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Research article

Core compactness of ordered topological spaces

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Abstract: In this paper, we investigate the property of core compactness of ordered topological spaces. Particularly, we give a series of characterizations of the core compactness for directed spaces. Several results obtained in this paper are closely related to a long-standing open problem in Open problems in Topology (J. van Mill, G. M. Reed Eds., North-Holland, 1990): Which distributive continuous lattice's spectrum is exactly a sober locally compact Scott space?

Keywords: directed space; core compactness; Scott topology; closed subsets; spectrum
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1. Introduction

A topological space is called core compact if its topology is a continuous lattice. There is a deep relationship between core compactness and function spaces of topological spaces [2, 4, 12]. For example, a famous characterization is that a T_0 space is core compact if and only if it is exponential in the category of T_0 spaces [7, 9]. A poset P endowed with the Scott topology $\sigma(P)$ is called a Scott space, denoted by ΣP. When all topological spaces are restricted to the Scott spaces, core compactness can be characterized simply by productions as follows: For a poset P, $\sigma(P)$ is a continuous lattice if and only if for any poset Q one has $\Sigma(P \times Q) = \Sigma(P) \times \Sigma(Q)$ [7], which is equivalent to $\sigma(\Gamma(P))$ being a continuous lattice with $\sigma(\Gamma(P)) = \upsilon(\Gamma(P))$ [3, 19, 20], where $\Gamma(P)$ is the lattice of all Scott closed subsets of P, and $\upsilon(\Gamma(P))$ is the upper topology of $\Gamma(P)$. These results show that the continuous Scott topologies of posets are a class of special distributively continuous lattices. In [3], it was guessed that these properties, like $\sigma(\Gamma(P)) = \upsilon(\Gamma(P))$, seem to hold only for the core compact Scott topology. Is there another class of T_0 space such that the continuous topology has the same features of the Scott spaces?

In this paper, we will investigate core compactness of posets endowed with special topologies. Particularly, we will show that these features for core compact Scott spaces can be extended to directed spaces and hence give a positive answer to the question. Here, a directed space [25] is a special T_0

space, which is a generalization of the Scott spaces. The idea is that *the priori is a T_0 space and its convergent directed subsets relative to the specialization order* rather than a poset and its existing directed suprema. Directed spaces are natural topological extensions of directed complete partial orders (dcpos) in domain theory. Several results obtained in this paper are closely related to a long-standing open problem: Which distributive continuous lattice's spectrum is exactly a sober locally compact Scott space (see [21, Problem 528])?

2. Preliminaries

We assume some basic knowledge of domain theory and topology, as in [1, 7].

Let P be a poset. We define $\upsilon(P)$ and $A(P)$ to be the upper topology and the Alexandroff topology on P, respectively. A subset U of P is called Scott open if U is an upper set, and for any directed subset $D \subseteq P$ with $\sup D \in U$, there exists some $d \in D$ such that $d \in U$. All Scott open subsets of P form a topology called the Scott topology, denoted by $\sigma(P)$.

Topological spaces will always be supposed to be T_0. For a topological space X, its topology is denoted by $\mathcal{O}(X)$ or τ. The partial order $\sqsubseteq$ defined on X by $x \sqsubseteq y \Leftrightarrow x \in \overline{\{y\}}$ is called the specialization order, where $\overline{\{y\}}$ is the closure of $\{y\}$. From now on, all order-theoretical statements about T_0 spaces, such as upper sets, lower sets, directed sets, and so on, always refer to the specialization order "$\sqsubseteq$".

For any two topological spaces X, Y, we define Y^X or $TOP(X, Y)$, the set of all continuous maps from X to Y, endowed with the pointwise order. Let H be a Scott open subset of $\mathcal{O}(X)$ and V be an open subset of Y. Set $N(H \leftarrow V) = \{f \in TOP(X, Y) : f^{-1}(V) \in H\}$. As H ranges over $\sigma(\mathcal{O}(X))$, and V ranges over $\mathcal{O}(Y)$, the sets $N(H \leftarrow V)$ form a subbasis for a topology on $TOP(X, Y)$, called the Isbell topology. Define $[X \to Y]_p$ and $[X \to Y]_I$ to be the topological space equipped with the topology of pointwise convergence and the Isbell topology on Y^X, respectively.

We now introduce the notion of a directed space.

Let $(X, \mathcal{O}(X))$ be a T_0 space. Every directed subset $D \subseteq X$ can be regarded as a monotone net $(d)_{d\in D}$. Set $DS(X) = \{D \subseteq X : D \text{ is directed}\}$ to be the family of all directed subsets of X. For an $x \in X$, we define $D \to x$ to mean that x is a limit of D, i.e., D converges to x with respect to the topology on X. Then, the following result is obvious.

Lemma 2.1. Let X be a T_0 space. For any $(D, x) \in DS(X) \times X$, $D \to x$ if and only if $D \cap U \neq \emptyset$ for any open neighborhood of x.

Set $DLim(X) = \{(D, x) \in DS(X) \times X : D \to x\}$ to be the set of all pairs of directed subsets and their limits in X. Then, $(\{y\}, x) \in DLim(X)$ iff $x \sqsubseteq y$ for all $x, y \in X$.

Definition 2.2. Let X be a T_0 space. A subset $U \subseteq X$ is called directed-open if for all $(D, x) \in DLim(X)$, $x \in U$ implies $D \cap U \neq \emptyset$.

Obviously, every open set of X is directed-open. Set $d(\mathcal{O}(X)) = \{U \subseteq X : U \text{ is directed-open}\}$, and then $\mathcal{O}(X) \subseteq d(\mathcal{O}(X))$.

Theorem 2.3. [25] Let X be a T_0 topological space. Then,

(1) For all $U \in d(\mathcal{O}(X))$, $U = \uparrow U$.

(2) X equipped with $d(\mathcal{O}(X))$ is a T_0 topological space such that $\sqsubseteq_d = \sqsubseteq$, where $\sqsubseteq_d$ is the specialization order relative to $d(\mathcal{O}(X))$.
(3) For a directed subset D of X, $D \to x$ iff $D \to_d x$ for all $x \in X$, where $D \to_d x$ means that D converges to x with respect to the topology $d(\mathcal{O}(X))$.
(4) $d(d(\mathcal{O}(X)) = d(\mathcal{O}(X))$.

