AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 5175–5180.
DOI: 10.3934/math.2023259
Received: 19 October 2022
Revised: 23 November 2022
Accepted: 29 November 2022
Published: 13 December 2022

Research article

An extension of the classical John-Nirenberg inequality of martingales

Changzheng Yao and Congbian Ma*

School of Mathematics and Statistics, Xinxiang University, Xinxiang 453000, China

* **Correspondence:** Email: congbianm@whu.edu.cn.

Abstract: In this paper, we prove the John-Nirenberg theorem of the bmo_p martingale spaces for the full range $0 < p < \infty$. We also consider the John-Nirenberg inequality on symmetric spaces of martingales.

Keywords: martingale; interpolation; Hardy spaces; symmetric spaces; inequality
Mathematics Subject Classification: 60G42, 60G46

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$ with the associated conditional expectations $(\mathbb{E}_n)_{n\geq 0}$. A sequence $f = (f_n)_{n\geq 0}$ adapted to $(\mathcal{F}_n)_{n\geq 0}$ is said to be a martingale if $\mathbb{E}(|f_n|) < \infty$ and $E_n(f_{n+1}) = f_n$ for every $n \geq 0$. For the sake of simplicity, we assume $f_0 = 0$. Let $1 \leq p < \infty$. The quasi-Banach spaces bmo_p are defined as follows:

$$bmo_p = \{f = (f_n)_{n\geq 0} : \|f\|_{bmo_p} = \sup_n \|\mathbb{E}_n(|f - f_n|^p)\|_\infty^{\frac{1}{p}} < \infty\}.$$

Here, the notation f in $|f - f_n|^p$ stands for f_∞. It follows from [7] that

$$\|f\|_{bmo_p} = \sup_n \sup_{a\in L_p(\mathcal{F}_n), \|a\|_p \leq 1} \|(f - f_n)a\|_p.$$

Before describing our main results, we recall the classical John-Nirenberg inequality in the martingale theory (see [6, 7]).

Theorem 1. *If the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then for $1 \leq p < \infty$ we have that*

$$bmo_p = bmo_1$$

with equivalent norms.

In 2014, Yi et al. [8] proved the John-Nirenberg inequality on the rearrangement-invariant Banach function space E with $1 \leq p_E \leq q_E < \infty$. In 2019, Li [4] considered the John-Nirenberg theorem on Lorentz space $bmo_{p,q}$ with $1 < p < \infty$ and $0 < q < \infty$.

In this paper, we first prove the John-Nirenberg inequality of bmo_p martingale spaces for $0 < p < \infty$, extending Theorem 1 via a new interpolation method. Then, we extend this result to a wider class of the symmetric quasi-Banach function space E with $0 < p_E \leq q_E < \infty$.

2. Preliminaries and notations

Let us first recall some basic facts on the symmetric quasi-Banach function spaces. Let $((0,\infty), \mathcal{F}, P)$ be the Lesbegue measure space and $L_0(0,\infty)$ be the space of all Lesbegue measurable real-valued functions defined on $(0,\infty)$. Let E be a quasi-Banach subspace of $L_0(0,\infty)$, simply called a quasi-Banach function space on $(0,\infty)$ in the sequel. A quasi-Banach function space E on $(0,\infty)$ is called symmetric if for any $g \in E$ and any measurable function f with $\mu_t(f) \leq \mu_t(g)$ ($\mu_t(f)$ and $\mu_t(g)$ respectively represent the non-increasing rearrangement of f and g) for all $t \geq 0$, $f \in E$ and $\|f\|_E \leq \|g\|_E$. E is said to have the Fatou property if for every net $(x_i)_{i\in I}$ in E satisfying $0 \leq x_i \uparrow$ and $\sup_{i\in I} \|x_i\|_E < \infty$ the supremum $x = \sup_{i\in I} x_i$ exists in E and $\|x_i\|_E \uparrow \|x\|_E$.

The Köthe dual of a symmetric Banach function space E on $(0,\infty)$ is given by

$$E^{\times} = \{f \in L_0(0,\infty) : \int_0^{\infty} |f(t)g(t)|dt < \infty : \forall g \in E\},$$

with the norm $\|f\|_{E^{\times}} := \sup\{\int_0^{\infty} |f(t)g(t)|dt : \|g\|_E \leq 1\}$. The space $E^{\times}$ is symmetric and has the Fatou property. Refer to [1, 5] for more details.

