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Research article

A novel decision aid approach based on spherical hesitant fuzzy Aczel-Alsina geometric aggregation information

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Abstract: Taking into account the significance of spherical hesitant fuzzy sets, this research concentrates on an innovative multi-criteria group decision-making technique for dealing with spherical hesitant fuzzy (SHF) situations. To serve this purpose, we explore SHF Aczel Alsina operational laws such as the Aczel-Alsina sum, Aczel-Alsina product and Aczel-Alsina scalar multiplication as well as their desirable characteristics. This work is based on the fact that aggregation operators have significant operative adaptability to aggregate the uncertain information under the SHF context. With the aid of Aczel-Alsina operators, we develop SHF Aczel-Alsina geometric aggregation operators to address the complex hesitant uncertain information. In addition, we describe and verify several essential results of the newly invented aggregation operators. Furthermore, a decision aid methodology based on the proposed operators is developed using SHF information. The applicability and viability of the proposed methodology is demonstrated by using a case study related to breast cancer treatment. Comprehensive parameter analysis and a systematic comparative study are also carried out to ensure the dependability and validity of the works under consideration.

Keywords: Aczel-Alsina t-norm and t-conorm; spherical hesitant fuzzy numbers; Aczel-Alsina weighted geometric aggregation operators; decision making
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

The act of evaluating, classifying, and choosing the best options based on decision support (DS) data and a particular DS model is known as multi-attribute decision making (MADM). Choosing the best decision support approaches and using expert information are the two main factors in determining a choice. In order to improve judgement on DS issues, the MADM approach [10, 11, 49] needs to be expanded and improved to consider competence and society's intricacies. Zadeh's [66] idea of fuzzy sets offers a very efficient method for addressing these problems. After that, intuitionistic fuzzy sets (IFSs) [12] were developed, using positive membership grades (PMGs) and negative membership grades (NMGs) to reflect uncertainty in DS processes. As time passed, decision-makers (DMs) started to express their choices for various possibilities when presented with a DM issue by using intuitionistic fuzzy numbers [51, 58, 61]. As a consequence, intuitionistic fuzzy information is starting to attract the attention of more and more scholars.

To compile the knowledge collected from specialists, we require the use of various aggregation operators (Agops). Many Agops have been developed, such as the intuitionistic fuzzy (IF) averaging operator [59] and certain Einstein Agops, such as the IF Einstein averaging/geometric operators developed by Wang and Liu [55]. Yu and Xu [63] created a list of prioritized Agops and talked about how they could be used to solve difficulties involving decision-making problems (DMPs). Under the linguistic IF technique, Liu and Wang [35] constructed some unique Agops and produced an approach to deal with the challenging uncertain DMP. The decision-making approach incorporating an IF Bonferroni means Agop was proposed by Xu and Yager [60]. According to the IF Agops implemented on a linguistic data set that was prioritized by Arora and Garg [6]. In order to deal with the uncertainty in DMP, Zhao et al. [67] proposed the generalized IF Agops, such as the generalized IF averaing/geometric operators. Yu [64] proposed a few confidence level-based IF Agops and addressed challenging real-world DMPs. Yu [65] developed the IF Agop and discussed its usefulness in DM by using the Heronian mean. The decision-support methodology was developed by Jiang et al. [25] based on the IF power Agop and entropy measure. Aczel-Alsina concept based Aczel-Alsina Agops were developed by Senapati et al. [45] and they were used in the IF multi-attribute decision support method. The unique generalized IF soft information-based Agops was created by Khan et al. [29], who also investigated its use in decision-making. Seikh and Mandal [47] presented the Dombi norm based Agops for IFSs and discussed their application to MADM. Akram et al. [4] introduced a novel outranking approach for decision-making in an environment of complex Pythagorean fuzzy information.

All of these methods are helpful for representing partial information, but in engineering practice, they are unable to deal with contradictory or ambiguous facts. Cuong [14] established the picture fuzzy set (PFS), which is represented by the degrees of membership of positive, neutral and negative, and the totality of such membership grades should not be larger than one. It is obvious that using PFSs rather than fuzzy sets or IFSs to explain dubious data tends to be more acceptable and accurate. A large number of scholars have begun working on the PFS after it was developed.

Information must be combined in order to create a synthesis of the achievement level of criterion. A number of Agops of picture fuzzy numbers have been created so far, for example, Ashraf et al. [7] presented a list of novel picture fuzzy (PF) algebraic Agops and a decision-support model to explain the complex unclear data in DMPs. PF geometric operators are one of the Agops that Garg [15] created. Wei [56] created a list of PF Agops and discussed how they could be utilized to solve decision-support issues. The unique modified PF soft information-based Agops was created by Khan et al. [30], who also investigated its use in decision-making. Many Einstein Agops, such as PF Einstein averaging/geometric operators, were proposed by Khan et al. [31]. A certain PF averaging/geometric Agops was created for algebraic rules and linguistic data sets by Qiyas et al. [42].

Seikh and Mandal [48] introduced PF Agops by using the Frank t-norm and t-conorm to tackle the uncertain information in DMPs. Jana et al. [24]. produced some Dombi Agops, like the PF Dombi averaging/geometric operators under the conditions of PF environments. Certain PF Agops, including PF Hamacher averaging and geometric operators utilizing the Hamacher t-norm and s-norm, were introduced by Wei [57]. New distance measure-based algebraic Agops were developed by Ashraf et al. [8] under a cubic PF context. The development of certain logarithmic PF Agops and discussion of their utility in decision-making was done by Khan et al. [32]. The flexible DM according to preferred priorities of alternatives was not fully taken into account in the MADM method, despite the fact that these operators offer some ideas for managing the MADM challenges. Akram et al. [3] presented the ELECTRE-I methodology under the conditions of Pythagorean fuzzy information with hesitancy. The t-norms and the corresponding t-conorms (e.g., the algebraic t-norm and t-conorm, the Einstein t-norm and t-conorm, and the Hamacher t-norm and t-conorm) are widely acknowledged as being essential operations in fuzzy sets and other fuzzy systems. The Aczel-Alsina t-norm and Aczel-Alsina t-conorm operations, which have the capability of variation by altering a parameter [2], were introduced by Aczel and Alsina in 1982. Much aggregation information using the Aczel-Alsina t-norm and s-norm has been developed based on the many structures of fuzzy information; for example, Senapati et al. [46] presented the Aczel-Alsina aggregation operations for interval-valued IFS and discussed their applications in DMPs. Mahmood et al. [36] presented the analysis and application of Aczel-Alsina norm based Agops under the conditions of bipolar complex fuzzy information. Ashraf et al. [9] presented the EDAS method by using single valued neutrosophic Aczel-Alsina aggregation information. Riaz et al. [44] presented the spherical fuzzy Aczel-Alsina operations to tackle the uncertain information in decision-making. Naeem et al. [39] developed the PF Aczel-Alsina Agops and discussed their applications in determining the factors affecting mango crops.

In order to solve the favoured priority of alternatives in multi-attribute DMPs, this work intends to offer the Aczel-Alsina t-norm and t-conorm operations as well as a list of new Agops under the conditions of an image fuzzy environment. To choose the best method for breast cancer therapy, an illustration involving breast cancer treatment is provided. The comparison demonstrates how the suggested strategy may support the ability to make flexible selections in line with the objectives of different possibilities. The following statements are made about our method's requirements:

(1) For spherical hesitant fuzzy numbers (SHFNs), we developed a few Aczel-Alsina operations that can overcome the absence of algebraic, Einstein, and Hamacher operations and represent the relationship between various SHFNs.

(2) Using spherical hesitant fuzzy (SHF) Aczel-Alsina weighted operators, we extended Aczel-Alsina operators. SHF data can be supported by the SHF Aczel-Alsina weighted geometric (SHFAWG), SHF Aczel-Alsina order weighted geometric (SHFAOWG), and SHF Aczel-Alsina hybrid weighted geometric (SHFAHWG) operators, which can overcome the drawbacks of the current operator.

(3) Using SHF data, we developed a method to address MADM difficulties.

(4) We applied the proposed SHF Aczel-Alsina Agops to a MADM issue to demonstrate the suitability and consistency of the proposed operators.

(5) The results show that the proposed method is more effective over time and produces results that are even more genuine than those produced by existing methods.

The arrangement of the rest of the text in the paper is as follows. In Section 2, some important details about t-norms, t-conorms, Aczel-Alsina t-norms, SHFSs and various operating rules in terms of SHFNs are described. Section 3 discusses the Aczel-Alsina operational guidelines and the characteristics of SHFNs. In Section 4, we explain several SHF Aczel-Alsina aggregation operations and examine a number of their beneficial characteristics. The MADM problem is addressed in the following section by using SHF Aczel-Alsina Agops. In Section 6, to study the most efficient method for breast cancer therapy, we give an illustrated example relating to breast cancer treatment. In Section 7, We examine how a parameter influences the results of decision-making. Section 8 compares and contrasts the potential Agops with the current Agops. In Section 9, the paper is concluded, which expounds on additional research.

2. Preliminaries

We shall examine a few crucial ideas that are essential to the progress of this work in this portion.

