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Research article

Exact expression of ultimate time survival probability in homogeneous discrete-time risk model

Andrius Grigutis*

Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania

* **Correspondence:** Email: andrius.grigutis@mif.vu.lt.

Abstract: In this work, we set up the generating function of the ultimate time survival probability $\varphi(u+1)$, where

$$\varphi(u) = \mathbb{P}\left(\sup_{n\geqslant 1}\sum_{i=1}^{n}(X_i-\kappa) < u\right),$$

$u \in \mathbb{N}_0$, $\kappa \in \mathbb{N}$ and the random walk $\{\sum_{i=1}^{n} X_i,\ n \in \mathbb{N}\}$ consists of independent and identically distributed random variables X_i, which are non-negative and integer-valued. We also give expressions of $\varphi(u)$ via the roots of certain polynomials. The probability $\varphi(u)$ means that the stochastic process

$$u + \kappa n - \sum_{i=1}^{n} X_i$$

is positive for all $n \in \mathbb{N}$, where a certain growth is illustrated by the deterministic part $u+\kappa n$ and decrease is given by the subtracted random part $\sum_{i=1}^{n} X_i$. Based on the proven theoretical statements, we give several examples of $\varphi(u)$ and its generating function expressions, when random variables X_i admit Bernoulli, geometric and some other distributions.

Keywords: homogeneous discrete-time risk model; random walk; survival probability; initial values; generating function; Vandermonde matrix; ruin theory
Mathematics Subject Classification: 60G50, 60J80, 91G05

1. Introduction and preliminaries

The study of the sum of independent and identically distributed random variables $\sum_{i=1}^{n} X_i$ is hardly avoidable in probability theory and related fields. This sequence of sums $\{\sum_{i=1}^{n} X_i,\ n \in \mathbb{N}\}$ is called the random walk. Let us define the stochastic process

$$W(n) := u + \kappa n - \sum_{i=1}^{n} X_i,\ n \in \mathbb{N}, \tag{1.1}$$

where $u \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\kappa \in \mathbb{N}$ and random variables X_i, $i \in \mathbb{N}$ are independent, identically distributed, non-negative and integer-valued. If $\kappa = 1$, the defined process (1.1) is known as a discrete-time risk model; see [1]. Allowing $\kappa \in \mathbb{N}$, we call the process (1.1) a generalized premium discrete-time risk model; see [2]. Such types of processes appear in insurance mathematics (ruin theory), arguing that they describe an insurer's wealth in time moments $n \in \mathbb{N}$, where u means the initial surplus (also called capital or reserve), κ denotes the premium rate (earnings per unit of time), i.e., $(n+1)\kappa - n\kappa = \kappa$, and the random walk $\{\sum_{i=1}^{n} X_i,\ n \in \mathbb{N}\}$ represents the expenses (payoffs) caused by random size claims. Then, one can become curious to know whether the initial surplus u plus the gained premiums κn are sufficient to cover the incurred random expenses $\sum_{i=1}^{n} X_i$. More precisely, one aims to know whether $W(n) > 0$ for all $n \in \{1, 2, \ldots, T\}$ when T is some fixed natural number or $T \to \infty$. The positivity of $W(n)$ is of course associated with the probability. For the model given in (1.1), we define the finite time survival probability:

$$\varphi(u, T) := \mathbb{P}\left(\bigcap_{n=1}^{T} \{W(n) > 0\}\right) = \mathbb{P}\left(\sup_{1 \leqslant n \leqslant T} \sum_{i=1}^{n} (X_i - \kappa) < u\right),\ T \in \mathbb{N},$$

and the ultimate time survival probability:

$$\varphi(u) := \mathbb{P}\left(\bigcap_{n=1}^{\infty} \{W(n) > 0\}\right) = \mathbb{P}\left(\sup_{n \geqslant 1} \sum_{i=1}^{n} (X_i - \kappa) < u\right). \tag{1.2}$$

Both $\varphi(u,\ T)$ and $\varphi(u)$ are nothing but distribution functions of the provided integer-valued sequence of sums of random variables; these functions are left-continuous, non-decreasing and step functions if we allow $u \in \mathbb{R}$. Also, $\varphi(\infty) = 1$ if $\mathbb{E}X < \kappa$; see Section 2. In particular, $\varphi(0)$ is interpreted as the ultimate time survival probability when an insurer starts the activity with no initial surplus, i.e., when $u = 0$. Then, the insurer maintains chances to "persist alive" if the payoff's size in the first moment of time $n = 1$ is less than κ, i.e., if $X_1 < \kappa$.

Calculation of $\varphi(u, T)$ is simple; see, for instance, [2, Theorem 1]. Let us turn to the ultimate time survival probability $\varphi(u)$. The law of total probability and rearrangements in (1.2) imply

$$\varphi(u) = \sum_{i=1}^{u+\kappa} x_{u+\kappa-i}\varphi(i); \tag{1.3}$$

see [2, page 3].

By setting $u = 0$ in (1.3), we get

$$\varphi(0) = x_{\kappa-1}\varphi(1) + x_{\kappa-2}\varphi(2) + \ldots + x_0\varphi(\kappa); \tag{1.4}$$

to calculate the probability $\varphi(\kappa)$ when $x_0 > 0$, we must know the initial ones $\varphi(0), \varphi(1), \ldots, \varphi(\kappa - 1)$. The calculation of $\varphi(u)$, when $u = \kappa, \kappa + 1, \ldots$, using the recurrence equation (1.3), requires

$\varphi(0), \varphi(1), \ldots, \varphi(\kappa-1)$ too. The needed quantity of these initial values is X distribution-dependent, as some of the probabilities $x_0, x_1, \ldots, x_{\kappa-1}$ may equal to zero, cf. (1.4) when $\mathbb{P}(X > j) = 1$ for some $j \geqslant 0$. The paper [2] deals with finding the mentioned initial values $\varphi(0), \varphi(1), \ldots, \varphi(\kappa-1)$, and it is shown there that they can be found by calculating the limits of certain recurrent sequences. For instance, if $\kappa = 2$ and $x_0 > 0$, then it follows by (1.4) that

$$\varphi(0) = x_1\varphi(1) + x_0\varphi(2),$$

where (see [3, pages 2 and 3])

$$\varphi(0) = \varphi(\infty)\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\begin{vmatrix}\beta_n & \gamma_n\\ \beta_{n+1} & \gamma_{n+1}\end{vmatrix}},\ \varphi(1) = \varphi(\infty)\lim_{n\to\infty}\frac{\beta_n-\beta_{n+1}}{\begin{vmatrix}\beta_n & \gamma_n\\ \beta_{n+1} & \gamma_{n+1}\end{vmatrix}}; \tag{1.5}$$

$$\beta_0 = 1,\ \beta_1 = 0,\ \beta_n = \frac{1}{x_0}\left(\beta_{n-2} - \sum_{i=1}^{n-1} x_{n-i}\beta_i\right),\ \text{for } n \geqslant 2,$$
$$\gamma_0 = 0,\ \gamma_1 = 1,\ \gamma_n = \frac{1}{x_0}\left(\gamma_{n-2} - \sum_{i=1}^{n-1} x_{n-i}\gamma_i\right),\ \text{for } n \geqslant 2,$$

and $\varphi(\infty) = 1$ if $\mathbb{E}X < 2$.

Calculating the limits in (1.5) and aiming to prove that the provided determinant 2×2 never vanishes; in paper [3], it was proved the connection to the solutions of $s^2 = G_X(s)$, where $s \in \mathbb{C}$, $|s| \leqslant 1$ and $G_X(s)$ is the probability-generating function of the random variable X. On top of that, it was realized in [3] that the values of $\varphi(0)$ and $\varphi(1)$ in (1.5) can be derived by using the classical stationarity property for the distribution of the maximum of a reflected random walk; see [4, Chapter VI, Section 9]. Using the mentioned stationarity property, the generating function of $\varphi(u+1)$, $u \in \mathbb{N}_0$ for $\kappa = 2$ was found in [3, Theorem 5]; however, this required the finiteness of the second moment of the random variable X, i.e., $\mathbb{E}X^2 < \infty$. In this article, we extend the work in [3] and find the generating function of $\varphi(u+1)$, $u \in \mathbb{N}_0$ for an arbitrary $\kappa \in \mathbb{N}$. Moreover, we show that the requirement of $\mathbb{E}X^2 < \infty$ is redundant and provide exact expressions of $\varphi(u)$, $u \in \mathbb{N}_0$ via solutions of systems of linear equations which are based on the roots of $s^\kappa = G_X(s)$ and Vandermonde-like matrices.

