Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 4645–4665.
DOI: 10.3934/math.2023229
Received: 25 September 2022
Revised: 07 November 2022
Accepted: 11 November 2022
Published: 06 December 2022

Research article

Existence, uniqueness and approximation of nonlocal fractional differential equation of sobolev type with impulses

M. Manjula[1], K. Kaliraj[1], Thongchai Botmart[2,*], Kottakkaran Sooppy Nisar[3] and C. Ravichandran[4]

[1] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600 005, Tamil Nadu, India.

[2] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand.

[3] Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia.

[4] Department of Mathematics, Kongunadu Arts and Science College, Coimbatore 641029, Tamil Nadu, India.

* **Correspondence:** Email: thongbo@kku.ac.th.

Abstract: This paper is concerned with the study of nonlocal fractional differential equation of sobolev type with impulsive conditions. An associated integral equation is obtained and then considered a sequence of approximate integral equations. By utilizing the techniques of Banach fixed point approach and analytic semigroup, we obtain the existence and uniqueness of mild solutions to every approximate solution. Then, Faedo-Galerkin approximation is used to establish certain convergence outcome for approximate solutions. In order to illustrate the abstract results, we present an application as a conclusion.

Keywords: fractional calculus; impulsive; fixed point techniques; sobolev type
Mathematics Subject Classification: 34B10, 34K40, 34K45, 47H10

1. Introduction

Research work in the area of fractional differential equation is multidisciplinary such as control systems, elasticity, circuit systems, heat transfer, fluid mechanics, signal analysis, traffic flow, pollution control, etc. It is considered as an alternative model to nonlinear differential equation. Fractional differential equations are a useful tool in modelling several events. In [1], controllability of Hilfer fractional neutral differential systems with infinite delay is obtained. An article obtained on

Neutral fractional stochastic partial differential equations with Clarke subdifferential is investigated using fractional calculus and fixed point theorems in [2]. Nisar et al. [3], in their publication, briefly discussed the analysis of controllability for nonlinear Hilfer neutral fractional derivatives via fractional calculus and Banach contraction principle. Numerous credible theoretical studies on fractional differential equation can be referred to books in [4–8] and the research articles are [9–15]. The fractional differential equation has many solutions with nonlocal conditions, impulsive type and sobolev type. In which, the nonlocal conditions is a generalization of the classical conditions, was motivated by the physical phenomena. The pioneering work on nonlocal condition is due to Byszewski [16]. Papers related on nonlocal conditions, we may refer [17–20].

Moreover, the theory of fractional impulsive differential equations has been entirely developed during the past decades. Since 1990's many mathematician have derived lots of results on differential equations undergoing impulsive effects. It focuses on the analysis of dynamic processes that experience abrupt changes. Impulses have a relatively shorter time difference between changes than the overall length of the process. The following may act as motivation to investigate such systems using impulsive differential equations. Consider the simplest scenario for a person's hemodynamic equilibrium. Some injectable medications, such as insulin, may be provided in the case of a de-compensation, such as high or low glucose levels. It is clear that the entrance of medications into the circulation and the body's subsequent absorption are slow and ongoing processes. This circumstance might be seen as an impulsive activity that begins suddenly and lasts for a set amount of time. For detailed information about the impulsive fractional differential equation and its applications, we refer to the readers [21–25].

On the other hand, the Sobolev type differential system is typically seen in the mathematical structure of numerous physical events, such as fluid flow through fractured rocks, thermodynamics. Additionally, Sobolev type differential equations are utilised to describe the attributes of systems and processes in mathematical modelling and simulations. For more literature on sobolev type differential equation, see [26–30] and references therein. In addition, one of the most effective methods for determining out approximations of solutions to a given differential equation in an abstract space is the Faedo-Galerkin approach. The Faedo-Galerkin method may be used within a vartional formulation in order to provide solutions of the equations under possibly weaker assumption on the data and may also prove a very useful tool for numerical approximation of equations. A detailed view on Faedo-Galerkin approximation we refer [31–35].

In accordance with the aforementioned literature survey, there are relatively few work outcomes that explore the existence and uniqueness of a mild solution to a Sobolev type fractional differential equation with Impulses applying a fixed point technique. This fact is the fundamental motivator behind our current progress. This article [36, 37], outlines the nonlocal Sobolev type fractional differential impulsive system as follows:

$$^{c}D^{\beta}_{\sigma}[\mathcal{M}x(\sigma)] = \mathcal{L}x(\sigma) + \mathcal{F}(\sigma, x(\sigma), x(h(\sigma))),\ \sigma \in [0, T], \tag{1.1}$$

$$\Delta x(\sigma_i) = I_i(x(\sigma_i)),\ i = 1, 2, ..., q,\ q \in \mathbb{N}, \tag{1.2}$$

$$\mathrm{g}(x) = \phi \in \mathbb{H}_1. \tag{1.3}$$

Where $0 < \beta < 1,\ T \in (0, \infty)$ $^{c}D^{\beta}_{\sigma}$ is the Caputo fractional derivative, $0 = \sigma_0 < \sigma_1 < ... < \sigma_q < \sigma_{q+1} = T$ are pre-fixed numbers, $\Delta x|_{\sigma=\sigma_i} = x(\sigma_i^+) - x(\sigma_i^-)$ and $x(\sigma_i^+) = \lim_{h\to 0+} x(\sigma_i + h)$ and $x(\sigma_i^-) = \lim_{h\to 0-} x(\sigma_i + h)$ denote the right and left limits of $x(\sigma)$ at $\sigma = \sigma_i$, respectively. From (1.1), assume

$\mathcal{L} : D(\mathcal{L}) \subset \mathbb{H}_1 \to \mathbb{H}_2$ $\mathcal{M} : D(\mathcal{M}) \subset \mathbb{H}_1 \to \mathbb{H}_2$ are closed (unbounded), positive and self-adjoint operators, where $\mathbb{H}_1$ and $\mathbb{H}_2$ are Hilbert spaces and the appropriate functions are $\mathcal{F} : [0, T] \times \mathbb{H}_1 \to \mathbb{H}_1$ and $\mathrm{g} : C([0, T], \mathbb{H}_1) \to \mathbb{H}_1$, $\mathrm{h} : [0, T] \to [0, T]$, $I_i : \mathbb{H}_1 \to \mathbb{H}_1$.

This articles is organized as:

Basic concepts and lemmas are covered in Section 2. In Section 3, the fixed point theorem is used to determine the existence and uniqueness of an approximate solution. In Section 4, the convergence of the approximate solutions is obtained. In Section 5, the convergence of approximate Faedo-Galerkin solutions is proved. Finally, we provide a theoretical application to assist in the effectiveness of our result.

2. Basic results

The upcoming segment recalls the necessary things to obtain the primary facts of our discussion.

Let $(\mathbb{H}_1, \|\cdot\|, <\cdot, \cdot>)$, $(\mathbb{H}_2, \|\cdot\|, <\cdot, \cdot>)$ be Separable Hilbert spaces. Assume $C([0, T], \mathbb{H}_1)$ from $[0, T]$ into $\mathbb{H}_1$ with $\|x\|_{[0,T]} := \sup\{\|x(\sigma)\| : \sigma \in [0, T]\}$ be a Banach space of continuous function and boundedness of linear operator $L(\mathbb{H}_1)$ equipped with $\|\mathrm{f}\|_{L(\mathbb{H}_1)} = \sup\{\|\mathrm{f}(x)\| : \|x\| = 1\}$.

Definition 2.1. *[37] The R-L integral of order $\beta > 0$ is*

$$\mathcal{J}_\sigma^\beta \mathcal{F}(\sigma) = \frac{1}{\Gamma(\beta)} \int_0^\sigma (\sigma - s)^{\beta-1} \mathcal{F}(s) ds, \tag{2.1}$$

where $\mathcal{F} \in L^1((0, T), \mathbb{H}_1)$.

Definition 2.2. *[37] The R-L derivative is*

$$^{RL}D_\sigma^\beta \mathcal{F}(\sigma) = D_\sigma^\delta \mathcal{J}_\sigma^{\delta-\beta} \mathcal{F}(\sigma), \ |\beta - \delta| \in (0, 1), \ \delta \in \mathbb{N}, \tag{2.2}$$

where $D_\sigma^\delta = \frac{d^\delta}{d\sigma^\delta}$, $\mathcal{F} \in L^1((0, T), \mathbb{H}_1)$, $\mathcal{J}_\sigma^{\delta-\beta} \mathcal{F} \in W^{\delta,1}((0, T), \mathbb{H}_1)$.

Definition 2.3. *[37] The Caputo derivative is*

$$^cD_\sigma^\beta \mathcal{F}(\sigma) = \frac{1}{\Gamma(\delta - \beta)} \int_0^\sigma (\sigma - s)^{\delta-\beta-1} \mathcal{F}^\delta(s) ds, \ \delta - 1 < \beta < \delta, \tag{2.3}$$

where $\mathcal{F} \in C^{\delta-1}((0, T), \mathbb{H}_1) \bigcap L^1((0, T), \mathbb{H}_1)$,

$$\mathcal{J}_\sigma^\beta ({}^cD_\sigma^\beta \mathcal{F}(\sigma)) = \mathcal{F}(\sigma) - \sum_{k=0}^{\delta-1} \frac{\sigma^k}{k!} \mathcal{F}^k(0) \ \text{ holds.}$$

Operators $\mathcal{L}$, $\mathcal{M}$ impose the following conditions:

(a) $\mathcal{L}$ and $\mathcal{M}$ are closed linear operators;
(b) $D(\mathcal{M}) \subset D(\mathcal{L})$ and $\mathcal{M}$ is bijective;
(c) $\mathcal{M}^{-1} : \mathbb{H}_2 \to D(\mathcal{M}) \subset \mathbb{H}_1$ is compact.