For a quasi-Banach function space E on $(0,\infty)$, the lower and upper Boyd indices p_E and q_E of E are respectively defined by

$$p_E := \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|} \quad \text{and} \quad q_E := \lim_{s\to 0^+} \frac{\log s}{\log \|D_s\|},$$

where the dilation operator D_s on $L_0(0,\infty)$ is defined by $(D_s f)(t) = f(t/s)$ for all $t \in (0,\infty)$. For a symmetric quasi-Banach function space E on $(0,\infty)$, D_s is a bounded linear operator on E for every $s > 0$ and $0 \leq p_E \leq q_E \leq \infty$ (see [2, Lemma 2.2]).

Given a quasi-Banach function space E on $(0,\infty)$, for $0 < r < \infty$, $E^{(r)}$ will denote the quasi-Banach function space on $(0,\infty)$ defined by $E^{(r)} = \{x : |x|^r \in E\}$ and equipped with the quasi-norm $\|x\|_{E^{(r)}} = \left\||x|^r\right\|_E^{\frac{1}{r}}$. Note that

$$p_{E^{(r)}} = rp_E, \quad q_{E^{(r)}} = rq_E. \tag{2.1}$$

Let E_i be a quasi-Banach function space on $(0,\infty)$ for $i = 1, 2$. The pointwise product space $E_1 \odot E_2$ is defined by

$$E_1 \odot E_2 = \{f \in L_2(0,\infty) : f = f_1 f_2, f_i \in E_i, i = 1, 2\}$$

with the functional $\|\cdot\|_{E_1\odot E_2}$ being defined by

$$\|f\|_{E_1\odot E_2} = \inf\{\|f\|_{E_1}\|f\|_{E_2} : f = f_1 f_2, f_i \in E_i, i = 1, 2\}.$$

We need the following lemmas (see Theorem 2.1 in [1]).

Lemma 1. *Let E and F be two symmetric Banach function spaces on $(0,\infty)$.*

(i) If $0<p<\infty$, then $(E\odot F)^{(p)}=E^{(p)}\odot F^{(p)}$.

(ii) $L_1(0,\infty)=E\odot E^{\times}$.

Lemma 2. *Let E be a symmetric quasi-Banach function space on $(0,\infty)$ with the Fatou property. If $p_E>p$, then $E^{(\frac{1}{p})}$ can be renormed as a symmetric Banach function space.*

Proof. By (2.1), we have that $p_{E^{(\frac{1}{p})}}=\frac{1}{p}p_E>1$. Thus $E^{(\frac{1}{p})}$ is an interpolation space for the couple $(L_1(0,\infty),L_\infty(0,\infty))$ (see [3, Lemma 3.6]). Therefore, according to Lemma 2.2 in [1], we get that $E^{(\frac{1}{p})}$ can be renormed as a symmetric Banach function space. □

Now we define the Hardy spaces and BMO spaces of martingales. For a martingale $f=(f_n)_{n\geq 0}$, we denote its martingale difference by $df_i=f_i-f_{i-1}$ (with convention $f_0=0$). Then the conditional quadratic variation and the square function are defined by

$$s_n(f)=(\sum_{i=1}^{n}\mathbb{E}_{i-1}|df_i|^2)^{1/2},\quad s(f)=(\sum_{i=1}^{\infty}\mathbb{E}_{i-1}|df_i|^2)^{1/2},$$

$$S_n(f)=(\sum_{i=1}^{n}|df_i|^2)^{1/2},\quad S(f)=(\sum_{i=1}^{\infty}|df_i|^2)^{1/2}.$$

Let $0<p<\infty$. Define

$$H_p^s=\{f=(f_n)_{n\geq 0}:\|f\|_{H_p^s}=\|s(f)\|_p<\infty\},$$

$$H_p^S=\{f=(f_n)_{n\geq 0}:\|f\|_{H_p^S}=\|S(f)\|_p<\infty\},$$

$$bmo_p=\{f=(f_n)_{n\geq 0}:\|f\|_{bmo_p}=\sup_n\sup_{a\in L_p(\mathcal{F}_n),\|a\|_p\leq 1}\|(f-f_n)a\|_p<\infty\},$$

$$BMO_p=\{f=(f_n)_{n\geq 0}:\|f\|_{BMO_p}=\sup_n\sup_{a\in L_p(\mathcal{F}_n),\|a\|_p\leq 1}\|(f-f_{n-1})a\|_p<\infty\}.$$

Here, the notation f in $|f-f_{n-1}|^p$ stands for f_∞.

A stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is said to be regular if, for $n\geq 0$ and $A\in\mathcal{F}_n$, there exists $B\in\mathcal{F}_{n-1}$ such that $A\subset B$ and $\mathbb{P}(B)\leq R\mathbb{P}(A)$, where R is a positive constant independent of n. A martingale is said to be regular if it is adapted to a regular σ-algebra sequence. This means that there exists a constant $R>0$ such that $f_n\leq Rf_{n-1}$ for all nonnegative martingales $(f_n)_{n\geq 0}$ adapted to the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$. We refer the reader to Long [6] and Weisz [7] for the theory of martingales.