For the short overview of the literature, we mention that the references [1, 5–13] are known as the classical ones on the wide subject of renewal risk models, while [14–16] might be mentioned as the recent ones. The main reason for so much literature is that the ruin theory, being random walk-based, is heavily dependent on the random walk's structural assumptions, such as the independence of random variables, their distributions, etc. This work is also closely related to branching and Galton-Watson processes and queueing theory; see [17] and related papers. See also [18] or [19, Figure 1] on random walk occurrence in number theory. Last but not least, it is worth mentioning that Vandermonde matrices have a broad range of occurrences, from pure mathematics to many other applied sciences; see [20] and related works.

2. Several auxiliary notations and the net profit condition

Let

$$\mathcal{M} := \sup_{n\geqslant 1}\left(\sum_{i=1}^{n}(X_i-\kappa)\right)^{+},$$

where $x^{+} = \max\{0, x\}$, $x \in \mathbb{R}$ is the positive part function and the random variables X_i and $\kappa \in \mathbb{N}$ are the same as in the model (1.1). Let us denote the probability mass function of the random variable $\mathcal{M}$ by

$$\pi_i := \mathbb{P}(\mathcal{M} = i),\ i \in \mathbb{N}_0.$$

Then, the ultimate time survival probability definition (1.2) implies that

$$\varphi(u+1) = \sum_{i=0}^{u}\pi_i = \mathbb{P}(\mathcal{M} \leqslant u) \text{ for all } u \in \mathbb{N}_0. \tag{2.1}$$

In general, the random variable $\mathcal{M}$ can be extended, i.e., $\mathbb{P}(\mathcal{M} = \infty) > 0$; however, the condition $\mathbb{E}X < \kappa$ ensures

$$\lim_{u\to\infty}\varphi(u) = \mathbb{P}(\mathcal{M} < \infty) = 1;$$

see [2, Lemma 1]. This condition $\mathbb{E}X < \kappa$ is called the net profit condition, and it is crucial because survival is impossible, i.e., $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$, if $\mathbb{E}X \geqslant \kappa$, except for a few trivial cases when $\mathbb{P}(X = \kappa) = 1$; see [2, Theorem 9]. Intuitively, it is clear that long-term survival by the model (1.1) is impossible if the threatening claim amount X on average is equal or greater to the collected premium κ per unit of time.

For $s \in \mathbb{C}$, let us denote the generating function of $\varphi(1), \varphi(2), \ldots$ as follows:

$$\Xi(s) := \sum_{i=0}^{\infty}\varphi(i+1)s^i,\ |s| < 1$$

and the probability-generating functions of the random variables X and $\mathcal{M}$:

$$G_X(s) := \sum_{i=0}^{\infty}x_i s^i,\ G_{\mathcal{M}}(s) := \sum_{i=0}^{\infty}\pi_i s^i,\ |s| \leqslant 1.$$

Then, $\Xi(s)$ and $G_{\mathcal{M}}(s)$, for $|s| < 1$, satisfy the relation

$$\Xi(s) = \sum_{i=0}^{\infty}\varphi(i+1)s^i = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_j s^i = \sum_{j=0}^{\infty}\pi_j\sum_{i=j}^{\infty}s^i = \frac{\sum_{j=0}^{\infty}\pi_j s^j}{1-s} = \frac{G_{\mathcal{M}}(s)}{1-s}. \tag{2.2}$$

In many examples, the radius of convergence of $G_X(s)$ or $G_{\mathcal{M}}(s)$ is larger than one. See [3, Lemma 8] for more properties of the probability-generating function in $|s| \leqslant 1$.

3. Main results

In this section, based on the previously introduced notations and relation $(1-s)\Xi(s) = G_{\mathcal{M}}(s)$ in (2.2), we formulate the main results of the work.

Theorem 3.1. *Let us consider the model defined in* (1.1) *and suppose that the net profit condition* $\mathbb{E}X < \kappa$ *holds. Then, the probability mass functions of the random variables* $\mathcal{M}$ *and* X *satisfy the following two equalities:*

$$G_{\mathcal{M}}(s)(s^{\kappa} - G_X(s)) = \sum_{i=0}^{\kappa-1} \pi_i \sum_{j=0}^{\kappa-1-i} x_j(s^{\kappa} - s^{i+j}),\ |s| \leqslant 1, \tag{3.1}$$

$$\kappa - \mathbb{E}X = \sum_{i=0}^{\kappa-1} \pi_i \sum_{j=0}^{\kappa-1-i} x_j(\kappa - i - j). \tag{3.2}$$

We prove Theorem 3.1 in Section 5.

Equality (3.1) implies the following relation among the probabilities $\pi_0, \pi_1, \ldots$.

Corollary 3.1. *Let* $\pi_i = \mathbb{P}(\mathcal{M} = i)$, $i \in \mathbb{N}_0$ *and* $F_X(u) = \sum_{i=0}^{u} x_i$, $u \in \mathbb{N}_0$ *be the distribution function of the random variable X. Then, for* $\kappa \in \mathbb{N}$, *the following equalities hold:*

$$\pi_{\kappa} x_0 = \pi_0 - \sum_{i=0}^{\kappa-1} \pi_i F_X(\kappa - i), \tag{3.3}$$

$$\pi_n x_0 = \pi_{n-\kappa} - \sum_{i=0}^{\kappa-1} \pi_i x_{n-i},\ n = \kappa+1,\ \kappa+2,\ \ldots\ .$$

Proof of Corollary 3.1. The n-th derivative of both sides of the equality (3.1) and $s \to 0$ gives

$$\pi_n x_0 = \pi_{n-\kappa} - \sum_{i=0}^{n-1} \pi_i x_{n-i} - \sum_{i=0}^{\kappa-1} \pi_i \sum_{j=0}^{\kappa-1-i} x_j \mathbb{1}_{\{n=\kappa\}},\ n = \kappa,\ \kappa+1,\ \ldots$$

or

$$\pi_n x_0 = \pi_{n-\kappa} - \sum_{i=0}^{n-1} \pi_i x_{n-i},\ n = \kappa+1,\ \kappa+2,\ \ldots\ .$$

□

Let us turn to the survival probabilities $\varphi(1)$, $\varphi(2)$, ... generating function $\Xi(s)$. It is easy to see that the equalities (2.2) and (3.1) imply

$$\Xi(s) = \frac{\sum_{i=0}^{\kappa-1} \pi_i \sum_{j=0}^{\kappa-1-i} x_j(s^{\kappa} - s^{i+j})}{(1-s)(s^{\kappa} - G_X(s))}. \tag{3.4}$$

Therefore, in a similar way that the recurrence equation (1.3) requires the initial values of $\varphi(0)$, $\varphi(1)$, ..., $\varphi(\kappa-1)$, the generating function $\Xi(s)$ in (3.4) (the equality (3.3) as well) requires $\pi_0, \pi_1, \ldots, \pi_{\kappa-1}$,

$\kappa \in \mathbb{N}$. These probabilities can be solved by using the relations (3.1) and (3.2) and this is achievable as provided in Items **(i)–(iv)** below:

(**i**) *We can choose* $|s| \leqslant 1$ *such that the left-hand side of* (3.1) *vanishes, i.e., the roots of* $s^\kappa = G_X(s)$. *If the net profit condition* $G'_X(1) = \mathbb{E}X < \kappa$ *holds and the greatest common divisor of powers of* s *in* $s^\kappa = G_X(s)$ *is one, there are exactly* $\kappa - 1$ *roots of* $s^\kappa = G_X(s)$ *in* $|s| < 1$ *when counted with their multiplicities. This fact is implied by Rouché's theorem and estimate* $|G_X(s)| \leqslant 1 < |\lambda s^\kappa|$ *when* $\lambda > 1$ *and* $|s| = 1$, *which means that, because of the fundamental theorem of algebra, both functions* λs^κ *and* $\lambda s^\kappa - G_X(s)$ *have* κ *zeros in* $|s| < 1$. *When* $\lambda \to 1^+$, *there is always one root out of those* κ *in* $|s| < 1$ *migrating to* $s = 1$ ($s = 1$ *is always the root of* $s^\kappa = G_X(s)$), *and some to other boundary points* $|s| = 1$ (*roots of unity*) *if the greatest common divisor of powers of* s *in* $s^\kappa = G_X(s)$ *is greater than one; see [21, Chapter 10], [22, Remark 10] and [3, Section 4, Lemmas 9 and 10 therein].*