Conditions (a)-(c) and closed graph theorem imply $\mathcal{L}\mathcal{M}^{-1} : \mathbb{H}_2 \to \mathbb{H}_2$ is the boundedness of the linear operator. Therefore, an infinitesimal generator $\mathcal{E} = \mathcal{L}\mathcal{M}^{-1}$ of semigroup $\mathcal{S}(\sigma) := e^{\mathcal{L}\mathcal{M}^{-1}}$ and so $\max_{\sigma\in I} \|\mathcal{S}(\sigma)\|$ is finite. We have the following integral as per prior definition,

$$\mathcal{M}x(\sigma) = \mathcal{M}x(0) + \sum_{i=1}^{q} I_i(x(\sigma_i)) + \int_0^{\sigma} \frac{(\sigma - s)^{\beta-1}}{\Gamma(\beta)}[\mathcal{L}x(s) + \mathcal{F}(s, x(s))]ds, \ \sigma \in [0, T]. \tag{2.4}$$

The above (2.4) exists a.e. As a result, aforementioned equalization is equal to the impulsive differential equation of Sobolev type. Therefore, there exists $N_0 \geq 1$ a positive constant such that $\|\mathcal{S}(\sigma)\| \leq N_0$. Let the resolvent set of $\mathcal{E}$ is $\rho(\mathcal{E})$. Hence, $\mathcal{E}^{\alpha}$, $0 < \alpha \leq 1$ be the fractional power which is a closed linear operator and $D(\mathcal{E}^{\alpha})$ is a subspace, in such a way its simple to show that it is a Banach space with supremum norm and is represented as $(\mathbb{H}_1)_{\alpha}$ with $(\|\cdot\|_{\alpha})$. We have $(\mathbb{H}_1)_{\eta} \hookrightarrow (\mathbb{H}_1)_{\alpha}, 0 < \alpha < \eta$ so the embedding is continuous. Then, we define $(\mathbb{H}_1)_{-\alpha} = ((\mathbb{H}_1)_{\alpha})^*, \alpha > 0$, dual space of $(\mathbb{H}_1)_{\alpha}$, is a Banach space equipped with $\|x\|_{-\alpha} = \|\mathcal{E}^{-\alpha}x\|, x \in (\mathbb{H}_1)_{-\alpha}$.

Proposition 2.1. *[38] Assume $\mathcal{E}$ of $\mathcal{S}(\sigma)$, $\sigma \geq 0$, $0 \in \rho(\mathcal{E})$ is an infinitesimal generator. We get*

(i) For $\sigma > 0$, $\alpha \geq 0$, $\mathcal{S}(\sigma)$ maps from $\mathbb{H}_1 \to D(\mathcal{E}^{\alpha})$.
(ii) For each $x \in D(\mathcal{E}^{\alpha})$, $\mathcal{S}(\sigma)\mathcal{E}^{\alpha}x = \mathcal{E}^{\alpha}\mathcal{S}(\sigma)x$.
(iii) Let $\left\|\frac{d^j}{d\sigma^j}\mathcal{S}(\sigma)\right\| \leq N_j$, $j = 1, 2$, $\sigma > 0$, where N_j is a positive constant.
(iv) A bounded operator $\mathcal{E}^{\alpha}\mathcal{S}(\sigma)$, $\|\mathcal{E}^{\alpha}\mathcal{S}(\sigma)\| \leq N_{\alpha}\sigma^{-\alpha}e^{-\delta\sigma}$, $\sigma > 0$.
(v) If $x \in D(\mathcal{E}^{\alpha})$, $\alpha \in (0, 1]$ implies $\|\mathcal{S}(\sigma)x - x\| \leq C_{\alpha}\sigma^{\alpha}\|\mathcal{E}^{\alpha}x\|$.

Remark 2.1. *[38] The boundedness of the linear operator $\mathcal{E}^{-\alpha}$ in $\mathbb{H}_1$ such that $D(\mathcal{E}^{\alpha}) = \mathrm{Im}(\mathcal{E}^{-\alpha})$. Let's denote $(\mathbb{H}_1)_{\alpha}(T) = C([0, T], (\mathbb{H}_1)_{\alpha})$ be Banach space of all $(\mathbb{H}_1)_{\alpha}$-valued continuous function equipped with $\|x\|_{(\mathbb{H}_1)_{\alpha}(T)} = \sup_{\sigma\in[0,T]} \|x(\sigma)\|_{\alpha}$, such that $x(\sigma)$ is continuous on $\sigma \neq \sigma_i$, left continuous at $\sigma = \sigma_i$ and right limit $x(\sigma_i^+)$ exists for $i = 1, 2, ..., q$.*

3. Existence of solutions

We inspect the existence of (1.1)–(1.3) as well as their uniqueness. The respective assumptions on $\mathcal{E}, \mathcal{F}, h, I_i(i = 1, 2, ..., q)$ is presented as:

(1) Let $\mathcal{E}$ be closed, positive definite and self adjoint linear operator : $D(\mathcal{E}) \subset \mathbb{H}_2 \to \mathbb{H}_2$. A pure point spectrum $\mathcal{E}$ has

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m \leq ...,$$

with $\lambda_m \to \infty$, $m \to \infty$ and complete orthonormal system $\{\phi_j\}$,

$$\mathcal{E}\phi_j = \lambda_j\phi_j \text{ and } \langle\phi_l, \phi_j\rangle = \delta_{lj}, \tag{3.1}$$

where

$$\delta_{lj} = \begin{cases} 1, & \text{j = 1} \\ 0, & \text{otherwise.} \end{cases}$$

(2) The continuous mapping $\mathcal{F} : [0,\infty) \times (\mathbb{H}_1)_\alpha \times (\mathbb{H}_1)_\alpha \to \mathbb{H}_2$ and $m_R : [0,\infty) \to (0,\infty)$ an increasing function exists, on $R > 0$ such that

$$\|\mathcal{F}(\sigma, z, w)\| \leq m_R(\sigma), \tag{3.2}$$

$$\|\mathcal{F}(\sigma_1, z_1, w_1) - \mathcal{F}(\sigma_2, z_2, w_2)\| \leq m_R(\sigma)[|\sigma_1 - \sigma_2|^{\theta_1} + \|z_1 - z_2\|_\alpha], \tag{3.3}$$

for all $(\sigma, z, w), (\sigma_1, z_1, w_1), (\sigma_2, z_2, w_2) \in [0,\infty) \times \mathbb{B}_R((\mathbb{H}_1)_\alpha) \times \mathbb{B}_R((\mathbb{H}_1)_\alpha)$ where $\mathbb{B}_R(\mathbb{H}_1) = \{z \in \mathbb{H}_1 : \|z\|_{\mathbb{H}_1} \leq R\}$ and $\theta_1 \in (0,1)$.

(3) Let nonlinear function $h : [0,T] \to [0,T]$ such that $h(\sigma) \leq \sigma$, $0 \leq \sigma \leq T$ and $\exists$ constants $L_h > 0$ such that

$$|h(\sigma_1) - h(\sigma_2)| \leq L_h|\sigma_1 - \sigma_2|, \ \sigma_1, \sigma_2 \in [0,T]. \tag{3.4}$$

(4) There exist $\chi \in C([0,T], \ (\mathbb{H}_1)_\alpha)$ such that $\mathrm{g}(\chi) = \phi$ and $\chi(\sigma)$ is locally Lipschitz continuous.

(5) All the function $I_i : \mathbb{H}_1 \to \mathbb{H}_1$ $(i = 1,2,...,q)$ are continuous function such that

$$\|I_i(x)\|_\alpha \leq \mathcal{O}_i, \forall \ \alpha \in (0,1),$$
$$\|I_i(x_1) - I_i(x_2)\|_\alpha \leq \mathcal{N}_i\|x_1 - x_2\|_\alpha, \forall \ x_1, x_2 \in \mathbb{H}_1,$$

where $\mathcal{O}_i, \mathcal{N}_i$, $i = 1,2,...,q$ are positive constants.

Definition 3.1. *[39] Let* $x : [0,T] \to (\mathbb{H}_1)_\alpha$ *be a continuous function , if* $x(0) = x_0$ *and* $x(\cdot)$ *satisfies the following integral equation*

$$x(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)[\mathcal{M}]\chi(0) + \sum\limits_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i)I_i(x(\sigma_i)) \\ + \int\limits_0^\sigma (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}(s, x(s), x(h(s)))ds, \ \sigma \in [0,T], \end{cases} \tag{3.5}$$

is known as mild solution of (1.1)–(1.3), where

$$\mathcal{S}_\beta(\sigma) = \int_0^\infty \mathcal{M}^{-1}\zeta_\beta(\xi)\mathcal{S}(\sigma^\beta\xi)d\xi,$$

$$\mathcal{T}_\beta(\sigma) = \int_0^\infty \mathcal{M}^{-1}\beta\xi\zeta_\beta(\xi)\mathcal{S}(\sigma^\beta\xi)d\xi,$$

$$\zeta_\beta(\xi) = \frac{1}{\beta}\xi^{-1-\frac{1}{\beta}}\psi_\beta(\xi^{-\frac{1}{\beta}}) \geq 0,$$

$$\psi_\beta(\xi) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\xi^{-n\beta-1}\frac{\Gamma(n\beta+1)}{n!}\sin(n\pi\beta), 0 < \xi < \infty,$$

and PDF $\zeta_\beta(\xi)$*. i.e.,* $\zeta_\beta(\xi) \geq 0$, $\int\limits_0^\infty \zeta_\beta(\xi)d\xi = 1$.