In what follows, unless otherwise specified, for two nonnegative quantities A and B, by $A\lesssim B$ we mean that there exists an absolute constant $C>0$ such that $A\leq CB$, and by $A\approx B$ that $A\lesssim B$ and $B\lesssim A$.

3. Main results

In this section, we first establish the John-Nirenberg theorem of the bmo_p spaces for $0<p<1$.

Theorem 2. *If the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is regular, then, for any $f\in bmo_1$*

$$\|f\|_{bmo_p}\approx\|f\|_{bmo_1}\quad 0<p<1. \tag{3.1}$$

Proof. From Hölder's inequality it follows that

$$\|f\|_{bmo_p} \leq \|f\|_{bmo_1}.$$

To prove the converse we choose $1 < p_1 < \infty$ and $0 < \theta < 1$ such that $1 = (1-\theta)/p + \theta/p_1$. Fix n, and for any $0 < r < \infty$, let $T_n : L_r(\mathcal{F}_n) \to L_p(\mathcal{F})$ be a linear operator with $T_n(a) = (f - f_n)a$. Then by the definition of bmo_p, we have the following inequalities:

$$\|T_n\|_{L_p \to L_p} = \sup_{a \in L_p(\mathcal{F}_n), \|a\|_p \leq 1} \|(f - f_n)a\|_p \leq \|f\|_{bmo_p},$$

$$\|T_n\|_{L_{p_1} \to L_{p_1}} = \sup_{a \in L_p(\mathcal{F}_n), \|a\|_{p_1} \leq 1} \|(f - f_n)a\|_{p_1} \leq \|f\|_{bmo_{p_1}}.$$

Thus by interpolation, we have that

$$\|T_n\|_{(L_p, L_{p_1})_\theta \to (L_p, L_{p_1})_\theta} \leq \|f\|_{bmo_p}^{1-\theta} \|f\|_{bmo_{p_1}}^{\theta}.$$

Noting that $(L_p, L_{p_1})_\theta = L_1$ with equal norms and using the inequality

$$\|f\|_{bmo_q} \leq C_q \|f\|_{bmo_1} \quad \text{for} \quad 1 \leq q < \infty,$$

(see[7, Corollory 2.51]) we reduce that

$$\|T_n\|_{L_1 \to L_1} \leq (C_{p_1})^\theta \|f\|_{bmo_p}^{1-\theta} \|f\|_{bmo_1}^{\theta}$$

which implies that

$$\|f\|_{bmo_1} \leq (C_{p_1})^{\frac{\theta}{1-\theta}} \|f\|_{bmo_p}.$$

□

Remark 1. *(i) If, in the proof of Theorem 2, we replace $f - f_n$ with $f - f_{n-1}$ and bmo_p and bmo_1 with BMO_p and BMO_1 then*

$$\|f\|_{BMO_p} \approx \|f\|_{BMO_1} \quad \textit{for} \quad 0 < p < 1.$$

(ii) According to Theorem 1, bmo_p coincides with bmo_1 for $1 \leq p < \infty$. While for $0 < p < 1$, if a priori we assume that $f \in bmo_1$. Theorem 2 tells us the norms of bmo_p and bmo_1 are also equivalent.

Recall that if $(\mathcal{F}_n)_{n\geq 0}$ is regular, then $H_1^s = H_1^S$ which follows that their dual spaces bmo_2 and BMO_2 are equivalent. Hence, by Theorem 2, Theorem 1, (i) of Remark 1 and [7, Theorem 2.50], we obtain the following result.

corollary 1. *Let $0 < p < \infty$. If the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is regular, then for any $f \in BMO_1$ and $f \in bmo_1$*

$$\|f\|_{bmo_p} \approx \|f\|_{bmo_1} \approx \|f\|_{BMO_p} \approx \|f\|_{BMO_1}.$$

Now we present the John-Nirenberg inequality of martingale spaces associated with symmetric quasi-Banach function spaces, generalizing the results obtained in [8, 4].