(**ii**) *Let* $\alpha \neq 1$ *be a root of* $s^\kappa = G_X(s)$ *in* $|s| \leqslant 1$ *and denote* $\boldsymbol{\pi} := (\pi_0, \pi_1, \ldots, \pi_{\kappa-1})^T$ *as the column vector. Then, by* (3.1) *and*

$$\left(\alpha^j + \alpha^{j+1} + \ldots + \alpha^{\kappa-1}\right)(\alpha - 1) = \alpha^\kappa - \alpha^j,\ j \in \{0, 1, \ldots, \kappa - 1\},$$

it holds that

$$\begin{aligned}
0 &= \left(\sum_{j=0}^{\kappa-1} x_j(\alpha^\kappa - \alpha^j),\ \sum_{j=0}^{\kappa-2} x_j(\alpha^\kappa - \alpha^{j+1}),\ \ldots,\ x_0(\alpha^\kappa - \alpha^{\kappa-1})\right)\boldsymbol{\pi}\\
&= \left(\sum_{j=0}^{\kappa-1} x_j \sum_{i=j}^{\kappa-1} \alpha^i,\ \sum_{j=0}^{\kappa-2} x_j \sum_{i=j+1}^{\kappa-1} \alpha^i,\ \ldots,\ x_0\alpha^{\kappa-1}\right)\boldsymbol{\pi}\\
&= \left(\sum_{j=0}^{\kappa-1} \alpha^j F_X(j),\ \sum_{j=0}^{\kappa-2} \alpha^{j+1} F_X(j),\ \ldots,\ \alpha^{\kappa-1}x_0\right)\boldsymbol{\pi} = \sum_{i=0}^{\kappa-1} \pi_i \sum_{j=0}^{\kappa-1-i} \alpha^{j+i} F_X(j),
\end{aligned}$$

where $F_X(u)$ *is the distribution function of* X.

(**iii**) *Let* $\alpha_1, \ldots, \alpha_{\kappa-1} \neq 1$ *be the roots of* $s^\kappa = G_X(s)$ *in* $|s| \leqslant 1$. *Then, by* **(i)**, **(ii)** *and* (3.2),

$$\begin{pmatrix}
\sum_{j=0}^{\kappa-1} \alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-2} \alpha_1^{j+1} F_X(j) & \ldots & \alpha_1^{\kappa-1} x_0\\
\vdots & \vdots & \ddots & \vdots\\
\sum_{j=0}^{\kappa-1} \alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-2} \alpha_{\kappa-1}^{j+1} F_X(j) & \ldots & \alpha_{\kappa-1}^{\kappa-1} x_0\\
\sum_{j=0}^{\kappa-1} x_j(\kappa - j) & \sum_{j=0}^{\kappa-2} x_j(\kappa - j - 1) & \ldots & x_0
\end{pmatrix}
\begin{pmatrix}\pi_0\\ \vdots\\ \pi_{\kappa-2}\\ \pi_{\kappa-1}\end{pmatrix}
=
\begin{pmatrix}0\\ \vdots\\ 0\\ \kappa - \mathbb{E}X\end{pmatrix}. \tag{3.5}$$

If $A\boldsymbol{\pi} = B$ *denotes the system* (3.5), $x_0 > 0$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_{\kappa-1} \neq 1$ *are the roots of multiplicity one, then, according to Lemma 4.2 proved in Section 4, the determinant* $|A| \neq 0$ *and, therefore,* $\boldsymbol{\pi} = A^{-1}B$.

(**iv**) *Suppose the root* $\alpha \neq 1$ *of* $s^\kappa = G_X(s)$ *in* $|s| \leqslant 1$ *is of multiplicity* $l \in \{2, 3, \ldots, \kappa - 1\}$, $\kappa \geqslant 3$. *Then, according to the equality* (3.1) *in Theorem 3.1 and* **(ii)**, *the derivatives*

$$\frac{d^m}{ds^m}\left(\sum_{i=0}^{\kappa-1} \pi_i \sum_{j=0}^{\kappa-1-i} s^{j+i} F_X(j)\right)\Bigg|_{s=\alpha} = 0 \textit{ for all } m \in \{0, 1, \ldots, l - 1\}, \tag{3.6}$$

and, to avoid identical lines in matrix A, *we can set up the modified system* (3.5) *by replacing its lines (except the last one) with the corresponding derivatives* (3.6). *If* $x_0 > 0$, *such a modified main matrix* A *remains non-singular, as proved in Lemma 4.3 of Section 4.*

Note 1: The condition $x_0 > 0$ does not lose generality. If $\mathbb{P}(X > j) = 1$ for some $j \in \{0, 1, \ldots, \kappa - 2\}$, $\kappa \geqslant 2$ and the net profit condition remains valid (note that $\mathbb{P}(X > \kappa - 1)$ implies $\mathbb{E}X \geqslant \kappa$), then there is a reduction the order of recurrence in (1.3) *and, consequently, some terms in the sums of* (3.1) *and* (3.2) *equal zero, causing corresponding adjustments in the system* (3.5) *or its modified version described in* (**iv**)*. We then end up dividing by some x_{j+1} instead of x_0 where needed. For instance, if $x_0 = 0$ and $x_1 > 0$, we then can express $\varphi(\kappa - 1)$ from* (1.4) *dividing by x_1. See also [2, Theorem 7] and Corollary 3.2 when $x_0 = 0$. Also, the both sides of $s^\kappa = G_X(s)$ can be canceled by some power of $s \neq 0$ if $\mathbb{P}(X > j) = 1$ for some $j \in \{0, 1, \ldots, \kappa - 2\}$, $\kappa \geqslant 2$.*

We further denote by $|A|$ the determinant of the matrix A where $M_{i,j}$, $i, j \in \{1, 2, \ldots, \kappa\}$, $\kappa \in \mathbb{N}$ are its minors and the matrix A is the main matrix in (3.5) *or its modification replacing the coefficients by derivatives, as described in (***iv***).*

The equality (3.4) *and thoughts listed in (***i***)–(***iv***) allow us to formulate the following statement.*

Theorem 3.2. *Let $|s| < 1$ and $s^\kappa - G_X(s) \neq 0$. If the net profit condition $\mathbb{E}X < \kappa$ holds, then the survival probability-generating function is given by*

$$\Xi(s) = \frac{\kappa - \mathbb{E}X}{G_X(s) - s^\kappa} \sum_{i=0}^{\kappa-1} \tilde{\pi}_i \sum_{j=0}^{\kappa-1-i} s^{j+i} F_X(j), \tag{3.7}$$

where $\tilde{\pi}_i = \pi_i/(\kappa - \mathbb{E}X)$,

$$\tilde{\pi}_0 = \frac{(-1)^{\kappa+1} M_{\kappa,1}}{|A|},\ \tilde{\pi}_1 = \frac{(-1)^{\kappa+2} M_{\kappa,2}}{|A|},\ \ldots,\ \tilde{\pi}_{\kappa-1} = \frac{M_{\kappa,\kappa}}{|A|}$$

and the matrix A is created as provided in (**i**)–(**iv**).

Moreover, the initial values for the recurrence equation (1.3)*, including $\varphi(\kappa)$, are*

$$\varphi(0) = \frac{\kappa - \mathbb{E}X}{|A|} \sum_{i=1}^{\kappa} (-1)^{\kappa+i} M_{\kappa,i} F_X(\kappa - i),$$
$$\varphi(u) = \frac{\kappa - \mathbb{E}X}{|A|} \sum_{i=1}^{u} (-1)^{\kappa+i} M_{\kappa,i},\ u = 1, 2, \ldots, \kappa.$$

We prove Theorem 3.2 in Section 5.

Note 2: We agree that, for $\kappa = 1$, the matrix $A = (x_0)$, its determinant $|A| = x_0$ and the minor $M_{1,1} = 1$. Recall that x_0 gets replaced by some x_{j+1} if $\mathbb{P}(X > j) = 1$ for some $j \in \{0, 1, \ldots, \kappa-2\}$, $\kappa \geqslant 2$ and the net profit condition holds; see Note 1.