Remark 3.1. *[38] Let* $0 \leq v \leq 1$*,*

$$\int_0^\infty \xi^v\zeta_\beta(\xi)d\xi = \int_0^\infty \xi^{-\beta v}\psi_\beta(\xi)d\xi = \frac{\Gamma(1+v)}{\Gamma(1+\beta v)}.$$

Proposition 3.1. *[26] Let $\mathcal{S}(\sigma)$ be a uniformly continuous semigroup and $\mathcal{E}$ be its infinitesimal generator. Then, $\mathcal{S}_\beta(\sigma)$ and $\mathcal{T}_\beta(\sigma)$ are boundedness of the linear operator such that*

(i) $\|\mathcal{S}_\beta(\sigma)x\| \le W_1N_0\|x\|$ *and* $\|\mathcal{T}_\beta(\sigma)x\| \le \frac{W_1N_0}{\Gamma(\beta)}\|x\|$, $x \in \mathbb{H}_1$.
(ii) The strong continuity of $\{\mathcal{T}_\beta(\sigma), \sigma \ge 0\}$ *and* $\{\mathcal{S}_\beta(\sigma), \sigma \ge 0\}$ $0 \le \tau_1 < \tau_2 \le T$, *for* $x \in \mathbb{H}_1$, *we have* $\|\mathcal{T}_\beta(\tau_2)x - \mathcal{T}_\beta(\tau_1)x\| \to 0$ *and* $\|\mathcal{S}_\beta(\tau_2)x - \mathcal{S}_\beta(\tau_1)x\| \to 0$ *as* $\tau_2 \to \tau_1$.
(iii) Suppose $\mathcal{S}(\sigma)$, $\sigma \ge 0$ *is compact, then* $\mathcal{S}_\beta(\sigma)$ *and* $\mathcal{T}_\beta(\sigma)$ *are compact operators.*
(iv) For each $x \in \mathbb{H}_1$, *we have* $\mathcal{E}\mathcal{T}_\beta(\sigma)x = \mathcal{E}^{1-\eta}\mathcal{T}_\beta\mathcal{E}^{\eta}x$, $\sigma \in [0, T]$, $\eta \in (0, 1)$. *We have* $\|\mathcal{E}^{\alpha}\mathcal{T}_\beta(\sigma)\| \le \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\sigma^{-\alpha\beta}$, $\sigma \in [0, T]$, $\alpha \in (0, 1)$.
(v) For any $x \in \mathbb{X}_\alpha$ *and fixed* $\sigma \ge 0$, *we have* $\|\mathcal{S}_\beta(\sigma)x\|_\alpha \le W_1N_0\|x\|_\alpha$ *and* $\|\mathcal{T}_\beta(\sigma)x\|_\alpha \le \frac{W_1N_0}{\Gamma(\beta)}\|x\|_\alpha$.

Arbitrarily fixed point $T_0 > 0$ such that $0 < T < T_0 < \infty$,

$$\psi = \frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0) \frac{T^{\beta(1-\alpha)}}{(1-\alpha)} < 1. \tag{3.6}$$

Let $\mathcal{H}_n$ be finite dimensional subspace, spanned by $\{\phi_0, \phi_1, ..., \phi_n\}$ and a projection operator $P^n : \mathbb{H}_1 \to \mathcal{H}_n$, $n = 0, 1, ...$ Assume $\mathcal{F}_n : [0, T] \times (\mathbb{H}_1)_\alpha \to \mathbb{H}_1$ and $I_{i,n} : \mathbb{H}_1 \to \mathbb{H}_1$, is defined by

$$\mathcal{F}_n(\sigma, x(\sigma), x(h(\sigma))) = \mathcal{F}(\sigma, P^n x(\sigma), P^n x(h(\sigma))), \tag{3.7}$$

$$I_{i,n}(x) = I_i(P^n x), \forall\, x \in \mathbb{H}_1,\ n = 0, 1, 2, ...,\ i = 1, 2, ...q, \tag{3.8}$$

and the operator $\mathbb{Q}_n$ on $\mathbb{B}$ as follows

$$(\mathbb{Q}_n x)(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \sum\limits_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i) I_{i,n}(x(\sigma_i)) \\ + \int\limits_0^\sigma (\sigma - s)^{\beta-1} \mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x(s), x(h(s)))ds,\ \sigma \in [0, T]. \end{cases} \tag{3.9}$$

Theorem 3.1. *Assume (1)–(5) holds, then* $x_n \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$ *be a unique fixed point of* $\mathbb{Q}_n$ *exists i.e.,* $\mathbb{Q}_n x_n = x_n$ *for each* $n = 0, 1, 2, ...$ *and* x_n *fulfills the approximate integral equation,*

$$x_n(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \sum\limits_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i) I_{i,n}(x_n(\sigma_i)) \\ + \int\limits_0^\sigma (\sigma - s)^{\beta-1} \mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x_n(s), x(h(s)))ds,\ \sigma \in [0, T]. \end{cases} \tag{3.10}$$

Proof. Let $\mathbb{Q}_n : \mathbb{B}_R((\mathbb{H}_1)_\alpha(T)) \to \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$ is defined by

$$(\mathbb{Q}_n x)(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \sum\limits_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i) I_{i,n}(x(\sigma_i)) \\ + \int\limits_0^\sigma (\sigma - s)^{\beta-1} \mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x(s), x(h(s)))ds,\ \sigma \in [0, T]. \end{cases}$$

We will demonstrate that $\mathbb{Q}_n$ is well defined. This is sufficient to demonstrate that the map $\sigma \longmapsto (\mathbb{Q}_n x)(\sigma) : [0, T]$ into $(\mathbb{H}_1)_\alpha$ w.r.t. α norm is continuous.

Let $\sigma_1, \sigma_2 \in [0, T]$ with $\sigma_2 > \sigma_1$, we get

$$
\begin{aligned}
\|[\mathbb{Q}_n x(\sigma_2) - \mathbb{Q}_n x(\sigma_1)]\|_\alpha &= \|[\mathcal{S}_\beta(\sigma_2) - \mathcal{S}_\beta(\sigma_1)]\mathcal{M}\chi(0)\|_\alpha \\
&+ \sum_{i=1}^{q} \left\| [\mathcal{S}_\beta(\sigma_2 - \sigma_i) - \mathcal{S}_\beta(\sigma_1 - \sigma_i)] I_{i,n}(x(\sigma_i)) \right\|_{\alpha-1} \\
&+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)^{\beta-1} \mathcal{T}_\beta(\sigma_2 - s)\mathcal{F}_n(s, x(s), x(h(s)))ds \right\|_\alpha \\
&+ \left\| \int_0^{\sigma_1} (\sigma_2 - s)^{\beta-1} \mathcal{T}_\beta(\sigma_2 - s)\mathcal{F}_n(s, x(s), x(h(s)))ds \right. \\
&- \left. \int_0^{\sigma_1} (\sigma_1 - s)^{\beta-1} \mathcal{T}_\beta(\sigma_1 - s)\mathcal{F}_n(s, x(s), x(h(s)))ds \right\|_\alpha \\
&\leq \|[\mathcal{S}_\beta(\sigma_2) - \mathcal{S}_\beta(\sigma_1)]\mathcal{M}\chi(0)\|_\alpha \\
&+ \sum_{i=1}^{q} \left\| [\mathcal{S}_\beta(\sigma_2 - \sigma_i) - \mathcal{S}_\beta(\sigma_1 - \sigma_i)] \mathcal{E}^{\alpha-1} I_{i,n}(x(\sigma_i)) \right\|_{\alpha-1} \\
&+ \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)^{\beta-1} \|\mathcal{E}^\alpha \mathcal{T}_\beta(\sigma_2 - s)\| \, \|\mathcal{F}_n(s, x(s), x(h(s)))\| ds \\
&+ \int_0^{\sigma_1} (\sigma_1 - s)^{\beta-1} \|\mathcal{E}^\alpha [\mathcal{T}_\beta(\sigma_1 - s) - \mathcal{T}_\beta(\sigma_2 - s)]\| \, \|\mathcal{F}_n(s, x(s), x(h(s)))\| ds \\
&+ \int_0^{\sigma_1} [(\sigma_1 - s)^{\beta-1} - (\sigma_2 - s)^{\beta-1}] \|\mathcal{E}^\alpha \mathcal{T}_\beta(\sigma_2 - s)\| \, \|\mathcal{F}_n(s, x(s), x(h(s)))\| ds \\
&\leq \|[\mathcal{S}_\beta(\sigma_2) - \mathcal{S}_\beta(\sigma_1)]\mathcal{M}\chi(0)\|_\alpha + W_1 N_\alpha \sum_{i=1}^{q} O_i \|\mathcal{E}^\alpha\| (\sigma_2 - \sigma_1) \\
&+ \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0) \frac{(\sigma_2 - \sigma_1)^{\beta(1-\alpha)}}{\beta(1-\alpha)} + \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0) \\
&\times \int_0^{\sigma_1} (\sigma_1 - s)^{\beta-1} [(\sigma_1 - s)^{-\alpha\beta} - (\sigma_2 - s)^{-\alpha\beta}] ds + \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \\
&\times m_R(T_0) \int_0^{\sigma_1} [(\sigma_1 - s)^{\beta-1} - (\sigma_1 - s)^{\beta-1}](\sigma_2 - s)^{-\alpha\beta} ds. \qquad (3.11)
\end{aligned}
$$

For $x \in \mathbb{H}_1$, we have,

$$
[\mathcal{S}(\sigma_2^\beta \xi) - \mathcal{S}(\sigma_1^\beta \xi)]x = \int_{\sigma_1}^{\sigma_2} \frac{d}{d\sigma} \mathcal{S}(\sigma^\beta \xi) x d\sigma = \int_{\sigma_1}^{\sigma_2} \beta \xi \sigma^{\beta-1} \mathcal{E} \mathcal{S}(\sigma^\beta \xi) d\sigma.
$$

Thus, we get

$$\int_0^\infty \mathcal{M}^{-1}\zeta_\beta(\xi)\|\mathcal{S}(\sigma_2^\beta\xi)-\mathcal{S}(\sigma_1^\beta\xi)\|\,\|\mathcal{E}^\alpha\mathcal{M}\chi(0)\|d\xi \le \int_0^\infty \mathcal{M}^{-1}\zeta_\beta(\xi)\Big[\int_{\sigma_1}^{\sigma_2}\Big\|\frac{d}{ds}\mathcal{S}(\sigma^\beta\xi)\Big\|d\sigma\Big]\mathcal{M}\Big\|\mathcal{E}^\alpha\chi(0)\Big\|d\xi$$
$$\le \int_0^\infty \mathcal{M}^{-1}\zeta_\beta(\xi)[N_1(\sigma_2-\sigma_1)]\|\mathcal{M}\|\,\|\chi(0)\|_\alpha d\xi$$
$$\le R_1(\sigma_2-\sigma_1), \tag{3.12}$$

where $R_1 = N_1W_1\|\mathcal{M}\|\,\|\chi(0)\|_\alpha$.

