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Research article

A study on controllability for Hilfer fractional differential equations with impulsive delay conditions

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Abstract: This article focuses on the controllability of a Hilfer fractional impulsive differential equation with indefinite delay. We analyze our major outcomes using fractional calculus, the measure of non-compactness and a fixed-point approach. Finally, an example is provided to show the theory.

Keywords: boundary condition; fractional calculus; implusive condition; integro-differential system; controllability; fixed-point theorem
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1. Introduction

Fractional calculus and its potential applications have grown in importance because fractional calculus has evolved into a powerful tool with more accurate and successful results in modeling various complex phenomena in a wide range of seemingly diverse and widespread fields of science and engineering. This technology could be used in physics, signal processing, wave propagation, robotics, and other fields [1–8] and there are research papers on the theory of fractional differential equations [9–38].

The evolution of a physical system in time is described by an initial and boundary value problem, i.e., a differential equation (ordinary or partial) and an initial or boundary condition. In many cases, it is better to have more information on the conditions. The local condition is replaced then by a non-local condition, which gives a better effect than the local initial or boundary condition, since the measurement given by a non-local condition is usually more precise than the only one measurement given by a local condition. The study of initial value problems with non-local conditions is of significance, since they have applications in problems in physics and other areas of applied mathematics.

Hilfer [24] developed a new sort of fractional derivative that combines Riemann-Liouville (R-L) and Caputo fractional derivative (FDs). Impulsive differential equaitions plays an important role in the real life applications. Many authors have examined the applications of this, see [19,20,22,23]. Inspired by this work, several scholars have recently expressed a strong interest in this area, and readers can consult past investigations [25, 29]. Various authors studied the outcomes of controllability results for linear and nonlinear integer-order differential equations in [10, 11, 14–17, 21, 26, 29–34].

Nonetheless, to the best of our knowledge, the topic of controllability addressed in this article has not been studied, which provides an impetus for our research. The Hilfer fractional implusive differential equation (H-FIDEs) have the following form:

$$D^{\zeta,\eta}_{0^+}u(\mathrm{t}) = Au(\mathrm{t}) + \mathfrak{F}(\mathrm{t}, u_{\mathrm{t}}) + Bv(\mathrm{t}), \quad \mathrm{t} \in \mathfrak{Y} = (0, p], \mathrm{t} \neq \mathrm{t}_k, \tag{1.1}$$

$$\Delta u|_{\mathrm{t}=\mathrm{t}_k} = I_k(u(\mathrm{t}_k^-)), \quad k = 1, 2, \ldots, m, \tag{1.2}$$

$$I^{(1-\zeta)(1-\eta)}_{0^+}u(\mathrm{t}) = \phi(\mathrm{t}) + \mathfrak{g}(y_{\mathrm{t}_1}, y_{\mathrm{t}_2}, y_{\mathrm{t}_3}, \cdots, y_{\mathrm{t}_m}) \in \mathcal{P}_{\mathfrak{g}}, \tag{1.3}$$

where $D^{\zeta,\eta}_{0^+}$ denotes the Hilfer FD of order ζ and type η. Also, $0 \le \zeta \le 1$; $\frac{1}{2} < \eta < 1$ and $(U, \|\cdot\|)$ is a Banach space and A denotes the infinitesimal generator of a strongly continuous functions of bounded linear operators $\{T(\mathrm{t})\}_{\mathrm{t}\ge 0}$ on U. A suitable function $\mathfrak{F} : \mathfrak{Y} \times \mathcal{P}_{\mathfrak{g}} \to U$ is connected with the phase space $u_\theta(\mathrm{t})$ with the mapping $u_{\mathrm{t}} : (-\infty, 0] \to U, u_\theta(\mathrm{t}) = u(\mathrm{t} + \theta), \theta \le 0$. Here, $v(\cdot)$ is provided in $L^2(\mathfrak{Y}, V)$, a Banach space of admissible control functions; $0 < \mathrm{t}_1 < \mathrm{t}_2 < \mathrm{t}_3 < \cdots, < \mathrm{t}_m \le p, \mathfrak{g} : \mathcal{P}_{\mathfrak{g}} \to \mathcal{P}_{\mathfrak{g}}$ denotes continuous functions.

The article is organized as follows: Section 2 introduces a few key notions and definitions related to our research that will be used throughout the discussion of this article. Section 3 is flipped to discuss the controllability results of the H-FIDEs. Finally, Section 4 provides an example to illustrate the theory.

2. Preliminaries

Now we recall some definitions, concepts, and lemmas chosen to achieve the desired outcomes. Let $PC(\mathfrak{Y}, U)$ be the Banach space of all continuous function spaces from $\mathfrak{Y} \to U$. Assume that $\gamma = \zeta + \eta - \zeta\eta$, In our case,$(1 - \gamma) = (1 - \zeta)(1 - \eta)$. Now, define $C_{1-\gamma}(\mathfrak{Y}, U) = \{u : \mathrm{t}^{1-\gamma}z(\mathrm{t}) \in PC(\mathfrak{Y}, U)\}$, along $\|\cdot\|_\gamma$ defined by $\|u\|_\gamma = \sup\{\mathrm{t}^{1-\gamma}\|u(\mathrm{t})\|,\ \mathrm{t} \in \mathfrak{Y}, \gamma = (\zeta + \eta - \zeta\eta)\}$. Clearly, $C_{1-\gamma}(\mathfrak{Y}, U)$ is a Banach space. We introduce $\mathfrak{F}$ with norm, $\|\mathfrak{F}\|_{L^\mu(\mathfrak{Y},R^+)}$, whenever $\mathfrak{F} \in L^\mu(\mathfrak{Y}, R^+)$ for some μ with $1 \le \mu \le \infty$.