The next statement provides possible expressions of $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_{\kappa-1}$ and $\varphi(0), \varphi(1), \ldots, \varphi(\kappa)$, $\kappa \in \mathbb{N}$.

Theorem 3.3. *Suppose that $x_0 > 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_{\kappa-1} \neq 1$ are the roots of multiplicity one of $s^\kappa = G_X(s)$ in $|s| \leqslant 1$. Then, the values $\tilde{\pi}_i = \pi_i/(\kappa - \mathbb{E}X)$ for $i = 0, 1, \ldots, \kappa - 1$ admit the following representation:*

$$\tilde{\pi}_0 = \frac{1}{x_0} \prod_{j=1}^{\kappa-1} \frac{\alpha_j}{\alpha_j - 1},$$

$$\begin{aligned}
\tilde{\pi}_1 &= -\frac{\sum_{1\leqslant j_1<\ldots<j_{\kappa-2}\leqslant\kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-2}}}{x_0\prod_{j=1}^{\kappa-1}(\alpha_j-1)} - \frac{F_X(1)}{x_0}\tilde{\pi}_0,\\
\tilde{\pi}_2 &= \frac{\sum_{1\leqslant j_1<\ldots<j_{\kappa-3}\leqslant\kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-3}}}{x_0\prod_{j=1}^{\kappa-1}(\alpha_j-1)} - \frac{F_X(2)}{x_0}\tilde{\pi}_0 - \frac{F_X(1)}{x_0}\tilde{\pi}_1,\\
&\vdots\\
\tilde{\pi}_{\kappa-1} &= \frac{(-1)^{\kappa+1}}{x_0}\prod_{j=1}^{\kappa-1}\frac{1}{\alpha_j-1} - \frac{1}{x_0}\sum_{i=0}^{\kappa-2}\tilde{\pi}_i F_X(\kappa-1-i),\ \kappa\geqslant 2,
\end{aligned}$$

and the initial values for the recurrence equation (1.3)*, including $\varphi(\kappa)$, are*

$$\begin{aligned}
\tilde{\varphi}(0) =&(-1)^{\kappa+1}\prod_{j=1}^{\kappa-1}\frac{1}{\alpha_j-1},\ \ \tilde{\varphi}(1) = \frac{1}{x_0}\prod_{j=1}^{\kappa-1}\frac{\alpha_j}{\alpha_j-1},\\
\tilde{\varphi}(2) =& -\frac{F_X(1)}{x_0}\tilde{\varphi}(1) + \prod_{j=1}^{\kappa-1}\frac{1/x_0}{\alpha_j-1}\left(\prod_{j=1}^{\kappa-1}\alpha_j - \sum_{1\leqslant j_1<\ldots<j_{\kappa-2}\leqslant\kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-2}}\right),\\
\tilde{\varphi}(3) =& -\frac{F_X(1)}{x_0}\tilde{\varphi}(2) - \frac{F_X(2)}{x_0}\tilde{\varphi}(1) + \prod_{j=1}^{\kappa-1}\frac{1/x_0}{\alpha_j-1}\\
&\times\left(\prod_{j=1}^{\kappa-1}\alpha_j - \sum_{1\leqslant j_1<\ldots<j_{\kappa-2}\leqslant\kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-2}} + \sum_{1\leqslant j_1<\ldots<j_{\kappa-3}\leqslant\kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-3}}\right),\\
&\vdots\\
\tilde{\varphi}(\kappa) =& -\frac{1}{x_0}\sum_{i=1}^{\kappa-1}F_X(\kappa-i)\tilde{\varphi}(i) + \prod_{j=1}^{\kappa-1}\frac{1/x_0}{\alpha_j-1}\\
&\times\left(\prod_{j=1}^{\kappa-1}\alpha_j - \sum_{1\leqslant j_1<\ldots<j_{\kappa-2}\leqslant\kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-2}} + \sum_{1\leqslant j_1<\ldots<j_{\kappa-3}\leqslant\kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-3}} + \ldots + (-1)^{\kappa+1}\right),
\end{aligned}$$

$\kappa\geqslant 2$*, where*

$$\tilde{\varphi}(i) = \frac{\varphi(i)}{(\kappa-\mathbb{E}X)},\ i\in\{0,\ 1,\ \ldots,\ \kappa\}.$$

Note that $\prod_{j=1}^{0}(\cdot) = \sum_{1\leqslant j_1<j_0\leqslant\ldots}(\cdot) = 1$ in Theorem 3.3, and we prove this theorem in Section 5.

In view of Theorems 3.2 and 3.3, we give several separate expressions on $\Xi(s)$.

Corollary 3.2. *If $\kappa = 1$, then*

$$\Xi(s) = \frac{1-\mathbb{E}X}{G_X(s)-s}.$$

If $\kappa = 2$ and $x_0 > 0$, then

$$\Xi(s) = \frac{2-\mathbb{E}X}{\alpha-1}\cdot\frac{\alpha-s}{G_X(s)-s^2},$$

where $\alpha \in [-1, 0)$ *is the unique root of* $G_X(s) = s^2$.

If $\kappa = 2$, $x_0 = 0$ *and* $x_1 > 0$, *then*

$$\Xi(s) = \frac{2 - \mathbb{E}X}{\tilde{G}_X(s) - s},$$

where $\tilde{G}_X(s) = \sum_{i=0}^{\infty} x_{i+1} s^i$, $|s| \leqslant 1$.

Proof of Corollary 3.2. The provided $\Xi(s)$ expressions are implied by Theorem 3.2. Recall that $s^2 = G_X(s)$, $x_0 > 0$ has the unique real root $\alpha \in [-1, 0)$. In addition, when $x_0 > 0$, then $\alpha = -1$ is the root of $s^2 = G_X(s)$ iff $\mathbb{P}(X \in 2\mathbb{N}_0) = 1$; see [3, Section 4 and Corollary 15 therein] and the description (**i**) in Section 3. □

4. Lemmas

In this section, we formulate and prove several auxiliary statements needed to derive the main results stated in Section 3.

Lemma 4.1. *The random variable*

$$\mathcal{M} = \sup_{n \geqslant 1} \left(\sum_{i=1}^{n} (X_i - \kappa) \right)^+,$$

where $x^+ = \max\{0, x\}$ *is the positive part of* $x \in \mathbb{R}$, *admits the following distribution property:*

$$(\mathcal{M} + X - \kappa)^+ \stackrel{d}{=} \mathcal{M}.$$

Proof. The proof is straightforward according to the definition of $\mathcal{M}$ and basic properties of the maximum. Indeed,

$$\begin{aligned}
(\mathcal{M} + X - \kappa)^+ &= \max \left\{ 0, \max \left\{ 0, \sup_{n \geqslant 1} \sum_{i=1}^{n} (X_i - \kappa) \right\} + X - \kappa \right\} \\
&\stackrel{d}{=} \max \left\{ 0, \max \left\{ X_1 - \kappa, \sup_{n \geqslant 2} \sum_{i=1}^{n} (X_i - \kappa) \right\} \right\} \stackrel{d}{=} \max \left\{ 0, \sup_{n \geqslant 1} \sum_{i=1}^{,n} (X_i - \kappa) \right\} = \mathcal{M}.
\end{aligned}$$

See also, [22, Lemma 5.2], [3, Lemma 25] and [4, page 198]. □

Lemma 4.2. *Let* $\alpha_1, \ldots, \alpha_{\kappa-1} \neq 1$ *be the roots of multiplicity one of* $s^\kappa = G_X(s)$ *in the region* $|s| \leqslant 1$, *and suppose that the probability* x_0 *is positive. Then, the determinant* $|A|$ *of the main matrix in* (3.5) *is*

$$|A| = \frac{x_0^\kappa}{(-1)^{\kappa+1}} \prod_{j=1}^{\kappa-1} (\alpha_j - 1) \prod_{1 \leqslant i < j \leqslant \kappa-1} (\alpha_j - \alpha_i) \neq 0.$$