Definition 2.4. [25] A topological space X is said to be a directed space if it is T_0 and every directed-open set is open; equivalently, $d(\mathcal{O}(X)) = \mathcal{O}(X)$.

One can see that *the idea to define a directed space is similar to define a sequential space and the Scott topology on a poset*. In T_0 topological spaces, the notion of a directed space is equivalent to a monotone determined space defined by Erné [5]. Given any space X, we denote $\mathcal{D}X$ to be the topological space $(X, d(\mathcal{O}(X)))$.

Theorem 2.5. [25] Let X be a T_0 space. We have the following.

(1) $\mathcal{D}X$ is a directed space.
(2) The following three conditions are equivalent to each other:
  (i) X is a directed space.
  (ii) For all $U \subseteq X$, U is open iff for any $(D, x) \in DLim(X)$, $x \in U$ implies $U \cap D \neq \emptyset$.
  (iii) For all $A \subseteq X$, A is closed iff for any directed subset $D \subseteq A$, $D \to x$ implies $x \in A$ for all $x \in X$.

Directed spaces include many important structures in domain theory. Let X be a topological space. X is called a c-space if for any $x \in X$ and any open subset U of X with $x \in U$, there exists some $y \in X$ such that $x \in (\uparrow y)^\circ \subseteq (\uparrow y) \subseteq U$. X is called a locally hypercompact space if for any $x \in X$ and any open subset U of X with $x \in U$, there exists a finite subset F of X such that $x \in (\uparrow F)^\circ \subseteq (\uparrow F) \subseteq U$.

Example 2.6.

(1) Every poset endowed with the Scott topology is a directed space.
(2) ([5]) Every poset endowed with the weak Scott topology is a directed space.
(3) Every c-space is a directed space. In particular, any poset endowed with the Alexandroff topology is a directed space.
(4) ([5, 6]) Every locally hypercompact space is a directed space.

We define **DTop** to be the category of all nonempty directed spaces with continuous maps as morphisms. It is easy to verify that if a directed space is T_1, then it must be a discrete space [25].

Theorem 2.7. [18, 25] **DTop** is Cartesian closed. Let X, Y be directed spaces.

(1) The categorical product $X \otimes Y$ of X and Y is homeomorphic to $\mathcal{D}(X \times Y)$.
(2) The exponential object $[X \to Y]$ is homeomorphic to $\mathcal{D}([X \to Y]_p)$.

Lemma 2.8. Given any two directed spaces X, Y, a subset U of $X \otimes Y$ is open iff the following conditions hold:

(1) For any directed subsets $(x_i)_I$ with $(x_i)_I \to x$ in X and any $(x, y) \in U$, we have $((x_i, y))_I \cap U \neq \emptyset$;

(2) for any directed subsets $(y_i)_I$ with $(y_i)_I \to y$ in Y and any $(x, y) \in U$, we have $((x, y_i))_I \cap U \neq \emptyset$.

Proof. Assume that U is open in $X \otimes Y$, and $(x, y) \in U$. For any directed set $(x_i)_I \to x$ in X and any $y \in Y$, $((x_i, y))_I$ is a directed subset of $X \otimes Y$, and $((x_i, y))_I \to (x, y)$ in $X \times Y$. By Theorem 2.3 (3), $((x_i, y))_I \to (x, y)$ in $X \otimes Y$. Thus, $((x_i, y))_I \cap U \neq \emptyset$. It is the same for $(y_i)_I$.

Conversely, assume that (1) and (2) are satisfied. We show that U is open in $X \otimes Y$. It is easily seen that U is an upper set relative to the specialization order of $X \times Y$. Let $D = ((x_i, y_i))_I$ be a directed subset of $X \times Y$ and converge to $(x, y) \in U$ in $X \times Y$. We have that $(x_i)_I \to x$ in X, and $(y_i)_I \to y$ in Y, respectively. Thus, $((x_i, y))_I \to (x, y)$ in $X \times Y$, and then there exists some $i_0 \in I$ such that $(x_{i_0}, y) \in U$. By $((x_{i_0}, y_i))_I \to (x_{i_0}, y)$, there exists some $i_1 \in I$ such that $(x_{i_0}, y_{i_1}) \in U$. Let $i_0, i_1 \leq i_2$, and then $(x_{i_2}, y_{i_2}) \in U$. □

3. Core compactness of directed spaces

Core compactness can be viewed as a weaker continuity property than quasicontinuity, where quasicontinuous spaces are exactly the locally hypercompact spaces [6, 26]. Although all quasicontinuous spaces are directed spaces, not all core compact spaces are directed spaces. All nontrivial compact T_2 spaces are the examples. $\mathbb{N}^{\top}$, the discrete natural numbers adding a top element, endowed with the upper topology, is a locally compact sober space, which is neither a directed space nor a T_1 space. In [15], Lawson gave some equivalent conditions for a T_0 topological space X to be quasicontinuous. One of the key equivalent conditions is that for any T_0 topological space Y, $X \times Y = X \bigotimes Y$, where $X \bigotimes Y$ is the tensor product of X and Y. In [7], some equivalent conditions for a T_0 topological space to be core compact were given. Recall that ΣP means a poset endowed with the Scott topology $\sigma(P)$, called a Scott space. In particular, it was proved that for a poset P, ΣP is core compact iff for any dcpo Q, $\Sigma(P \times Q) = \Sigma P \times \Sigma Q$.

In this section, we investigate the core compactness of a directed space. We show that for any two directed spaces X and Y, their tensor product $X \bigotimes Y$ is the same as their categorical product $X \otimes Y$ in **DTop**. Similar to Scott spaces, a directed space is core compact iff for any directed spaces Y, $X \times Y = X \otimes Y$. Finally, we give more equivalent conditions for a directed space to be core compact.

Definition 3.1. [14] For any two topological spaces X, Y, the tensor product $X \bigotimes Y$ of X and Y has the same carrier set of $X \times Y$. A set W is open in $X \bigotimes Y$ if for all $(x, y) \in X \times Y$, the slices $W_x = \{y \in Y : (x, y) \in W\}$ and $W^y = \{x \in X : (x, y) \in W\}$ are open in Y and X, respectively.

