 0, 0)| \\ &\leq 2\mu_2 \|w\| + \Lambda_2 \leq 2\mu_2 r + \Lambda_2, \end{aligned}$$

and

$$|\varphi(w)| \leq \sigma \|w\| \leq \sigma r.$$

Using these estimates, we obtain

$$\begin{aligned} \|O(w)\| \leq\ & \sup_{\tau\in[0,1]} \left\{ \frac{|A|}{\Gamma(\delta+\vartheta+\theta)} \int_0^\tau (\tau - s)^{\delta+\vartheta+\theta-1} |\psi(s, w(s), w(\omega s))| \, ds \right. \\ &+ \frac{|B|}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^\tau (\tau - s)^{\delta+\vartheta+\theta+\alpha-1} |\phi(s, w(s), w(\varpi s))| \, ds \\ &+ \frac{\tau^{\delta+\vartheta+\theta-1}}{\left|\lambda_1 - \lambda_2\eta^{\delta+\vartheta+\theta-1}\right|} \left[\frac{|\lambda_2| |A|}{\Gamma(\delta+\vartheta+\theta)} \int_0^\eta (\eta - s)^{\delta+\vartheta+\theta-1} |\psi(s, w(s), w(\omega s))| \, ds \right. \\ &+ \frac{|\lambda_2| |B|}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^\eta (\eta - s)^{\delta+\vartheta+\theta+\alpha-1} |\phi(s, w(s), w(\varpi s))| \, ds \\ &+ \frac{|\lambda_1| |A|}{\Gamma(\delta+\vartheta+\theta)} \int_0^1 (1 - s)^{\delta+\vartheta+\theta-1} |\psi(s, w(s), w(\omega s))| \, ds \\ &\left. + \frac{|\lambda_1| |B|}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^1 (1 - s)^{\delta+\vartheta+\theta+\alpha-1} |\phi(s, w(s), w(\varpi s))| \, ds \right] \\ &\left. + \frac{\tau^{\delta+\vartheta+\theta-1}}{\left|\lambda_1 - \lambda_2\eta^{\delta+\vartheta+\theta-1}\right|} |\varphi(w)| \right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(2\mu r+\Lambda)|A|}{\Gamma(\delta+\vartheta+\theta+1)}\left[1+\frac{1}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\left(|\lambda_2|\,\eta^{\delta+\vartheta+\theta}+|\lambda_1|\right)\right]\\
&\quad+\frac{(2\mu r+\Lambda)|B|}{\Gamma(\delta+\vartheta+\theta+\alpha+1)}\left[1+\frac{1}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\left(|\lambda_2|\,\eta^{\delta+\vartheta+\theta}+|\lambda_1|\right)\right]\\
&\quad+\frac{\sigma r}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\\
&= \left(2\mu(\Delta_1+\Delta_2)+\frac{\sigma}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\right)r+\Lambda(\Delta_1+\Delta_2)\leq r
\end{aligned}
$$

$\Rightarrow OB_r \subset B_r$. Now, for $w, z \in B_r$, we obtain

$$
\begin{aligned}
&\|O(w)-O(z)\|\\
\leq\;& \sup_{\tau\in[0,1]}\left\{\frac{|A|}{\Gamma(\delta+\vartheta+\theta)}\int_0^\tau(\tau-s)^{\delta+\vartheta+\theta-1}\left|\psi(s,w(s),w(\omega s))-\psi(s,z(s),z(\omega s))\right|ds\right.\\
&+\frac{|B|}{\Gamma(\delta+\vartheta+\theta+\alpha)}\int_0^\tau(\tau-s)^{\delta+\vartheta+\theta+\alpha-1}\left|\phi(s,w(s),w(\varpi s))-\phi(s,z(s),z(\varpi s))\right|ds\\
&+\frac{\tau^{\delta+\vartheta+\theta-1}}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\left[+\frac{|\lambda_2||A|}{\Gamma(\delta+\vartheta+\theta)}\int_0^\eta(\eta-s)^{\delta+\vartheta+\theta-1}\left|\psi(s,w(s),w(\omega s))\right.\right.\\
&\left.-\psi(s,z(s),z(\varpi s))\right|ds\\
&+\frac{|\lambda_2||B|}{\Gamma(\delta+\vartheta+\theta+\alpha)}\int_0^\eta(\eta-s)^{\delta+\vartheta+\theta+\alpha-1}\left|\phi(s,w(s),w(\varpi s))-\phi(s,z(s),z(\varpi s))\right|ds\\
&+\frac{|\lambda_1||A|}{\Gamma(\delta+\vartheta+\theta)}\int_0^1(1-s)^{\delta+\vartheta+\theta-1}\left|\psi(s,w(s),w(\omega s))-\psi(s,z(s),z(\omega s))\right|ds\\
&\left.+\frac{|\lambda_1||B|}{\Gamma(\delta+\vartheta+\theta+\alpha)}\int_0^1(1-s)^{\delta+\vartheta+\theta+\alpha-1}\left|\phi(s,w(s),w(\varpi s))-\phi(s,z(s),z(\varpi s))\right|ds\right]\\
&\left.+\frac{\tau^{\delta+\vartheta+\theta-1}}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\left|\varphi(w)-\varphi(z)\right|\right\}\\
\leq\;& \left(2\mu(\Delta_1+\Delta_2)+\frac{\sigma}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\right)\|w-z\|,
\end{aligned}
$$

in context of condition $2\mu(\Delta_1+\Delta_2)+\frac{\sigma}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}<1$, this demonstrates that O is a contraction. The operator O possesses an unique fixed point that corresponds to an unique solution to the problem according to Banach's fixed point theorem (1.1). The Leray-Schauder alternative yielded the second main result.