2.1. Aczel-Alsina norm

Definition 1. *[2] A mapping* $\mathcal{X} : [0,1] \times [0,1] \to [0,1]$ *is a t-norm for every* $p, q, r \in [0,1]$ *if it fulfills the below requirements.*
(1) $\mathcal{X}(p,q) = \mathcal{X}(q,p)$;
(2) $\mathcal{X}(p,q) \leq \mathcal{X}(p,r)$ *if* $q \leq r$;
(3) $\mathcal{X}(p, \mathcal{X}(q,r)) = \mathcal{X}(\mathcal{X}(p,q), r)$;
(4) $\mathcal{X}(p,1) = p$.

Definition 2. *[2] A mapping* $\mathcal{A} : [0,1] \times [0,1] \to [0,1]$ *is an s-norm if it fulfills the below requirements.*
(1) $\mathcal{A}(p,q) = \mathcal{A}(q,p)$;
(2) $\mathcal{A}(p,q) \leq \mathcal{A}(p,r)$ *if* $q \leq r$;
(3) $\mathcal{A}(p, \mathcal{A}(q,r)) = \mathcal{A}(\mathcal{A}(p,q), r)$;
(4) $\mathcal{A}(p,0) = p$.

Aczél-Alsina norms are two practical processes that benefit from the flexibility that comes with parameter activity [2].

Definition 3. *[2] A mapping* $\left(\mathcal{X}_{\beta}^{\rho}\right)_{\rho \in [0,\infty]}$ *is an Aczel-Alsina t-norm, if it fulfills the below requirements.*

$$\mathcal{X}_{\beta}^{\rho}(p,q) = \begin{cases} \mathcal{X}_{\mathfrak{D}}(p,q), & if \quad \rho = 0 \\ \min(p,q), & if \quad \rho = \infty \\ e^{-((-\ln p)^{\rho} + (-\ln q)^{\rho})^{\frac{1}{\rho}}}, & otherwise \end{cases}$$

where $p, q \in [0,1]$, ρ *is positive constant and* $\mathcal{X}_{\mathfrak{D}}$ *is a drastic t-norm, defined as*

$$\mathcal{X}_{\mathfrak{D}}(p,q) = \begin{cases} p, & if \quad q = 1 \\ q, & if \quad p = 1 \\ 0, & otherwise \end{cases}.$$

Definition 4. *[2] A mapping* $\left(\mathcal{A}_{\beta}^{\rho}\right)_{\rho \in [0,\infty]}$ *is an Aczel-Alsina s-norm, if it fulfills the below requirements.*

$$\mathcal{A}_{\beta}^{\rho}(p,q)=\begin{cases}\mathcal{A}_{\mathfrak{D}}(p,q), & if \quad \rho=0\\ \max(p,q), & if \quad \rho=\infty\\ 1-e^{-((-\ln(1-p))^{\rho}+(-\ln(1-q))^{\rho})^{\frac{1}{\rho}}}, & otherwise\end{cases}$$

where $p,q\in[0,1]$, *ρ is positive constant and $\mathcal{A}_{\mathfrak{D}}$ is drastic s-norm, defined as*

$$\mathcal{A}_{\mathfrak{D}}(p,q)=\begin{cases}p, & if \quad q=0\\ q, & if \quad p=0\\ 0, & otherwise\end{cases}.$$

For every $\rho\in[0,\infty]$, the t-norm $\mathcal{X}_{\beta}^{\rho}$ and s-norm $\mathcal{A}_{\beta}^{\rho}$ are dual to each other.

2.2. Spherical hesitant fuzzy sets

Definition 5. *[10, 11] A spherical fuzzy set $\mathbb{N}$ in F is defined as*

$$\mathbb{N}=\left\{\ddot{A}_{\mathbb{N}}(t),\ddot{E}_{\mathbb{N}}(t),\ddot{E}_{\mathbb{N}}(t)\in[0,1]\,|t\in F\right\},$$

where positive grade $\ddot{A}_{\mathbb{N}}$,neutral grade$\ddot{E}_{\mathbb{N}}$and negative grade $\ddot{O}_{\mathbb{N}}$ of the element t to the spherical fuzzy set $\mathbb{N}$, fulfilled that $0\leq\left(\ddot{A}_{\mathbb{N}}\right)^2+\left(\ddot{E}_{\mathbb{N}}\right)^2+\left(\ddot{O}_{\mathbb{N}}\right)^2\leq 1$, *for each $t\in F$.*

Definition 6. *[27, 40] An SHF sets $\mathbb{N}$ in F is defined as*

$$\mathbb{N}=\left\{\ddot{A}_{\mathbb{N}}(t),\ddot{E}_{\mathbb{N}}(t),\ddot{O}_{\mathbb{N}}(t)\in[0,1]\,|t\in F\right\},$$

where

$$\ddot{A}_{\mathbb{N}}(t)=\{\ddot{a}|\ddot{a}\in\ddot{A}_{\mathbb{N}}(t)\},\ \ddot{E}_{\mathbb{N}}(t)=\{\ddot{e}|\ddot{e}\in\ddot{E}_{\mathbb{N}}(t)\}\ and\ \ddot{O}_{\mathbb{N}}(t)=\{\ddot{o}|\ddot{o}\in\ddot{O}_{\mathbb{N}}(t)\}$$

are the three sets of some values in $[0,1]$, *denoted as the positive, neutral and negative grades with the condition* $0\leq(\ddot{a}^+)^2+(\ddot{e}^+)^2+(\ddot{o}^+)^2\leq 1$, *for all $t\in F$ such that*

$$\ddot{a}^+=\bigcup_{\ddot{a}\in\ddot{A}_{\mathbb{N}}(t)}\max\{\ddot{a}\},\ \ddot{e}^+=\bigcup_{\ddot{e}\in\ddot{E}_{\mathbb{N}}(t)}\max\{\ddot{e}\},\ and\ \ddot{o}^+=\bigcup_{\ddot{o}\in\ddot{O}_{\mathbb{N}}(t)}\max\{\ddot{o}\}.$$

Definition 7. *[27] Let* $\mathbb{N}_{\mathcal{K}}=\left\{\ddot{A}_{\mathbb{N}_{\mathcal{K}}},\ddot{E}_{\mathbb{N}_{\mathcal{K}}},\ddot{O}_{\mathbb{N}_{\mathcal{K}}}\right\}$ *be two SHFNs, where* $(\mathcal{K}=1,2)$.
(1) $\mathbb{N}_1\subseteq\mathbb{N}_2$ *iff* $\ddot{A}_{\mathbb{N}_1}\leq\ddot{A}_{\mathbb{N}_2},\ddot{E}_{\mathbb{N}_1}\geq\ddot{E}_{\mathbb{N}_2}$*and* $\ddot{O}_{\mathbb{N}_1}\geq\ddot{O}_{\mathbb{N}_2}$ *for all $t\in F$;*
(2) $\mathbb{N}_1=\mathbb{N}_2$ *if* $\mathbb{N}_1\subseteq\mathbb{N}_2$ *and* $\mathbb{N}_2\subseteq\mathbb{N}_1$;
(2) $\mathbb{N}_1\bigcap\mathbb{N}_2=\bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)}\left\{\min\left(\ddot{A}_t\right),\min\left(\ddot{E}_t\right),\max\left(\ddot{O}_t\right)\right\}$;
(3) $\mathbb{N}_1\bigcup\mathbb{N}_2=\bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)}\left\{\max\left(\ddot{A}_t\right),\min\left(\ddot{E}_t\right),\min\left(\ddot{O}_t\right)\right\}$;
(4) $(\mathbb{N}_1)^c=\bigcup_{(\ddot{a}_1,\ddot{e}_1,\ddot{o}_1)\in(\ddot{A}_1,\ddot{E}_1,\ddot{O}_1)}\left\{\ddot{O}_1,\ddot{E}_1,\ddot{A}_1\right\}$.

Definition 8. *[27] Let* $\mathbb{N}_{\mathcal{K}}=\left\{\ddot{A}_{\mathbb{N}_{\mathcal{K}}},\ddot{E}_{\mathbb{N}\mathcal{K}},\ddot{O}_{\mathbb{N}_{\mathcal{K}}}\right\}$ *be two SHFNs, where* $(\mathcal{K}=1,2)$. *The operations about any two SHFNs are introduced as follows:*
(1) $\mathbb{N}_1\oplus\mathbb{N}_2=\bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)}\left\{\sqrt{\ddot{a}_1^2+\ddot{a}_2^2-\ddot{a}_1^2.\ddot{a}_2^2},\ddot{e}_1.\ddot{e}_2,\ddot{o}_1.\ddot{o}_2\right\}$;

(2) $\mathbb{N}_1 \otimes \mathbb{N}_2 = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)} \left\{\ddot{a}_1.\ddot{a}_2, \ddot{e}_1.\ddot{e}_2, \sqrt{\ddot{o}_1^2 + \ddot{o}_2^2 - \ddot{o}_1^2.\ddot{o}_2^2}\right\}$;

(3) $\psi \cdot \mathbb{N}_1 = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)} \left\{\sqrt{1 - \left(1 - \ddot{a}_1^2\right)^{\psi}}, (\ddot{e}_1)^{\psi}, (\ddot{o}_1)^{\psi}\right\}, \psi > 0$;

(4) $(\mathbb{N}_1)^{\psi} = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)} \left\{(\ddot{a}_1)^{\psi}, (\ddot{e}_1)^{\psi}, \sqrt{1 - \left(1 - \ddot{o}_1^2\right)^{\psi}}\right\}, \psi > 0.$

Now, we used the operational rules of SHF sets to prove the following identities.