From (3.11), we have

$$\int_0^{\sigma_1}(\sigma_1-s)^{\beta-1}[(\sigma_1-s)^{-\alpha\beta}-(\sigma_2-s)^{-\alpha\beta}]ds \le vd_1^{v-1}(1-h)^{-p_1(1-v)-1}(\sigma_2-\sigma_1)^{p_1(1-v)}, \tag{3.13}$$

where $h = [1-(\frac{v}{p_1})^{\frac{1}{vp_1}}]$, $p_1 = 1-\beta\alpha$, $v = \frac{(1-\beta)}{1-\beta\alpha}$ and $0 < d_1 \le 1$.

$$\int_0^{\sigma_1}[(\sigma_1-s)^{\beta-1}-(\sigma_1-s)^{\beta-1}](\sigma_2-s)^{-\alpha\beta}ds \le \frac{N_{1+\alpha}}{\alpha}b_1^{\alpha-1}(1-h_1)^{-\beta(1-\alpha)-1}(\sigma_2-\sigma_1)^{\beta(1-\alpha)}, \tag{3.14}$$

where $h_1 = (1-(\frac{\alpha}{\beta})^{\frac{1}{\alpha\beta}})$, $0 < b_1 \le 1$ and $N_{1+\alpha}$ is some positive constant with $\|\mathcal{E}^{\alpha+1}\mathcal{S}(\sigma)\|\le N_{1+\alpha}\sigma^{-1-\alpha}$, $\forall\ \sigma > 0$. Thus, from the inequalities (3.11)–(3.14), (2).

We conclude that $\sigma \longmapsto \mathcal{F}_n(\sigma, x(\sigma))$ map is uniformly *Hölder′s* continuous. We justify $\mathbb{Q}_n(\mathbb{B}_R((\mathbb{H}_1)_\alpha(T))) \subseteq \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$. Let $x \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$, $0 \le \sigma \le T$. We get

$$\|(\mathbb{Q}_nx)(\sigma)\|_\alpha \le \|\mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0)\|_\alpha + \sum_{i=1}^q\|\mathcal{S}_\beta(\sigma-\sigma_i)I_{i,n}(x(\sigma_i))\|_\alpha$$
$$+\Big\|\int_0^\sigma(\sigma-s)^{\beta-1}\mathcal{T}_\beta(\sigma-s)\mathcal{F}_n(s,x(s),x(h(s)))ds\Big\|_\alpha$$
$$\le W_1\|\mathcal{M}\|N_0\|\chi(0)\|_\alpha + W_1N_\alpha\sum_{i=1}^q\mathcal{O}_i + \frac{\beta W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^\sigma(\sigma-s)^{\beta(1-\alpha)-1}m_R(T_0)ds$$
$$\le W_1\|\mathcal{M}\|N_0\|\chi(0)\|_\alpha + W_1N_\alpha\sum_{i=1}^q\mathcal{O}_i + \frac{W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}m_R(T_0)\frac{T^{\beta(1-\alpha)}}{(1-\alpha)}. \tag{3.15}$$

We may now take R as a positive integer such that,

$$R = W_1\|\mathcal{M}\|N_0\|\chi(0)\|_\alpha + W_1N_\alpha\sum_{i=1}^q\mathcal{O}_i + \frac{W_1N_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}m_R(T_0)\frac{T^{\beta(1-\alpha)}}{(1-\alpha)}.$$

Therefore, we deduce that $\mathbb{Q}_n(\mathbb{B}_R((\mathbb{H}_1)_\alpha(T))) \subseteq \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$. Lastly, we demonstrate $\mathbb{Q}_n$ is a strict contraction map. For $x_1, x_2 \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$, $0 \le \sigma \le T$.

$$\begin{aligned}\|(\mathbb{Q}_n x_1)(\sigma) - (\mathbb{Q}_n x_2)(\sigma)\|_\alpha &\le \Big\| \int_0^\sigma (\sigma - s)^{\beta-1} \mathcal{E}^\alpha \mathcal{T}_\beta(\sigma - s) ds \Big\| \\ &\times \|\mathcal{F}_n(s, x_1(s), x_1(h(s))) - \mathcal{F}_n(s, x_2(s), x_2(h(s)))\|_\alpha \\ &+ \sum_{i=1}^q \|\mathcal{S}_\beta(\sigma - \sigma_i)\| \, \|I_{i,n}(x_1(\sigma_i)) - I_{i,n}(x_2(\sigma_i))\|_\alpha \\ &\le \frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0) \frac{T^{\beta(1-\alpha)}}{(1-\alpha)} \|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(T)} \\ &+ W_1 N_\alpha \sum_{i=1}^q \mathcal{N}_i \|x_1 - x_2\|_{T,\alpha} \le \wedge \|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(T)}. \end{aligned} \tag{3.16}$$

In Eq (3.16), $\wedge = \frac{W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} m_R(T_0) \frac{T^{\beta(1-\alpha)}}{(1-\alpha)} + W_1 N_\alpha \sum_{i=1}^q \mathcal{N}_i < 1$.

As a result, $\mathbb{Q}_n$ is determined to be a contraction mapping. Thus, a unique $x_n \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$ exists such that $\mathbb{Q}_n x_n = x_n$. □

Lemma 3.1. *Assume (1)–(5) holds.*

(i) Let $\chi(0) \in D(\mathcal{E}^\alpha)$, $\alpha \in (0, 1)$ *implies* $x_n(\sigma) \in D(\mathcal{E}^\upsilon)$ $\forall\, 0 < \sigma \le T, \upsilon \in [0, 1)$.

(ii) If $\chi(0) \in D(\mathcal{E})$, *implies* $x_n(\sigma) \in D(\mathcal{E}^\upsilon)$ $\forall\, 0 \le \sigma \le T, \upsilon \in [0, 1)$.

Proof. We get a unique $x_n \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$ that satisfies (3.10) by using the previous theorem. In [38], for $\sigma > 0$, $0 \le \upsilon < 1$ we get $\mathcal{S}(\sigma) : \mathbb{H}_1 \to D(\mathcal{E}^\upsilon)$, $D(\mathcal{M}^\alpha) \subseteq D(\mathcal{M}^\upsilon)$. Also $\mathcal{S}(\sigma)x \in D(\mathcal{E})$. As a outcome, of all these facts, $D(\mathcal{E}) \subseteq D(\mathcal{E}^\upsilon)$, $0 \le \upsilon \le 1$. □

Lemma 3.2. *If (1)–(5) holds.*

(i) Let $\chi(0) \in D(\mathcal{E}^\alpha)$, $\alpha \in (0, 1)$, $0 < \sigma_0 \le T$, *then a constant* S_{σ_0} *exist,*

$$\|x_n(\sigma)\|_\upsilon \le S_{\sigma_0},\ 0 \le \upsilon < 1,\ \sigma \in [\sigma_0, T], \textit{independent of } n.$$

(ii) Let $\chi(0) \in D(\mathcal{E})$, *then a constant* $S_0 > 0$ *exists,*

$$\|x_n(\sigma)\|_\upsilon \le S_0,\ 0 \le \upsilon < 1,\ \sigma \in [0, T], \textit{independent of } n. \tag{3.17}$$

Proof. Let $\chi(0) \in D(\mathcal{A}^\alpha)$. In (3.10), we apply $\mathcal{E}^\upsilon$ on both sides, we have

$$\begin{aligned}\|\mathcal{E}^\upsilon x_n(\sigma)\| &\le \|\mathcal{E}^\upsilon \mathcal{M} \mathcal{S}_\beta(\sigma)\chi(0)\| + \sum_{i=1}^q \|\mathcal{S}_\beta(\sigma - \sigma_i)\mathcal{E}^\upsilon I_i(x(\sigma_i))\| \\ &+ \int_0^\sigma (\sigma - s)^{\beta-1} \|\mathcal{E}^\upsilon \mathcal{T}_\beta(\sigma - s)\| \, \|\mathcal{F}_n(s, x_n(s), x(h(s)))\| ds \\ &\le N_\upsilon W_1 \|\mathcal{M}\| \sigma_0^{-\upsilon} \|\chi(0)\| + N_\upsilon W_1 \sum_{i=1}^q O_i + \frac{\beta N_\upsilon W_1 \Gamma(2-\upsilon)}{\Gamma(1+\beta(1-\upsilon))} m_R(T_0) \frac{T^{\beta(1-\upsilon)}}{\beta(1-\upsilon)} \le S_{\sigma_0}. \end{aligned} \tag{3.18}$$

Again, if $\chi(0) \in D(\mathcal{E}) \Rightarrow \chi(0) \in D(\mathcal{E}^{\upsilon}), \quad 0 \leq \upsilon \leq 1$ and we get

$$\begin{aligned}\|x_n(\sigma)\|_{\upsilon} &\leq N_0 W_1 \|\mathcal{M}\| \, \|\chi(0)\|_{\upsilon} + N_{\upsilon} W_1 \sum_{i=1}^{q} O_i + \frac{\beta N_{\upsilon} W_1 \Gamma(2-\upsilon)}{\Gamma(1+\beta(1-\upsilon))} m_R(T_0) \frac{T^{\beta(1-\upsilon)}}{\beta(1-\upsilon)} \\ &\leq S_0. \end{aligned} \tag{3.19}$$

□

4. Convergence of solutions

Now, to investigate the convergence of solution $x_n(\sigma)$ of the approximate integral Eq (3.10) to a unique solution $x(\cdot)$ of the Eq (3.5).