Given any two topological spaces X, Y, a map from a topological space $(X \times Y, \tau)$ to a topological space Z is called separately continuous if it is continuous at each argument, i.e., for any $(x_0, y_0) \in X \times Y$, the maps $f_{x_0} : Y \to Z$ and $f^{y_0} : X \to Z$ are continuous, where $f_{x_0}(y) = f(x_0, y), f^{y_0}(x) = f(x, y_0)$. The topology of the tensor product is also called the topology of separate continuity [15], which is the weak topology determined by all separately continuous functions from the product. A map from $X \bigotimes Y$ is continuous iff it is separately continuous [14].

Lemma 3.2. For any two directed spaces X and Y, $X \bigotimes Y = X \otimes Y$.

Proof. Assume that W is open in $X \otimes Y$. Given any $(x, y) \in W$, we show that the slice W^y is open in X. For any directed subset $(x_i)_I$ of X with $(x_i)_I \to x \in W^y$ in X, since $((x_i, y))_I \to (x, y)$ in $X \otimes Y$, $((x_i, y))_I$

is finally in W. It follows that $(x_i)_I$ is finally in W^y. So, W^y is open in X. Similarly, W_x is open in Y. Therefore, W is open in $X \bigotimes Y$.

Conversely, assume that W is open in $X \bigotimes Y$. Given any directed subset $(x_i)_I \to x$ in X, then for $D = ((x_i, y))_I$ we have $D \to (x, y) \in W$ in $X \times Y$. Since W^y is open in X, and X is a directed space, there exists some $i_0 \in I$ such that $x_{i_0} \in W^y$. Then, $(x_{i_0}, y) \in W \cap D$. Similarly, for any directed subsets $((x, y_i))_I \to (x, y) \in W$, there exists some y_i such that $(x, y_i) \in W \cap D$. Therefore, W is open in $X \otimes Y$ by Lemma 2.8. □

Corollary 3.3. Let X, Y, Z be directed spaces. A map $f : X \otimes Y \to Z$ is continuous iff it is separately continuous.

We recall the following condition for a T_0 topological space to be core compact.

Theorem 3.4. [7] Let X be a T_0 space. Then, the following statements are equivalent.

(1) $\mathcal{O}(X)$ is a continuous lattice.
(2) The set $\{(U, x) \in \mathcal{O}(X) \times X : x \in U\}$ is open in $\Sigma\mathcal{O}(X) \times X$.

Theorem 3.5. Let X be a directed space. Then, the following conditions are equivalent.

(1) X is core compact.
(2) $X \otimes Y = X \times Y$ for any directed space Y.
(3) $\Sigma\mathcal{O}(X) \times X = \Sigma\mathcal{O}(X) \otimes X$.

Proof. (3) ⇒ (1). Assume that $\Sigma\mathcal{O}(X) \times X = \Sigma\mathcal{O}(X) \otimes X$. $\Sigma\mathcal{O}(X)$ is a directed space. To show that X is core compact, we need only to show that $E = \{(U, x) \in \mathcal{O}(X) \times X : x \in U\}$ is open in $\Sigma\mathcal{O}(X) \times X$ by Lemma 3.4. Then, it is equivalent to showing that E is open in $\Sigma\mathcal{O}(X) \otimes X$. Assume that a directed set $((U_i, x_i))_{i \in I}$ converges to $(U, x) \in E$ in $\Sigma\mathcal{O}(X) \times X$. Then, $(x_i)_I \to x$, $(U_i)_I \to U$. Therefore, there exists some $i_0 \in I$ such that $x_{i_0} \in U$ and $U \subseteq \bigcup_{i \in I} U_i$. Then, $\exists i_1 \in I$ such that $x_{i_0} \in U_{i_1}$. Letting $i_0, i_1 \leq i_2$, we have $(U_{i_2}, x_{i_2}) \in E$.

(1) ⇒ (2). Suppose that X is core compact. We show that $X \otimes Y = X \times Y$ for any directed space Y. Since $X \otimes Y$ is finer than $X \times Y$, we need only to show that every open subset U of $X \otimes Y$ is open in $X \times Y$. Given $(x_0, y_0) \in U$, let $V = \{x \in X : (x, y_0) \in U\}$. By Lemma 3.2, V is an open subset of X. Since X is core compact, there exists a family of open subsets $\{V_n : n \in N\}$ such that

$$x_0 \in V_0 \ll \cdots \ll V_{n+1} \ll V_n \ll \cdots \ll V_2 \ll V_1 \ll V.$$

Let $W = \bigcup_{1 \leq n} \{y \in Y : V_n \times \{y\} \subseteq U\}$. Obviously, $y_0 \in W$. Since $V_0 \subseteq V_n$ for any $n \geq 1$, then $V_0 \times W \subseteq U$. We need only to show that W is an open subset of Y. Given any directed subset $D \to y \in W$ in Y, there exists some n such that $V_n \times \{y\} \subseteq U$. For any $x \in V_n$, $((x, d))_{d \in D} \to (x, y) \in U$ in $X \otimes Y$. Thus, there is some $d_x \in D$ such that $(x, d_x) \in U$. Then, there exists an open neighborhood V_x of x such that $V_x \times \{d_x\} \subseteq U$. Since $V_{n+1} \ll V_n \subseteq \bigcup_{x \in V_n} V_x$, we have that $V_{n+1} \subseteq \bigcup_{i=1}^{n} V_{x_i}$ for some finite subset $B = \{x_1, \cdots, x_n\}$ of V_n. Since B is finite, there exists some $d_0 \in D$ such that $d_{x_i} \leq d_0$. Then, $V_{n+1} \subseteq \{x \in X : (x, d_0) \in U\}$, and thus $V_{n+1} \times \{d_0\} \subseteq U$, i.e., $d_0 \in W$. Therefore, W is open in Y. Then, $(x_0, y_0) \in V_0 \times W \subseteq U$, i.e., U is open in $X \times Y$.