Definition 9. *[27] Let* $\mathbb{N}_{\mathcal{K}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{K}}}, \ddot{E}_{\mathbb{N}_{\mathcal{K}}}, \ddot{O}_{\mathbb{N}_{\mathcal{K}}}\right\}$ *be a collection of SHFNs, where* $(\mathcal{K} = 1, 2, ..., d)$ *and* $\psi_1, \psi_1 > 0$; *then,*
(1) $\mathbb{N}_1 \oplus \mathbb{N}_2 = \mathbb{N}_2 \oplus \mathbb{N}_1$;
(2) $\mathbb{N}_1 \otimes \mathbb{N}_2 = \mathbb{N}_2 \otimes \mathbb{N}_1$;
(3) $\psi_1 (\mathbb{N}_1 \oplus \mathbb{N}_2) = \psi_1\mathbb{N}_1 \oplus \psi_1\mathbb{N}_2$;
(4) $(\mathbb{N}_1 \otimes \mathbb{N}_2)^{\psi_1} = \mathbb{N}_1^{\psi_1} \otimes \mathbb{N}_2^{\psi_1}$;
(5) $\psi_1\mathbb{N}_1 \oplus \psi_2\mathbb{N}_1 = (\psi_1 + \psi_2)\,\mathbb{N}_1$;
(6) $\mathbb{N}_1^{\psi_1} \otimes \mathbb{N}_1^{\psi_2} = \mathbb{N}_1^{(\psi_1+\psi_2)}$;
(7) $\left(\mathbb{N}_1^{\psi_1}\right)^{\psi_2} = \mathbb{N}_1^{\psi_1\psi_2}$.

Definition 10. *[27] Let* $\mathbb{N} = \left\{\ddot{A}_{\mathbb{N}}, \ddot{E}_{\mathbb{N}}, \ddot{O}_{\mathbb{N}}\right\}$ *be an SHFN. The score* $\alpha(\mathbb{N})$ *and accuracy* $\xi(\mathbb{N})$ *are given as follows:*
(1) $\alpha(\mathbb{N}) = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)} \left\{\frac{1}{l(\ddot{A}_{\mathbb{N}_{\mathcal{K}}})} \sum \ddot{a}_{\mathcal{K}} - \frac{1}{l(\ddot{E}_{\mathbb{N}_{\mathcal{K}}})} \sum \ddot{e}_{\mathcal{K}} - \frac{1}{l(\ddot{O}_{\mathbb{N}_{\mathcal{K}}})} \sum \ddot{o}_{\mathcal{K}}\right\}.$

Definition 11. *(2)* $\xi(\mathbb{N}) = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)} \{\ddot{a}_{\mathcal{K}} + \ddot{e}_{\mathcal{K}} + \ddot{o}_{\mathcal{K}}\}.$

Definition 12. *[27] Let* $\mathbb{N} = \left\{\ddot{A}_{\mathbb{N}}, \ddot{E}_{\mathbb{N}}, \ddot{O}_{\mathbb{N}}\right\}$ *be two SHFNs, where* $(\mathcal{K} = 1, 2)$. *Then, the comparison technique of SHFNs can be defined as:*
(1) $\alpha(\mathbb{N}_1) > \alpha(\mathbb{N}_2)$ *implies that* $\mathbb{N}_1 > \mathbb{N}_2$;
(2) $\alpha(\mathbb{N}_1) = \alpha(\mathbb{N}_2)$ *and* $\xi(\mathbb{N}_1) > \xi(\mathbb{N}_2)$ *implies that* $\mathbb{N}_1 > \mathbb{N}_2$;
(3) $\alpha(\mathbb{N}_1) = \alpha(\mathbb{N}_2)$ *and* $\xi(\mathbb{N}_1) = \xi(\mathbb{N}_2)$ *implies that* $\mathbb{N}_1 = \mathbb{N}_2$.

Definition 13. *[27] Let* $\mathbb{N}_{\mathcal{K}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{K}}}, \ddot{E}_{\mathbb{N}_{\mathcal{K}}}, \ddot{O}_{\mathbb{N}_{\mathcal{K}}}\right\}$ *be a collection of SHFNs, where* $(\mathcal{K} = 1, 2, ..., d)$. *An SHF weighted geometric (SHFWG) Agop with the dimension r is a mapping* $\mathcal{D}^g \to \mathcal{D}$ *with a weight vector* $\mu = \left(\mu_1, \mu_2, ..., \mu_g\right)^T$ *such that* $\mu_{\mathcal{K}} > 0$ *and* $\sum_{\mathcal{K}=1}^{g} \mu_{\mathcal{K}} = 1$ *as*

$$
\begin{aligned}
SHFWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_r) &= \prod_{\mathcal{K}=1}^{g} (\mathbb{N}_{\mathcal{K}})^{\mu_{\mathcal{K}}} \\
&= \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)} \left\{\prod_{t=1}^{g} \left(\ddot{a}_t^2\right)^{\mu_t}, \prod_{t=1}^{g} \left(\ddot{e}_t^2\right)^{\mu_t}, \sqrt{1 - \prod_{t=1}^{g} (1 - \ddot{o}_t^2)^{\mu_t}}\right\}.
\end{aligned}
$$

3. Aczel-Alsina operation for SHFNs

We discussed Aczel-Alsina techniques in relation to SHFNs taking into account the t-norm and t-conorm of Aczel-Alsina.

Definition 14. *Let* $\mathbb{N}_{\mathcal{K}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{K}}}, \ddot{E}_{\mathbb{N}_{\mathcal{K}}}, \ddot{O}_{\mathbb{N}_{\mathcal{K}}}\right\}$ *be two SHFNs, where* $(\mathcal{K} = 1, 2)$ *and* ρ *is a positive constant. Then, operations for SHFNs based on Aczel-Alsina norms are described as follows:*

(1) $$\mathbb{N}_1 \oplus \mathbb{N}_2 = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left\{ \begin{array}{c} \sqrt{1 - e^{-\left(\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho + \left(-\ln\left(1-\ddot{a}_2^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\left(-\ln \ddot{e}_1^2\right)^\rho + \left(-\ln \ddot{e}_2^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\left(-\ln \ddot{o}_1^2\right)^\rho + \left(-\ln \ddot{o}_2^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\};$$

(2) $$\mathbb{N}_1 \otimes \mathbb{N}_2 = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\left(-\ln \ddot{a}_1^2\right)^\rho + \left(-\ln \ddot{a}_2^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\left(-\ln\left(1-\ddot{e}_1^2\right)\right)^\rho + \left(-\ln\left(1-\ddot{e}_2^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\left(-\ln\left(1-\ddot{o}_1^2\right)\right)^\rho + \left(-\ln\left(1-\ddot{o}_2^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\};$$

(3) $$\psi \cdot \mathbb{N}_1 = \bigcup_{(\ddot{a}_1,\ddot{e}_1,\ddot{o}_1)\in(\ddot{A}_1,\ddot{E}_1,\ddot{O}_1)} \left\{ \begin{array}{c} \sqrt{1 - e^{-\left(\psi\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\psi\left(-\ln \ddot{e}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi\left(-\ln \ddot{o}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\}, \psi > 0;$$

(4) $$(\mathbb{N}_1)^\psi = \bigcup_{(\ddot{a}_1,\ddot{e}_1,\ddot{o}_1)\in(\ddot{A}_1,\ddot{E}_1,\ddot{O}_1)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\psi\left(-\ln \ddot{a}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \sqrt{1 - e^{-\left(\psi\left(-\ln\left(1-\ddot{e}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\psi\left(-\ln\left(1-\ddot{o}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\}, \psi > 0.$$

Theorem 1. *Let* $\mathbb{N}_{\mathcal{K}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{K}}}, \ddot{E}_{\mathbb{N}_{\mathcal{K}}}, \ddot{O}_{\mathbb{N}_{\mathcal{K}}}\right\}$ *be a collection of SHFNs, where* $(\mathcal{K} = 1, 2, ..., d)$ *and* $\psi_1, \psi_1 > 0$; *then,*

(1) $\mathbb{N}_1 \oplus \mathbb{N}_2 = \mathbb{N}_2 \oplus \mathbb{N}_1$;

(2) $\mathbb{N}_1 \otimes \mathbb{N}_2 = \mathbb{N}_2 \otimes \mathbb{N}_1$;

(3) $\psi_1 (\mathbb{N}_1 \oplus \mathbb{N}_2) = \psi_1\mathbb{N}_1 \oplus \psi_1\mathbb{N}_2$;

(4) $(\mathbb{N}_1 \otimes \mathbb{N}_2)^{\psi_1} = \mathbb{N}_1^{\psi_1} \otimes \mathbb{N}_2^{\psi_1}$;

(5) $\psi_1\mathbb{N}_1 \oplus \psi_2\mathbb{N}_1 = (\psi_1 + \psi_2)\mathbb{N}_1$;

(6) $\mathbb{N}_1^{\psi_1} \otimes \mathbb{N}_1^{\psi_2} = \mathbb{N}_1^{(\psi_1+\psi_2)}$;

(7) $\left(\mathbb{N}_1^{\psi_1}\right)^{\psi_2} = \mathbb{N}_1^{\psi_1\psi_2}$.