Lemma 10. [5] Let $Q : X \to X$ be a completely continuous operator. Let $G(Q) = \{u \in X : u = \rho Q(u)$ for some $0 < \rho < 1\}$. Then either the set $G(Q)$ is unbounded, or Q has at least one fixed point (Leray-Schauder alternative).

For the forthcoming result, we suppose that

(H_3) $\psi, \phi : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous and $\exists$ real constants $\pi_i, \gamma_i \geq 0, i = 1, 2$ and $\pi_0 > 0, \gamma_0 > 0$

such that for any $w_i \in \mathbb{R}, i = 1, 2$, we have

$$|\varphi(\tau, w_1, w_2)| \leq \pi_0 + \pi_1 |w_1| + \pi_2 |w_2|,$$

and

$$|\phi(\tau, w_1, w_2)| \leq \gamma_0 + \gamma_1 |w_1| + \gamma_2 |w_2|.$$

(H_4) $\varphi : C([0, 1], \mathbb{R}) \longrightarrow \mathbb{R}$ is continuous function with $\varphi(0) = 0$ and $\exists$ constant $\epsilon > 0$ such that

$$|\varphi(w)| \leq \epsilon \|w\| \text{ for all } w \in C([0, 1], \mathbb{R}).$$

Theorem 11. If $(H_i), i = 3, 4$, are satisfied and also

$$(\pi_1 + \pi_2)\Delta_1 + (\gamma_1 + \gamma_2)\Delta_2 + \frac{\epsilon}{\left|\lambda_1 - \lambda_2 \eta^{\delta+\vartheta+\theta-1}\right|} < 1, \tag{2.4}$$

then on $[0, 1]$ the problem (1.1) has at least one solution.

Proof. We demonstrate that the operator $O : Z \longrightarrow Z$ is completely continuous in the first step. Since the functions ψ, ϕ and φ are continuous, the operator O is also continuous.

Let $\Theta \subset W$ be bounded. Then $\exists$ positive constants $M_i, (i = 1, 2)$ such that

$$|\varphi(\tau, w, z)| \leq M_1, |\phi(\tau, w, z)| \leq M_2,$$

for each $w, z \in \Theta$ and constants N such that $|\psi(w)| \leq N$ for all $z \in C([0, 1], \mathbb{R})$. Then for any $w \in \Theta$ and by $(H_i), i = 3, 4$, we have

$$\begin{aligned} \|Ow\| \leq\ & \frac{M_1 |A|}{\Gamma(\delta + \vartheta + \theta + 1)} \left[1 + \frac{1}{\left|\lambda_1 - \lambda_2 \eta^{\delta+\vartheta+\theta-1}\right|} \left(|\lambda_2| \eta^{\delta+\vartheta+\theta} + |\lambda_1|\right)\right] \\ & + \frac{M_2 |B|}{\Gamma(\delta + \vartheta + \theta + \alpha + 1)} \left[1 + \frac{1}{\left|\lambda_1 - \lambda_2 \eta^{\delta+\vartheta+\theta-1}\right|} \left(|\lambda_2| \eta^{\delta+\vartheta+\theta+\alpha} + |\lambda_1|\right)\right] \\ & + \frac{N}{\left|\lambda_1 - \lambda_2 \eta^{\delta+\vartheta+\theta-1}\right|}, \end{aligned}$$

which implies that

$$\|O(z)\| \leq M_1\Delta_1 + M_2\Delta_2 + \frac{N}{\left|\lambda_1 - \lambda_2 \eta^{\delta+\vartheta+\theta-1}\right|}.$$

The operator O is uniformly bounded, as shown in the above Eq (2.1). As a follow-up, we demonstrate that O is equicontinuous sets of Z. Let $\tau_1, \tau_2 \in [0, 1]$ with $\tau_1 < \tau_2$. Next, we obtain

$$\begin{aligned} & |Ow(\tau_2) - Ow(\tau_1)| \\ \leq\ & \frac{M_1 |A|}{\Gamma(\delta + \vartheta + \theta + 1)} \left[(\tau_2 - \tau_1)^{\delta+\vartheta+\theta} + \left|\tau_2^{\delta+\vartheta+\theta} - \tau_1^{\delta+\vartheta+\theta}\right|\right] \\ & + \frac{M_2 |B|}{\Gamma(\delta + \vartheta + \theta + \alpha + 1)} \left[(\tau_2 - \tau_1)^{\delta+\vartheta+\theta+\alpha} + \left|\tau_2^{\delta+\vartheta+\theta+\alpha} - \tau_1^{\delta+\vartheta+\theta+\alpha}\right|\right] \end{aligned}$$

$$+\frac{\left|\tau_2^{\delta+\vartheta+\theta-1}-\tau_1^{\delta+\vartheta+\theta-1}\right|}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\left(\frac{|\lambda_2|\,|A|\,\eta^{\delta+\vartheta+\theta}}{\Gamma(\delta+\vartheta+\theta+1)}+\frac{|\lambda_2|\,|B|\,\eta^{\delta+\vartheta+\theta+\alpha}}{\Gamma(\delta+\vartheta+\theta+\alpha+1)}\right.$$
$$\left.+\frac{|\lambda_1|\,|A|}{\Gamma(\delta+\vartheta+\theta+1)}+\frac{|\lambda_1|\,|B|}{\Gamma(\delta+\vartheta+\theta+\alpha+1)}+N\right),$$

which does not depend on w and tends to 0 as $\tau_2-\tau_1\to 0$. Thus, O is equicontinuous. Thus, by using the Arzelá-Ascoli theorem, $O:Z\to Z$ is completely continuous.