(2) ⇒ (3). It is obvious. □

Let X, Y be two topological spaces. It is easy to verify that any open subset of the topological product of X and Y is open in the tensor product. By Lemma 3.2 and Theorem 3.5, we can see that for directed spaces X, Y, the tensor product $X \bigotimes Y$ is equal to the topological product if either X or Y is core compact. Conversely, if X is not core compact, then there exists some directed space Y such that $X \bigotimes Y \neq X \times Y$. There exist directed spaces that are not core compact: for example, the famous Johnstone space [13], which we will investigate more detailedly in Section 4. Therefore, the tensor product $X \bigotimes Y$ of topological spaces X, Y is strictly finer than the topological product $X \times Y$ in general. By the definition of directed topology, it is easy to check that $\mathcal{D}(X \times Y) = \mathcal{D}(\mathcal{D}X \times \mathcal{D}Y)$. Therefore, $\mathcal{D}(X \times Y) = \mathcal{D}X \bigotimes \mathcal{D}Y$, which is finer than $X \bigotimes Y$. Denote by $\mathbb{N}^\top$ the flat domain, i.e., the set of all natural numbers adding a top element $\top$, and $x \leq y$ in $\mathbb{N}^\top$ iff $x = y$ or $y = \top$. Then, consider topological space $Z = (\mathbb{N}^\top, \upsilon(\mathbb{N}^\top))$. It is easily seen that $\mathcal{D}Z = (\mathbb{N}^\top, A(\mathbb{N}^\top))$. Hence, $\{(\top, \top)\}$ is an open subset of $\mathcal{D}(Z \times Z)$. However, $\{(\top, \top)\}$ is not open in $Z \bigotimes Z$. Thus, the tensor product $X \bigotimes Y$ is strictly coarser than $\mathcal{D}(X \times Y)$ in general.

Now, we give some more equivalent conditions for a directed space to be core compact. We define $\Sigma 2$ to be the Sierpinski space, i.e., the set $\{0, 1\}$ endowed with the topology $\{\emptyset, \{1\}, \{0, 1\}\}$.

Lemma 3.6. [15] The topology of pointwise convergence on $[X \to \Sigma 2]_p$ is the upper topology, which corresponds to the upper topology on $\mathcal{O}(X)$.

Proposition 3.7. [7] If X is a space such that $\mathcal{O}(X)$ is a continuous lattice, and Y is an injective space, then $[X \to Y]_I$ is injective. In particular, the Isbell topology on $TOP(X, Y)$ is the Scott topology.

Theorem 3.8. Let X be a directed space. The following conditions are equivalent.

(1) $[X \to Y]$ is injective for all injective T_0 spaces Y.
(2) $[X \to \Sigma 2]$ is injective.
(3) $\mathcal{O}(X)$ is continuous.
(4) $\{(U, x) : x \in U\}$ is open in $\Sigma\mathcal{O}(X) \times X$.
(5) The evaluation map $ev : [X \to \Sigma 2] \times X \to \Sigma 2$ is continuous.
(6) For all directed spaces Y, $X \bigotimes Y = X \otimes Y = X \times Y$.
(7) For all directed spaces Y, Z, if a map $f : X \times Y \to Z$ is separately continuous, then it is jointly continuous.
(8) For any directed space Y, the evaluation map $[X \to Y] \times X \to Y$ is continuous.
(9) The natural map $[Z \times X \to Y] \to [Z \to [X \to Y]]$ is onto (and a homeomorphism) for all directed spaces Y and Z.

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Leftrightarrow$ (3). Since $[X \to \Sigma 2] = \mathcal{D}([X \to \Sigma 2]_p) = \Sigma\mathcal{O}(X)$, and an injective space is a continuous lattice endowed with the Scott topology, $\mathcal{O}(X)$ is continuous iff $[X \to \Sigma 2]$ is injective.

(3) $\Rightarrow$ (1). Assume that Y is an injective space. By Proposition 3.7, $[X \to Y]_I = \Sigma(TOP(X, Y))$ is an injective space. Thus, $TOP(X, Y)$ is a continuous lattice. Since the topology of pointwise convergence is coarser than the Isbell topology, $[X \to Y]_p$ is coarser than $\Sigma(TOP(X, Y))$, and then $[X \to Y] = \mathcal{D}([X \to Y]_p) = \Sigma(TOP(X, Y)) = [X \to Y]_I$.

(3) $\Leftrightarrow$ (4). By Theorem 3.4.

(4) $\Leftrightarrow$ (5). It is a direct conclusion by the fact that $[X \to \Sigma 2] = \Sigma\mathcal{O}(X)$.

(3) ⇔ (6). By Theorem 3.5 and Lemma 3.2.

(6) ⇒ (7). That f is separately continuous is equivalent to f being continuous from $X \bigotimes Y$ to Z. Thus f is jointly continuous.

(7) ⇒ (6). Given any directed space Y, let $Z = X \otimes Y$. Then, Z is also a directed space. The identity map $id : X \times Y \to X \otimes Y$ is separately continuous since $X \otimes Y = X \bigotimes Y$. Thus, $id : X \times Y \to X \otimes Y$ is continuous. Then, $X \times Y = X \otimes Y$.

(7) ⇒ (8) ⇒ (5). Since $ev : [X \to Y] \otimes X \to Y$ is continuous, i.e., $ev : [X \to Y] \times X \to Y$ is separately continuous, ev is continuous from $[X \to Y] \times X$. (8) ⇒ (5) is obvious.

(8) ⇒ (9). Since the natural map $[Z \otimes X \to Y] \to [Z \to [X \to Y]]$ is a homeomorphism for all directed spaces Y and Z (see [18, 25]), we need only to show (6). This has been proved.

(9) ⇒ (8). Let $Z = [X \to Y]$. Then, the inverse of identity map $id : [[X \to Y] \to [X \to Y]]$ is $ev : [[X \to Y] \times X \to Y]$. □

4. Core compactness vs the lattice of closed sets

It is well known that the spectrum with the hull-kernel topology of a completely distributive lattice (resp., a distributive hypercontinuous lattice) is exactly a continuous (resp., quasicontinuous) dcpo endowed with the Scott topology [8, 10, 16]. In this section, some conclusions for Scott spaces are extended to directed spaces. These conclusions are closely related to the long-standing open problem of which distributive continuous lattice's spectrum is exactly a sober locally compact Scott space (see [21, Problem 528]).