Proof. (1) Since $\mathbb{N}_{\mathcal{K}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{K}}}, \ddot{E}_{\mathbb{N}_{\mathcal{K}}}, \ddot{O}_{\mathbb{N}_{\mathcal{K}}}\right\}$ is a collection of SHFNs, where $(\mathcal{K} = 1, 2, ..., d)$ and $\psi_1, \psi_1 > 0$, then by the Definition 14, we have

$$\begin{aligned} & \mathbb{N}_1 \oplus \mathbb{N}_2 \\ = & \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left\{ \begin{array}{c} \sqrt{1 - e^{-\left(\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho + \left(-\ln\left(1-\ddot{a}_2^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\left(-\ln \ddot{e}_1^2\right)^\rho + \left(-\ln \ddot{e}_2^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\left(-\ln \ddot{o}_1^2\right)^\rho + \left(-\ln \ddot{o}_2^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\} \\ = & \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left\{ \begin{array}{c} \sqrt{1 - e^{-\left(\left(-\ln\left(1-\ddot{a}_2^2\right)\right)^\rho + \left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\left(-\ln \ddot{e}_2^2\right)^\rho + \left(-\ln \ddot{e}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\left(-\ln \ddot{o}_2^2\right)^\rho + \left(-\ln \ddot{o}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\} \\ = & \mathbb{N}_2 \oplus \mathbb{N}_1. \end{aligned}$$

(2) By Definition 14, we have

$$\begin{aligned}
\mathbb{N}_1 \otimes \mathbb{N}_2 &= \left\{ \begin{array}{c} \sqrt{e^{-\left(\left(-\ln \ddot{a}_1^2\right)^{\rho}+\left(-\ln \ddot{a}_2^2\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\left(-\ln \ddot{e}_1^2\right)^{\rho}+\left(-\ln \ddot{e}_2^2\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\left(-\ln\left(1-\ddot{o}_1^2\right)\right)^{\rho}+\left(-\ln\left(1-\ddot{o}_2^2\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \sqrt{e^{-\left(\left(-\ln \ddot{a}_2^2\right)^{\rho}+\left(-\ln \ddot{a}_1^2\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\left(-\ln \ddot{e}_2^2\right)^{\rho}+\left(-\ln \ddot{e}_1^2\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\left(-\ln\left(1-\ddot{o}_2^2\right)\right)^{\rho}+\left(-\ln\left(1-\ddot{o}_1^2\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \\
&= \mathbb{N}_2 \otimes \mathbb{N}_1.
\end{aligned}$$

(3) By Definition 14, we have

$$\begin{aligned}
\psi_1\left(\mathbb{N}_1 \oplus \mathbb{N}_2\right) &= \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \psi_1 \left\{ \begin{array}{c} \sqrt{1-e^{-\left(\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^{\rho}+\left(-\ln\left(1-\ddot{a}_2^2\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\left(-\ln \ddot{e}_1^2\right)^{\rho}+\left(-\ln \ddot{e}_2^2\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\left(-\ln \ddot{o}_1^2\right)^{\rho}+\left(-\ln \ddot{o}_2^2\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \\
&= \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left\{ \begin{array}{c} \sqrt{1-e^{-\left(\psi_1\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^{\rho}+\psi_1\left(-\ln\left(1-\ddot{a}_2^2\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{e}_1^2\right)^{\rho}+\psi_1\left(-\ln \ddot{e}_2^2\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{o}_1^2\right)^{\rho}+\psi_1\left(-\ln \ddot{o}_2^2\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \\
&= \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left(\begin{array}{c} \left\{ \begin{array}{c} \sqrt{1-e^{-\left(\psi_1\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{e}_1^2\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{o}_1^2\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \oplus \\ \left\{ \begin{array}{c} \sqrt{1-e^{-\left(\psi_1\left(-\ln\left(1-\ddot{a}_2^2\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-(\psi_1(-\ln \ddot{e}_2)^{\rho})^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{o}_2^2\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \end{array} \right) \\
&= \psi_1\mathbb{N}_1 \oplus \psi_1\mathbb{N}_2.
\end{aligned}$$

(4) It is obvious given (3).

(5) By Definition 14, we have

$$\psi_1\mathbb{N}_1 \oplus \psi_2\mathbb{N}_1 = \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left(\left\{ \begin{array}{c} \sqrt{1-e^{-\left(\psi_1\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{e}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{o}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\} \oplus \left\{ \begin{array}{c} \sqrt{1-e^{-\left(\psi_2\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_2\left(-\ln \ddot{e}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_2\left(-\ln \ddot{o}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\} \right)$$

$$= \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left\{ \begin{array}{c} \sqrt{1-e^{-\left(\psi_1\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho+\psi_2\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{e}_1^2\right)^\rho+\psi_2\left(-\ln \ddot{e}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1\left(-\ln \ddot{o}_1^2\right)^\rho+\psi_2\left(-\ln \ddot{o}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\}$$

$$= \bigcup_{(\ddot{a}_t,\ddot{e}_t,\ddot{o}_t)\in(\ddot{A}_t,\ddot{E}_t,\ddot{O}_t)(t=1,2)} \left\{ \begin{array}{c} \sqrt{1-e^{-\left(\psi_1+\psi_2\left(-\ln\left(1-\ddot{a}_1^2\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1+\psi_2\left(-\ln \ddot{e}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\psi_1+\psi_2\left(-\ln \ddot{o}_1^2\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\}$$

$$= (\psi_1 + \psi_2)\,\mathbb{N}_1.$$

(6) & (7) They can be proven in a similar way as (5). □

4. Aczel-Alsina geometric Agops for SHFNs

This section develops a list of innovative Agops using Aczel-Alsina norms in SHF environments.

Definition 15. *Let* $\mathbb{N}_\mathcal{K} = \left\{\ddot{A}_{\mathbb{N}_\mathcal{K}}, \ddot{E}_{\mathbb{N}_\mathcal{K}}, \ddot{O}_{\mathbb{N}_\mathcal{K}}\right\}$ *be a collection of SHFNs, where* $(\mathcal{K} = 1, 2, ..., d)$. *An SHFAWG Agop with the dimension r is a mapping* $\mathcal{P}^\ell \to \mathcal{P}$ *with a weight vector* $\mu = (\mu_1, \mu_1, ..., \mu_\ell)^T$ *such that* $\mu_\mathcal{J} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \mu_\mathcal{J} = 1$ *it is defined as*

$$SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) = \prod_{\mathcal{J}=1}^{\ell} (\mathbb{N}_\mathcal{J})^{\mu_\mathcal{J}}.$$

Theorem 2. *Suppose* $\mathbb{N}_\mathcal{J} = \left\{\ddot{A}_{\mathbb{N}_\mathcal{J}}, \ddot{E}_{\mathbb{N}_\mathcal{J}}, \ddot{O}_{\mathbb{N}_\mathcal{J}}\right\}$ *is a collection of SHFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$. *An SHFAWG Agop with the dimension* ℓ *is a mapping* $\mathcal{P}^\ell \to \mathcal{P}$ *with the weight vector* $\mu = (\mu_1, \mu_1, ..., \mu_\ell)^T$ *such that* $\mu_\mathcal{J} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \mu_\mathcal{J} = 1$ *it is defined as:*

$$SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) = \prod_{\mathcal{J}=1}^{\ell} (\mathbb{N}_\mathcal{J})^{\mu_\mathcal{J}}$$

$$= \bigcup_{\left(\ddot{a}_{\mathbb{N}_\mathcal{J}}, \ddot{e}_{\mathbb{N}_\mathcal{J}}, \ddot{E}_{\mathbb{N}_\mathcal{J}}\right) \in (\ddot{A}_{\mathbb{N}_\mathcal{J}}, \ddot{E}_{\mathbb{N}_\mathcal{J}}, \ddot{O}_{\mathbb{N}_\mathcal{J}})(\mathcal{J}=1,2)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \mu_\mathcal{J}\left(-\ln \ddot{a}^2_{\mathbb{N}_\mathcal{J}}\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \mu_\mathcal{J}\left(-\ln\left(1-\ddot{e}^2_{\mathbb{N}_\mathcal{J}}\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \mu_\mathcal{J}\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_\mathcal{J}}\right)\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\}.$$

Proof. The following results are obtained by applying mathematical induction to the proof of Theorem 2:

Step-1: For $\ell = 2$, we get

$$SHFAWG(\mathbb{N}_1, \mathbb{N}_2) = (\mathbb{N}_1)^{\mu_1} \otimes (\mathbb{N}_2)^{\mu_2}.$$