Finally, we demonstrate that the set, $\chi=\{w\in Z: w=\varrho O(w),\ 0<\varrho<1\}$, is bounded. Let $w\in\chi$, then $w=\varrho O(w)$ .For each $\tau\in[0,1]$, we have

$$w(\tau)=\varrho Ow(\tau).$$

Then,

$$\begin{aligned}|w(\tau)| \leq\ & \frac{[\pi_0+(\pi_1+\pi_2)\|w\|]\,|A|}{\Gamma(\delta+\vartheta+\theta+1)}\left[1+\frac{1}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\left(|\lambda_2|\,\eta^{\delta+\vartheta+\theta}+|\lambda_1|\right)\right]\\ &+\frac{[\gamma_0+(\gamma_1+\gamma_2)\|w\|]\,|B|}{\Gamma(\delta+\vartheta+\theta+\alpha+1)}\left[1+\frac{1}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\left(|\lambda_2|\,\eta^{\delta+\vartheta+\theta+\alpha}+|\lambda_1|\right)\right]\\ &+\frac{\epsilon}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\|w\|\end{aligned}$$

which implies

$$\begin{aligned}\|w\| &\leq [\pi_0+(\pi_1+\pi_2)\|w\|]\,\Delta_1+[\gamma_0+(\gamma_1+\gamma_2)\|w\|]\,\Delta_2\\ &\quad+\frac{\epsilon}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\|w\|\\ &=\left[(\pi_1+\pi_2)\,\Delta_1+(\gamma_1+\gamma_2)\,\Delta_2+\frac{\epsilon}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\right]\|w\|+\pi_0\Delta_1+\gamma_0\Delta_2.\end{aligned}$$

Consequently,

$$\|w\|\leq\frac{\pi_0\Delta_1+\gamma_0\Delta_2}{1-\left[(\pi_1+\pi_2)\,\Delta_1+(\gamma_1+\gamma_2)\,\Delta_2+\frac{\epsilon}{\left|\lambda_1-\lambda_2\eta^{\delta+\vartheta+\theta-1}\right|}\right]}.$$

This implies that χ is bounded. According to Lemma 10, this indicates that the operator O contains at least one fixed point. The problem (1.1) on $[0,1]$ has hence at least one solution in this case. Hence, the proof is completed.

□

3. MLU-stability (MLUS) of pantograph problem (1.1)

The MLUS of the sequential fractional problem (1.1) will be defined and studied in this section. For $\tau\in[0,1]$, we provide the following fractional inequalities:

$$\left|{}^{RL}D^{\delta}\left({}^{C}D^{\vartheta}\left({}^{C}D^{\theta}z(\tau)\right)\right)-A\psi(\tau,z(\tau),z(\omega\tau))-BI^{\alpha}\left[\phi(\tau,z(\tau),z(\varpi\tau))\right]\right|\leq\kappa, \tag{3.1}$$

and

$$\left|{}^{RL}D^{\delta}\left({}^{C}D^{\vartheta}\left({}^{C}D^{\theta}z(\tau)\right)\right)-A\psi(\tau,z(\tau),z(\omega\tau))-BI^{\alpha}\left[\phi(\tau,z(\tau),z(\varpi\tau))\right]\right|\leq\kappa m(\tau),\tag{3.2}$$

where $\kappa\in\mathbb{R}^{+}$ and $m:[0,1]\to\mathbb{R}^{+}$ is continuous function.

Definition 12. The problem (1.1) is MLU-Hyers stable, with respect to $E_{\delta+\vartheta+\theta}$ if $\exists$ a real number υ such that for each $\kappa>0$ and for each solution $z\in Z$ of the inequality (3.1), $\exists$ a solution $w\in Z$ of the problem (1.1) with

$$|z(\tau)-w(\tau)|\leq\upsilon\kappa E_{\delta+\vartheta+\theta}[\tau],\ \tau\in[0,1].$$

Definition 13. The problem (1.1) is MLU-Hyers-Rassias stable, with respect to $mE_{\delta+\vartheta+\theta}$ if $\exists$ a real number $\upsilon_m>0$ such that for each $\kappa>0$ and for each solution $z\in Z$ of the inequality (3.2), $\exists$ a solution $w\in Z$ of problem (1.1) with

$$|z(\tau)-w(\tau)|\leq\upsilon_m\kappa m(\tau)E_{\delta+\vartheta+\theta}[\tau],\ \tau\in[0,1].$$

Remark 14. A function $z\in Z$ is a solution of the inequality (3.1) if and only if $\exists$ a function $f\in C([0,1],\mathbb{R})$ (which depend on z) such that

$$|f(\tau)|\leq\kappa,\quad\tau\in[0,1],$$

and

$$\begin{aligned}&{}^{C}D^{\delta}\left({}^{C}D^{\vartheta}\left({}^{C}D^{\theta}z(\tau)\right)\right)-A\psi(\tau,z(\tau),z(\omega\tau))+BI^{\alpha}\left[\phi(\tau,z(\tau),z(\varpi\tau))\right]\\=\;&f(\tau),\ \tau\in[0,1].\end{aligned}$$

Theorem 15. If hypotheses (H_i), $i=1,2$, are satisfied, then the problem (1.1) is MLU-Hyers stable.