Given any poset P, $\upsilon(\Gamma(P)) = \sigma(\Gamma(P))$ is a necessary condition for ΣP to be core compact [3]. Deonte by $C(X)$ the lattice of closed subsets of a topological space X. We show that for any directed space X, X is core-compact iff $(C(X), \sigma(C(X)))$ is sober and locally compact with $\sigma(C(X)) = \upsilon(C(X))$.

Given a topological space X, we define $\prod^n X$ to be the topological product of n copies of X. For any $n \in \mathbb{N}$, define a map $s_n : \prod^n X \to \Sigma(C(X))$ as follows: $\forall (x_1, x_2, \ldots, x_n) \in \prod^n X$,

$$s_n(x_1, x_2, \ldots, x_n) = \downarrow\{x_1, x_2, \ldots, x_n\}.$$

Proposition 4.1. For a topological space X, $\sigma(C(X)) = \upsilon(C(X))$ holds iff s_n is continuous for all $n \in \mathbb{N}$.

Proof. Assume that $\sigma(C(X)) = \upsilon(C(X))$. Given any $F \in C(X)$,

$$s_n^{-1}(\downarrow_{C(X)} F) = \{\{x_1, x_2, \ldots, x_n\} \in \prod^n X : \overline{\{x_1, x_2, \ldots, x_n\}} \subseteq F\} = \prod^n F$$

is a closed subset of $\prod^n X$. Thus, $s_n : \prod^n X \to (C(X), \upsilon(C(X)))$ is continuous. Then, $s_n : \prod^n X \to \Sigma(C(X))$ is continuous.

For the converse, assume that s_n is continuous for all $n \in \mathbb{N}$. Let $\mathcal{U}$ be an open subset of $\Sigma(C(X))$, and $A \in \mathcal{U}$. Assume $A \neq \emptyset$. Note that since $A = \bigcup\{\downarrow F : F \subseteq_f A\}$, and $\{\downarrow F : F \subseteq_f A\}$ is a directed family in $C(X)$, there exists a non-empty finite subset F of A such that $\downarrow F \in \mathcal{U}$. Let $F = \{x_1, x_2, \ldots, x_n\}$, and then $s_n(x_1, x_2, \ldots, x_n) = \downarrow F \in \mathcal{U}$. It follows that $(x_1, x_2, \ldots, x_n) \in s_n^{-1}(\mathcal{U})$. By the continuity of s_n, there exists a family of open subsets $U_k (1 \leq k \leq n)$ such that $U_1 \times U_2 \times \cdots \times U_n$ is open in $\prod^n X$, and

$$(x_1, x_2, \ldots, x_n) \in U_1 \times U_2 \times \cdots \times U_n \subseteq s_n^{-1}(\mathcal{U}).$$

Since $x_k \in A$ for $1 \leq k \leq n$, we have $A \in \Diamond U_k = \{B \in C(X) : B \cap U_k \neq \emptyset\}$. It follows that $A \in \bigcap_{k=1}^{n} \Diamond U_k \in \upsilon(C(X))$. For any $B \in \bigcap_{k=1}^{n} \Diamond U_k$, there exists $y_k \in B \cap U_k$ for $1 \leq k \leq n$. Since $(y_1, y_2, \ldots, y_n) \in s_n^{-1}(\mathcal{U})$, we have $\bigcup_{k=1}^{n} \downarrow y_k \in \mathcal{U}$. It follows that $B \in \mathcal{U}$, i.e., $A \in \bigcap_{k=1}^{n} \Diamond U_k \subseteq \mathcal{U}$. Thus, $\sigma(C(X)) = \upsilon(C(X))$. □

Lemma 4.2. [8, 22] Let L be a complete lattice. L is a quasicontinuous lattice iff $\omega(L)$ is a continuous lattice.

Proposition 4.3. [3] Let L be a continuous lattice. If L satisfies the condition that $\upsilon(L^{op}) = \sigma(L^{op})$, then $(L^{op}, \sigma(L^{op}))$ is a sober and locally compact space.

Proposition 4.4. Let X be a directed space. If X is core compact, then $\sigma(C(X)) = \nu(C(X))$. Moreover, $(C(X), \sigma(C(X)))$ is sober and locally compact.

Proof. Let X be a core compact directed space. Then, for every $n \in \mathbb{N}$, $\mathcal{D}(\prod^{n} X) = \prod^{n} X$ by Theorem 3.5. Thus, s_n is continuous from $\prod^{n} X$ to $\Sigma(C(X))$ iff it is continuous from $\mathcal{D}(\prod^{n} X)$ to $\Sigma(C(X))$.

We show that $s_n : \mathcal{D}(\prod^{n} X) \to \Sigma(C(X))$ is continuous, i.e., s_n preserves $D \to x$ for every $(D, x) \in DLim(\prod^{n} X)$. Let $\{(x_{1i}, x_{2i}, \ldots, x_{ni}) : i \in I\}$ be a directed subset of $\prod^{n} X$ converging to $(x_1, x_2, \ldots, x_n)$ in $\prod^{n} X$. Then, for each $1 \leq k \leq n$, $\{(x_{ki}) : i \in I\}$ converges to x_k by the definition of the topological product. We have

$$s_n((x_1, x_2, \ldots, x_n)) = \bigcup_{k=1}^{n} \downarrow x_k \subseteq \overline{\bigcup_{k=1}^{n} \bigcup_{i \in I} \downarrow x_{ki}} = \overline{\bigcup_{i \in I} \bigcup_{k=1}^{n} \downarrow x_{ki}} = \bigvee_{i \in I} s_n(x_{1i}, x_{2i}, \ldots, x_{ni}).$$

Thus, s_n is a continuous map from $\mathcal{D}(\prod^{n} X)$ into $\Sigma(C(X))$.