By Definition 14, we have

$$(\mathbb{N}_1)^{\mu_1} = \bigcup_{\left(\ddot{a}_{\mathbb{N}_1}, \ddot{e}_{\mathbb{N}_1}, \ddot{o}_{\mathbb{N}_1}\right) \in (\ddot{A}_{\mathbb{N}_1}, \ddot{E}_{\mathbb{N}_1}, \ddot{O}_{\mathbb{N}_1})} \left\{ \begin{array}{c} \sqrt{e^{-\left(\mu_1\left(-\ln \ddot{a}^2_{\mathbb{N}_1}\right)^\rho\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\mu_1\left(-\ln\left(\ddot{e}^2_{\mathbb{N}_1}\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\mu_1\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_1}\right)\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\}$$

and

$$(\mathbb{N}_2)^{\mu_2} = \bigcup_{\left(\ddot{a}^2_{\mathbb{N}_2}, \ddot{e}_{\mathbb{N}_2}, \ddot{o}_{\mathbb{N}_2}\right) \in (\ddot{A}_{\mathbb{N}_2}, \ddot{E}_{\mathbb{N}_2}, \ddot{O}_{\mathbb{N}_2})} \left\{ \begin{array}{c} \sqrt{e^{-\left(\mu_2\left(-\ln \ddot{a}^2_{\mathbb{N}_2}\right)^\rho\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\mu_2\left(-\ln\left(\ddot{e}^2_{\mathbb{N}_2}\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\mu_2\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_2}\right)\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right\}$$

therefore

$$SHFAWG(\mathbb{N}_1, \mathbb{N}_2) = \bigcup_{\left(\ddot{a}_{\mathbb{N}_\mathcal{J}}, \ddot{e}^2_{\mathbb{N}_\mathcal{J}}, \ddot{o}_{\mathbb{N}_\mathcal{J}}\right) \in (\ddot{A}_{\mathbb{N}_\mathcal{J}}, \ddot{E}_{\mathbb{N}_\mathcal{J}}, \ddot{O}_{\mathbb{N}_\mathcal{J}})(\mathcal{J}=1,2)} \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{e^{-\left(\mu_1\left(-\ln \ddot{a}^2_{\mathbb{N}_1}\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\mu_1\left(-\ln\left(\ddot{e}^2_{\mathbb{N}_1}\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\mu_1\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_1}\right)\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right) \otimes \\ \left(\begin{array}{c} \sqrt{e^{-\left(\mu_2\left(-\ln \ddot{a}^2_{\mathbb{N}_2}\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\mu_2\left(-\ln\left(\ddot{e}^2_{\mathbb{N}_2}\right)\right)^\rho\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\mu_2\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_2}\right)\right)^\rho\right)^{\frac{1}{\rho}}}} \end{array} \right) \end{array} \right\}$$

$$= \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1,2)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\mu_1\left(-\ln \ddot{a}^2_{\mathbb{N}_1}\right)^{\rho}+\mu_2\left(-\ln \ddot{a}^2_{\mathbb{N}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\mu_1\left(-\ln \ddot{e}^2_{\mathbb{N}_1}\right)^{\rho}+\mu_2\left(-\ln \ddot{e}^2_{\mathbb{N}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\mu_1\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_1}\right)\right)^{\rho}+\mu_2\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}$$

$$= \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1,2)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{2}\mu_{\mathcal{J}}\left(-\ln \ddot{a}^2_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{2}\mu_{\mathcal{J}}\left(-\ln \ddot{e}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{2}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}.$$

Thus Theorem 2 is true for $\ell = 2$.

Let us assume that Theorem 2 is true for $\ell = d$; we have

$$SHFAWG\left(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_d\right) = \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln \ddot{a}^2_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln \ddot{e}^2_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}.$$

We are to prove that Theorem 2 is true for $\ell = d + 1$.

$$SHFAWG\left(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_d, \mathbb{N}_{d+1}\right) = \prod_{\mathcal{J}=1}^{\ell}\left(\mathbb{N}_{\mathcal{J}}\right)^{\mu_{\mathcal{J}}} \otimes \left(\mathbb{N}_{d+1}\right)^{\mu_{d+1}}$$

$$\prod_{\mathcal{J}=1}^{\ell}\left(\mathbb{N}_{\mathcal{J}}\right)^{\mu_{\mathcal{J}}} \otimes \left(\mathbb{N}_{d+1}\right)^{\mu_{d+1}}$$

$$= \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1...d)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln \ddot{a}^2_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln \ddot{e}^2_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \otimes$$

$$\bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})} \left\{ \begin{array}{c} \sqrt{e^{-\left(\mu_{d+1}\left(-\ln \ddot{a}^2_{\mathbb{N}_{d+1}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\mu_{d+1}\left(-\ln\left(\ddot{e}^2_{\mathbb{N}_{d+1}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\mu_{d+1}\left(-\ln\left(1-\ddot{o}^2_{\mathbb{N}_{d+1}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}$$

$$= \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..d)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln \ddot{a}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}+\mu_{d+1}(-\ln \ddot{a}_{\mathbb{N}d+1})^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln \ddot{e}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}+\mu_{d+1}(-\ln \ddot{e}_{\mathbb{N}d+1})^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{d}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}+\mu_{d+1}(-\ln(1-\ddot{o}_{\mathbb{N}d+1}))^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}$$

$$= \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}}(\mathcal{J}=1...d+1)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d+1}\mu_{\mathcal{J}}\left(-\ln \ddot{a}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{d+1}\mu_{\mathcal{J}}\left(-\ln \ddot{e}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{d+1}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}.$$

Hence, Theorem 2 is true $\forall\ \ell$. □

By using the operator SHFAWG, we can clearly explain the related features.

Theorem 3. *(Idempotency) Let* $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}(\mathcal{J} = 1, 2, ..., \ell)$ *be a collection of equivalent SHFNs, i.e.,* $\mathbb{N}_{\mathcal{J}} = \mathbb{N}$ *for each* $(\mathcal{J} = 1, 2, ..., \ell)$ *. Then*

$$SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) = \mathbb{N}.$$

Proof. We have

$$SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) = \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln \ddot{a}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln \ddot{e}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}.$$

Put $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\} = \mathbb{N}\ (\mathcal{J} = 1, 2, ..., \ell)$; *then*, we have

$$\begin{aligned} & SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) \\ = & \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln \ddot{a}_{\mathbb{N}}^{2}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}(-\ln \ddot{e}_{\mathbb{N}})^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}(-\ln(1-\ddot{o}_{\mathbb{N}}))^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\} \\ = & \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})} \left\{ \begin{array}{c} \sqrt{e^{-\left(\left(-\ln \ddot{a}_{\mathbb{N}}^{2}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{e^{-((-\ln \ddot{e}_{\mathbb{N}})^{\rho})^{\frac{1}{\rho}}}}, \\ \sqrt{1-e^{-((-\ln(1-\ddot{o}_{\mathbb{N}}))^{\rho})^{\frac{1}{\rho}}}} \end{array} \right\} \\ = & \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})} (\ddot{a}_{\mathbb{N}}, \ddot{e}_{\mathbb{N}}, \ddot{o}_{\mathbb{N}}) = \mathbb{N}. \end{aligned}$$

Thus, $SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) = \mathbb{N}$ holds. □

Theorem 4. *(Boundedness) Let* $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\} (\mathcal{J} = 1, 2, ..., \ell)$ *be a collection of SHFNs. Let* $\mathbb{N}^{-}_{\mathcal{J}} = \left(\min_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right\}, \max_{\mathcal{J}}\left\{\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}\right)$ *and* $\mathbb{N}^{+}_{\mathcal{J}} = \left(\max_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}\right)$ $(\mathcal{J} = 1, 2, ..., \ell)$. *Then,*

$$\mathbb{N}^{-}_{\mathcal{J}} \leq SHFAWG\left(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_{\ell}\right) \leq \mathbb{N}^{+}_{\mathcal{J}}.$$

Proof. We have $\min_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\} \leq \ddot{A}_{\mathbb{N}_{\mathcal{J}}} \leq \max_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\}$, i.e.,

$$\begin{aligned}
& \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(\min\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\} \\
\leq & \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\} \\
\leq & \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(\max\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\};
\end{aligned}$$

similarly, we have

$$\begin{aligned}
& \bigcup_{\left(\ddot{e}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{E}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(\min\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\} \\
\leq & \bigcup_{\left(\ddot{e}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{E}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\} \\
\leq & \bigcup_{\left(\ddot{e}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{E}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(\min\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\}.
\end{aligned}$$

Now, we also have

$$\begin{aligned}
& \bigcup_{\left(\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(\max\left(1-\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right)\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\} \\
\leq & \bigcup_{\left(\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \sqrt{\left\{1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}\right\}} \\
\leq & \bigcup_{\left(\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)} \left\{\sqrt{1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(\min\left(1-\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right)\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right\}.
\end{aligned}$$

Therefore,

$$\mathbb{N}_{\mathcal{J}}^{-} \leq SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) \leq \mathbb{N}_{\mathcal{J}}^{+}.$$

□

Theorem 5. *Let* $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ *and* $\mathbb{N}_{\mathcal{J}}^{*} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}^{*}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}^{*}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}^{*}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be two collections of SHFNs, if* $\mathbb{N}_{\mathcal{J}} \leq \mathbb{N}_{\mathcal{J}}^{*}$ *for* $(\mathcal{J} = 1, 2, ..., \ell)$ *. Then,*

$$SHFAWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) \leq SHFAWG(\mathbb{N}_1^{*}, \mathbb{N}_2^{*}, ..., \mathbb{N}_\ell^{*}).$$

Proof. The proof is obvious. □

Definition 16. *Let* $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ *be collection of SHFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$*. An SHFAOWG Agop with the dimension* ℓ *is a mapping* $\mathcal{P}^\ell \to \mathcal{P}$ *with a weight vector* $\mu = (\mu_1, \mu_1, ..., \mu_\ell)^T$ *such that* $\mu_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \mu_{\mathcal{J}} = 1$ *it is defined as*

$$SHFAOWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) = \prod_{\mathcal{J}=1}^{\ell} (\mathbb{N}_{\tau(\mathcal{J})})^{\mu_{\mathcal{J}}},$$

where $(\tau(1), \tau(2), ..., \tau(\ell))$ *are the permutations in such a way as* $\mathbb{N}_{\tau(\mathcal{J})} \leq \mathbb{N}_{\tau(\mathcal{J}-1)}$*.*

Theorem 6. *Suppose* $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ *is a collection of SHFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$*. An SHFAOWG Agop with the dimension* ℓ *is a mapping* $\mathcal{P}^\ell \to \mathcal{P}$ *with a weight vector* $\mu = (\mu_1, \mu_1, ..., \mu_\ell)^T$ *such that* $\mu_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \mu_{\mathcal{J}} = 1$ *it is defined as:*

$$\begin{aligned} & SHFAOWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) \\ = & \prod_{\mathcal{J}=1}^{\ell} (\mathbb{N}_{\tau(\mathcal{J})})^{\mu_{\mathcal{J}}} \\ = & \bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}}, \ddot{e}_{\mathbb{N}_{\mathcal{J}}}, \ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right) \in \left(\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right)(\mathcal{J}=1..\ell)} \left\{ \begin{array}{c} \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \mu_{\mathcal{J}}\left(-\ln \ddot{a}^2_{\mathbb{N}_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \mu_{\mathcal{J}}\left(-\ln \ddot{E}_{\mathbb{N}_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \\ \sqrt{1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{O}_{\mathbb{N}_{\tau(\mathcal{J})}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}} \end{array} \right\}, \end{aligned}$$

where $(\tau(1), \tau(2), ..., \tau(\ell))$ *are the permutations in such a way as* $\mathbb{N}_{\tau(\mathcal{J})} \leq \mathbb{N}_{\tau(\mathcal{J}-1)}$*.*

By using the operator SHFAOWG, we can clearly explain the related features.

