Mathematics

Research article

On dual surfaces in Galilean 3-space

Nural Yüksel[*]

Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey

* **Correspondence:** Email: yukseln@erciyes.edu.tr.

Abstract: In the present paper, we describe dual translation surfaces in the Galilean 3-space having the constant Gaussian and mean curvatures as well as Weingarten and linear Weingarten dual translation surfaces. We also study dual translation surfaces in $\mathbb{G}_3$ under the condition $\Delta^{II} r = \lambda_i r_i$, where $\lambda_i \in R$ and Δ^{II} denotes the Laplacian operator respect to the second fundamental form.

1. Introduction

A surface that arises when a curve $\alpha(u)$ is translated over another curve $\beta(v)$ is called a translation surface. A translation surface is defined to be the sum of two curves $\alpha(u)$ and $\beta(v)$, which are called generating curves. Therefore, translation surfaces are made up of quadrilateral, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures. A translation surface in Euclidean 3-space by two curves in orthogonal planes, is given in [2, 3] as a graph of the function $z(u, v) = f(u) + g(v)$, such that $f(u)$ and $g(v)$ are smooth functions defined on an interval of $\mathbb{R}$.

The translation surfaces in $\mathbb{E}^3$ studied by H. F. Scherk [1] are obtained as a graph of the sum $z(u, v) = f(u) + g(v)$, and where he proved that the only minimal translation surfaces are the those denoted by

$$z(u, v) = \frac{1}{a} \log \left| \frac{\cos(au)}{\cos(av)} \right| = \frac{1}{a} \log |\cos(au)| - \frac{1}{a} \log |\cos(av)| \,,$$

where a is a non-zero constant. These surfaces [12, 13] are nowadays said to be Scherk's minimal surfaces .

Constant mean curvature and constant Gaussian curvature surfaces have been the main problem for the geometers for a long time. Translation surfaces with constant mean curvature, in particular,

zero mean curvature, in the Euclidean space and the Minkowski space are stated in [7]. Lone and Karacan yield categorization of dual translation surface, which has constant dual isotropic mean curvature or constant dual isotropic Gaussian curvature [8]. Dillen, Goemans and Woestyne obtained a categorization of translation surfaces in the 3-dimensional Euclidean and Minkowski space such that the Weingarten circumstance is satisfying [5].

Sipus and Divjak [11] explained translation surfaces in the Galilean space having constant Gaussian and mean curvatures together with translation Weingarten surfaces. Yoon searched coordinate finite type translation surfaces in $\mathbb{G}_3$ and explained some examples of new classes of translation surfaces [12]. He also studied a translation surface in $\mathbb{G}_3$ with a log-linear density and classified such a surface with vanishing weighted mean curvature [13].

LN-surfaces, which were taken into account in [6] with enough elasticity to model smooth surfaces. Peternell and Odehnal generalized the idea of LN-surfaces to $\mathbb{R}^4$ [9]. Bulca calculated the Gaussian, normal and mean curvatures of LN- surfaces in $\mathbb{E}^4$. Further, she pointed out the flat and minimal points of the surfaces in [3]. Sampoli, Peternell and Jüttler indicate that even the convolution surface of an LN-surface and any rational surface admits rational parametrization [10]. Moreover, LN and LCN translation surfaces have been characterized in 3-dimensional affine and Euclidean spaces in [14, 15].

In this study, the dual translation surfaces defined by $z = f(u) + g(v)$, are considered in the three dimensional Galilean space $\mathbb{G}_3$.