Proof. Let $z\in Z$ represent the inequality's (3.1) solution and let $w\in Z$ represents the unique solution of the problem

$$\begin{cases}{}^{RL}D^{\delta}\left({}^{C}D^{\vartheta}\left({}^{C}D^{\theta}w(\tau)\right)\right)=A\psi(\tau,w(\tau),w(\omega\tau))+BI^{\alpha}\left[\phi(\tau,w(\tau),w(\varpi\tau))\right],\\ w(0)=0,\lambda_1w(1)-\lambda_2w(\eta)=\varphi(w),{}^{C}D^{\theta}w(0)=0,0<\eta<1,\beta,\lambda_1,\lambda_2\in\mathbb{R},\\ \tau\in[0,1],0<\delta,\vartheta,\theta\leq1,\alpha\geq0,0<\omega,\varpi<1,A,B\in\mathbb{R},\lambda_1\neq\lambda_2\eta^{\delta+\vartheta+\theta-1}.\end{cases}$$

According to Lemma 8, we have

$$w(\tau)=I^{\delta+\vartheta+\theta}y(\tau)+\frac{\Gamma(\delta)c_1\tau^{\delta+\vartheta+\theta-1}}{\Gamma(\delta+\vartheta+\theta)}+\frac{c_2\tau^{\theta}}{\Gamma(\theta+1)}+c_3,c_i\in\mathbb{R},i=1,2,3.$$

By integrating the inequality (3.1), we obtain

$$\begin{aligned}&\left|z(\tau)-I^{\delta+\vartheta+\theta}y_z(\tau)-\frac{\Gamma(\delta)a_1\tau^{\delta+\vartheta+\theta-1}}{\Gamma(\delta+\vartheta+\theta)}-\frac{a_2\tau^{\theta}}{\Gamma(\theta+1)}-a_3\right|\\\leq\;&\frac{\kappa\tau^{\delta+\vartheta+\theta}}{\Gamma(\delta+\vartheta+\theta+1)}\leq\frac{\kappa}{\Gamma(\delta+\vartheta+\theta+1)},\end{aligned}\tag{3.3}$$

where

$$y_z(\tau) = A\psi(\tau, z(\tau), z(\omega\tau)) + BI^{\alpha}[\phi(\tau, z(\tau), z(\varpi\tau))].$$

Latter, if $w(0) = z(0), w(\omega) = z(\omega)$ and ${}^{C}D^{\theta}w(0) = {}^{C}D^{\theta}z(0)$, then $c_1 = a_1, c_2 = a_2$ and $c_3 = a_3$.

For each $\tau \in [0, 1]$, we have

$$\begin{aligned}
|z(\tau) - w(\tau)| &= \Bigg| z(\tau) - I^{\delta+\vartheta+\theta} y_z(\tau) - \frac{\Gamma(\delta) a_1 \tau^{\delta+\vartheta+\theta-1}}{\Gamma(\delta+\vartheta+\theta)} \\
&\quad - \frac{a_2 \tau^{\theta}}{\Gamma(\theta+1)} - a_3 + I^{\delta+\vartheta+\theta}[y_z(\tau) - y_w(\tau)] \Bigg| \\
&\leq \left| z(\tau) - I^{\delta+\vartheta+\theta} y_z(\tau) - \frac{\Gamma(\delta) a_1 \tau^{\delta+\vartheta+\theta-1}}{\Gamma(\delta+\vartheta+\theta)} - \frac{a_2 \tau^{\theta}}{\Gamma(\theta+1)} - a_3 \right| \\
&\quad + \left| I^{\delta+\vartheta+\theta}[y_z(\tau) - y_w(\tau)] \right|,
\end{aligned}$$

then

$$\begin{aligned}
&\left| I^{\delta+\vartheta+\theta}[y_z(\tau) - y_w(\tau)] \right| \\
&\leq \frac{|A|}{\Gamma(\delta+\vartheta+\theta)} \int_0^{\tau} (\tau - s)^{\delta+\vartheta+\theta-1} |\psi(\tau, z(\tau), z(\omega\tau)) - \psi(\tau, w(\tau), w(\omega\tau))| ds \\
&+ \frac{|B|}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^{\tau} (\tau - s)^{\delta+\vartheta+\theta+\alpha-1} |\phi(s, z(s), z(\varpi s)) - \phi(s, w(s), w(\varpi s))| ds.
\end{aligned}$$