Theorem 4.1. *Suppose (1)–(5) holds,* $\chi(0) \in D(\mathcal{A}^{\alpha}), \quad \alpha \in (0,1)$. *Then,*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ \sigma\in[\sigma_0,T]\}} \|x_n(\sigma) - x_m(\sigma)\|_{\alpha} = 0, \textit{ for each } 0 < \sigma_0 \leq T. \tag{4.1}$$

Proof. Let $n \geq m$, we have

$$\begin{aligned}&\|\mathcal{F}_n(\sigma, x_n(\sigma), x_n(h(s))) - \mathcal{F}_m(\sigma, x_m(\sigma), x_m(h(s)))\| \\ \leq &\|\mathcal{F}_n(\sigma, x_n(\sigma), x_n(h(s))) - \mathcal{F}_n(\sigma, x_m(\sigma), x_m(h(s)))\| \\ + &\|\mathcal{F}_n(\sigma, x_m(\sigma), x_m(h(s))) - \mathcal{F}_m(\sigma, x_m(\sigma), x_m(h(s)))\| \\ \leq &2m_R(T_0)\|x_n(\sigma) - x_m(\sigma)\|_{\alpha} + m_R(T_0)\|(P^n - P^m)x_m(\sigma)\|_{\alpha} \\ + &\|(P^n - P^m)x_m(h(\sigma))\|_{\alpha}. \end{aligned} \tag{4.2}$$

For $0 < \alpha < \upsilon < 1$, we get

$$\begin{aligned}\|\mathcal{E}^{\alpha}(P^n - P^m)x_m(\sigma)\| &\leq \|\mathcal{E}^{\alpha-\upsilon}(P^n - P^m)\mathcal{E}^{\upsilon} x_m(\sigma)\| \\ &\leq \frac{1}{\lambda_m^{\upsilon-\alpha}} \|x_m(\sigma)\|_{\upsilon}. \end{aligned} \tag{4.3}$$

Thus, from (4.2), (4.3) we obtain

$$\|\mathcal{F}_n(\sigma, x_n) - \mathcal{F}_m(\sigma, x_m)\| \leq 2m_R(T_0)[\|x_n(\sigma) - x_m(\sigma)\|_{\alpha} + \frac{1}{\lambda_m^{\upsilon-\alpha}} \|x_m(\sigma)\|_{\upsilon}].$$

Similarly , we estimate

$$\|I_{i,n}(x_n(\sigma_i)) - I_{i,m}(x_m(\sigma_i))\| \leq \mathcal{N}_i[\|x_n(\sigma_i) - x_m(\sigma_i)\|_{\alpha} + \frac{1}{\lambda_m^{\upsilon-\alpha}} \|\mathcal{E}^{\upsilon} x_m(\sigma_i)\|_{\upsilon}].$$

We choose σ_0' such that $0 < \sigma_0' < \sigma_0 < T$,

$$\|x_n(\sigma) - x_m(\sigma)\|_\alpha \le \Big(\int_0^{\sigma_0'} + \int_{\sigma_0'}^{\sigma}\Big)(\sigma - s)^{\beta-1} ds \|\mathcal{E}^\alpha \mathcal{T}_\beta(\sigma - s)\|$$
$$\times \Big[\|\mathcal{F}_n(s, x_n(s), x_n(h(s))) - \mathcal{F}_m(s, x_m(s), x_m(h(s)))\|\Big]$$
$$+ \sum_{i=0}^{q} \|\mathcal{S}_\beta(\sigma - \sigma_i)\| \, \|I_{i,n}(x_n(\sigma_i)) - I_{i,m}(x_m(\sigma_i))\|_\alpha. \tag{4.4}$$

1^{st} integral of inequality (4.4),

$$\int_0^{\sigma_0'} (\sigma - s)^{\beta-1} \|\mathcal{E}^\alpha \mathcal{T}_\beta(\sigma - s)\| \times \|\mathcal{F}_n(s, x_n(s), x_n(h(s))) - \mathcal{F}_m(s, x_m(s), x_m(h(s)))\| ds$$
$$\le \frac{2\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} 2m_R(T_0)[(T - \sigma_0')^{\beta(1-\alpha)-1} \sigma_0']. \tag{4.5}$$

2^{nd} integral of (4.4) is evaluated as

$$\int_{\sigma_0'}^{\sigma} (\sigma - s)^{\beta-1} \|\mathcal{E}^\alpha \mathcal{T}_\beta(\sigma - s)\| \, \|\mathcal{F}_n(s, x_n(s), x_n(h(s))) - \mathcal{F}_m(s, x_m(s), x_m(h(s)))\| ds$$
$$\le \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} 2m_R(T_0)\Big[\frac{U_{\sigma_0'} T^{\beta(1-\alpha)}}{\beta(1-\alpha)\lambda_m^{\upsilon-\alpha}} + \int_{\sigma_0'}^{\sigma} (\sigma - s)^{\beta(1-\alpha)-1} \|x_n - x_m\|_{(\mathbb{H}_1)_\alpha(s)} ds\Big]. \tag{4.6}$$

Thus, from (4.4)–(4.6) we conclude

$$\|x_n(\sigma) - x_m(\sigma)\| \le D_1 \sigma_0' + \frac{D_2}{\lambda_m^{\upsilon-\alpha}} + D_3 \|x_n(\sigma) - x_m(\sigma)\|_\alpha + D_4 \int_{\sigma_0'}^{\sigma} (\sigma - s)^{\beta(1-\alpha)-1} \|x_n - x_m\|_{(\mathbb{H}_1)_\alpha(s)} ds,$$

where

$$D_1 = \frac{2\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} 2m_R(T_0)(T - \sigma_0')^{\beta(1-\alpha)-1},$$
$$D_2 = \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} 2m_R(T_0) \frac{U_{\sigma_0'} T^{\beta(1-\alpha)}}{\beta(1-\alpha)} + W_1 N_\alpha \sum_{i=1}^{q} \mathcal{N}_i,$$
$$D_3 = W_1 N_\alpha \sum_{i=1}^{q} \mathcal{N}_i,$$
$$D_4 = \frac{\beta W_1 N_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} 2m_R(T_0).$$

Since $1 - W_1 N_\alpha \sum_{i=1}^{q} \mathcal{N}_i > 0$, we have

$$\|x_n(\sigma) - x_m(\sigma)\|_\alpha \leq \frac{1}{1 - D_3}\Big[D_1 \sigma_0' + \frac{D_2}{\lambda_m^{\upsilon-\alpha}} + D_4 \int_{\sigma_0'}^{\sigma} (\sigma - s)^{\beta(1-\alpha)-1} \|x_n - x_m\|_{(\mathbb{H}_1)_\alpha(s)} ds\Big].$$

By lemma 5.6.7 in [38], we have that there exist a constant $\mathcal{K}$ such that

$$\|x_n(\sigma) - x_m(\sigma)\|_\alpha \leq \frac{1}{1 - D_3}\Big[D_1 \sigma_0' + \frac{D_2}{\lambda_m^{\upsilon-\alpha}}\Big]\mathcal{K}.$$

Taking supremum over $[\sigma_0, T]$ and let $m \to \infty$, we obtain

$$\lim_{m\to\infty} \sup_{\{n\geq m, \sigma_0 \leq \sigma \leq T\}} \|x_n(\sigma) - x_m(\sigma)\|_\alpha \leq \frac{D_1}{(1 - D_3)} \sigma_0' \mathcal{K}.$$

Because σ_0' is arbitrary, the right side of the expression could be made as tiny as required simply reducing σ_0'. □

Proposition 4.1. *Assume $\chi(0) \in D(\mathcal{E})$. Then*

$$\lim_{m\to\infty,\ \sigma\in[0,T]} \sup \|x_n(\sigma) - x_m(\sigma)\|_\alpha = 0.$$

Theorem 4.2. *Assume (1)–(5) holds, $\chi(0) \in D(\mathcal{E}^\alpha)$. Then, $\exists\ x_n(\sigma) \in (\mathbb{H}_1)_\alpha(T)$ a unique function satisfying,*

$$x_n(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \int_0^\sigma (\sigma - s)^{\beta-1} \mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x_n(s), x_n(h(s)))ds \\ + \sum_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i) I_{i,n}(x_n(\sigma_i)), \quad \sigma \in [0, T], \end{cases} \tag{4.7}$$

and $x \in (\mathbb{H}_1)_\alpha(T)$ satisfying

$$x(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \int_0^\sigma (\sigma - s)^{\beta-1} \mathcal{T}_\beta(\sigma - s)\mathcal{F}(s, x(s), x(h(s)))ds \\ + \sum_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i) I_i(x(\sigma_i)),\ \sigma \in [0, T]. \end{cases} \tag{4.8}$$

such that x_n converges to x in $(\mathbb{H}_1)_\alpha(T)$ i.e., $x_n \to x$ as $n \to \infty$.