2. Preliminaries

We emphasize that the Galilean space $\mathbb{G}_3$ is a Cayley-Klein space associated with a 3-dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure. The reader is referred to [11, 12] for more details of

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1| & , \ if \ \ x_1 \neq x_2 \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2} & , \ if \ \ x_1 = x_2. \end{cases} \tag{2.1}$$

The group of motions of G_3 is a six-parameter group given in affine coordinates by

$$\begin{aligned} \overline{x} &= a + x, \\ \overline{y} &= b + cx + y\cos\theta + z\sin\theta, \\ \overline{z} &= d + ex - y\sin\theta + z\cos\theta, \end{aligned}$$

with respect to the absolute figure, there are two types of lines in the Galilean space - isotropic lines which intersect the absolute line f and non-isotropic lines which do not. A plane is called Euclidean if it includes f, whereas it is said isotropic. In the given affine coordinates, isotropic vectors are of the form $(0, y, z)$, but Euclidean planes are of the form $x = k, k \in \mathbb{R}$. The induced geometry of a Euclidean plane is Euclidean, and of an isotropic plane isotropic (i.e., 2-dimensional Galilean or flag-geometry).

A C^r-surface S, $r \geq 1$, immersed in the Galilean space, $\Psi : U \rightarrow \mathbb{G}_3, U \subset \mathbb{R}^2$, $\Psi(u, v) = (x(u, v), y(u, v), z(u, v))$ has the following first fundamental form

$$I = (g_1 du + g_2 dv)^2 + \epsilon \left(h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2 \right),$$

where the symbols $g_i = x_i$, $h_{ij} = \tilde{\Psi}_i.\tilde{\Psi}_j$ stand for derivatives of the first coordinate function $x(u, v)$ with respect to u, v and for the Euclidean scalar product of the projections $\tilde{\Psi}_k$ of vectors Ψ_k onto the

yz-plane, respectively. Furthermore,

$$\epsilon = \begin{cases} 0\,, & \text{if direction } du : dv \text{ is non-isotropic,} \\ 1\,, & \text{if direction } du : dv \text{ is isotropic.} \end{cases}$$

At every point of a surface, there exists a unique isotropic direction defined by $g_1 du + g_2 dv = 0$. In that direction, the arc length is measured by

$$\begin{aligned} ds^2 &= h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2 \\ &= \frac{h_{11} g_2^2 - 2h_{12} g_1 g_2 + h_{22} g_1^2}{g_1^2} \\ &= \frac{W^2}{g_1^2} dv^2, \end{aligned}$$

where $g_1 \neq 0$. A surface is called admissible if it has no Euclidean tangent planes. Therefore, for an admissible surface either $g_1 \neq 0$ or $g_2 \neq 0$ holds. The Gaussian curvature K and mean curvature H are C^{r-2} functions, $r \geq 2$, defined by

$$K = \frac{LN - M^2}{W^2}, \ H = \frac{g_2^2 L - 2g_1 g_2 M + g_1^2 N}{2W^2},$$

where

$$L_{ij} = \frac{x_1 \Psi_{ij} - x_{ij} \Psi_1}{x_1} U, \ x_1 = g_1 \neq 0.$$

We will use L_{ij}, $i, j = 1, 2$, for L, M, N. The vector U defines a normal vector to a surface and given by

$$U = \frac{1}{W} \left(0, x_1 z_2 - x_2 z_1, x_2 y_1 - x_1 y_2 \right),$$

where $W^2 = (x_2 \Psi_1 - x_1 \Psi_2)^2$ [11]. It is well known that in terms of local coordinates $\{u, v\}$ of S, the Laplacian operator Δ^{II} according to the second fundamental form on S is defined by [4]

$$\Delta^{II} \Psi = -\frac{1}{\sqrt{LN - M^2}} \left[\frac{\partial}{\partial u} \left(\frac{N\Psi_u - M\Psi_v}{\sqrt{LN - M^2}} \right) - \frac{\partial}{\partial v} \left(\frac{M\Psi_u - L\Psi_v}{\sqrt{LN - M^2}} \right) \right]. \tag{2.2}$$