Using (H_1), we get

$$\begin{aligned}
&\left| I^{\delta+\vartheta+\theta}[y_z(\tau) - y_w(\tau)] \right| \\
\leq\ & \frac{2|A|\mu_1}{\Gamma(\delta+\vartheta+\theta)} \int_0^{\tau} (\tau - s)^{\delta+\vartheta+\theta-1} |z(\tau) - w(\tau)| ds \\
&+ \frac{2|B|\mu_2}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^{\tau} (\tau - s)^{\delta+\vartheta+\theta+\alpha-1} |z(\tau) - w(\tau)| ds.
\end{aligned} \tag{3.4}$$

By using (3.3) and (3.4), we have got

$$\begin{aligned}
|z(\tau) - w(\tau)| &\leq \frac{\kappa}{\Gamma(\delta+\vartheta+\theta+1)} \\
&\quad + \frac{2|A|\mu_1}{\Gamma(\delta+\vartheta+\theta)} \int_0^{\tau} (\tau - s)^{\delta+\vartheta+\theta-1} |z(\tau) - w(\tau)| ds \\
&\quad + \frac{2|B|\mu_2}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^{\tau} (\tau - s)^{\delta+\vartheta+\theta+\alpha-1} |z(\tau) - w(\tau)| ds.
\end{aligned}$$

According to Remarks 6 and 7, we have got

$$\begin{aligned}
&|z(\tau) - w(\tau)| \\
\leq\ & \frac{\kappa}{\Gamma(\delta+\vartheta+\theta+1)} \left(E_{\delta+\vartheta+\theta}\left[2|A|\mu_1 \tau^{\delta+\vartheta+\theta}\right] + E_{\delta+\vartheta+\theta+\alpha}\left[2|B|\mu_2 \tau^{\delta+\vartheta+\theta+\alpha}\right] \right) \\
=\ & \upsilon\kappa \left(E_{\delta+\vartheta+\theta}\left[2|A|\mu_1 \tau^{\delta+\vartheta+\theta}\right] + E_{\delta+\vartheta+\theta+\alpha}\left[2|B|\mu_2 \tau^{\delta+\vartheta+\theta+\alpha}\right] \right).
\end{aligned}$$

As a result, the problem (1.1) is MLU-Hyers stable.

Theorem 16. If hypotheses $(H_i), i = 1, 2$, are satisfied. Suppose there exists $\upsilon_m > 0$ such that

$$\frac{1}{\Gamma(\delta+\vartheta+\theta)}\int_0^t (t-s)^{\delta+\vartheta+\theta-1} m(s)ds \le \upsilon_m m(t), t \in [0,1], \tag{3.5}$$

where $m \in C([0,1], \mathbb{R}_+)$ is increasing. Then the problem (1.1) is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to $mE_{\delta+\vartheta+\theta}$.

Proof. By integrating the inequality (3.2), we have

$$\begin{aligned} &\left| z(\tau) - I^{\delta+\vartheta+\theta} y_z(\tau) - \frac{\Gamma(\delta) a_1 t^{\delta+\vartheta+\theta-1}}{\Gamma(\delta+\vartheta+\theta)} - \frac{a_2 \tau^\theta}{\Gamma(\theta+1)} - a_3 \right| \\ \le\ & \frac{\kappa}{\Gamma(\delta+\vartheta+\theta)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta-1} m(s)ds, \end{aligned}$$

where the inequality (3.2) has $z \in Z$ as a solution. Let's call the unique solution to the problem $w \in Z$

$$\begin{cases} {}^C D^\delta \left({}^C D^\vartheta \left({}^C D^\theta w(\tau) \right) \right) = A\psi(\tau, w(\tau), w(\omega\tau)) + BI^\alpha \left[\phi(\tau, w(\tau), w(\varpi\tau)) \right], \tau \in [0,1] \\ w(1) = z(1), w(0) = z(0), w(\eta) = z(\eta), {}^C D^\theta w(0) = {}^C D^\theta z(0), 0 < \eta < 1. \end{cases}$$

We have

$$w(\tau) = I^{\delta+\vartheta+\theta} y_w(\tau) + \frac{\Gamma(\delta) c_1 \tau^{\delta+\vartheta+\theta-1}}{\Gamma(\delta+\vartheta+\theta)} + \frac{c_2 \tau^\theta}{\Gamma(\theta+1)} + c_3.$$

Then

$$\begin{aligned} |z(\tau) - w(\tau)| \le\ & \frac{\kappa}{\Gamma(\delta+\vartheta+\theta)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta-1} m(s)ds \\ & + \frac{2|A|\mu_1}{\Gamma(\delta+\vartheta+\theta)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta-1} |z(\tau) - w(\tau)| ds \\ & + \frac{2|B|\mu_2}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta+\alpha-1} |z(\tau) - w(\tau)| ds. \end{aligned}$$

From (3.5), as it can be observed,

$$\begin{aligned} |z(\tau) - w(\tau)| \le\ & \kappa \upsilon_m m(\tau) \\ & + \frac{2|A|\mu_1}{\Gamma(\delta+\vartheta+\theta)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta-1} |z(\tau) - w(\tau)| ds \\ & + \frac{2|B|\mu_2}{\Gamma(\delta+\vartheta+\theta+\alpha)} \int_0^\tau (\tau-s)^{\delta+\vartheta+\theta+\alpha-1} |z(\tau) - w(\tau)| ds. \end{aligned}$$