We will now discuss some significant fractional calculus results (see Hilfer [24]).

Definition 2.1. *Let* $\mathfrak{F} : [p, +\infty) \rightarrow \mathbb{R}$ *and the integral*

$$I^{\eta}_{p^+}\mathfrak{F}(\mathrm{t}) = \frac{1}{\Gamma(\eta)} \int_p^{\mathrm{t}} \mathfrak{F}(\theta)(\mathrm{t} - \theta)^{\eta - 1} d\theta, \quad \mathrm{t} > p, \eta > 0$$

be called the left-sided R-L fractional integral of order η *having a lower limit* p *of a continuous function, where* $\Gamma(\cdot)$ *denotes the gamma function provided that the right-hand side exists.*

Definition 2.2. *Let* $\mathfrak{F} : [p, +\infty) \rightarrow \mathbb{R}$ *and the integral*

$$^{(R-L)}D^{\eta}_{p^+}\mathfrak{F}(\mathrm{t}) = \frac{1}{\Gamma(k - \eta)} \left(\frac{d}{d\mathrm{t}}\right)^k \int_p^{\mathrm{t}} \frac{\mathfrak{F}(\mathrm{t})}{(\mathrm{t} - \theta)^{k - \eta - 1}} d\mathrm{t}, \quad \mathrm{t} > p, k - 1 < \eta < k$$

be called the left-sided (R-L) fractional derivative of order $\eta \in [k - 1, k)$*, where* $k \in \mathbb{R}$*.*

Definition 2.3. *Let* $\mathfrak{F} : [p, +\infty) \rightarrow \mathbb{R}$ *and the integral*

$$D^{\zeta,\eta}_{p^+}\mathfrak{F}(\mathrm{t}) = \left(I^{\zeta(1-\eta)}_{p^+} D \left(I^{(1-\zeta)(1-\eta)}_{p^+}\mathfrak{F}\right)\right)(\mathrm{t})$$

be called the left-sided Hilfer-fractional derivative of order $0 \leq \zeta \leq 1$ *and* $0 < \eta < 1$ *function of* $\mathfrak{F}(t)$ *.*

Definition 2.4. *Let* $\mathfrak{F} : [p, +\infty) \rightarrow \mathbb{R}$ *and the integral*

$$^{C}D^{\mu}_{p^+}\mathfrak{F}(\mathrm{t}) = \frac{1}{\Gamma(k - \eta)} \frac{d^k}{d\mathrm{t}^k} \int_p^{\mathrm{t}} \mathfrak{F}^{(k)}(\mathrm{t})(\mathrm{t} - \theta)^{k - \eta - 1} d\mathrm{t}, \quad \mathrm{t} > p, k - 1 < \eta < k$$

be called the left-sided Caputo's derivative type of order $\eta \in (k - 1, k)$*, where* $k \in \mathbb{R}$*.*

Remark 2.5. *(i) The Hilfer FD coincides with the standard (R-L) FD; if* $\zeta = 0, 0 < \eta < 1$ *and* $p = 0$*, then*

$$D^{0,\eta}_{0^+}\mathfrak{F}(\mathrm{t}) = \frac{d}{d\mathrm{t}} I^{1-\eta}_{0^+}\mathfrak{F}(\mathrm{t}) =^{(R-L)} D^{\eta}_{0^+}\mathfrak{F}(\mathrm{t});$$

(ii) The Hilfer FD coincides with the standard Caputo derivative; if $\zeta = 1, 0 < \eta < 1$ *and* $p = 0$*, then*

$$D^{1,\eta}_{0^+}\mathfrak{F}(\mathrm{t}) = I^{1-\eta}_{0^+} \frac{d}{d\mathrm{t}}\mathfrak{F}(\mathrm{t}) =^{C} D^{\eta}_{0^+}\mathfrak{F}(\mathrm{t}).$$

Let us characterize the abstract phase space $\mathcal{P}_{\mathfrak{g}}$ and refer to [35] for more details. Consider that $\mathfrak{g} : (-\infty, 0] \rightarrow (0, +\infty)$ is continuous along $j = \int_{-\infty}^{0} \mathfrak{g}(\lambda) d\lambda < +\infty$. For each $k > 0$,

$$\mathcal{P} = \{\psi \ : [-i, 0] \rightarrow U \text{ such that } \psi(\lambda) \text{ is bounded and measurable}\},$$

along

$$\|\psi\|_{[-i,0]} = \sup_{\delta \in [-i,0]} \|\psi(\delta)\|$$

for all $\psi \in \mathcal{P}$.