Consider a surface in G_3 as the graph of a function $z = h(u, v)$. We will consider translation surfaces of Type 1 that are obtained by translating two planar curves. Translation surfaces of Type 1 in the Galilean 3-space can be locally represented by

$$z = f(u) + g(v), \tag{2.3}$$

which yields the parametrization

$$\Psi(u, v) = (u, v, f(u) + g(v)). \tag{2.4}$$

One translated curve is a non-isotropic curve in the plane $y = 0$ and given as

$$\alpha(u) = (u, 0, f(u)),$$

and the other is an isotropic curve in the plane $x = 0$ and it is expressed as

$$\beta(v) = (0, v, g(v)).$$

Let S be a Monge patch parametrized by $z = h(u, v)$, then the surface

$$\Psi^* = (-z_u, -z_v, -z + uz_u + vz_v), \tag{2.5}$$

is called a dual surface of S^* in $\mathbb{G}_3$ and the normal vector of S^* is given by

$$U^* = \left(\pm \frac{\left(z_{uu}z_{vv} - (z_{uv})^2 \right)}{W} \right)(0, v, 1). \tag{2.6}$$

The Gaussian curvature of the dual surface is

$$K^* = \frac{1}{\left(z_{uu}z_{vv} - (z_{uv})^2 \right)\left(1 + v^2 \right)^2}. \tag{2.7}$$

Definition 1. *A surface S in $\mathbb{G}_3$ is said to be Weingarten surface if and only if there exists a non-trivial functional relation $\Phi(K, H) = 0$ between the Gaussian curvature and mean curvature of S. The condition $\Phi(K, H) = 0$ is equivalent to the vanishing of the corresponding Jacobian determinant, i.e., $\left| \frac{\partial(K,H)}{\partial(u,v)} \right| = 0$. A linear Weingarten surface in Galilean 3-space is a surface where there exists a relation $aK + bH = c$ for some non-zero real numbers a, b, c [5, 10].*

In [11], Sipus and Divjak analyzed the translation surface of Type 1, which has the constant Gaussian and mean curvatures in the Galilean 3-space $\mathbb{G}_3$ and they showed the following theorems:

Theorem 1. *If S is a translation surface of Type 1 of constant Gaussian curvature in the Galilean 3-space, then S is congruent to a particular surface with*

$$f(x) = ax^2 + bx + c,$$

$a, b, c \in \mathbb{R}$, and

$$
\begin{aligned}
k(u) &= \frac{1}{2}Au^2 + Bu + C, \\
h(u) &= \frac{Au + B}{A}\sqrt{1 - (Au + B)^2} + \frac{1}{2A}\arcsin(Au + B) + C_1
\end{aligned}
$$

for $A, B, C, C_1 \in \mathbb{R}$, where $K = f''(x)k''(u)$ and $h'^2 + k'^2 = 1$.

Theorem 2. *If S is a translation surface of Type 1 of constant mean curvature $H \neq 0$ in the Galilean 3-space, then S is congruent to a surface*

$$z = f(u) - \frac{1}{2H}\sqrt{1 - (2Hv + c_1)^2} + c_2,$$

where $c_1, c_2 \in \mathbb{R}$.

3. Dual translation surfaces in $\mathbb{G}_3$

The main aim of this section is to give a classification of dual translation surfaces with constant Gaussian and mean curvatures in the Galilean 3-space. Let S^* be a dual translation surface. Using (2.3) and (2.5), we can define dual translation surfaces as

$$\Psi^*(u, v) = \left(-f'(u), -g'(v), uf'(u) + vg'(v) - f(u) - g(v)\right). \tag{3.1}$$

The unit normal vector field U^* of the surface (3.1) is

$$U^* = -\frac{1}{\sqrt{1 + v^2}}(0, v, 1). \tag{3.2}$$

By a straightforward computation, the Gaussian and mean curvatures of the dual translation surface are given by

$$K^* = \frac{1}{f'' g'' \left(1 + v^2\right)^2}, \qquad H^* = -\frac{1}{2\left(1 + v^2\right)^{\frac{3}{2}} g''}, \tag{3.3}$$

where $f'', g'' \neq 0$. Thus we have the following result.