Now, by Remarks 6 and 7, we have got

$$\begin{aligned} &|z(\tau) - w(\tau)| \\ \le\ & \kappa \upsilon_m m(\tau) \left(E_{\delta+\vartheta+\theta} \left[2|A|\mu_1 \tau^{\delta+\vartheta+\theta} \right] + E_{\delta+\vartheta+\theta+\alpha} \left[2|B|\mu_2 \tau^{\delta+\vartheta+\theta+\alpha} \right] \right). \end{aligned}$$

The problem (1.1) is therefore MLU-Hyers-Rassias stable. □

4. Application

Take the fractional pantograph problem below into consideration:

$$\begin{cases} {}^{RL}D^{\frac{4}{5}}\,{}^{C}\left({}^{C}D^{\frac{1}{2}}\left({}^{C}D^{\frac{2}{3}}w(\tau)\right)\right)=\frac{e^{-2}}{37\pi^{2}}\left(\frac{1}{5}+\frac{e^{-2\tau}}{45\pi e^{2}}\sin w(\tau)+\frac{e^{-2\tau}}{45\pi e^{2}}\cos w\left(\frac{10}{11}\tau\right)\right) \\ +\frac{1}{47e^{3}}I^{\frac{\sqrt{5}}{2}}\left[\frac{11}{13}+\frac{1}{\sqrt{40^{2}+\tau^{2}}}w(\tau)+\frac{e^{-\tau}}{40+\tau^{2}}\sin w\left(\frac{13}{12}\tau\right)\right],\tau\in[0,1], \\ w(0)=0,\frac{4}{3}w(1)-\frac{3}{2}w\left(\frac{5}{6}\right)=\frac{1}{45}w(\tau),{}^{C}D^{\theta}w(0)=\frac{\sqrt{3}}{4}, \end{cases} \tag{4.1}$$

and the following fractional inequalities

$$\left|{}^{RL}D^{\frac{4}{5}}\left({}^{C}D^{\frac{1}{2}}\left({}^{C}D^{\frac{2}{3}}z(\tau)\right)\right)-\frac{e^{-2}}{37\pi^{2}}\psi(\tau,z(\tau),z(\omega\tau))\right.$$

$$\left.-\frac{1}{47e^{3}}I^{\frac{\sqrt{5}}{2}}\left[\phi(\tau,z(\tau),z(\varpi\tau))\right]\right|\leq\kappa,$$

and

$$\left|{}^{RL}D^{\frac{4}{5}}\left({}^{C}D^{\frac{1}{2}}\left({}^{C}D^{\frac{2}{3}}z(\tau)\right)\right)-\frac{e^{-2}}{37\pi^{2}}\psi(\tau,z(\tau),z(\omega\tau))\right.$$

$$\left.-\frac{1}{47e^{3}}I^{\frac{\sqrt{5}}{2}}\left[\phi(\tau,z(\tau),z(\varpi\tau))\right]\right|\leq\kappa m(\tau),$$

where

$$\begin{aligned}\psi(\tau,z(\tau),z(\omega\tau)) &= \frac{1}{5}+\frac{e^{-2\tau}}{45\pi e^{2}}\sin z(\tau)+\frac{e^{-2\tau}}{45\pi e^{2}}\cos z(\omega\tau),\\ \phi(\tau,z(\tau),z(\varpi\tau)) &= \frac{11}{13}+\frac{1}{\sqrt{(40e^{2})^{2}+\tau^{2}}}z(\tau)+\frac{e^{-\tau}}{40e^{2}+\tau^{2}}\sin z(\varpi\tau),\end{aligned}$$

and

$$\delta=\frac{4}{5},\vartheta=\frac{1}{2},\theta=\frac{2}{3},\alpha=\frac{\sqrt{5}}{2},$$

$$A=\frac{e^{-2}}{37\pi^{2}},\varphi(w)=\frac{1}{45}w(\tau),B=\frac{1}{47e^{3}},$$

$$\lambda_{1}=\frac{4}{3},\lambda_{2}=\frac{3}{2},\omega=\frac{10}{11},\varpi=\frac{13}{12},$$

$$\eta=\frac{5}{6},\beta=\frac{\sqrt{3}}{4},\lambda_{1}\neq\lambda_{2}\eta^{\delta+\vartheta+\theta-1}.$$

For each $\tau\in[0,1]$ and $w_{j},z_{j}\in\mathbb{R}$, $j=1,2$, we have

$$|\psi(\tau,w_{1}(\tau),w_{2}(\lambda\tau))-\psi(\tau,z_{1}(\tau),z_{2}(\lambda\tau))| \leq \frac{1}{45\pi e^{2}}(|w_{1}-z_{1}|+|w_{2}-z_{2}|),$$

$$|\phi(\tau, w_1(\tau), w_2(\lambda\tau)) - \phi(\tau, z_1(\tau), z_2(\lambda\tau))| \leq \frac{1}{40e^2}(|w_1 - z_1| + |w_2 - z_2|),$$

and

$$|\varphi(w_1) - \varphi(z_1)| \leq \frac{1}{45}|w_1 - z_1|,$$

hence conditions (H_1) and (H_2) hold with $\mu_1 = \frac{1}{45\pi e^2}, \mu_2 = \frac{1}{40e^2}$ and $\sigma = \frac{1}{45}$ respectively.