By Proposition 4.1, we have $\sigma(C(X)) = \nu(C(X))$. Letting $L = O(X)$, L is a continuous lattice, and $C(X) = L^{op}$. $(C(X), \sigma(C(X)))$ is sober and locally compact by Proposition 4.3. □

In [3], an adjunction between $\sigma(P)$ and $\sigma(\Gamma(P))$ serves as a useful tool in studying the relation between P and $\Gamma(P)$. It can be extended to directed spaces as well. Given two posets P, Q, P is called a retract of Q if there is a pair of Scott continuous maps $f : P \to Q$ and $g : Q \to P$ such that $g \circ f = 1_P$.

Definition 4.5. Given a directed space X, we define a map $\eta : X \to \Sigma C(X)$ and a map $\Diamond : O(X) \to \sigma(C(X))$ as follows: $\forall x \in X$, $\forall U \in O(X)$,

$$\eta(x) = \downarrow x, \ \Diamond(U) = \{A \in C(X) : A \cap U \neq \emptyset\}.$$

Define $\eta^{-1} : \sigma(C(X)) \to O(X)$ as $\eta^{-1}(\mathcal{U}) = \{x \in X : \downarrow x \in \mathcal{U}\}$.

Then, we have the following result.

Proposition 4.6. For a directed space X, both η^{-1} and $\Diamond$ preserve arbitrary sups. Moreover, $\Diamond \circ \eta^{-1} \leq 1_{\sigma(C(X))}$, and $\eta^{-1} \circ \Diamond = 1_{O(X)}$. Thus, $(\eta^{-1}, \diamond)$ forms a pair of adjunction. $O(X)$ is a retract of $\sigma(C(X))$.

Proof. Let $\{U_i : i \in I\}$ be any subset of $\sigma(C(X))$. Then, $\Diamond(\bigcup_{i\in I} U_i) = \{A \in C(X) : A \cap \bigcup_{i\in I} U_i \neq \emptyset\} = \{A \in C(X) : \exists i \in I, A \cap U_i \neq \emptyset\} = \bigvee_{i\in I} \Diamond(U_i)$. η is the special case of s_n for $n = 1$. Thus, it is continuous. Then, η^{-1} preserves arbitrary sups. Given any $U \in \mathcal{O}(X)$, $x \in \eta^{-1}(\Diamond(U)) \Leftrightarrow \eta(x) \in \Diamond(U) \Leftrightarrow \downarrow x \cap U \neq \emptyset \Leftrightarrow x \in U$; hence, $\eta^{-1} \circ \Diamond = 1_{\mathcal{O}(X)}$. For any $\mathcal{U} \in \sigma(C(X))$, $A \in \Diamond \circ \eta^{-1}(\mathcal{U}) \Leftrightarrow A \cap \eta^{-1}(\mathcal{U}) \neq \emptyset \Rightarrow A \in \mathcal{U}$, i.e., $\Diamond \circ \eta^{-1} \leq 1_{\sigma(C(X))}$. □

Lemma 4.7. [1] A retract of a continuous domain is continuous.

Theorem 4.8. Let X be a directed space. Then, X is core-compact iff $(C(X), \sigma(C(X)))$ is core compact iff $(C(X), \sigma(C(X)))$ is sober and locally compact with $\sigma(C(X)) = \upsilon(C(X))$.

Proof. Suppose that X is core compact. By Proposition 4.4, $(C(X), \sigma(C(X)))$ is sober and locally compact with $\sigma(C(X)) = \upsilon(C(X))$. Conversely, suppose that $(C(X), \sigma(C(X)))$ is core compact, i.e., $\sigma(C(X))$ is continuous. By Proposition 4.6 and Lemma 4.7, $\mathcal{O}(X)$ is continuous, i.e., X is core compact. □

In [3], the example $Y = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$ of the real line $\mathbb{R}$, endowed with the subspace topology, is a core compact space, but $(C(Y), \sigma(C(Y)))$ is not core compact. Given any space X, considering $\eta' : X \to \upsilon(C(X))$, $\eta'(x) = \downarrow x$, η is continuous. $(\eta'^{-1}, \diamond)$ forms an adjunction between $\mathcal{O}(X)$ and $\upsilon(C(X))$, and $\mathcal{O}(X)$ is a retract of $\upsilon(C(X))$. Thus, $\upsilon(C(X))$ is core compact iff X is core compact by Lemma 4.2 and Lemma 4.7. We have the following question.

Problem 4.9. Let X be a T_0 space and $\Sigma C(X)$ be core compact. Must X be core compact? Equivalently, must $\upsilon(C(X)) = \sigma(C(X))$?

The adjunction $(\eta^{-1}, \Diamond)$ seems only to hold for directed spaces. A natural question that arises is whether a topological space X that makes the map η in Definition 4.5 continuous is a directed space? When L is a complete lattice endowed with a topology coarser than $\sigma(L)$, the answer is positive. However, for other cases, we still do not know the answer.

Lemma 4.10. Any retract of a directed space is a directed space.

Proof. Let X be a topological space and Y be a directed space. Suppose that $i : X \to Y$ and $r : Y \to X$ are continuous maps, and $r \circ i = id_X$. We need only to check that any directed open subset U of X is open in X. Noticing that $U = (r \circ i)^{-1}(U) = i^{-1}(r^{-1}(U))$, we need only to show that $r^{-1}(U)$ is open in Y. Given any directed subset D of Y and $D \to y \in r^{-1}(U)$, $r(D) \to r(y) \in U$. There exists some $d \in D$ such that $r(d) \in U$, i.e., there exists some $d \in r^{-1}(U)$. Thus, $r^{-1}(U)$ is open in Y. □

Proposition 4.11. Let L be a complete lattice, and $X = (L, \tau)$ with $\tau \subsetneq \sigma(L)$. If $\eta : X \to \Sigma(C(X))$ is continuous, then X is a Scott space.