Corollary 1. *If S^* is a dual translation surface in the Galilean 3-space, then there are no flat and minimal dual translation surfaces.*

Proof. The proof is obvious by Eq (3.3). $\qquad\qquad\square$

Theorem 3. *Let S^* be a dual translation surface which has constant Gaussian curvature in $\mathbb{G}_3$. Then it is congruent to an open part of the surface (3.1) with*

$$z = f(u) + \left(c_1 + c_2 v + \frac{c_3}{2} v \arctan v\right).$$

Proof. Dual translation surface with constant curvature K_0^* is described by the ordinary differential equation

$$K_0^* f'' g'' \left(1 + v^2\right)^2 - 1 = 0, \ K_0^* \in \mathbb{R}. \tag{3.4}$$

By partial differentiation of the previous expression with respect to v, we get

$$f'' \left(4v\left(1 + v^2\right) g'' + \left(1 + v^2\right)^2 g'''\right) = 0.$$

Therefore either $f'' = 0$ or

$$4v\left(1 + v^2\right) g'' + \left(1 + v^2\right)^2 g''' = 0. \tag{3.5}$$

If $f'' = 0$, then we have a contradiction. The general solution of (3.5) is given by

$$g(v) = c_1 + c_2 v + \frac{c_3}{2} v \arctan v,$$

where $c_i \in \mathbb{R}$. Therefore, S is determined by

$$z = f(u) + \left(c_1 + c_2 v + \frac{c_3}{2} v \arctan v\right), \tag{3.6}$$
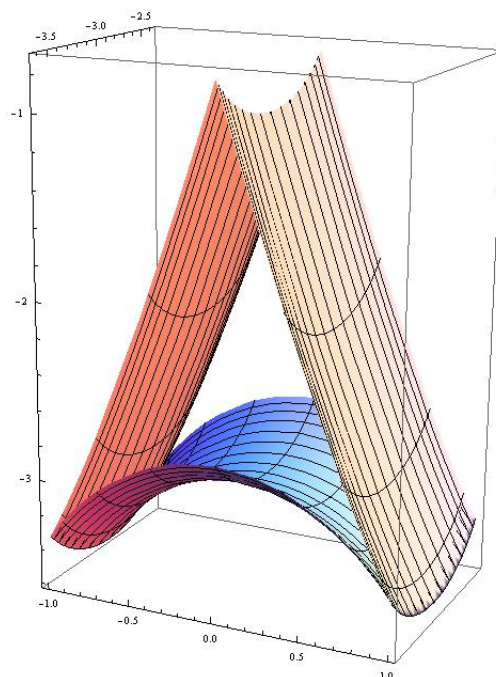
where $f'' \neq 0$ (see Figure 1).

Figure 1. $z = \cos u + \left(c_1 + c_2 v + \frac{c_3}{2} v \arctan v \right)$.

$\square$

Theorem 4. *Let S^* be a dual translation surface which has constant mean curvature in $\mathbb{G}_3$. Then it is congruent to an open part of the surface (3.1) with*

$$z = f(u) + \left(c_1 + c_2 v - \frac{\sqrt{1 + v^2}}{2H_0^*} \right).$$

Proof. We suppose that dual translation surface has constant mean curvature. Then we obtain

$$2 \left(1 + v^2 \right)^{\frac{3}{2}} g'' H_0^* - 1 = 0, \ H_0^* \in \mathbb{R}. \tag{3.7}$$