Theorem 7. *(1) (Idempotency) Let* $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be a collection of equivalent SHFNs, i.e.,* $\mathbb{N}_{\mathcal{J}} = \mathbb{N}$ *for each* $(\mathcal{J} = 1, 2, ..., \ell)$*. Then*

$$SHFAOWG(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell) = \mathbb{N}.$$

(2) (Boundedness) Let $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be a collection of SHFNs. Let* $\mathbb{N}_{\mathcal{J}}^{-} = \left(\min_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right\}, \max_{\mathcal{J}}\left\{\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}\right)$ *and*

$$\mathbb{N}_{\mathcal{J}}^{+} = (\max_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\})$$

$(\mathcal{J}=1,2,...,\ell)$. *Then,*

$$\mathbb{N}_{\mathcal{J}}^{-}\leq PHFAOWG(\mathbb{N}_1,\mathbb{N}_2,...,\mathbb{N}_\ell)\leq \mathbb{N}_{\mathcal{J}}^{+}.$$

(3) Let $\mathbb{N}_{\mathcal{J}}=\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ *and* $\mathbb{N}_{\mathcal{J}}^{*}=\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}^{*},\ddot{E}_{\mathbb{N}_{\mathcal{J}}}^{*},\ddot{O}_{\mathbb{N}_{\mathcal{J}}}^{*}\right\}$ $(\mathcal{J}=1,2,...,\ell)$ *be two collections of SHFNs. If* $\mathbb{N}_{\mathcal{J}}\leq\mathbb{N}_{\mathcal{J}}^{*}$ *for* $(\mathcal{J}=1,2,...,\ell)$. *Then,*

$$SHFAOWG(\mathbb{N}_1,\mathbb{N}_2,...,\mathbb{N}_\ell)\leq SHFAOWG(\mathbb{N}_1^{*},\mathbb{N}_2^{*},...,\mathbb{N}_\ell^{*}).$$

Proof. Proof of this theorem is similarly done by using Theorems 3–5. □

Definition 17. *Let* $\mathbb{N}_{\mathcal{J}}=\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ *be a collection of SHFNs, where* $(\mathcal{J}=1,2,...,\ell)$. *An SHFAHWG Agop with the dimension* ℓ *is a mapping* $\mathcal{P}^{\ell}\rightarrow\mathcal{P}$ *with a weight vector* $\mu=(\mu_1,\mu_2,...,\mu_\ell)^T$ *such that* $\mu_{\mathcal{J}}>0$ *and* $\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}=1$ *it is defined as*

$$SHFAHWG(\mathbb{N}_1,\mathbb{N}_2,...,\mathbb{N}_\ell)=\prod_{\mathcal{J}=1}^{\ell}\left(\mathbb{N}_{\tau(\mathcal{J})}^{*}\right)^{\mu_{\mathcal{J}}},$$

where $\mu=(\mu_1,\mu_2,...,\mu_\ell)^T$ *are the associated weights such that* $\mu_{\mathcal{J}}>0$ *and* $\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}=1$; *also,* $\mathbb{N}_{\tau(\mathcal{J})}^{*}=\left(\mathbb{N}_{\tau(\mathcal{J})}^{*}=n\mu_{\mathcal{J}}\mathbb{N}_{\tau(\mathcal{J})}\right)$ $(\mathcal{J}=1,2,...,\ell)$ *and* $(\tau(1),\tau(2),...,\tau(\ell))$ *are the permutations in such a way as* $\mathbb{N}_{\tau(\mathcal{J})}^{*}\leq\mathbb{N}_{\tau(\mathcal{J}-1)}^{*}$.

Theorem 8. *Suppose* $\mathbb{N}_{\mathcal{J}}=\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ *is a collection of SHFNs, where* $(\mathcal{J}=1,2,...,\ell)$. *An SHFAHWG Agop with the dimension* ℓ *is a mapping* $\mathcal{P}^{\ell}\rightarrow\mathcal{P}$ *with a weight vector* $\mu=(\mu_1,\mu_1,...,\mu_\ell)^T$ *such that* $\mu_{\mathcal{J}}>0$ *and* $\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}=1$ *it is defined as*

$$\begin{aligned}&SHFAHWG(\mathbb{N}_1,\mathbb{N}_2,...,\mathbb{N}_\ell)\\=&\prod_{\mathcal{J}=1}^{\ell}\left(\mathbb{N}_{\tau(\mathcal{J})}^{*}\right)^{\mu_{\mathcal{J}}}\\=&\bigcup_{\left(\ddot{a}_{\mathbb{N}_{\mathcal{J}}},\ddot{e}_{\mathbb{N}_{\mathcal{J}}},\ddot{o}_{\mathbb{N}_{\mathcal{J}}}\right)\in(\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}})(\mathcal{J}=1..\ell)}\left\{\begin{array}{c}\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\ddot{a}^2_{\mathbb{N}^{*}_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}},\\\sqrt{e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\ddot{E}_{\mathbb{N}^{*}_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}},\\\sqrt{1-e^{-\left(\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}\left(-\ln\left(1-\ddot{O}_{\mathbb{N}^{*}_{\tau(\mathcal{J})}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}}\end{array}\right\},\end{aligned}$$

where $\mu=(\mu_1,\mu_2,...,\mu_\ell)^T$ *are the associated weights such that* $\mu_{\mathcal{J}}>0$ *and* $\sum_{\mathcal{J}=1}^{\ell}\mu_{\mathcal{J}}=1$; *also,* $\mathbb{N}_{\tau(\mathcal{J})}^{*}=\left(\mathbb{N}_{\tau(\mathcal{J})}^{*}=\mu_{\mathcal{J}}\mathbb{N}_{\tau(\mathcal{J})}\right)$ $(\mathcal{J}=1,2,...,\ell)$ *and* $(\tau(1),\tau(2),...,\tau(\ell))$ *are the permutations in such a way as* $\mathbb{N}_{\tau(\mathcal{J})}^{*}\leq\mathbb{N}_{\tau(\mathcal{J}-1)}^{*}$.

By using the operator SHFAHWG, we can clearly explain the related features.

Theorem 9. *(1) (Idempotency) Let* $\mathbb{N}_{\mathcal{J}}=\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}},\ddot{E}_{\mathbb{N}_{\mathcal{J}}},\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ $(\mathcal{J}=1,2,...,\ell)$ *be a collection of equivalent SHFNs, i.e.,* $\mathbb{N}_{\mathcal{J}}=\mathbb{N}$ *for each* $(\mathcal{J}=1,2,...,\ell)$. *Then*

$$SHFAHWG(\mathbb{N}_1,\mathbb{N}_2,...,\mathbb{N}_\ell)=\mathbb{N}.$$

(2) (Boundedness) Let $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\} (\mathcal{J} = 1, 2, ..., \ell)$ *be a collection of SHFNs. Let* $\mathbb{N}_{\mathcal{J}}^{-} = \left(\min_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right\}, \max_{\mathcal{J}}\left\{\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}\right)$ *and*

$$\mathbb{N}_{\mathcal{J}}^{+} = \left(\max_{\mathcal{J}}\left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{E}_{\mathbb{N}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}\right)$$

$(\mathcal{J} = 1, 2, ..., \ell)$. *Then,*

$$\mathbb{N}_{\mathcal{J}}^{-} \leq SHFAHWG\left(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell\right) \leq \mathbb{N}_{\mathcal{J}}^{+}.$$

(3) Let $\mathbb{N}_{\mathcal{J}} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}\right\}$ *and* $\mathbb{N}_{\mathcal{J}}^{*} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}}}^{*}, \ddot{E}_{\mathbb{N}_{\mathcal{J}}}^{*}, \ddot{O}_{\mathbb{N}_{\mathcal{J}}}^{*}\right\} (\mathcal{J} = 1, 2, ..., \ell)$ *be two collections of SHFNs, if* $\mathbb{N}_{\mathcal{J}} \leq \mathbb{N}_{\mathcal{J}}^{*}$ *for* $(\mathcal{J} = 1, 2, ..., \ell)$. *Then,*

$$SHFAHWG\left(\mathbb{N}_1, \mathbb{N}_2, ..., \mathbb{N}_\ell\right) \leq SHFAHWG\left(\mathbb{N}_1^{*}, \mathbb{N}_2^{*}, ..., \mathbb{N}_\ell^{*}\right).$$

Proof. This theorem can be prove by utilizing Theorems 3–5. □

5. Decision support algorithm

A novel MADM method has been created to deal with the complicated ambiguous data in real-world DS issues in order to confirm the efficacy of the SHF Aczel-Alsina geometric Agop in this study.