Proof. Let us calculate the determinant

$$|A| = \begin{vmatrix}
\sum_{j=0}^{\kappa-1} \alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-2} \alpha_1^{j+1} F_X(j) & \ldots & \alpha_1^{\kappa-2} x_0 + \alpha_1^{\kappa-1} F_X(1) & \alpha_1^{\kappa-1} x_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sum_{j=0}^{\kappa-1} \alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-2} \alpha_{\kappa-1}^{j+1} F_X(j) & \ldots & \alpha_{\kappa-1}^{\kappa-2} x_0 + \alpha_{\kappa-1}^{\kappa-1} F_X(1) & \alpha_{\kappa-1}^{\kappa-1} x_0 \\
\sum_{j=0}^{\kappa-1} x_j (\kappa - j) & \sum_{j=0}^{\kappa-2} x_j (\kappa - j - 1) & \ldots & 2x_0 + x_1 & x_0
\end{vmatrix}.$$

We first put forward x_0 from the last column. Then, multiplying the last column by $F_X(\kappa-1)$, $F_X(\kappa-2)$, $\dots$, $F_X(1)$, respectively, and subtracting it from the first, the second, etc., columns, we obtain

$$|A| = x_0 \begin{vmatrix} \sum_{j=0}^{\kappa-2}\alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-3}\alpha_1^{j+1}F_X(j) & \dots & \alpha_1^{\kappa-2}x_0 & \alpha_1^{\kappa-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{j=0}^{\kappa-2}\alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-3}\alpha_{\kappa-1}^{j+1}F_X(j) & \dots & \alpha_{\kappa-2}^{\kappa-2}x_0 & \alpha_{\kappa-1}^{\kappa-1} \\ \sum_{j=0}^{\kappa-2}x_j(\kappa-j-1) & \sum_{j=0}^{\kappa-3}x_j(\kappa-j-2) & \dots & x_0 & 1 \end{vmatrix}.$$

Proceeding the similar with the penultimate column of the last determinant (to put forward x_0 and rearrange) and so on and applying the basic determinant properties, we obtain that

$$|A| = x_0^\kappa \begin{vmatrix} 1 & \alpha_1 & \dots & \alpha_1^{\kappa-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{\kappa-1} & \dots & \alpha_{\kappa-1}^{\kappa-1} \\ 1 & 1 & \dots & 1 \end{vmatrix} = \frac{x_0^\kappa}{(-1)^{\kappa+1}} \begin{vmatrix} \alpha_1-1 & \alpha_1^2-1 & \dots & \alpha_1^{\kappa-1}-1 \\ \alpha_2-1 & \alpha_2^2-1 & \dots & \alpha_2^{\kappa-1}-1 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\kappa-1}-1 & \alpha_{\kappa-1}^2-1 & \dots & \alpha_{\kappa-1}^{\kappa-1}-1 \end{vmatrix}$$

$$= \frac{x_0^\kappa}{(-1)^{\kappa+1}} \prod_{j=1}^{\kappa-1}(\alpha_j-1) \begin{vmatrix} 1 & \alpha_1 & \dots & \alpha_1^{\kappa-2} \\ 1 & \alpha_2 & \dots & \alpha_2^{\kappa-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{\kappa-1} & \dots & \alpha_{\kappa-1}^{\kappa-2} \end{vmatrix}.$$

The last determinant is nothing but the well-known Vandermonde determinant; see for example [23, Section 6.1]. Thus,

$$|A| = \frac{x_0^\kappa}{(-1)^{\kappa+1}} \prod_{j=1}^{\kappa-1}(\alpha_j-1) \prod_{1\leqslant i<j\leqslant\kappa-1}(\alpha_j-\alpha_i) \neq 0,$$

because the roots $\alpha_1, \alpha_2, \dots, \alpha_{\kappa-1}$ are distinct and lie in the region $|s| \leqslant 1$, $s \neq 1$. □

Lemma 4.3. *Let* $|s| \leqslant 1$. *Suppose some roots* $\alpha_1, \dots, \alpha_{\kappa-1} \neq 1$ *of* $G_X(s) = s^\kappa$ *are multiple, and assume that the probability* x_0 *is positive. Then, the modified main matrix in* (3.5)*, after replacing its lines (except the last one) by the derivatives* (3.6)*, remains non-singular.*

Proof. In short, the statement follows because the derivative is a linear mapping. More precisely, let us assume that α_1 is of multiplicity two. Then, there exists such sufficiently close to zero $\delta \in \mathbb{R} \setminus \{0\}$ that the matrix with the replaced second line

$$\begin{pmatrix} \sum_{j=0}^{\kappa-1}\alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_1^{j+1}F_X(j) & \dots & \alpha_1^{\kappa-1}x_0 \\ \sum_{j=0}^{\kappa-1}(\alpha_1+\delta)^j F_X(j) & \sum_{j=0}^{\kappa-2}(\alpha_1+\delta)^{j+1}F_X(j) & \dots & (\alpha_1+\delta)^{\kappa-1}x_0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{\kappa-1}\alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_{\kappa-1}^{j+1}F_X(j) & \dots & \alpha_{\kappa-1}^{\kappa-1}x_0 \\ \sum_{j=0}^{\kappa-1}x_j(\kappa-j) & \sum_{j=0}^{\kappa-2}x_j(\kappa-j-1) & \dots & x_0 \end{pmatrix} \tag{4.1}$$

is non-singular, see the expression of the determinant in Lemma 4.2. Then, subtracting the second line from the first in (4.1), dividing the first line by δ afterward and letting $\delta \to 0$, we get the desired line replacement using the derivative.

The proof is analogous for higher derivatives and/or more multiple roots. □

5. Proofs of the main results

In this section, we prove the statements formulated in Section 3. Let us start with the proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 4.1 and the rule of total expectation,

$$G_{\mathcal{M}}(s) = \mathbb{E}s^{(\mathcal{M}+X-\kappa)^+} = \mathbb{E}\left(\mathbb{E}\left(s^{(\mathcal{M}+X-\kappa)^+}|\mathcal{M}\right)\right) = \sum_{i=0}^{\kappa-1}\pi_i\mathbb{E}s^{(i+X-\kappa)^+} + s^{-\kappa}G_X(s)\sum_{i=\kappa}^{\infty}\pi_i s^i$$
$$= \sum_{i=0}^{\kappa-1}\pi_i\left(\mathbb{E}s^{(X+i-\kappa)^+} - s^{i-\kappa}G_X(s)\right) + s^{-\kappa}G_X(s)G_{\mathcal{M}}(s),$$

which implies the equality (3.1):

$$G_{\mathcal{M}}(s)(s^\kappa - G_X(s)) = \sum_{i=0}^{\kappa-1}\pi_i(\mathbb{E}s^{(X+i-\kappa)^+ +\kappa} - s^iG_X(s)) = \sum_{i=0}^{\kappa-1}\pi_i\sum_{j=0}^{\kappa-1-i}x_j(s^\kappa - s^{i+j}).$$

To prove the second equality (3.2) in Theorem 3.1, we take the derivative of both sides of (3.1) with respect to s:

$$S_1 + S_2 := G'_{\mathcal{M}}(s)(s^\kappa - G_X(s)) + G_{\mathcal{M}}(s)(\kappa s^{\kappa-1} - G'_X(s))$$
$$= \sum_{i=0}^{\kappa-1}\pi_i\sum_{j=0}^{\kappa-1-i}x_j(\kappa s^{\kappa-1} - (i+j)s^{i+j-1}) =: S_3.$$

We now let $s \to 1^-$ in the last equality. It is easy to see that

$$\lim_{s\to 1^-} S_3 = \sum_{i=0}^{\kappa-1}\pi_i\sum_{j=0}^{\kappa-1-i}x_j(\kappa - i - j)$$

and

$$\lim_{s\to 1^-} S_2 = \kappa - \mathbb{E}X,$$

because the net profit condition $\mathbb{E}X < \kappa$ holds. Before calculating $\lim_{s\to 1^-} S_1$, we observe that $\mathbb{E}X^2 = \infty \Leftrightarrow \mathbb{E}\mathcal{M} = \infty$ and $\mathbb{E}X^2 < \infty \Leftrightarrow \mathbb{E}\mathcal{M} < \infty$; see [24, Theorems 5 and 6]. Therefore, the requirement $\mathbb{E}X^2 < \infty$ implies $\lim_{s\to 1^-} S_1 = 0$ immediately. However, $\lim_{s\to 1^-} S_1 = 0$ despite $\mathbb{E}\mathcal{M} = \infty$. Indeed, if $G'_{\mathcal{M}}(s) \to \infty$ as $s \to 1^-$, then

$$\lim_{s\to 1^-} S_1 = \lim_{s\to 1^-}\frac{s^\kappa - G_X(s)}{1/G'_{\mathcal{M}}(s)} = \lim_{s\to 1^-}\frac{\kappa s^{\kappa-1} - G'_X(s)}{-G''_{\mathcal{M}}(s)/\left(G'_{\mathcal{M}}(s)\right)^2},$$

where

$$\limsup_{s\to 1^-}\frac{\left(G'_{\mathcal{M}}(s)\right)^2}{G''_{\mathcal{M}}(s)} \leqslant \frac{N}{N-1}\sum_{i=N}^{\infty}\pi_i$$

for any $N \in \{2, 3, \ldots\}$; see [22, Lemma 5.5]. Thus, the equality (3.2) follows and the theorem is proved. □