After solving (3.7), we get

$$g(v) = c_1 + c_2 v - \frac{\sqrt{1 + v^2}}{2H_0^*}.$$

Therefore S is determined by

$$z = f(u) + \left(c_1 + c_2 v - \frac{\sqrt{1 + v^2}}{2H_0^*} \right), \tag{3.8}$$
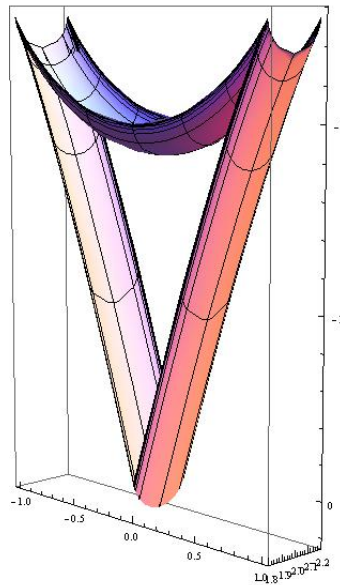
where $c_i \in \mathbb{R}$ (see Figure 2).

Figure 2. $z = \cos u + \left(c_1 + c_2 v - \frac{\sqrt{1+v^2}}{2H_0^*}\right)$.

□

Theorem 5. *Let S^* be a Weingarten dual translation surface in $\mathbb{G}_3$. Then it is congruent to an open part of the surface (3.1) with*

$$z = \left(c_1 + c_2 u + c_3 u^2\right) + g(v),$$

or

$$z = f(u) + \left(c_4 + c_5 v + c_6 \sqrt{1 + v^2}\right).$$

Proof. The condition $\frac{\partial(K^*, H^*)}{\partial(u,v)} = 0$ must be satisfied for the Weingarten surface S^*, it leads to $K_u^* H_v^* - K_v^* H_u^* = 0$. Since the mean curvature H_v^* is a function of v only, the previous equation reduces $K_u^* H_v^* = 0$. Therefore, either $K_u^* = 0$ or $H_v^* = 0$. The first condition $K_u^* = 0$ describes that a dual translation surface satisfies

$$-\frac{f'''}{(1 + v^2)^2 \, f''^2 g''} = 0.$$

Therefore $f''' = 0$ which implies

$$f(u) = c_1 + c_2 u + c_3 u^2.$$

If the second condition $H_v^* = 0$, then we have

$$\frac{3vg'' + \left(1 + v^2\right) g'''}{2 \left(1 + v^2\right)^{\frac{5}{2}} g''^2} = 0,$$