Proof. Suppose $\chi(0) \in D(\mathcal{E})$. From previous preposition, there is a $x \in (\mathbb{H}_1)_\alpha(T)$ such that $\lim_{n\to\infty} x_n(\sigma) = x(\sigma)$. Since $x_n \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))\ \forall\ n$, we get $x \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))\ \forall\ \sigma_0 \in (0, T]$,

$$\begin{aligned} \|\mathcal{F}_n(\sigma, x_n(\sigma), x_n(h(s))) - \mathcal{F}(\sigma, x(\sigma), x(h(s)))\| &= \|\mathcal{F}(\sigma, P^n x_n(\sigma), x_n(h(s))) - \mathcal{F}(\sigma, x(\sigma), x(h(s)))\| \\ &\leq 2m_R(T_0)[\|x_n(\sigma) - x(\sigma)\|_\alpha + \|(P^n - I)x(\sigma)\|_\alpha]. \end{aligned}$$

Taking supremum over $[0, T]$, we have

$$\sup_{\sigma\in[0,T]} \|\mathcal{F}_n(\sigma, x_n(\sigma)) - \mathcal{F}(\sigma, x(\sigma))\| \leq 2m_R(T_0)[\|x_n - x\|_{(\mathbb{H}_1)_\alpha(T)} + \|(P^n - I)x\|_{(\mathbb{H}_1)_\alpha(T)}] \to 0, \text{ as } n \to \infty.$$

Thus, we get

$$x(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}x_0 + \int\limits_0^\sigma (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}(s, x(s), x(h(s)))ds \\ + \sum\limits_{i=1}^q \mathcal{S}_\beta(\sigma - \sigma_i)I_i(x(\sigma_i)),\ \sigma \in [0, T]. \end{cases}$$

Now, let $\chi(0) \in D(\mathcal{E}^\alpha)$. Since $\mathcal{E}^\alpha x_n(\sigma)$ converges to $\mathcal{E}^\alpha x(\sigma)$ for each $\sigma \in (0, T]$ and $x_n(0) = x(0) = \chi(0)$. Then, $\mathcal{E}^\alpha x_n(\sigma)$ converges to $\mathcal{E}^\alpha x(\sigma)$ in $\mathbb{H}_1$. Furthermore, we have that $x_n \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$ for each n. Also $x \in \mathbb{B}_R((\mathbb{H}_1)_\alpha(T))$. From previous theorem, we get

$$\lim_{n\to\infty} \sup_{\sigma\in[\sigma_0,T]} \|x_n(\sigma) - x(\sigma)\|_\alpha = 0.$$

Also, we have

$$\begin{aligned} &\sup_{\sigma\in[0,T]} \|\mathcal{F}_n(\sigma, x_n(\sigma), x_n(h(s))) - \mathcal{F}(\sigma, x(\sigma), x(h(s)))\| \\ \leq &2m_R(T_0)[\|x_n - x\|_{(\mathbb{H}_1)_\alpha(T)} + \|(P^n - I)x\|_{(\mathbb{H}_1)_\alpha(T)}] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $0 < \sigma_0 < \sigma$, Eq (3.10) can be reframed as

$$x_n(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \Big(\int\limits_0^{\sigma_0} + \int\limits_{\sigma_0}^{\sigma}\Big)(\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x_n(s), x_n(h(s)))ds \\ + \sum\limits_{i=1}^q \mathcal{S}_\beta(\sigma - \sigma_i)I_{i,n}(x_n(\sigma_i)),\ \sigma \in [0, T]. \end{cases} \tag{4.9}$$

we estimate the 1^{st} integral of (4.9) as

$$\Big\|\int\limits_0^{\sigma_0} (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x_n(s), x_n(h(s)))ds\Big\| \leq \frac{N_0 W_1}{\Gamma(\beta)} 2m_R(T_0)\frac{\sigma_0^\beta}{\beta}.$$

Thus, we deduce that

$$\begin{aligned} &\|x_n(\sigma) - \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) - \sum_{i=1}^q \mathcal{S}_\beta(\sigma - \sigma_i)I_{i,n}(x_n(\sigma_i)) - \int\limits_{\sigma_0}^{\sigma} (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x_n(s), x_n(h(s)))ds\| \\ \leq &\frac{N_0 W_1}{\Gamma(\beta)} 2m_R(T_0)\frac{\sigma_0^\beta}{\beta}. \end{aligned}$$

Letting $n \to \infty$ and getting

$$\begin{aligned} &\|x(\sigma) - \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) - \sum_{i=1}^q \mathcal{S}_\beta(\sigma - \sigma_i)I_i(x(\sigma_i)) - \int\limits_{\sigma_0}^{\sigma} (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}(s, x(s), x(h(s)))ds\| \\ \leq &\frac{N_0 W_1}{\Gamma(\beta)} 2m_R(T_0)\frac{\sigma_0^\beta}{\beta}. \end{aligned}$$

Since, σ_0 is arbitrary, we deduce $x(\sigma)$ satisfies the integral Eq (3.5). Now, we shall show the uniqueness. Let x_1 and x_2 be the two solutions of integral Eq (3.5). Thus, we have

$$\begin{aligned}\|x_1(\sigma) - x_2(\sigma)\|_\alpha \leq & \int_0^\sigma (\sigma - s)^{\beta-1}\|\mathcal{E}^\alpha \mathcal{T}_\beta(\sigma - s)\| \, \|\mathcal{F}(s, x_1(s), x_1(h(s))) \\ & - \mathcal{F}(s, x_2(s), x_2(h(s)))\|ds + \sum_{i=1}^{q} \|\mathcal{S}_\beta(\sigma - \sigma_i)\| \, \|I_i(x_1(\sigma_i)) - I_i(x_2(\sigma_i))\| \\ & \leq \frac{\beta N_\alpha W_1 \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^\sigma (\sigma - s)^{\beta(1-\alpha)-1} 2m_R(T_0)\|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(s)} ds \\ & + N_\alpha W_1 \sum_{i=1}^{q} N_i \|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(s)}.\end{aligned}$$

Taking supremum on $[0, \sigma]$ and obtaining

$$\begin{aligned}\|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(T)} \leq & \frac{\beta N_\alpha W_1 \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \int_0^\sigma (\sigma - s)^{\beta(1-\alpha)-1} 2m_R(T_0)\|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(s)} ds \\ & + N_\alpha W_1 \sum_{i=1}^{q} N_i \|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(s)}.\end{aligned}$$

From Gronwall's inequality and the fact that

$$\|x_1(\sigma) - x_2(\sigma)\| \leq \frac{1}{\lambda_0^\alpha}\|x_1 - x_2\|_{(\mathbb{H}_1)_\alpha(T)}.$$

We deduce that $x_1 = x_2$ on $[0, T]$. □

5. Faedo-Galerkin approximation

Additionally, the convergence findings were accomplished using the Faedo-Galerkin approximation technique.

There is a unique $x \in (\mathbb{H}_1)_\alpha(T)$, $T \in (0, T_0)$, that fulfils the integral equation,

$$x(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \int_0^\sigma (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}(s, x(s), x(h(s)))ds \\ + \sum_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i)I_i(x(\sigma_i)), \ \sigma \in [0, T]. \end{cases} \tag{5.1}$$

An approximate integral equation has an unique solution $x_n \in (\mathbb{H}_1)_\alpha(T)$,

$$x_n(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}\chi(0) + \int_0^\sigma (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)\mathcal{F}_n(s, x_n(s), x_n(h(s)))ds \\ + \sum_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i)I_{i,n}(x_n(\sigma_i)), \ \sigma \in [0, T]. \end{cases} \tag{5.2}$$

The Faedo-Galerkin Approximation is produced by applying the projection on (5.2) as $v_n(\sigma) = P^n x_n(\sigma)$,

$$P^n x_n(\sigma) = v_n(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}P^n\chi(0) + \int_0^\sigma (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)P^n\mathcal{F}(s, P^n x_n(s), P^n x_n(h(s)))ds \\ + \sum_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i)P^n I_{i,n}(x_n(\sigma_i)) \end{cases}$$

$$v_n(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}P^n\chi(0) + \int_0^\sigma (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)P^n\mathcal{F}(s, v_n(s), v_n(h(s)))ds \\ + \sum_{i=1}^{q} \mathcal{S}_\beta(\sigma - \sigma_i)P^n I_{i,n}(v_n(\sigma_i)),\ \sigma \in [0, T]. \end{cases} \tag{5.3}$$

Let solution $x(\cdot)$ of (5.1) and $v_n(\cdot)$ of (5.3) have the following representation:

$$x(\sigma) = \sum_{i=0}^{\infty} \alpha_i(\sigma)\phi_i,\ \alpha_i(\sigma) = (x(\sigma), \phi_i),\ i = 0, 1, 2, ..., \tag{5.4}$$

$$v_n(\sigma) = \sum_{i=0}^{n} \alpha_i^n(\sigma)\phi_i,\ \alpha_i^n(\sigma) = (v_n(\sigma), \phi_i),\ i = 0, 1, 2, ... \tag{5.5}$$

Using (5.5) in (5.3) and taking inner product with ϕ_i, we obtain a system of fractional order integro-differential equation of the form.

$$\begin{aligned} \frac{d^\beta}{d\sigma^\beta}\alpha_i^n(\sigma) + \lambda_i\alpha_i^n(\sigma) &= \mathcal{F}_i^n(\sigma, \alpha_0^n(\sigma),\ \alpha_1^n(\sigma), ..., \alpha_n^n(\sigma)), \\ \Delta\alpha_i^n(\sigma_k) &= I_i^n(\alpha_i^n(\sigma_k)),\ k = 1, 2, ...q, \\ \alpha_i^n(0) &= Z_i, \end{aligned} \tag{*}$$

Where,

$$\begin{aligned} \mathcal{F}_i^n(\sigma, \alpha_0^n(\sigma), \alpha_1^n(\sigma), ..., \alpha_n^n(\sigma)) &= \Big(\mathcal{M}^{-1}\mathcal{F}\Big(\sigma, \sum_{i=0}^{n} \alpha_i^n(\sigma)\phi_i, \sum_{i=0}^{n} \alpha_i^n(\sigma)\phi_i\Big), \phi_i\Big), \\ I_i^n &= \Big(I_k\Big(\sum_{k=1}^{q}\sum_{i=1}^{n} \alpha_i^n(\sigma_k)\phi_i\Big), \phi_i\Big), \\ Z_i &= (\chi(0), \phi_i),\ i = 1, 2, ..., n. \end{aligned}$$

We also have the following convergence theorem.