Proof of Theorem 3.2. For $s^\kappa - G_X(s) \neq 0$, the equality (3.4) and division by $1-s$ (see (**ii**) in Section 3) imply

$$\Xi(s) = \frac{\sum_{i=0}^{\kappa-1}\pi_i\sum_{j=0}^{\kappa-1-i}s^{j+i}F_X(j)}{G_X(s)-s^\kappa} = \frac{1}{G_X(s)-s^\kappa}\left(\sum_{j=0}^{\kappa-1}s^jF_X(j),\ \sum_{j=0}^{\kappa-2}s^{j+1}F_X(j),\ \ldots,\ s^{\kappa-1}x_0\right)\begin{pmatrix}\pi_0\\ \pi_1\\ \vdots\\ \pi_{\kappa-1}\end{pmatrix}.$$

By the system (3.5), including its modified version described in (**iv**) in Section 3, and the recalled notations $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{\kappa-1})^T$ and $\tilde{\pi}_i = \pi_i/(\kappa - \mathbb{E}X)$, we obtain

$$\begin{aligned}\boldsymbol{\pi} &= \frac{1}{|A|}\begin{pmatrix} M_{1,1} & -M_{1,2} & \ldots & (-1)^{1+\kappa}M_{1,\kappa}\\ -M_{2,1} & M_{2,2} & \ldots & (-1)^{2+\kappa}M_{2,\kappa}\\ \vdots & \vdots & \ddots & \vdots\\ (-1)^{\kappa+1}M_{\kappa,1} & (-1)^{\kappa+2}M_{\kappa,2} & \ldots & M_{\kappa,\kappa}\end{pmatrix}^T\begin{pmatrix}0\\ \vdots\\ 0\\ \kappa-\mathbb{E}X\end{pmatrix}\\ &= \frac{\kappa-\mathbb{E}X}{|A|}\begin{pmatrix}(-1)^{\kappa+1}M_{\kappa,1}\\ (-1)^{\kappa+2}M_{\kappa,2}\\ \vdots\\ M_{\kappa,\kappa}\end{pmatrix} = (\kappa-\mathbb{E}X)\begin{pmatrix}\tilde{\pi}_0\\ \tilde{\pi}_1\\ \vdots\\ \tilde{\pi}_{\kappa-1}\end{pmatrix}.\end{aligned}$$

Thus, the expression of $\Xi(s)$ in (3.7) follows.

The claimed equalities on $\varphi(u)$ for $u = 1, \ldots, \kappa$ are evident due to the obtained expression of $\boldsymbol{\pi}$ and $\varphi(u+1) = \sum_{i=0}^{u}\pi_i$, $u \in \mathbb{N}_0$ provided in (2.1). It can be seen that the recurrence equation (1.3) yields

$$\varphi(0) = \sum_{i=1}^{\kappa}x_{\kappa-i}\varphi(i) = \sum_{i=0}^{\kappa-1}\pi_iF_X(\kappa-1-i).$$

□

Proof of Theorem 3.3. We calculate the minors $M_{\kappa,1}, M_{\kappa,2}, \ldots, M_{\kappa,\kappa}$ of the following matrix:

$$A = \begin{pmatrix} \sum_{j=0}^{\kappa-1}\alpha_1^jF_X(j) & \sum_{j=0}^{\kappa-2}\alpha_1^{j+1}F_X(j) & \ldots & \alpha_1^{\kappa-1}x_0\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{j=0}^{\kappa-1}\alpha_{\kappa-1}^jF_X(j) & \sum_{j=0}^{\kappa-2}\alpha_{\kappa-1}^{j+1}F_X(j) & \ldots & \alpha_{\kappa-1}^{\kappa-1}x_0\\ \sum_{j=0}^{\kappa-1}x_j(\kappa-j) & \sum_{j=0}^{\kappa-2}x_j(\kappa-j-1) & \ldots & x_0\end{pmatrix}.$$

Following the calculation of determinant $|A|$ in the proof of Lemma 4.2, we get

$$\begin{aligned}M_{\kappa,1} &= \begin{vmatrix} \sum_{j=0}^{\kappa-2}\alpha_1^{j+1}F_X(j) & \sum_{j=0}^{\kappa-3}\alpha_1^{j+2}F_X(j) & \ldots & \alpha_1^{\kappa-1}x_0\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{j=0}^{\kappa-2}\alpha_{\kappa-1}^{j+1}F_X(j) & \sum_{j=0}^{\kappa-3}\alpha_{\kappa-1}^{j+2}F_X(j) & \ldots & \alpha_{\kappa-1}^{\kappa-1}x_0\end{vmatrix}\\ &= x_0^{\kappa-1}\begin{vmatrix}\alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{\kappa-1}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_{\kappa-1} & \alpha_{\kappa-1}^2 & \ldots & \alpha_{\kappa-1}^{\kappa-1}\end{vmatrix} = x_0^{\kappa-1}\prod_{i=1}^{\kappa-1}\alpha_i\prod_{1\leqslant i<j\leqslant\kappa-1}\left(\alpha_j-\alpha_i\right).\end{aligned}$$

Note that $M_{\kappa,1}$ is defined for $\kappa \geqslant 1$ and $M_{1,1} = 1$ by agreement. The next one

$$
\begin{aligned}
M_{\kappa,2} &= \begin{vmatrix} \sum_{j=0}^{\kappa-1}\alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-3}\alpha_1^{j+2} F_X(j) & \dots & \alpha_1^{\kappa-1}x_0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{\kappa-1}\alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-3}\alpha_{\kappa-1}^{j+2} F_X(j) & \dots & \alpha_{\kappa-1}^{\kappa-1}x_0 \end{vmatrix} \\
&= x_0^{\kappa-2}\begin{vmatrix} x_0+\alpha_1 F_X(1) & \alpha_1^2 & \dots & \alpha_1^{\kappa-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0+\alpha_{\kappa-1}F_X(1) & \alpha_{\kappa-1}^2 & \dots & \alpha_{\kappa-1}^{\kappa-1} \end{vmatrix} \\
&= x_0^{\kappa-1}\prod_{1\leqslant i<j\leqslant \kappa-1}(\alpha_j-\alpha_i)\sum_{1\leqslant j_1<\ldots<j_{\kappa-2}\leqslant \kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-2}} + \frac{F_X(1)}{x_0}M_{\kappa,1}.
\end{aligned}
$$

Similarly as before, $M_{\kappa,2}$ is defined for $\kappa \geqslant 2$ only, and $M_{2,2} = x_0 + F_X(1)\alpha$, where $\alpha \in [-1, 0)$ is the unique root of $s^2 = G_X(s)$; see (**i**) in Section 3 and [3, Section 4 and Corollary 15 therein].