The algorithm is described in the below steps.

Suppose that there is an arrangement of ℓ choices $\{\eta_1, \eta_2, ..., \eta_\ell\}$ that are suitably evaluated by a set of m criteria that are $\{F_1, F_2, ..., F_m\}$. A weight vector $\mu = (\mu_1, \mu_1, ..., \mu_m)^T$ is then used to specify the usefulness of different attributes F_ι $(\iota = 1, 2, ..., m)$ such that $\mu_\iota > 0$ and $\sum_{\iota=1}^{m} \mu_\iota = 1$.

Let $\mathbb{N}_{\mathcal{J}\iota} = \left\{\ddot{A}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}\iota}}\right\}$ for $\ddot{A}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}\iota}} \in [0, 1]$ be the suitable evaluation of each property for every option, where $\ddot{A}_{\mathbb{N}_{\ell m}}$ denotes the PMG that the alternative $\eta_{\mathcal{J}}$ $(\mathcal{J} = 1, 2, \ldots, \ell)$. $\ddot{E}_{\mathbb{N}_{\ell m}}$ and $\ddot{O}_{\mathbb{N}_{\ell m}}$ indicate the NMG and the NeMG, respectively. According to all of the evaluation results, we can build the decision matrix of SHFNs: $\mathbb{N} = (\mathbb{N}_{\mathcal{J}\iota})_{\ell m}$.

The MADM problem was solved in this work by using the newly discovered SHF Aczel-Alsina geometric operators, and the various stages are given for selecting the most suitable alternative:

Step-1. Establish a group of characteristics that are suitable for the evaluation issue under discussion: An expert committee is put together to screen the features and come up with an acceptable set of appraisal features after gathering probable appraisal properties.

$$D_{\mathcal{J}\times\iota} = \begin{matrix} \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_\ell \end{matrix} \begin{pmatrix} F_1 & F_2 & & F_m \\ \left(\ddot{A}_{\mathbb{N}_{11}}, \ddot{E}_{\mathbb{N}_{11}}, \ddot{O}_{\mathbb{N}_{11}}\right) & \left(\ddot{A}_{\mathbb{N}_{12}}, \ddot{E}_{\mathbb{N}_{12}}, \ddot{O}_{\mathbb{N}_{12}}\right) & ... & \left(\ddot{A}_{\mathbb{N}_{1m}}, \ddot{E}_{\mathbb{N}_{1m}}, \ddot{O}_{\mathbb{N}_{1m}}\right) \\ \left(\ddot{A}_{\mathbb{N}_{21}}, \ddot{E}_{\mathbb{N}_{21}}, \ddot{O}_{\mathbb{N}_{21}}\right) & \left(\ddot{A}_{\mathbb{N}_{22}}, \ddot{E}_{\mathbb{N}_{22}}, \ddot{O}_{\mathbb{N}_{22}}\right) & ... & \left(\ddot{A}_{\mathbb{N}_{2m}}, \ddot{E}_{\mathbb{N}_{2m}}, \ddot{O}_{\mathbb{N}_{2m}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\ddot{A}_{\mathbb{N}_{\ell 1}}, \ddot{E}_{\mathbb{N}_{\ell 1}}, \ddot{O}_{\mathbb{N}_{\ell 1}}\right) & \left(\ddot{A}_{\mathbb{N}_{\ell 2}}, \ddot{E}_{\mathbb{N}_{\ell 2}}, \ddot{O}_{\mathbb{N}_{\ell 2}}\right) & ... & \left(\ddot{A}_{\mathbb{N}_{\ell m}}, \ddot{E}_{\mathbb{N}_{\ell m}}, \ddot{O}_{\mathbb{N}_{\ell m}}\right) \end{pmatrix}.$$

Step-2. Using the following normalization, we get the normalized decision matrix:

$$\mathcal{N}_{\mathcal{J}\times\iota} = \begin{cases} \left(\ddot{A}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{O}_{\mathbb{N}_{\mathcal{J}\iota}}\right) & if \quad C_I \\ \left(\ddot{O}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{E}_{\mathbb{N}_{\mathcal{J}\iota}}, \ddot{A}_{\mathbb{N}_{\mathcal{J}\iota}}\right) & if \quad C_{II} \end{cases} \tag{5.1}$$

where C_I refers to "if $F_\imath$ $(\imath = 1, 2, ..., m)$ is a benefit criterion" and C_{II} refers to "if $F_\imath$ $(\imath = 1, 2, ..., m)$ is a cost criterion".

Step-3. SHFWG Agops are used to accumulate the acquired expert data of DS situations.

Step-4. Utilizing newly designed SHF Aczel-Alsina geometric operators, decision support problems' expert ambiguous data are aggregated..

Step-5(a). SHFAWG operator is used to combine the aggregated data.

Step-5(b). SHFAOWG operator is used to combine the aggregated data.

Step-5(c). SHFAHWG operator is used to combine the aggregated data.

Step-6. The score values of $\eta_{\mathcal{J}}$ $(\mathcal{J} = 1, 2, \ldots, \ell)$ are computed using the score function in Definition 10.

Step-7. All options are ordered in descending order according to their score values, and the option with the highest score value is chosen as the best.

6. Mathematical illustration

A breast cancer treatment related case study is presented in this section to verify the usefulness and viability of the proposed methodology.

Case study: The question of choosing a treatment for breast cancer is covered in the study's implementation of a plan. In order to achieve this, four breast cancer treatment options, such as surgery (η_1), radiation treatment (η_2), drug therapy (η_3), and hormonal treatment (η_4) are appraised using four factors: The type of illness or tumor (F_1), the phase of the disease (F_2), the kind of of patient (F_3) and the adverse effects (F_4).

The following are the alternatives:

η_1 Operation: A surgical procedure or course of therapy for cancer entails the surgical removal of a tumor and possibly some neighboring cells. Breast cancer surgery might improve both the oncologic and efficiency of life outcomes for the patient [26]. η_2 Radiotherapy: This technique utilizes highly energetic radiation to eradicate any potential cancer cells. Breast cancer treatment must include radiotherapy, which has been demonstrated to increase both local control and ultimate success rates [53]. η_3 Chemotherapy: Conventional chemotherapy is a crucial component of cancer treatment plans for a range of cancer types. This course of action uses toxic drugs to kill cancer cells [43]. η_4 Hormone replacement: A type of cancer therapy known as hormone therapy slows or stops the growth of the disease. Despite its impressive effects, it is frequently used as a therapeutic adjunct and in severe malignancies. Hormone therapy is a widely used method in the treatment of breast cancer [34].

The following are the attributes:

F_1 Illness or tumor form: One of the key determinants of the technique and extent of cancer therapy is the kind of cancer [1]. F_2 Phase of illness: The cancer's stage and where it is located in the body are both taken into account when determining the best course of treatment [1]. F_3 Patient kind: Despite the lack of a generally acknowledged strategy for treating breast cancer at any stage, each patient's general health, degree of fitness and medical characteristics are unique, demanding the formulation of a tailored treatment plan [20]. F_4 Adverse effect: The life expectancy of cancer patients can be significantly impacted by a number of undesirable adverse effects from cancer treatments, both emotionally and cognitively [21]. It is crucial to consider for any potential negative effects while developing a treatment plan. Weight = $\mu = (0.15, 0.25, 0.35, 0.25)$, $\rho = 4$.

Step-1. Table 1 summarizes the SHFN expert informations:

Table 1. Expert Matrix-1 of SHFNs.

	F_1	F_2	F_3	F_4
η_1	$\{(0.32, 0.33, 0.48)\}$	$\{(0.31, 0.37, 0.43), (0.26, 0.43, 0.51)\}$	$\{(0.20, 0.27, 0.50)\}$,	$\{(0.38, 0.42, 0.48)\}$
η_2	$\{(0.31, 0.37, 0.43), (0.27, 0.43, 0.53)\}$	$\{(0.28, 0.39, 0.41)\}$	$\{(0.36, 0.49, 0.69)\}$	$\{(0.17, 0.34, 0.54)\}$
η_3	$\{(0.28, 0.26, 0.39)\}$	$\{(0.25, 0.48, 0.27)\}$	$\{(0.43, 0.65, 0.62)\}$	$\{(0.19, 0.41, 0.27), (0.27, 0.23, 0.35)\}$
η_4	$\{(0.33, 0.34, 0.35)\}$	$\{(0.32, 0.27, 0.61)\}$	$\{(0.11, 0.13, 0.24), (0.12, 0.22, 0.25)\}$	$\{(0.41, 0.65, 0.44)\}$

Step-2. The normalized decision matrices are evaluated in Table 2:

Table 2. Normalized Expert Matrix-1 of SHFNs.