Theorem 5.1. *If the hypothesis (1)–(5) holds. The results follows:*
(i) If $x_0 \in D(\mathcal{E}^\alpha)$*, then for any* $\sigma_0 \in (0, T]$

$$\lim_{m\to\infty} \sup_{\{n\geq m, \sigma\in[\sigma_0,T]\}} \|\mathcal{E}^\alpha[v_n(\sigma) - v_m(\sigma)]\| = 0. \tag{5.6}$$

(ii) If $x_0 \in D(\mathcal{E})$*, then*

$$\lim_{m\to\infty} \sup_{\{n\geq m, \sigma\in[0,T]\}} \|\mathcal{E}^\alpha[v_n(\sigma) - v_m(\sigma)]\| = 0. \tag{5.7}$$

Proof. If $n \geq m,\ 0 \leq \alpha < \upsilon$. Then, we have

$$\begin{aligned}\|v_n(\sigma) - v_m(\sigma)\|_\alpha &= \|P^n x_n(\sigma) - P^m x_m(\sigma)\|_\alpha \\ &\leq \|P^n[x_n(\sigma) - x_m(\sigma)]\|_\alpha + \|(P^n - P^m)x_m\|_\alpha \\ &\leq \|x_n(\sigma) - x_m(\sigma)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}}\|x_m(\sigma)\|_\upsilon.\end{aligned}$$

By the Theorem 4.1 and Proposition 4.1, we have that $x_n \to x_m$ and $\lambda_m \to \infty$ as $m \to \infty$. □

Theorem 5.2. *Suppose (1)–(5) is fulfilled, $x_0 \in D(\mathcal{E}^\alpha)$, unique function $v_n \in (\mathbb{H}_1)_\alpha(T)$ exists, satisfying the following equation:*

$$v_n(\sigma) = \begin{cases} \mathcal{S}_\beta(\sigma)\mathcal{M}P^n x_0 + \int\limits_0^\sigma (\sigma - s)^{\beta-1}\mathcal{T}_\beta(\sigma - s)P^n\mathcal{F}(s, v_n(s))ds \\ + \sum\limits_{i=1}^q \mathcal{S}_\beta(\sigma - \sigma_i)P^n I_{i,n}(v_n(\sigma_i)),\ \sigma \in [0, T]. \end{cases} \tag{5.8}$$

Proof. For $x_0 \in D(\mathcal{E}^\alpha)$ and $\sigma \in [0, T]$. We have

$$\begin{aligned}\|v_n(\sigma) - x(\sigma)\|_\alpha &= \|P^n x_n(\sigma) - P^n x(\sigma) + P^n x(\sigma) - x(\sigma)\|_\alpha \\ &\leq \|P^n(x_n(\sigma) - x(\sigma))\|_\alpha + \|(P^n - I)x(\sigma)\|_\alpha.\end{aligned}$$

We have $v_n \to x$ as $n \to \infty$ according to the Theorem 4.2. As a result, the Theorem 4.2 leads to the conclusion. The preceding theorem can be used to show α_i^n to α_i's convergence. □

Theorem 5.3. *Suppose (1)–(5) holds. Then,*
(i) If $x_0 \in D(\mathcal{E}^\alpha)$, then for any $0 < \sigma_0 \leq T$

$$\lim_{n\to\infty} \sup_{\sigma\in[\sigma_0,T]} [\sum_{i=0}^n \lambda_i^{2\alpha}(\alpha_i(\sigma) - \alpha_i^n(\sigma))^2] = 0. \tag{5.9}$$

(ii) If $x_0 \in D(\mathcal{E})$, then

$$\lim_{n\to\infty} \sup_{\sigma\in[0,T]} [\sum_{i=0}^n \lambda_i^{2\alpha}(\alpha_i(\sigma) - \alpha_i^n(\sigma))^2] = 0. \tag{5.10}$$

Proof. The system (*) determines the α_i^n's. It can be easily investigated that

$$\begin{aligned}\mathcal{E}^\alpha[x(\sigma) - v(\sigma)] &= \mathcal{E}^\alpha[\sum_{i=0}^\infty (\alpha_i(\sigma) - \alpha_i^n(\sigma))\phi_i] \\ &= \sum_{i=0}^\infty \lambda_i^\alpha(\alpha_i(\sigma) - \alpha_i^n(\sigma))\phi_i.\end{aligned}$$

Thus,

$$\|\mathcal{E}^\alpha[x(\sigma) - v(\sigma)]\|^2 \geq \sum_{i=0}^n \lambda_i^{2\alpha}[\alpha_i(\sigma) - \alpha_i^n(\sigma)]^2.$$

As a result, the Theorems 5.1 and 5.2 lead to the conclusion. □

6. Applications

Let the fractional impulsive differential system of sobolev type is of the form:

$$^{c}D_{\sigma}^{\beta}[w(\sigma,x)-w_{xx}(\sigma,x)]+\frac{\partial^2 w(\sigma,x)}{\partial x^2}=f(\sigma,w(\sigma,x)),\ x\in\mathcal{S},\ \sigma\in[0,T], \tag{6.1}$$

$$\Delta w(\sigma_i,x)=\frac{2w(\sigma_i,x)}{2+w(\sigma_i,x)},\ i=1,2,...,q,\ x\in(0,\pi), \tag{6.2}$$

$$w(0,x)=w_0(x), \tag{6.3}$$

$$w(\sigma,0)=w(\sigma,\pi)=0,\ 0\le\sigma\le T,\ 0<T<\infty. \tag{6.4}$$

Where $^{c}D_{\sigma}^{\beta}$ is Caputo derivative, $\beta\in(0,1)$. Suppose $w(\sigma)(x)=w(\sigma,x)$ and $f(\sigma,\cdot)=\mathcal{F}(\sigma,\cdot)$. Let $\Delta w(\sigma_i,x)=w(\sigma_i^+,x)-w(\sigma_i^-,x)$, where $w(\sigma_i^+,x)$ and $w(\sigma_i^-,x)$ are respectively the right and the left hand limit of w at $(\sigma,x)=(\sigma_i,x)$.

Now, we take $\mathbb{H}_1=\mathbb{H}_2=L^2(0,\pi)$ and consider the operator $\mathcal{L},\mathcal{M}$ on domains and ranges contained in $L^2(0,\pi)$ defined by

$$\mathcal{M}y=y-y''$$
$$\mathcal{L}y=-y''$$

An infinitesimal generator of an analytic semigroup is denoted by $\mathcal{E}=\mathcal{L}\mathcal{M}^{-1}$, such that

$$\mathcal{E}y=-y'',$$

with the domain

$$D(\mathcal{E})=\{y\in\mathbb{H}_1: y,\ y'\text{are absolutely continuous } y''\in\mathbb{H}_1,\ y(0)=y(\pi)=0\}$$

If we take $\beta=\frac{1}{2}$, then $D(\mathcal{E}^{\frac{1}{2}})$ which is denoted by $\beta_{\frac{1}{2}}$ is the Banach space endowed with the norm,

$$||x||_{\frac{1}{2}}=||\mathcal{E}^{\frac{1}{2}}x||,\ x\in D(\mathcal{E}^{\frac{1}{2}}).$$

Also, for $\sigma\in[0,T]$. we define $D_{\sigma}^{\frac{1}{2}}=\{y: y$ is a map from $[0,T]$ into $\beta_{\frac{1}{2}}\ni x(\sigma)$ is continuous at $\sigma\neq\sigma_i$ left continuous at $\sigma=\sigma_i$ and right limit $x(\sigma_i^+)$ exists for $i=1,2,...,q\}$.

The spectrum of $\mathcal{E}$ is given by $\mathcal{E}y=-y''=\alpha y$. The general solution y of $\mathcal{E}y=\alpha y$ is

$$y(x)=C\cos(\sqrt{\alpha}x)+D\sin(\sqrt{\alpha}x).$$

Using the boundary conditions $y(0)=y(\pi)=0$. we obtain $C=0,\ \alpha=\alpha_n=n^2,\ n\in\mathbb{N}$. Thus for each $n\in\mathbb{N}$, the solution is given by $y_n(x)=D\sin nx$. If we take $D=\frac{\sqrt{2}}{\sqrt{\pi}}$, then $<y_n,y_m>=0,\ n\neq m$ and $<y_n,y_m>=1,\ n=m$. Thus $\mathcal{E}$ has pure point spectrum and eigenvalues y_n are orthonormal.

Suppose, $I_i(w(\sigma_i,x))=\frac{2w(\sigma_i,x)}{2+w(\sigma_i,x)}$. Let us define $y(\sigma)(x)=w(\sigma,x)$ and $I_i(w(\sigma_i,x))=I_i(x(\sigma_i))(y)$ then $I_i(y(\sigma_i))=\frac{2y(\sigma_i)}{2+y(\sigma_i)}$. For $y_1,y_2\in D(\mathcal{A}^{\frac{1}{2}})$, we have

$$||I_i(y_1)-I_i(y_2)||_{\frac{1}{2}}\le||y_1-y_2||_{\frac{1}{2}},$$

$$I_i(y_1)\|_{\frac{1}{2}} \leq \|y_1\|_{\frac{1}{2}}.$$

Now, we define

$$\begin{aligned} f(\sigma, x(\sigma)) &= \mathcal{F}(\sigma, w(\sigma, x)),\\ I_i(w(\sigma_i, x)) &= I_i(x(\sigma_i))(y),\\ \phi(\sigma)(x) &= g(\sigma, x), \end{aligned}$$

then problem (6.1)–(6.4) reduces to

$$\begin{aligned} {}^cD^{\beta}_{\sigma}[\mathcal{M}x(\sigma)] &= \mathcal{L}x(\sigma) + \mathcal{F}(\sigma, x(\sigma), x(h(\sigma))),\ \sigma \in [0, T],\\ \Delta x(\sigma_i) &= I_i(x(\sigma_i)),\ i = 1, 2, ..., q,\ q \in \mathbb{N},\\ \mathrm{g}(x) &= \phi. \end{aligned}$$

It is easy to see that the operator $\mathcal{E}$ fulfils (1). Also, by *Hölder* continuity of h, fulfils (3). Then, from the definitions, it can be easily shown that χ and I_i are satisfies (4) and (5).