Theorem 3. *Let E be a symmetric quasi-Banach function space on $(0, \infty)$ with $0 < p_E \le q_E < \infty$ that has the Fatou property. If $(\mathcal{F}_n)_{n\ge0}$ is regular, then for any $f \in bmo_1$,*

$$\|f\|_{bmo_E} \approx \|f\|_{bmo_1}, \tag{3.2}$$

where

$$bmo_E = \{f = (f_n)_{n\ge0} : \|f\|_{bmo_E} = \sup_n \sup_{a\in E(\mathcal{F}_n),\|a\|_E\le1} \|(f - f_n)a\|_E < \infty\}.$$

Proof. Choose p and q such that $0 < p < p_E \le q_E < q < \infty$. Then by Lemma 2, $E^{(\frac{1}{p})}$ can be renormed as a symmetric Banach function space; so, we assume that $E^{(\frac{1}{p})}$ is a symmetric Banach function space. By (ii) of Lemma 1, we have that $L_1(0, \infty) = E^{(\frac{1}{p})} \odot E^{(\frac{1}{p})\times}$. It follows that

$$L_p(0, \infty) = E \odot F, \tag{3.3}$$

where $F = (E^{(\frac{1}{p})\times})^p$ (see (i) of Lemma 1). Fix n. Take $a \in L_p(\mathcal{F}_n)$ with $\|a\|_p \le 1$. Then by (3.3), there exist $a_1 \in E$ and $a_2 \in F$ such that $a = a_1a_2$ and $\|a_1\|_E, \|a_2\|_F \le 1$. Thus we have that

$$\begin{aligned} \|(f - f_n)a\|_p &= \|(f - f_n)a_1a_2\|_p \\ &\le \|a_2\|_F \|(f - f_n)a_1\|_E \\ &\le \|f\|_{bmo_E}, \end{aligned}$$

which implies $\|f\|_{bmo_p} \le \|f\|_{bmo_E}$. Therefore, by Theorem 2, $\|f\|_{bmo_1} \le \|f\|_{bmo_E}$.

Now we turn to the converse inequality. Fix n. Similar to the definition of the operator T_n in Theorem 3.1, we can view $f - f_n$ as an operator from $L_p(\mathcal{F}_n)$ to $L_p(\mathcal{F})$ and from $L_q(\mathcal{F}_n)$ to $L_q(\mathcal{F})$; then, we get that

$$\|f - f_n\|_{L_p\to L_p} \le \|f\|_{bmo_p} \text{ and } \|f - f_n\|_{L_q\to L_q} \le \|f\|_{bmo_q}. \tag{3.4}$$

By Lemma 3.6 in [3], we have that E is an interpolation space in $(L_p(0, \infty), L_q(0, \infty))$ which implies that

$$\|f - f_n\|_{E\to E} \le C \max\{\|f - f_n\|_{L_p\to L_p}, \|f - f_n\|_{L_q\to L_q}\}, \tag{3.5}$$

where $C > 0$ is a constant depending only on p and q. Putting (3.4) and (3.5) together and using Corollary 1, we obtain that

$$\|f - f_n\|_{E\to E} \le C \max\{\|f\|_{bmo_p}, \|f\|_{bmo_q}\} \le C\|f\|_{bmo_1}.$$

It follows that $\|f\|_{bmo_E} \le C\|f\|_{bmo_1}$. This completes the proof. □

Remark 2. *When $E = L_p(0, \infty)$ for $0 < p < \infty$, (3.2) implies that*

$$\|f\|_{bmo_p} \approx \|f\|_{bmo_1}.$$

Acknowledgments

The second author was supported in part by NSFC No.11801489.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. T. Bekjan, Z. Chen, M. Raikhan, M. Sun, Interpolation and John-Nirenberg inequality on symmetric spaces of noncommutative martingales, *Studia Math.,* **262** (2021), 241–273. https://doi.org/10.4064/sm200508-11-12
2. S. Dirksen, Noncommutative Boyd interpolation theorems, *T. Am. Math. Soc.*, **367** (2015), 4079–4110.
3. S. Dirksen, B. dePagter, D. Potapov, F. Sukochev, Rosenthal inequalities in noncommutative symmetric spaces, *J. Funct. Anal.*, **261** (2011), 2890–2925. https://doi.org/10.1016/j.jfa.2011.07.015
4. L. Li, A remark John-Nirenberg inequalities for martingales, *Ukrainian Math. J.*, **770** (2019), 1571–1577.
5. J. Lindenstrauss, L. Tzafriri, *Classical banach spaces*, Berlin: Springer, 1979.
6. R. Long, *Martingale spaces and inequalities*, Bei Jing: Peking University Press, 1993.
7. F. Weisz, *Martingale Hardy spaces and their applications in fourier analysis*, Berlin: Springer, 1994.
8. R. Yi, L. Wu, Y. Jiao, New John-Nirenberg inequalities for martingales, *Statist. Probab. Lett.*, **86** (2014), 68–73. https://doi.org/10.1016/j.spl.2013.12.010