	F_1	F_2	F_3	F_4
η_1	$\{(0.48, 0.33, 0.32)\}$	$\{(0.43, 0.37, 0.31), (0.51, 0.43, 0.26)\}$	$\{(0.50, 0.27, 0.20)\}$,	$\{(0.48, 0.42, 0.38)\}$
η_2	$\{(0.43, 0.37, 0.31), (0.53, 0.43, 0.27)\}$	$\{(0.41, 0.39, 0.28)\}$	$\{(0.69, 0.49, 0.36)\}$	$\{(0.54, 0.34, 0.17)\}$
η_3	$\{(0.39, 0.26, 0.28)\}$	$\{(0.27, 0.48, 0.25)\}$	$\{(0.62, 0.65, 0.43)\}$	$\{(0.27, 0.41, 0.19), (0.35, 0.23, 0.27)\}$
η_4	$\{(0.35, 0.34, 0.33)\}$	$\{(0.61, 0.27, 0.32)\}$	$\{(0.24, 0.13, 0.11), (0.25, 0.22, 0.12)\}$	$\{(0.44, 0.65, 0.41)\}$

Step-3. Utilize the SHFAWG operator to integrate the aggregated data enclosed in Table 3:

Table 3. SHF aggregated data (SHFAWG).

η_1	$\{(0.7466, 0.3736, 0.3339), (0.7785, 0.3951, 0.3297)\}$
η_2	$\{(0.7350, 0.4436, 0.3254), (0.7503, 0.4475, 0.3226)\}$
η_3	$\{(0.4846, 0.5896, 0.3824), (0.5294, 0.5890, 0.3828)\}$
η_4	$\{(0.4530, 0.5674, 0.3572), (0.4729, 0.5674, 0.3572)\}$

Step-5(b). Utilize the SHFAOWG operator to integrate the aggregated data enclosed in Table 4:

Table 4. SHF aggregated data (SHFAOWG).

η_1	$\{(0.7364, 0.3572, 0.3384), (0.7770, 0.4044, 0.3330)\}$
η_2	$\{(0.7178, 0.4315, 0.3212), (0.7487, 0.4424, 0.3138)\}$
η_3	$\{(0.4846, 0.5896, 0.3824), (0.5294, 0.5891, 0.3828)\}$
η_4	$\{(0.4759, 0.5674, 0.3597), (0.4937, 0.5675, 0.3597)\}$

Step-5(c). Utilize the SHFAHWG operator (under the associated weights $(0.15, 0.25, 0.35, 0.25)^T$ to integrate the aggregated data enclosed in Table 5:

Table 5. SHF aggregated data (SHFAHWG).

η_1	$\{(0.7364, 0.3783, 0.3384), (0.7770, 0.3783, 0.3330)\}$
η_2	$\{(0.7263, 0.4441, 0.3279), (0.7499, 0.4504, 0.3234)\}$
η_3	$\{(0.484, 0.5896, 0.3824), (0.5294, 0.5891, 0.3828)\}$
η_4	$\{(0.4759, 0.5674, 0.3597), (0.4937, 0.5675, 0.3597)\}$

Step-6. According to the score function in Definition 10, the score values of $\eta_{\mathcal{J}}$ $(\mathcal{J} = 1, 2, 3, 4)$ are enclosed in Table 6:

Table 6. Score and ranking of SHFNs.

Operators	Score				Ranking
	$\xi(\eta_1)$	$\xi(\eta_2)$	$\xi(\eta_3)$	$\xi(\eta_4)$	
$SHFAWG$	0.0464	-0.0212	-0.4649	-0.4616	$\eta_1 > \eta_2 > \eta_4 > \eta_3$
$SHFAOWG$	0.0402	-0.0212	-0.4624	-0.4423	$\eta_1 > \eta_2 > \eta_4 > \eta_3$
$SHFAHWG$	0.0427	-0.0348	-0.4652	-0.4423	$\eta_1 > \eta_2 > \eta_4 > \eta_3$

Step-7. With regard to the features given in the factors that affect cancer treatment, η_1 (**operation**) has the greatest score value among all of the recommended Aczel-Alsina operators, making it our best option.

7. Comparison analysis

This section presents a comparison analysis of the proposed SHF Aczel-Alsina Agops based methodology with the existing DS method developed in the literature.

Comparison with wang and Li [54]:

To determine the optimum option, wang et al. [54] presented a list of innovative PF weighted interaction aggregation operations. The comparative results are shown in Tables 9 and 10. We utilized the proposed method to verify and check the validity of the proposed methodology; we used the SHFWG operator to rank the alternatives, having the attribute weight vector $w = (0.2, 0.1, 0.3, 0.4)$ with $\rho = 3$. The results are shown below.

The collected expert data [54] are presented in Table 7:

Table 7. Expert evaluation information.

	F_1	F_2	F_3	F_4
η_1	$\left\{\{0.43, 0.53\}, \{0.33\}, \{0.06, 0.09\}\right\}$	$\left\{\{0.76, 0.89\}, \{0.05, 0.08\}, \{0.03\}\right\}$	$\left\{\{0.42\}, \{0.35\}, \{0.12, 0.18\}\right\}$	$\left\{\{0.08\}, \{0.75, 0.89\}, \{0.02\}\right\}$
η_2	$\left\{\{0.53, 0.65, 0.73\}, \{0.10, 0.12\}, \{0.08\}\right\}$	$\left\{\{0.13\}, \{0.53, 0.64\}, \{0.12, 0.21\}\right\}$	$\left\{\{0.03\}, \{0.77, 0.82\}, \{0.10, 0.13\}\right\}$	$\left\{\{0.58, 0.73\}, \{0.15\}, \{0.08\}\right\}$
η_3	$\left\{\{0.72, 0.86, 0.91\}, \{0.03\}, \{0.02\}\right\}$	$\left\{\{0.07\}, \{0.05, 0.09\}, \{0.05\}\right\}$	$\left\{\{0.04\}, \{0.65, 0.72, 0.85\}, \{0.05, 0.10\}\right\}$	$\left\{\{0.45, 0.68\}, \{0.18, 0.26\}, \{0.06\}\right\}$
η_4	$\left\{\{0.77, 0.85\}, \{0.09\}, \{0.05\}\right\}$	$\left\{\{0.65, 0.74\}, \{0.10, 0.16\}, \{0.10\}\right\}$	$\left\{\{0.02\}, \{0.78, 0.89\}, \{0.05\}\right\}$	$\left\{\{0.08\}, \{0.65, 0.84\}, \{0.06\}\right\}$

Table 8. Collected expert data under SHFNs.

η_1	$\left\{0.1509, 0.1509, 0.1519, 0.1519\right\}, \left\{0.6757, 0.8282, 0.6757, 0.8282\right\}, \left\{0.09849, 0.1476, 0.100, 0.1480\right\}$
η_2	$\left\{\{0.0903, 0.0905, 0.09054, 0.09076, 0.09058, 0.09085\}, \{0.6737, 0.7262, 0.6776, 0.7282, 0.6737, 0.7262, 0.6776, 0.7282\}, \{0.09539, 0.1114, 0.1453, 0.1480\}\right\}$
η_3	$\left\{\{0.1007, 0.1019, 0.1007, 0.1019, 0.1007, 0.1019\}, \left\{0.5546, 0.5548, 0.6219, 0.6220, 0.7590, 0.7590, 0.5546, 0.5548, 0.6219, 0.6220, 0.7590, 0.7590\right\}, \{0.05385, 0.08307\}\right\}$
η_4	$\left\{0.05494, 0.05494, 0.05494, 0.05497\right\}, \left\{0.7021, 0.7934, 0.8107, 0.8424, 0.7021, 0.7934, 0.8107, 0.8424\right\}, \{0.07071\}$

Comparative studies the expert data collected in [54] are enclosed in Table 9:

Table 9. Score and ranking.

wang and Li [54]	Score				Ranking
	$\xi(\eta_1)$	$\xi(\eta_2)$	$\xi(\eta_3)$	$\xi(\eta_4)$	
PHFWG	-0.6792	-0.6822	-0.5961	-0.7788	$\eta_3 > \eta_1 > \eta_2 > \eta_4$
Proposed Method	**Score**				**Ranking**
	$\xi(\eta_1)$	$\xi(\eta_2)$	$\xi(\eta_3)$	$\xi(\eta_4)$	
SHFAWG	-0.7240	-0.7358	-0.6123	-0.8029	$\eta_3 > \eta_1 > \eta_2 > \eta_4$

8. Conclusions

In the current study, we studied the features and connections of these systems while also generalizing the Aczel-Alsina t-norm and t-conorm to SHF situations. The SHFAWG operator, SHFAOWG operator, and SHFAHWG operator have all been added as additional Agops to deal with scenarios when the given conflicts are in SHFNs. The interactions among these operators, along with many alluring features and individual instances of those operators, have all been carefully examined. In multi-attribute group decision-making (MAGDM) scenarios with SHF data, the suggested operators were used, and a mathematical formulation was offered to show the DMP. The impact of factors on the outcomes of decision-making has been studied. By correctly selecting the parameter, the suggested operators can be used to gain the best option. The indicated Agops provide DMs with a newly adaptable strategy for lowering SHF MAGDM issues as a result. To put it differently, we can quickly characterize fuzzy data by giving it a parameter, which makes the information aggregation system more apparent than some other existing approaches. On the contrary, existing Agops, such as those developed by Wang and Li [54] do not make data aggregation more flexible. As a consequence, our suggested Agops are more knowledgeable and reliable when making decisions using SHF data.

In future projects, we will look into the applications of Aczel-Alsina weighted Agops of SHFNs in more domains like industrial automation, pattern recognition, and data analysis.

Conflict of interest

The authors declare no conflict of interest.

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