Proof. Since L is a complete lattice, we have $\downarrow(\inf A) = \bigcap_{x_i\in A} \downarrow x_i$, that is, $\eta : L \to C(X)$ preserves all infs. Then, there exists a right adjoint $d : C(X) \to L$ such that $d(F) = \inf \eta^{-1}(\uparrow_{C(X)} F) = \sup F$. Thus, $(\eta, \sup)$ forms a pair of adjunction, and $\sup : C(X) \to L$ preserves all sups. Then, the map $\sup : \Sigma(C(X)) \to X$ is continuous. It is easy to check that $\sup \circ \eta = id_X$. Thus, X is a retract of $\Sigma(C(X))$ and a directed space by Lemma 4.10. Then, X is a Scott space since the Scott topology is the coarsest topology on L such that it is a directed space. □

By Theorem 4.8, if the spectrum space of a distributive continuous lattice L is a directed space, then $\sigma(L^{op}) = \upsilon(L^{op})$ must hold. Particularly, the reverse holds when L is algebraic [3, 5]. So, we emphasize the following open question:

Problem 4.12. Is the hull-kernel topology of the spectrum Spec$L$ equal to the Scott topology when L is a distributive continuous lattice with $\sigma(L^{op}) = \upsilon(L^{op})$?

Equivalently, let X be a sober and core compact space with $\upsilon(C(X)) = \sigma(C(X))$. Is X a directed space?

Another related problem is the following:

Problem 4.13. Is the soberification of a core compact directed space a directed space (Scott space)?

Obviously, so Problem 4.13 must be if Problem 4.12 is affirmative. There exists a non-continuous spatial complete lattice L with $\sigma(L^{op}) = \upsilon(L^{op})$, but its spectrum is not a Scott space.

Example 4.14. Let $\mathbb{J}$ be the classical non-sober dcpo given by Johnstone [13]. Define $\mathbb{J} = \mathbb{N}\times(\mathbb{N}\cup\{\omega\})$. Given any two element $(m_1, n_1), (m_2, n_2)$ of $\mathbb{J}$, define $(m_1, n_1) \leq (m_2, n_2)$ iff one of the following two conditions holds: (i) $m_1 = m_2$; $n_1 \leq n_2$ in $\mathbb{N}$ or $n_2 = \omega$. (ii) $n_2 = \omega$, $n_1 \leq m_2$.

It satisfies that $\sigma(\Gamma(\mathbb{J})) = \upsilon(\Gamma(\mathbb{J}))$. Set $L = \sigma(\mathbb{J})$. The spectrum of L is created by adding a top element to $\mathbb{J}$, i.e., Spec$L = \mathbb{J} \cup \{\top\}$, which is not sober with its Scott topology. Hence, the hull-kernel topology of Spec$L$ is not equal to the Scott topology. This is also an example that the soberification of a directed space is not a directed space.

(1) $\sigma(\Gamma(\mathbb{J})) = \nu(\Gamma(\mathbb{J}))$. Given any closed subset $\mathcal{A}$ of $\Sigma(\Gamma(\mathbb{J}))$, $\bigcup\mathcal{A}$ is a lower subset of $\mathbb{J}$. We show that $\bigcup\mathcal{A}$ is closed in $\Sigma\mathbb{J}$. Given any directed subset D in $\bigcup\mathcal{A}$, either D contains a largest element x of D or is cofinal with one chain $\{m\}\times\mathbb{N}$ of $\mathbb{J}$ and has a maximal element (m, ω) of $\mathbb{J}$ as the supremum. For the first case, $\sup D = x \in \bigcup\mathcal{A}$; for the second case, since $\mathcal{A}$ is a Scott closed subset of $\Gamma(\mathbb{J})$, $\{\downarrow d : d \in D\} \subseteq \mathcal{A}$. Then, $\downarrow(m, \omega)$ must be in $\mathcal{A}$, and $(m, \omega) \in \bigcup\mathcal{A}$. Thus, $\bigcup\mathcal{A}$ is Scott closed. Then, let $A = \eta^{-1}(\mathcal{A}) = \{x \in \mathbb{J} : \downarrow x \in \mathcal{A}\} = \bigcup\mathcal{A}$. A is closed in $\Sigma(\mathbb{J})$. If A contains infinite maximal points of $\mathbb{J}$, then for each $(m, \omega) \in A$, let $B_m = \mathbb{N}\times\{1, 2, \ldots, m\}$. Then, $\mathcal{B} = \{B_m : (m, \omega) \in A\} \subseteq \mathcal{A}$ forms a directed subset of $\Gamma(\mathbb{J})$, and $\bigvee\mathcal{B} = \overline{\bigcup\mathcal{B}} = \mathbb{J}$. Thus, $\mathbb{J} \in \mathcal{A}$, $\mathcal{A} = \Gamma(\mathbb{J})$.

Now, we consider that A contains only finite maximal points of $\mathbb{J}$. It is easy to see that the topology on A inherent from $\Sigma\mathbb{J}$ is equal to the Scott topology. Then, given any open subset U of ΣA and $x \in U$, there must be a compact open subset K such that $x \in K \subseteq U$. We need only to let $K = \uparrow x \cup \uparrow \{x_1, \ldots, x_m\}$ in A, where $x_i (1 \leq i \leq m)$ is a picked element that is lower than each maximal element $(n_i, \omega) \in \uparrow x \cap U \cap A$. Then, K is compact and open. So, $(A, \sigma(A))$ is a locally compact space and hence a core compact space. By Proposition 4.4, $\upsilon(\Gamma(A)) = \sigma(\Gamma(A))$. Since $\mathcal{A}$ is a closed subset of $\Sigma(\Gamma(\mathbb{J}))$, $\mathcal{A}$ is closed in $\Sigma(\Gamma(A))$ and then closed in $(\Gamma(A), \upsilon(\Gamma(A))$. Thus, $\mathcal{A}$ can be considered as an intersection of a family of finitely generated lower sets in $\Gamma(A)$ and then also an intersection of a family of finitely generated lower sets in $\Gamma(\mathbb{J})$. Thus, $\sigma(\Gamma(\mathbb{J})) = \nu(\Gamma(\mathbb{J}))$.

(2) Spec$L = \mathbb{J} \cup \{\top\}$. It is easy to see that a closed subset of $\Sigma\mathbb{J}$ either contains finite maximal elements, or is equal to the whole space. For the first case, it is not irreducible. For the second case, it is irreducible, since any closed subset that contains infinite maximal points of $\mathbb{J}$ must be equal to $\mathbb{J}$. Thus, the only non-trivial irreducible closed subset of $\Sigma\mathbb{J}$ is $\mathbb{J}$. Then, Spec$L$ is order isomorphic to $\mathbb{J} \cup \{\top\}$.