and

$$g(v) = c_4 + c_5 v + c_6 \sqrt{1 + v^2},$$

where $c_i \in \mathbb{R}$. Thus S is determined by either

$$z = \left(c_1 + c_2 u + c_3 u^2\right) + g(v), \tag{3.9}$$

see Figure 3, or

$$z = f(u) + \left(c_4 + c_5 v + c_6 \sqrt{1 + v^2}\right), \tag{3.10}$$

see Figure 4.
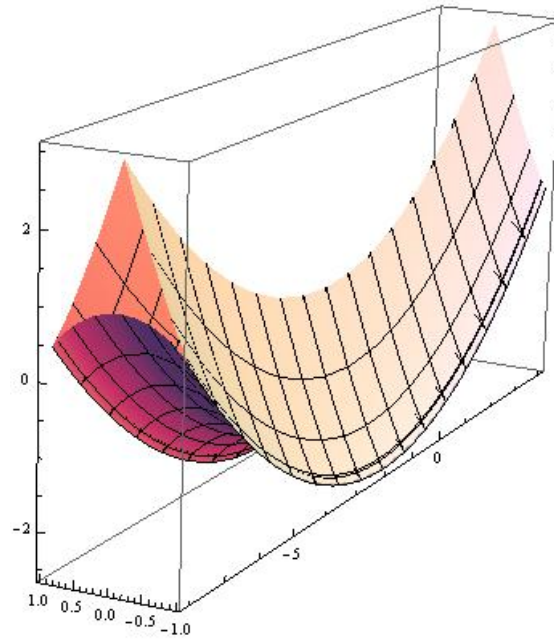
Figure 3. $z = \cos v + \left(c_1 + c_2 u + c_3 u^2\right)$.
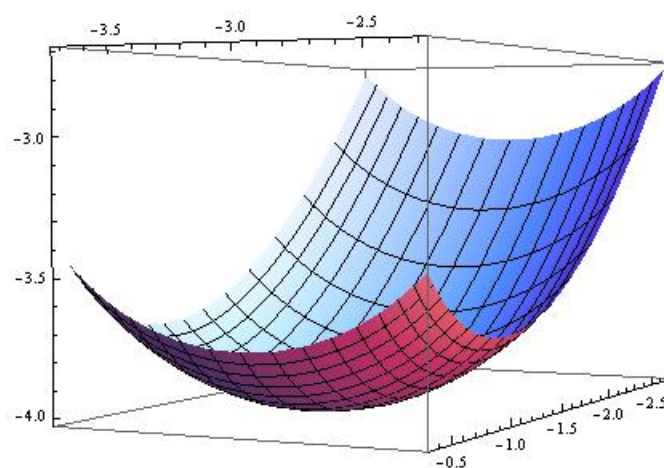
Figure 4. $z = e^u + \left(c_4 + c_5 v + c_6 \sqrt{1 + v^2}\right)$.

$\square$

Theorem 6. *Let S^* be a linear Weingarten dual translation surface satisfying $aK^* + bH^* = c$, where a, b, c are real constants. Then S^* is congruent to (3.1) with*

$$z = c_1 + c_2 u + \frac{m}{2} u^2 + c_3 + c_4 v + \frac{-bm \sqrt{1 + v^2} + av \ \arctan v}{2cm}.$$

Proof. Let us consider surface with $a \neq 0, b \neq 0$. A dual translation surface is linear Weingarten if and only if

$$2a \sqrt{1 + v^2} + f'' \left(-b - bv^2 - 2cg'' \left(1 + v^2\right)\left(1 + 2v^2 + v^4\right)\right) = 0, \qquad (3.11)$$

which can be written as

$$f'' = -\frac{2a \sqrt{1 + v^2}}{\left(-b - bv^2 - 2cg'' \left(1 + v^2\right)\left(1 + 2v^2 + v^4\right)\right)} = m,$$

where $m \in \mathbb{R}$. Therefore, we get

$$f(u) = c_1 + c_2 u + \frac{m}{2} u^2, \qquad (3.12)$$

$$g(v) = c_3 + c_4 v + \frac{-bm \sqrt{1 + v^2} + av \ \arctan v}{2cm},$$

where $c_i, m \in \mathbb{R}$ and they satisfy the functional relationship $aK^* + bH^* = c$ for any constants a, b, c, (see Figure 5).

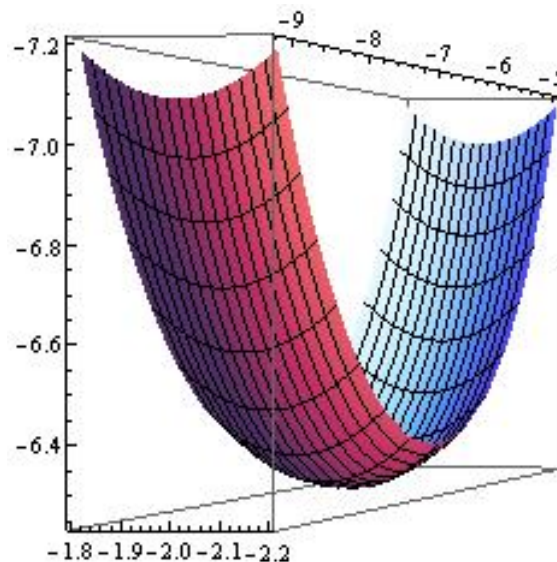**Figure 5.** $z = c_1 + c_2 u + \frac{m}{2} u^2 + c_3 + c_4 v + \frac{-bm \sqrt{1+v^2}+av \ \arctan v}{2cm}$.