Proceeding,

$$
\begin{aligned}
M_{\kappa,3} &= \begin{vmatrix} \sum_{j=0}^{\kappa-1}\alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_1^{j+1} F_X(j) & \sum_{j=0}^{\kappa-4}\alpha_1^{j+3} F_X(j) & \dots & \alpha_1^{\kappa-1}x_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{\kappa-1}\alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_{\kappa-1}^{j+1} F_X(j) & \sum_{j=0}^{\kappa-4}\alpha_{\kappa-1}^{j+3} F_X(j) & \dots & \alpha_{\kappa-1}^{\kappa-1}x_0 \end{vmatrix} \\
&= x_0^{\kappa-2}\begin{vmatrix} x_0+\alpha_1^2 F_X(2) & \alpha_1 & \alpha_1^3 & \dots & \alpha_1^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0+\alpha_{\kappa-1}^2 F_X(2) & \alpha_{\kappa-1} & \alpha_{\kappa-1}^3 & \dots & \alpha_{\kappa-1}^{\kappa-1} \end{vmatrix} + \frac{F_X(1)}{x_0}M_{\kappa,2} \\
&= x_0^{\kappa-1}\prod_{1\leqslant i<j\leqslant \kappa-1}(\alpha_j-\alpha_i)\sum_{1\leqslant j_1<\ldots<j_{\kappa-3}\leqslant \kappa-1}\alpha_{j_1}\cdots\alpha_{j_{\kappa-3}} - \frac{F_X(2)}{x_0}M_{\kappa,1} + \frac{F_X(1)}{x_0}M_{\kappa,2},\ \kappa \geqslant 3,
\end{aligned}
$$

and so on until the last minor:

$$
\begin{aligned}
M_{\kappa,\kappa} &= \begin{vmatrix} \sum_{j=0}^{\kappa-1}\alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_1^{j+1} F_X(j) & \dots & x_0\alpha_1^{\kappa-2}+\alpha_1^{\kappa-1}F_X(1) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{\kappa-1}\alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_{\kappa-1}^{j+1} F_X(j) & \dots & x_0\alpha_{\kappa-1}^{\kappa-2}+\alpha_{\kappa-1}^{\kappa-1}F_X(1) \end{vmatrix} \\
&= \begin{vmatrix} \sum_{j=0}^{\kappa-1}\alpha_1^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_1^{j+1} F_X(j) & \dots & x_0\alpha_1^{\kappa-2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{\kappa-1}\alpha_{\kappa-1}^j F_X(j) & \sum_{j=0}^{\kappa-2}\alpha_{\kappa-1}^{j+1} F_X(j) & \dots & x_0\alpha_{\kappa-1}^{\kappa-2} \end{vmatrix} + \frac{F_X(1)}{x_0}M_{\kappa,\kappa-1} \\
&= x_0^{\kappa-1}\prod_{1\leqslant i<j\leqslant \kappa-1}(\alpha_j-\alpha_i) + (-1)^{\kappa}\frac{F_X(\kappa-1)}{x_0}M_{\kappa,1} + (-1)^{\kappa+1}\frac{F_X(\kappa-2)}{x_0}M_{\kappa,2} \\
&\quad + \ldots + (-1)^{2\kappa-1}\frac{F_X(2)}{x_0}M_{\kappa,\kappa-2} + \frac{F_X(1)}{x_0}M_{\kappa,\kappa-1}.
\end{aligned}
$$

The statement on expressions of $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_{\kappa-1}$ follows dividing the obtained minors (multiplied by -1 where needed) by the determinant $|A|$.

We now prove the claimed formulas of $\varphi(0), \varphi(1), \ldots, \varphi(\kappa), \kappa \in \mathbb{N}$. By the recurrence equation (1.3) with $u = 0$, $\varphi(u+1) = \sum_{i=0}^{u} \pi_i$, $u \in \mathbb{N}_0$ and the already proved expression of $\pi_{\kappa-1}$, $\kappa \in \mathbb{N}$ in Theorem 3.3,

$$\varphi(0) = \sum_{i=0}^{\kappa-1} \varphi(i+1) x_{\kappa-1-i} = \sum_{i=0}^{\kappa-1} \pi_i F_X(\kappa - 1 - i) = \frac{\kappa - \mathbb{E}X}{(-1)^{\kappa+1}} \prod_{j=1}^{\kappa-1} \frac{1}{\alpha_j - 1}.$$

The formula for $\varphi(1)$ is evident because $\varphi(1) = \pi_0$, where the expression of π_0 is already proved in Theorem 3.3, too. The rest is clear by calculating the sum $\varphi(u + 1) = \sum_{i=0}^{u} \pi_i$, $u \in \mathbb{N}_0$, where π_i are given in the first part of Theorem 3.3. □

6. Particular examples

In this section, we give several examples illustrating the applicability of theoretical statements formulated in Section 3. The required numerical computations were performed by using Wolfram Mathematica [25]. As mentioned in Section 1, in [2], it has been proved that the required initial values for the recurrence equation (1.3) can be approximately found by calculating certain recurrent limits, while results of this work in Section 3, in many instances, provide exact closed-form expressions of the survival probabilities. Therefore, in some considered examples here, we check if the calculated exact value of φ matches the previously known approximate one.

Example 6.1. *Suppose the random claim amount X is Bernoulli-distributed, i.e.,* $1 - \mathbb{P}(X = 0) = p = \mathbb{P}(X = 1)$, $0 < p < 1$ *and the premium $\kappa \in \mathbb{N}$. We find the ultimate time survival probability-generating function $\Xi(s)$ and calculate $\varphi(u)$, $u \in \mathbb{N}_0$.*

If $\kappa = 1$, in view of the first part of Corollary 3.2 and the recurrence equation (1.3), it is trivial that $\Xi(s) = 1/(1 - s)$, $|s| < 1$ and $\varphi(0) = x_0\varphi(1) = 1 - p$, $\varphi(u) = 1$, $u \in \mathbb{N}$. In other words, the ultimate time survival is guaranteed if the initial surplus $u \in \mathbb{N}$ and the maximal claim size is one in the model $u + n - \sum_{i=1}^{n} X_i$.

If $\kappa \geqslant 2$, it is easy to understand that $u + \kappa n - \sum_{i=1}^{n} X_i > 0$ for all $n \in \mathbb{N}$, $u \in \mathbb{N}_0$, regardless of the size of X_i; consequently, $\varphi \equiv 1$, $\Xi(s) = 1/(1 - s)$, $|s| < 1$.

Example 6.2. *Suppose that the random claim amount X is distributed geometrically with the parameter $p \in (0, 1)$, i.e., $\mathbb{P}(X = i) = p(1 - p)^i$, $i = 0, 1, \ldots$, and the premium rate equals two, i.e., $\kappa = 2$. We find the ultimate time survival probability-generating function $\Xi(s)$ and calculate $\varphi(0)$ and $\varphi(1)$ when the net profit condition is satisfied, i.e., $\mathbb{E}X < 2$.*

We start with an observation on the net profit condition:

$$\mathbb{E}X = \frac{1-p}{p} < 2 \quad \Leftrightarrow \quad \frac{1}{3} < p < 1.$$

Then, according to Theorem 3.1 and the description (**i**) in Section 3,

$$G_X(s) = \frac{p}{1 - (1 - p)s} = s^2 \Rightarrow \alpha := s = \frac{p - \sqrt{4p - 3p^2}}{2(1 - p)} \in (-1, 0)$$

when $1/3 < p < 1$, and by Corollary 3.2 with $\kappa = 2$ and $x_0 = p > 0$,

$$\Xi(s) = \frac{(3p - 1)(p - \sqrt{4p - 3p^2})}{p(3p - 2 - \sqrt{4p - 3p^2})} \cdot \frac{1 - (1 - p)s}{(1 - s)s^2 + (s^3 - 1)p}, \ \frac{1}{3} < p < 1.$$

For $\kappa = 2$, $u = 0$ and $1/3 < p < 1$, the recurrence equation (1.3) or Theorem 3.3 yields

$$\varphi(0) = x_1\varphi(1) + x_0\varphi(2) = (1-p)p\,\Xi(0) + p\,\Xi'(0) = \frac{2-\mathbb{E}X}{1-\alpha} = \frac{3p-2+\sqrt{4p-3p^2}}{2p},$$
$$\varphi(1) = \Xi(0) = \frac{2-\mathbb{E}X}{x_0}\frac{\alpha}{\alpha-1} = \frac{3p-\sqrt{4p-3p^2}}{2p^2}.$$

One may check that, for $p = 101/300$,

$$\varphi(0) = \frac{\sqrt{90597}-297}{202} = 0.0197691\ldots,$$
$$\varphi(1) = \frac{45450-150\sqrt{90597}}{10201} = 0.0295066\ldots,$$

and that coincides with the approximate values of $\varphi(0)$ and $\varphi(1)$ in [2, page 12] obtained via recurrent sequences.