With the given data, it is found that

$$\begin{aligned}\mu &= \max\{\mu_i, i = 1, 2\} = \frac{1}{40e^2},\ \lambda_1 - \lambda_2\eta^{\delta+\vartheta+\theta-1} = 7.571\,3 \times 10^{-2},\\ \Delta_1 &= 6.200 \times 10^{-3},\ \Delta_2 = 4.7412 \times 10^{-3}.\end{aligned}$$

Thus condition

$$2\mu(\Delta_1 + \Delta_2) = 7.403\,7 \times 10^{-5} < 1 - \frac{\sigma}{\left|\lambda_1 - \lambda_2\eta^{\delta+\vartheta+\theta-1}\right|} = 0.706\,49,$$

is satisfied. Therefore, the problem (4.1) has a unique solution on [0, 1], according to Theorem 9, and is MLU-Hyers stable with

$$\begin{aligned}&|z(\tau) - w(\tau)|\\ \leq\ & \frac{\kappa}{\Gamma\left(\frac{89}{30}\right)}\left(E_{\frac{59}{30}}\left[\frac{2e^{-2}}{1665\pi^3e^2}\tau^{\frac{59}{30}}\right] + E_{\frac{118+30\sqrt{5}}{60}}\left[\frac{1}{940e^5}\tau^{\frac{118+30\sqrt{5}}{60}}\right]\right), \tau \in [0, 1].\end{aligned}$$

Let $m(\tau) = \tau^2$, then

$$I^{\frac{4}{5}+\frac{1}{2}+\frac{2}{3}}m(\tau) = I^{\frac{4}{5}+\frac{1}{2}+\frac{2}{3}}\left(\tau^2\right) = \frac{2}{\Gamma\left(\frac{149}{30}\right)}\tau^{2+\frac{4}{5}+\frac{1}{2}+\frac{2}{3}} \leq \frac{2}{\Gamma\left(\frac{149}{30}\right)}\tau^2 = \upsilon_m m(\tau).$$

Thus condition (3.4) is satisfied with $m(\tau) = \tau^2$ and $\upsilon_m = \frac{2}{\Gamma(\frac{149}{30})}$. Thus, Theorem 16 demonstrates that problem (4.1) is MLU-Hyers-Rassias stable with

$$\begin{aligned}&|z(\tau) - w(\tau)|\\ \leq\ & \frac{\kappa\tau^2}{\Gamma\left(\frac{179}{30}\right)}\left(E_{\frac{59}{30}}\left[\frac{2e^{-2}}{1665\pi^3e^2}\tau^{\frac{59}{30}}\right] + E_{\frac{118+30\sqrt{5}}{60}}\left[\frac{1}{940e^5}\tau^{\frac{118+30\sqrt{5}}{60}}\right]\right), \tau \in [0, 1].\end{aligned}$$

5. Conclusions

In this work, we considered pantograph equations with three sequential fractional derivatives. The existence and Mittag-Leffler-Ulam stability of solutions have been discussed. The existence results of the solutions for the mentioned problem were investigated by the contraction mapping principle and Leray-Schauder's alternative. The Mittag-Leffler-Ulam-Hyers stability and Mittag-Leffler-Ulam-Hyers-Rassias stability results have been proved by applying generalized singular Gronwall's inequality. The main results were illustrated with the aid of an example.

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Conflict of interest

The publication of this paper, according to the author, is free from any conflicts of interest.

References

1. M. Ahmad, J. Jiang, A. Zada, Z. Ali, Z. Fu, J. Xu, Hyers-Ulam-Mittag-Leffler stability for a system of fractional neutral differential equations, *Discrete Dyn. Nat. Soc.*, **2020** (2020), 2786041. https://doi.org/10.1155/2020/2786041
2. I. Ahmad, J. J. Nieto, G. U. Rahman, K. Shah, Existence and stability for fractional order pantograph equations with nonlocal conditions, *Electron. J. Differ. Equ.*, **132** (2020), 1–16.
3. G. Ali, K. Shah, G. ur Rahman, Investigating a class of pantograph differential equations under multi-points boundary conditions with fractional order, *Int. J. Appl. Comput. Math.*, **7** (2021), 2. https://doi.org/10.1007/s40819-020-00932-0
4. K. Balachandran, S. Kiruthika, J. J. Trujillo, Existence of solutions of nonlinear fractional pantograph equations, *Acta Math. Sci.*, **33** (2013), 712–720. https://doi.org/10.1016/S0252-9602(13)60032-6
5. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer-Verlag, 2003. https://doi.org/10.1007/978-0-387-21593-8
6. Y. Gouari, Z. Dahmani, I. Jebri, Application of fractional calculus on a new differential problem of Duffing type, *Adv. Math.: Sci. J.*, **9** (2020), 10989–11002. https://doi.org/10.37418/amsj.9.12.82
7. M. Houas, Existence and stability of fractional pantograph differential equations with Caputo-Hadamard type derivative, *Turkish J. Ineqal.*, **4** (2020), 29–38.
8. M. Houas, M, Bezziou, Existence of solutions for neutral Caputo-type fractional integro-differential equations with nonlocal boundary conditions, *Commun. Optim. Theory*, **2021** (2021), 10.
9. M. Houas, Z. Dahmani, On existence of solutions for fractional differential equations with nonlocal multi-point boundary conditions, *Lobachevskii J. Math.*, **37** (2016), 120–127. https://doi.org/10.1134/S1995080216020050
10. A. Iserles, Exact and discretized stability of the pantograph equation, *Appl. Numer. Math.*, **24** (1997), 295–308. https://doi.org/10.1016/S0168-9274(97)00027-5
11. A. Iserles, On the generalized pantograph functional-differential equation, *Eur. J. Appl. Math.*, **4** (1993), 1–38. https://doi.org/10.1017/S0956792500000966

12. E. T. Karimov, B. Lopez, K. Sadarangani, About the existence of solutions for a hybrid nonlinear generalized fractional pantograph equation, *Fract. Differ. Calculus*, **6** (2016), 95–110. https://doi.org/10.7153/fdc-06-06

13. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V. Amsterdam, 2006.