$\square$

Theorem 7. *Let S^* be a dual translation surface given by (3.1) in the three dimensional Galilean space $\mathbb{G}_3$. Then there is no surface S^* satisfies the condition $\Delta^{II} \Psi = \lambda_i \Psi_i$, where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$.*

Proof. We classify dual translation surfaces in $\mathbb{G}_3$ satisfying the equation

$$\Delta^{II}\Psi = \lambda_i x_i, \tag{3.13}$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^{II}\Psi = \left(\Delta^{II}\Psi_1, \Delta^{II}\Psi_2, \Delta^{II}\Psi_3\right). \tag{3.14}$$

By a straightforward computation, using Eq (2.2), the Laplacian operator on S^* turns out to be

$$\Delta^{II}\Psi = \begin{pmatrix} -\frac{\sqrt{1+v^2}f'''}{2f''}, \\ -\frac{\sqrt{1+v^2}g'''}{2g''}, \\ 2\sqrt{1+v^2} + u\frac{\sqrt{1+v^2}f'''}{2f''} + v\frac{\sqrt{1+v^2}g'''}{2g''} \end{pmatrix}. \tag{3.15}$$

So Eq (3.13) by means of (3.15) gives the following system of ordinary differential equations

$$\frac{\sqrt{1+v^2}f'''}{2f''} = \lambda_1 f', \tag{3.16}$$

$$\frac{\sqrt{1+v^2}g'''}{2g''} = \lambda_2 g', \tag{3.17}$$

$$2\sqrt{1+v^2} + u\frac{\sqrt{1+v^2}f'''}{2f''} + v\frac{\sqrt{1+v^2}g'''}{2g''} = \lambda_3\left(uf'(u) + vg'(v) - f(u) - g(v)\right), \tag{3.18}$$

where $\lambda_i \in \mathbb{R}$. On combining Eqs (3.16)–(3.18), we have

$$2\sqrt{1+v^2} + \lambda_1 uf' + v\lambda_2 g' = \lambda_3\left(uf' + vg' - f - g\right), \tag{3.19}$$

which can be written as

$$\begin{aligned} \lambda_1 uf' - \lambda_3\left(uf' + f\right) &= m, \\ \lambda_3\left(vg' - g\right) - \lambda_2 vg' - 2\sqrt{1+v^2} &= m, \end{aligned} \tag{3.20}$$

where $\lambda_i, m \in \mathbb{R}$. If $\{\lambda_1, \lambda_2, \lambda_3\} = \{0, 0, 0\}$, hence we have a contradiction. In the cases $\{\lambda_1, \lambda_2 \neq 0, \lambda_3 \neq 0\}$, there are no analytical solutions for the function $g(v)$. We discuss three cases with respect to constants $\lambda_1, \lambda_2, \lambda_3$. We summarized the solutions of (3.20) in the following:

$$\lambda_1 \neq 0, \quad \lambda_2 = 0, \quad \lambda_3 \neq 0 \Rightarrow \left\{ \begin{aligned} f(u) &= \frac{m}{\lambda_3} + c_1\left(u\left(\lambda_1 - \lambda_3\right)\right)^{-\frac{\lambda_3}{\lambda_1 - \lambda_3}} \\ g(v) &= \frac{c_2 v \lambda_3 - m - 2\sqrt{1+v^2} + 2v\arcsin v}{\lambda_3} \end{aligned} \right\},$$

$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \lambda_3 = 0 \Rightarrow \left\{ \begin{aligned} f(u) &= c_1 + \frac{m\ln|u|}{\lambda_1} \\ g(v) &= \frac{c_1\lambda_2 - 2\sqrt{1+v^2} - (2+m)\ln|v| + 2\ln\left|1 + \sqrt{1+v^2}\right|}{\lambda_2} \end{aligned} \right\},$$