Example 6.3. *Suppose that X attains the natural values only, i.e., $x_0 = 0$, $x_1 > 0$, $\kappa = 2$ and the net profit condition is satisfied $\mathbb{E}X < 2$. We provide the ultimate time survival probability $\varphi(u)$ formulas for all $u \in \mathbb{N}_0$.*

Let us recall that

$$\tilde{G}_X(s) = \sum_{i=0}^{\infty} x_{i+1}s^i,\ |s| \leqslant 1.$$

The recurrence equation (1.3) and Corollary 3.2 for $x_0 = 0$ and $x_1 > 0$ imply

$$\varphi(0) = x_1\varphi(1) = 2-\mathbb{E}X,\ \varphi(1) = \Xi(0) = \frac{2-\mathbb{E}X}{x_1},$$
$$\varphi(u) = \frac{2-\mathbb{E}X}{(u-1)!}\frac{d^{u-1}}{ds^{u-1}}\left(\frac{1}{\tilde{G}_X(s)-s}\right)\bigg|_{s=0} = \frac{1}{x_1}\left(\varphi(u-1) - \sum_{i=1}^{u-1} x_{u-i+1}\varphi(i)\right),\ u \geqslant 2,$$

which echoes and widens the statement of Theorem 3 in [2], providing another method of $\varphi(u)$, $u \geqslant 2$ calculation.

Example 6.4. *Suppose that the random claim amount X is distributed geometrically with the parameter $p = 101/300$, i.e., $\mathbb{P}(X = i) = p(1-p)^i$, $i = 0, 1, \ldots$, and the premium rate equals three, i.e., $\kappa = 3$. We set up the ultimate time survival probability-generating function $\Xi(s)$ and calculate or provide formulas for $\varphi(u)$, $u \in \mathbb{N}_0$.*

First, we observe that the net profit condition is satisfied, i.e., $\mathbb{E}X = 199/101 < 3$. We now follow the statement of Theorem 3.1 and the surrounding comments beneath it. Then, for $p = 101/300$, the equation

$$G_X(s) = \frac{p}{1-(1-p)s} = s^3$$

has two complex conjugate solutions $\alpha_1 := -0.368094 + 0.522097\boldsymbol{i}$ and $\alpha_2 := -0.368094 - 0.522097\boldsymbol{i}$ inside the unit circle $|s| < 1$. Then, by Theorem 3.2,

$$\Xi(s) = \frac{\sum_{i=0}^{2} \pi_i \sum_{j=0}^{2-i} s^{i+j} F_X(j)}{s^3 - G_X(s)},$$

where $(\pi_0, \pi_1, \pi_2) = (0.582072, 0.0818989, 0.0658497)$ is the unique solution of

$$\begin{pmatrix} x_0 + F_X(1)\alpha_1 + F_X(2)\alpha_1^2 & x_0\alpha_1 + F_X(1)\alpha_1^2 & x_0\alpha_1^2 \\ x_0 + F_X(1)\alpha_2 + F_X(2)\alpha_2^2 & x_0\alpha_2 + F_X(1)\alpha_2^2 & x_0\alpha_2^2 \\ 3x_0 + 2x_1 + x_2 & 2x_0 + x_1 & x_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 - \mathbb{E}X \end{pmatrix},$$

with appropriate numerical characteristics of the provided distribution. Theorem 3.3 and the recurrence equation (1.3) imply

$$\begin{aligned}
\varphi(0) &= \sum_{i=1}^{3} x_{3-i}\varphi(i) = \frac{3 - \mathbb{E}X}{(1-\alpha_1)(1-\alpha_2)} = 0.480212\ldots, \\
\varphi(1) &= \pi_0 = \frac{(3 - \mathbb{E}X)\alpha_1\alpha_2}{x_0(1-\alpha_1)(1-\alpha_2)} = 0.582072\ldots, \\
\varphi(2) &= \pi_0 + \pi_1 = (3 - \mathbb{E}X)\frac{x_0(\alpha_1 + \alpha_2) + x_1\alpha_1\alpha_2}{x_0^2(\alpha_1 - 1)(1-\alpha_2)} = 0.663971\ldots, \\
\varphi(3) &= \pi_0 + \pi_1 + \pi_2 = 0.729821\ldots, \\
&\text{where } \pi_2 = \frac{3 - \mathbb{E}X}{x_0}\left(\frac{1}{(\alpha_1 - 1)(\alpha_2 - 1)} - \sum_{i=0}^{1} \pi_i F_X(2-i)\right), \\
\varphi(u) &= \frac{1}{x_0}\left(\varphi(u-3) - \sum_{i=1}^{u-1} x_{u-i}\varphi(i)\right) = \frac{d^{u-1}}{ds^{u-1}} \frac{\Xi(s)}{(u-1)!}\bigg|_{s=0}, \ u \geqslant 3.
\end{aligned}$$

The provided values of $\varphi(0)$, $\varphi(1)$, $\varphi(2)$ and $\varphi(3)$ coincide with the ones given in [2, page 14], where they are obtained approximately from certain recurrent sequences.

Example 6.5. *Suppose* $x_0 = 0.128$, $x_1 = 0.576$, $x_2 = 0.264$, $x_3 = 0.032$, $\sum_{i=0}^{3} x_i = 1$ *and* $\kappa = 3$. *We set up the ultimate time survival probability-generating function* $\Xi(s)$ *and calculate* $\varphi(u)$, $u \in \mathbb{N}_0$.

For the provided distribution $\mathbb{E}X = 1.2 < 3$, the equation

$$0.128 + 0.576s + 0.264s^2 + 0.032s^2 = s^3$$

has one root $s = -4/11 =: \alpha$ of multiplicity two. Then, according to Theorem 3.1 and the comments (**i**)–(**iv**) beneath it, we create the modified system, replacing the second line with the corresponding derivatives:

$$\begin{pmatrix} x_0 + F_X(1)\alpha + F_X(2)\alpha^2 & x_0\alpha + F_X(1)\alpha^2 & x_0\alpha^2 \\ F_X(1) + 2F_X(2)\alpha & x_0 + 2F_X(1)\alpha & 2x_0\alpha \\ 3x_0 + 2x_1 + x_2 & 2x_0 + x_1 & x_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 - \mathbb{E}X \end{pmatrix},$$

which implies $(\pi_0, \pi_1, \pi_2) = (1, 0, 0)$; consequently

$$\varphi(0) = 0.968, \ \varphi(u) = 1, \ u \in \mathbb{N}, \ \Xi(s) = \frac{1}{1-s}, \ |s| < 1.$$

One may observe that the obtained result is expected, because $u + 3n - \sum_{i=1}^{n} X_i > 0$ for all $n \in \mathbb{N}$, except when $u = 0$ and X_i attains the value of 3.

7. Conclusions

This work shows that, if certain conditions are met, there exist exact closed-form expressions of the ultimate time survival probability

$$\varphi(u) = \mathbb{P}\left(\bigcap_{n=1}^{\infty}\left\{u + \kappa n - \sum_{i=1}^{n} X_i > 0\right\}\right),$$

where the roots of $s^\kappa = G_X(s)$, the distance $\kappa - \mathbb{E}X > 0$ and the distribution function $F_X(s)$ are involved; see Theorem 3.3. Moreover, having the values of the probability mass function

$$\mathbb{P}\left(\sup_{n\geqslant 1}\left(\sum_{i=1}^{n} X_i - \kappa\right)^+ = u\right), \ u = 0, \ 1, \ \ldots, \ \kappa - 1,$$

we can get the exact expression of the survival probability-generating function

$$\Xi(s) = \sum_{i=0}^{\infty} \varphi(i+1)s^i, \ |s| < 1;$$

see Theorem 3.2. As mentioned in Section 1, the expression of survival $\varphi(u)$ or ruin $1-\varphi(u)$ probability is heavily dependent on what type of random variables generate the random walk

$$\left\{\sum_{i=1}^{n} (X_i - \kappa), \ n \in \mathbb{N}\right\}. \tag{7.1}$$

The random variables X_i in the sequence (7.1) can be discrete/continuous or dependent/independent, and their quantity for each $n \in \mathbb{N}$ can be deterministic/random, etc. Equally, the premium, or just an intercept technically, $\kappa \in \mathbb{N}$ in (7.1) influences the sequence's distribution, too. As demonstrated, the ultimate time survival probabilities $\varphi(u)$ are solutions of systems of linear equations, which are based on the roots of $s^\kappa = G_X(s)$. The recent work in [22] shows that similar systems can be used to find $\varphi(u)$ when X_i are distributed differently. Thus, it is of interest to study the sequence (7.1), assuming various other mentioned options for X_i and κ; see, for instance, [26–29]. Also, the broadness of a random walk's occurrence in mathematics and other applied sciences indicates that this work and referenced research should not be applicable to ruin theory only.

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Conflict of interest

The author declares no conflict of interest.

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