14. V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear. Anal.*, **69** (2008), 2677–2682. https://doi.org/10.1016/j.na.2007.08.042

15. S. Y. Lin, Generalized Gronwall inequalities and their applications to fractional differential equations, *J. Inequal. Appl.*, **2013** (2013), 549. https://doi.org/10.1186/1029-242X-2013-549

16. S. K. Ntouyas, A. Alsaedi, B. Ahmad, Existence theorems for mixed Riemann-Liouville and Caputo fractional differential equations and inclusions with nonlocal fractional integro-differential boundary conditions, *Fractal Fract.*, **3** (2019), 21. https://doi.org/10.3390/fractalfract3020021

17. S. K. Ntouyas, D. Vivek, Existence and uniqueness results for sequential Hilfer fractional differential equations with multi-point boundary conditions, *Acta Math. Univ. Comen.*, **90** (2021), 171–185.

18. L. Podlubny, *Fractional differential equations*, New York: Academic Press, 1999.

19. K. Shah, D. Vivek, K. Kanagarajan, Dynamics and stability of $\psi-$fractional pantograph equations with boundary conditions, *Bol. Soc. Paran. Mat.*, **39** (2021), 43–55. https://doi.org/10.5269/bspm.41154

20. M. Sezer, S. Yalçinbaş, N. Şahin, Approximate solution of multi-pantograph equation with variable coefficients, *J. Comput. Appl. Math.*, **214** (2008), 406–416. https://doi.org/10.1016/j.cam.2007.03.024

21. I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.*, **26** (2010), 103–107.

22. D. Vivek, K. Kanagarajan, S. Sivasundaram, Dynamics and stability of pantograph equations via Hilfer fractional derivative, *Nonlinear Stud.*, **23** (2016), 685–698.

23. D. Vivek, E. M. Elsayed, K. Kanagarajan, Existence and Ulam stability results for a class of boundary value problem of neutral pantograph equations with complex order, *SeMA J.*, **77** (2021), 243–256. https://doi.org/10.1007/s40324-020-00214-1

24. J. Wang, Y. Zhang, Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations, *Optimization*, **63** (2014), 1181–1190. https://doi.org/10.1080/02331934.2014.906597

25. Z. H. Yu, Variational iteration method for solving the multi-pantograph delay equation, *Phys. Lett. A*, **372** (2008), 6475–6479. https://doi.org/10.1016/j.physleta.2008.09.013

26. M. R. Fatehi, M. Samavat, M. A. Vali, F. Khaleghi, State analysis and optimal control of linear time-invariant scaled systems using the Chebyshev wavelets, *Contemp. Eng. Sci.*, **5** (2012), 91–105.

27. F. Ghomanjani, M. H. Farahi, A. V. Kamyad, Numerical solution of some linear optimal control systems with pantograph delays, *IMA J. Math. Control Inf.*, **32** (2015), 225–243. https://doi.org/10.1093/imamci/dnt037

28. S. M. Hoseini, Optimal control of linear pantograph-type delay systems via composite Legendre method, *J. Franklin Inst.*, **357** (2020), 5402–5427. https://doi.org/10.1016/j.jfranklin.2020.02.051
29. Z. Gong, C. Liu, K. L. Teo, X. Yi, Optimal control of nonlinear fractional systems with multiple pantograph-delays, *Appl. Math. Comput.*, **425** (2022), 127094. https://doi.org/10.1016/j.amc.2022.127094
30. J. F. Gómez-Aguilar, Space-time fractional diffusion equation using a derivative with nonsingular and regular kernel, *Phys. A: Stat. Mech. Appl.*, **465** (2017), 562–572. https://doi.org/10.1016/j.physa.2016.08.072
31. J. F. Gómez-Aguilar, A. Atangana, Time-fractional variable-order telegraph equation involving operators with Mittag-Leffler kernel, *J. Electromagnet. Waves Appl.*, **33** (2019), 165–177. https://doi.org/10.1080/09205071.2018.1531791
32. P. Pandey, J. F. Gómez-Aguilar, On solution of a class of nonlinear variable order fractional reaction-diffusion equation with Mittag-Leffler kernel, *Numer. Methods Partial Differ. Equ.*, **37** (2021), 998–1011. https://doi.org/10.1002/num.22563
33. J. F. Gómez-Aguilar, H. Yépez-Martínez, J. Torres-Jiménez, T. Córdova-Fraga, R. F. Escobar-Jiménez, V. H. Olivares-Peregrino, Homotopy perturbation transform method for nonlinear differential equations involving to fractional operator with exponential kernel, *Adv. Differ. Equ.*, **2017** (2017), 68. https://doi.org/10.1186/s13662-017-1120-7

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