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 \neq 0 \Rightarrow \left\{ \begin{aligned} f(u) &= c_1 u + \frac{m}{\lambda_3} \\ g(v) &= \frac{c_2 v \lambda_3 - m - 2\sqrt{1+v^2} + 2v\arcsin v}{\lambda_3} \end{aligned} \right\},$$

for some constants $c_i \neq 0$ and $\lambda_i \neq 0$. In the third row, we have a contradiction for the function $f(u)$, because substituting the first and second rows into (3.14)–(3.16), respectively we can easily see that they do not satisfy these equations. □

Theorem 8. *The dual translation surface $H^{*^2} - K^* = 0$ is locally a generalized umbilical surface and it can be stated to be a surface (3.1) by*

$$z = c_1 + c_2 u + \frac{m}{2} u^2 + c_3 + c_4 v + \frac{1}{4} m \left(v \ \arctan v - \frac{1}{2} \ln \left| 1 + v^2 \right| \right).$$

Proof. Next, we aim to classify the dual translation surfaces given by (3.1) in $\mathbb{G}_3$ that satisfy $H^{*^2} - K^* = 0$. By considering Eq (3.3), we have

$$\frac{f'' - 4g'' - 4v^2 g''}{2 \left(1 + v^2 \right)^3 f'' g''^2} = 0. \tag{3.21}$$

We can rewrite (3.21) as

$$f'' - 4g'' - 4v^2 g'' = 0,$$

which implies

$$f'' = 4g'' + 4v^2 g'' = m.$$

Solving this, we obtain

$$
\begin{aligned}
f(u) &= c_1 + c_2 u + \frac{m}{2} u^2, \\
g(v) &= c_3 + c_4 v + \frac{1}{4} m \left(v \ \arctan v - \frac{1}{2} \ln \left| 1 + v^2 \right| \right),
\end{aligned}
\tag{3.22}
$$

where $m, c_i \in \mathbb{R}$ (see Figure 6).
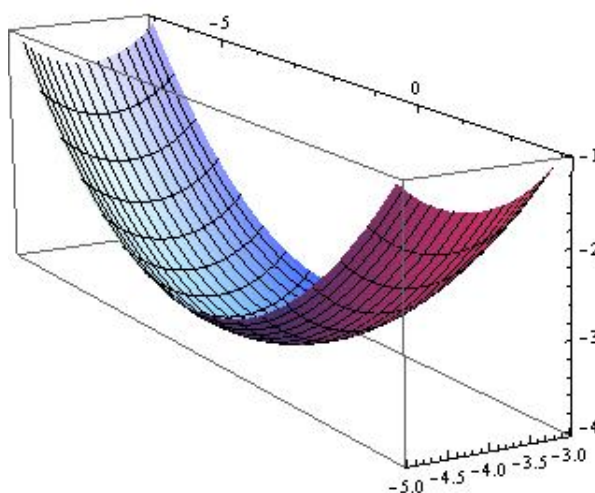
Figure 6. $z = c_1 + c_2 u + \frac{m}{2} u^2 + c_3 + c_4 v + \frac{1}{4} m \left(v \ \arctan v - \frac{1}{2} \ln \left| 1 + v^2 \right| \right)$.

□

4. Results

A rational surface S^* in $\mathbb{R}^3$ is called LN-surface if its tangent planes admit the representation $T(u, v) = ux + vy + z = h(u, v)$ with a rational function $h(u, v) = f(u) + g(v)$. In the present study, we have defined these surfaces in Galilean 3-space and called these surfaces as dual translation surfaces.

We also have calculated the Gaussian and mean curvatures of these surfaces. Further, we have pointed out the constant Gaussian and mean curvatures of these surfaces.

We have discussed whether these surfaces satisfy the Laplacian operator with respect to the second fundamental form.

Conflict of interest

The author states that she has no conflict of interests.

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