Now we prove that $\mathcal{F}$ satisfies the condition (2):

$$\begin{aligned} &\|\mathcal{F}(\sigma_1, z_1, w_1) - \mathcal{F}(\sigma_2, z_2, w_2)\|_{L^2}\\ \leq &L\Big[\int_0^{\pi} |\mathcal{F}(\sigma_1(x,\sigma), z_1(x,\sigma), w_1(x,\sigma)) - \mathcal{F}(\sigma_2(x,\sigma), z_2(x,\sigma), w_2(x,\sigma))|^2 dx\Big]^{\frac{1}{2}}\\ \leq &L\Big[\int_0^{\pi} \{|(\sigma_1(x,\sigma) - \sigma_2(x,\sigma))| + \|(z_1(x,\sigma) - z_2(x,\sigma))\|^2\} dx\Big]^{\frac{1}{2}}\\ \leq &2L[|\sigma_1 - \sigma_2| + \|z_1 - z_2\|_{L^2}]. \end{aligned}$$

Hence (2) holds.

Thus, all the assumptions of Theorem 5.2 are satisfied. So, Theorem 5.2 guarantees the existence of Faedo-Galerkin approximations and their convergence to the unique solution of (6.1)–(6.4).

7. Conclusions

The Faedo-Galerkin approximation outcomes for Caputo fractional impulsive derivative of Sobolev type with nonlocal condition are the main subject of this paper. The major ideas were developed by utilising the analytic semigroup and the Banach fixed point theorem. Finally, we give examples to back up our abstract conclusion. In future, it might be used to find the generalization in fractional differential equations.

Acknowledgments

This research received funding support from the NSRF via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation (Grant number B05F650018).

Conflict of interest

The authors declare no conflicts of interest.

References

1. K. Kavitha, V. Vijayakumar, R. Udhayakumar, Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness, *Chaos Solitons Fractals*, **139** (2020), 110035. https://doi.org/10.1016/j.chaos.2020.110035
2. H. M. Ahmed, H. M. El-Owaidy, M. A. AL-Nahhas, Neutral fractional stochastic partial differential equations with Clarke subdifferential, *Appl. Anal.*, **100** (2021), 3220–3232. https://doi.org/10.1080/00036811.2020.1714035
3. K. S. Nisar, K. Jothimani, K. Kaliraj, C. Ravichandran, An analysis of controllability results for nonlinear Hilfer neutral fractional derivatives with non-dense domain, *Chaos Solitons Fractals*, **146** (2021), 110915. https://doi.org/10.1016/j.chaos.2021.110915
4. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.
5. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, In: *North-holland mathematics studies*, Amsterdam: Elsevier, **204** (2006), 1–523.
6. I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, In: *Mathematics in Science and Engineering*, San Diego: Academic Press, **198** (1999), 1–340.
7. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach Science Publishers, 1993.
8. V. Daftardar-Gejji, *Fractional calculus: theory and applications*, Narosa Publishing House, 2014.
9. P. Agarwal, D. Baleanu, Y. Q. Chen, S. Momani, J. A. T. Machado, Fractional Calculus, In: *Springer proceedings in mathematics and statistics*, 2019.
10. V. Vijayaraj, C. Ravichandran, T. Botmart, K. S. Nisar, K. Jothimani, Existence and data dependence results for neutral fractional order integro-differential equations, *AIMS Mathematics*, **8** (2023), 1055–1071. https://doi.org/10.3934/math.2023052
11. K. S. Nisar, C. Ravichandran, A. H. Abdel-Aty, I. S. Yahia, C. Park, Case study on total controllability and optimal control of Hilfer netural non-instantaneous fractional derivative, *Fractals*, **30** (2022), 2240187. https://doi.org/10.1142/S0218348X22401879
12. K. Kaliraj, K. S. Viswanath, K. Logeswari, C. Ravichandran, Analysis of fractional integro-differential equation with Robin boundary conditions using topological degree method, *Int. J. Appl. Comput. Math.*, **8** (2022), 176. https://doi.org/10.1007/s40819-022-01379-1
13. K. S. Nisar, K. Logeswari, V. Vijayaraj, H. M. Baskonus, C. Ravichandran, Fractional order modeling the Gemini virus in capsicum annuum with optimal control, *Fractal Fract.*, **6** (2022), 61. https://doi.org/10.3390/fractalfract6020061

14. K. Logeswari, C. Ravichandran, K. S. Nisar, Mathematical model for spreading of COVID-19 virus with the Mittag-Leffler kernel, *Numer. Methods Partial Differential Equations*, **24** (2020). https://doi.org/10.1002/num.22652
15. S. Belmor, C. Ravichandran, F. Jarad, Nonlinear generalized fractional differential equations with generalized fractional integral conditions, *J. Taibah Univ. Sci.*, **14** (2020), 114–123. https://doi.org/10.1080/16583655.2019.1709265
16. L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.*, **162** (1991), 494–505. https://doi.org/10.1016/0022-247X(91)90164-U
17. C. Ravichandran, K. Munusamy, K. S. Nisar, N. Valliammal, Results on neutral partial integrodifferential equations using Monch-Krasnosel'Skii fixed point theorem with nonlocal conditions, *Fractal Fract.*, **6** (2022), 75. https://doi.org/10.3390/fractalfract6020075
18. K. Kumar, R. Patel, V. Vijayakumar, A. Shukla, C. Ravichandran, A discussion on boundary controllability of nonlocal impulsive neutral integrodifferential evolution equations, *Math. Methods Appl. Sci.*, **45** (2022), 8193–8215. https://doi.org/10.1002/mma.8117
19. K. Kaliraj, M. Manjula, C. Ravichandran, New existence results on nonlocal neutral fractional differential equation in concepts of Caputo derivative with impulsive conditions, *Chaos Solitons Fractals*, **161** (2022), 112284. https://doi.org/10.1016/j.chaos.2022.112284
20. Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal. Real. Appl.*, **11** (2010), 4465–4475. https://doi.org/10.1016/j.nonrwa.2010.05.029
21. P. K. L. Priya, K. Kaliraj, An application of fixed point technique of Rothe's-type to interpret the controllability criteria of neutral nonlinear fractional ordered impulsive system, *Chaos Solitons Fractals*, **164** (2022), 112647. https://doi.org/10.1016/j.chaos.2022.112647
22. K. Kaliraj, E. Thilakraj, C. Ravichandran, K. S. Nisar, Controllability analysis for impulsive integro-differential equation via Atangana-Baleanu fractional derivative, *Mathe. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7693
23. X. P. Zhang, Y. X. Li, P. Y. Chen, Existence of extremal mild solutions for the initial value problem of evolution equations with non-instantaneous impulses, *J. Fixed Point Theory Appl.*, **19** (2017), 3013–3027. https://doi.org/10.1007/s11784-017-0467-4
24. A. Shukla, V. Vijayakumar, K. S. Nisar, A new exploration on the existence and approximate controllability for fractional semilinear impulsive control systems of order $r \in (1,2)$, *Chaos Solitons Fractals*, **154** (2022), 111615. https://doi.org/10.1016/j.chaos.2021.111615
25. K. D. Kucche, P. U. Shikhare, On impulsive delay integrodifferential equations with integral impulses, *Mediterr. J. Math.*, **17** (2020), 103.
26. A. Debbouche, J. J. Nieto, Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls, *Appl. Math. Comput.*, **245** (2014), 74–85. https://doi.org/10.1016/j.amc.2014.07.073
27. K. Liu, M. Feckan, J. R. Wang, A class of (ω, T)-periodic solutions for impulsive evolution equations of Sobolev type, *Bull. Iran. Math. Soc.*, **48** (2022), 2743–2763. https://doi.org/10.1007/s41980-021-00666-9

28. F. Li, J. Liang, H. K. Xu, Existence of mild solutions for fractional integro-differential equations of Sobolev type with nonlocal conditions, *J. Math. Anal. Appl.*, **391** (2012), 510–525. https://doi.org/10.1016/j.jmaa.2012.02.057

29. H. M. Ahmed, M. A. Ragusa, Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential, *B. Malays. Math. Sci. Soc.*, **45** (2022), 3239–3253. https://doi.org/10.1007/s40840-022-01377-y

30. H. M. Ahmed, Sobolev-type fractional stochastic integrodifferential equations with nonlocal conditions in Hilbert space, *J. Theoret. Probab.*, **30** (2017), 771–783.

31. R. Göthel, D. S. Jones, Faedo-Galerkin approximations in equations of evolution, *Math. Methods Appl. Sci.*, **6** (1984), 41–54. https://doi.org/10.1002/mma.1670060104

32. P. D. Miletta, Approximation of solutions to evolution equations, *Math. Methods Appl. Sci.*, **17** (1994), 753–763. https://doi.org/10.1002/mma.1670171002

33. M. Muslim, R. P. Agarwal, Approximation of solutions to impulsive functional differential equations, *J. Appl. Math. Comput.*, **34** (2010), 101–112. http://doi.org/10.1007/s12190-009-0310-1

34. A. Raheem, M. Kumar, Approximate solutions of nonlinear nonlocal fractional impulsive differential equations via Faedo-Galerkin method, *J. Fract. Calc. Appl.*, **12** (2021), 172–187.

35. M. M. Raja, V. Vijayakumar, R. Udhayakumar, Y. Zhou, A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces, *Chaos Solitons Fractals*, **141** (2020), 110310. https://doi.org/10.1016/j.chaos.2020.110310

36. A. Chaddha, D. N. Pandey, Approximations of solutions for an impulsive fractional differential equation with a deviated argument, *Int. J. Appl. Comput. Math.*, **2** (2016), 269–289. http://doi.org/10.1007/s40819-015-0059-1

37. A. Chadha, D. Bahuguna, D. N. Pandey, Faedo-Galerkin approximate solutions for nonlocal fractional differential equation of Sobolev type, *Fract. Differential Calc.*, **8** (2018), 205–222. https://doi.org/10.7153/fdc-2018-08-13

38. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New York: Springer, 1983.

39. K. Kaliraj, M. Manjula, C. Ravichandran, K. S. Nisar, Results on neutral differential equation of Sobolev type with nonlocal conditions, *Chaos Solitons Fractals*, **158** (2022), 112060. https://doi.org/10.1016/j.chaos.2022.112060

AIMS Press