(3) The hull-kernel topology of Spec$L$ is not equal to the Scott topology. By definition, a nonempty set U is an open set of the hull-kernel topology iff $U = V \cup \{\top\}$, where V is a non-empty open set of $\Sigma\mathbb{J}$. There is no directed subset of $\mathbb{J}$ in Spec$L$ whose supremum is $\top$. Thus, $\{\top\}$ is Scott open in Spec$L$. So, the two topologies are not equal.

Theorem 4.15. [3, 5] Let L be a continuous lattice. Consider the following conditions:

(1) $\sigma(L) = \upsilon(L)$,
(2) $\sigma(L^{op}) = \upsilon(L^{op})$,
(3) every upper set closed in the dual Scott topology $\sigma(L^{op})$ is compact in the Scott $\sigma(L)$, and
(4) the hull-kernel topology of the spectrum Spec$L$ is equal to its Scott topology.

Then, (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). When L is distributive, one has (4) $\Rightarrow$ (2). Additionally, if L is distributive and algebraic, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

In Theorem 4.15, condition (1) is equivalent to L being hypercontinuous. J. Lawson gave an important example of a meet-continuous non-continuous lattice W such that the Scott topology $\sigma(W)$ is continuous (see [7, Theorem VI-4.5]). Let $L = \sigma(W)$. Then, $\sigma(L^{op}) = \upsilon(L^{op})$. However, since W is not quasicontinuous, it follows that $\sigma(L) \neq \upsilon(L)$. Thus, for general continuous lattices, the condition (1) is strictly stronger than condition (2).

A complete lattice L is said to be lean if condition (3) of Theorem 4.15 holds [11]. In the end, we give an equivalent condition for $\sigma(L^{op}) = \upsilon(L^{op})$.

Lemma 4.16. [23] Let L be a complete lattice, $F \subseteq L$. F is closed in $(L, \sigma(L))$ iff it is compact saturated in $(L^{op}, \upsilon(L^{op}))$.

Lemma 4.17. [3, 24] Let L be a complete lattice. Then, $(L, \upsilon(L))$ is sober, and $(L, \sigma(L))$ is well-filtered.

Lemma 4.18. [17] A topological space X is core compact and well-filtered iff X is locally compact and sober.

Theorem 4.19. For a continuous lattice L, the following two conditions are equivalent to each other:

(1) $\sigma(L^{op}) = \upsilon(L^{op})$, i.e., L is lean;
(2) L^{op} is lean.

Proof. (1) $\Rightarrow$ (2). Given any closed subset F of $(L, \sigma(L))$, it is a compact saturated subset of $(L^{op}, \upsilon(L^{op}))$ by Lemma 4.16. Thus, F is compact saturated in $(L^{op}, \sigma(L^{op}))$, i.e., L^{op} is lean.

(2) $\Rightarrow$ (1). Assume that L^{op} is lean. For any space X, denote by $Q(X)$ the set of nonempty compact saturated subsets of X with the reverse inclusion order. Given any closed subset F of $(L, \sigma(L))$, it is compact saturated in $(L^{op}, \sigma(L^{op}))$. By Lemma 4.16 and Lemma 4.2, $Q(\Sigma L^{op}) = Q((L^{op}, \upsilon(L^{op})))$, and $\upsilon(L^{op}) = \omega(L)$ is continuous. Thus, $(L^{op}, \upsilon(L^{op}))$ is core compact. Since L^{op} is a complete lattice, $(L^{op}, \upsilon(L^{op}))$ is locally compact and sober by Lemma 4.17 and Lemma 4.18.

We claim that ΣL^{op} is core compact. Define $\eta : L^{op} \to Q(\Sigma L^{op})$, $\eta(a) = \uparrow a$, and $\Box : \sigma(L^{op}) \to \sigma(Q(\Sigma L^{op}))$, $\Box(U) = \{K \in Q(\Sigma L^{op}) : K \subseteq U\}$. It is easy to see that η is Scott continuous. η^{-1} preserves arbitrary sups. L^{op} is a complete lattice, so ΣL^{op} is well-filtered. Then, $\Box$ is well defined and Scott continuous. $\eta^{-1} \circ \Box\,(U) = \eta^{-1}(\{K \in Q(\Sigma L^{op}) : K \subseteq U\}) = U$. Thus, $\eta^{-1} \circ \Box = id_{\sigma(L^{op})}$, $\sigma(L^{op})$ is a retract of $\sigma(Q(\Sigma L^{op}))$. Then, $\sigma(L^{op})$ is continuous, i.e., ΣL^{op} is core compact. So, it is locally compact and sober by Lemma 4.18.

By the Hofmann-Mislove Theorem [7, Theorem II-2.14], OFlit($Q(X)$) is isomorphic to $O(X)$ for any locally compact sober space X under the maps g : OFilt($Q(X)$) $\to O(X)$, $g(\mathcal{F}) = \cup\mathcal{F}$ and $f : O(X) \to$ OFilt($Q(X)$), $f(U) = \Box U$. Since $(L^{op}, \sigma(L^{op}))$ and $(L^{op}, \upsilon(L^{op}))$ are both locally compact sober, and they have the same compact saturated subsets, $(L^{op}, \sigma(L^{op}))$ is equal to $(L^{op}, \upsilon(L^{op}))$. □

5. Conclusions

Directed spaces are natural topological extensions of dcpos in domain theory. We showed that for directed spaces, the tensor products are equal to the categorical products and gave out a series of characterizations of core compactness of directed spaces. Some special properties of Scott spaces can be extended to directed spaces. For example, the upper topology and the Scott topology on the lattice of closed subsets of a core compact directed space coincide. We showed that the example $L = \sigma(\mathbb{J})$ is a non-continuous spatical complete lattice with $\sigma(L^{op}) = \upsilon(L^{op})$, but its spectrum is not a Scott space. These results can help us understand more about the long-standing open problem of which distributive lattice's spectrum is a sober locally compact Scott space. The answers of Problem 4.12 and Problem 4.13 are still unknown. It is also interesting to investigate if these results, like Proposition 4.4 and Proposition 4.6, only hold for directed spaces.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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