Now, we define

$$\mathcal{P}_{\mathfrak{g}} = \left\{\psi : (-\infty, 0] \rightarrow U \text{ such that for any } i > 0, \psi|_{[-i,0]} \in \mathcal{P} \text{ and } \int_{-\infty}^{0} \mathfrak{g}(\delta) \|\psi\|_{[\delta,0]} d\delta < +\infty\right\},$$

provided that $\mathcal{P}_{\mathfrak{g}}$ is endowed along

$$\|\psi\|_{\mathcal{P}_{\mathfrak{g}}} = \int_{-\infty}^{0} \mathfrak{g}(\delta)\|\psi\|_{[\delta,0]}d\delta$$

for all $\psi \in \mathcal{P}_{\mathfrak{g}}$; therefore, $(\mathcal{P}_{\mathfrak{g}}.\|\cdot\|_{\mathcal{P}_{\mathfrak{g}}})$ is a Banach space.

Now, we discuss

$$\mathcal{P}_{\mathfrak{g}}{}' = \{u : (-\infty, p) \to U \text{ such that } u|_{\mathfrak{Y}} \in C(\mathfrak{Y}, U), u_0 = \psi \in \mathcal{P}_{\mathfrak{g}}, k = 0, 1, \ldots, n\},$$

where u_k is a limitation of u to $\mathfrak{Y} = (\lambda_k, \lambda_{k+1}]$ for $k = 0, 1, \ldots, n$.

Set $\|\cdot\|_p$ as semi-norm in $\mathcal{P}_{\mathfrak{g}}{}'$ defined by

$$\|u\|_p = \|\phi\|_{\mathcal{P}_{\mathfrak{g}}} + \sup \|u(\chi)\| : \chi \in [0, p]\}, u \in \mathcal{P}'_{\mathfrak{g}}.$$

Lemma 2.6. *Assuming $u \in \mathcal{P}_{\mathfrak{g}}{}'$; then, for $\lambda \in \mathfrak{Y}, u \in \mathcal{P}'_{\mathfrak{g}}$. Moreover,*

$$j|u(\lambda)| \leq \|u_\lambda\|_{\mathcal{P}_{\mathfrak{g}}} \leq \|\phi\|_{\mathcal{P}_{\mathfrak{g}}} + j \sup_{\delta\in[0,\lambda]} \|u(\delta)\|,$$

where

$$j = \int_{-\infty}^{0} \mathfrak{g}(\lambda)d\lambda < +\infty.$$

Lemma 2.7. *A continuous function $u : (-\infty, p] \to U$ is said to be an integral solution of H-FIDEs (1.1)–(1.3) if*

(i) $u : [0, p] \to U$ *is continuous;*
(ii) $I^{b}_{o^+}u(\mathrm{t}) \in D(A)$ *for* $\mathrm{t} \in [0, p]$*; and*
(iii) For $[0, p]$, *the system* $u(\mathrm{t})$ *satisfies*

$$\begin{aligned} u(\mathrm{t}) = &\frac{\phi_0}{\Gamma(\zeta(1-\eta)+\eta)}\mathrm{t}^{(\zeta-1)(1-\eta)} \\ &+ \mathfrak{g}(y_{\mathrm{t}_1}, y_{\mathrm{t}_2}, y_{\mathrm{t}_3}, \cdots, y_{\mathrm{t}_m}) \\ &+ \frac{1}{\Gamma(\eta)}\int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{(\eta-1)}\mathfrak{F}(\varrho, u_\varrho)d\varrho \\ &+ \frac{1}{\Gamma(\eta)}\int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{(\eta-1)}Bv(\varrho)d\varrho, \\ &+ \sum_{0<\mathrm{t}_i<\mathrm{t}} S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_i)I_i(u(\mathrm{t}_i^-)) \end{aligned}$$

for $\mathrm{t} \in \mathfrak{Y}$.

Remark 2.8. *We introduce the mild solution of the H-FIDEs by introducing the Wright function to $M(\psi)$.* (1.1)–(1.3) *as follows:*

$$M(\psi) = \sum_{k=1}^{\infty} \frac{(-\psi)^{k-1}}{(k-1)!\Gamma(1-k\eta)}, \quad 0 < \eta < 1, \psi \in C$$

and it satisfies

$$\int_0^\infty \psi^\varrho M(\psi)d\psi = \frac{\Gamma(1+\varrho)}{\Gamma(1+\eta\varrho)}$$

for $\psi \geq 0$.

Lemma 2.9. *If the H-FIDEs* (1.1)–(1.3) *are satisfied, then* $\exists\, \mathfrak{F} : \mathfrak{Y} \times \mathcal{P}_h \to U$*; we get*

$$u(\mathrm{t}) = S_{\zeta,\eta}(\mathrm{t})[\phi_0 + \mathrm{g}(y_{\mathrm{t}_1}, y_{\mathrm{t}_2}, y_{\mathrm{t}_3}, \cdots, y_{\mathrm{t}_m})] + \int_0^{\mathrm{t}} P_\eta(\mathrm{t})\mathfrak{F}(\mathrm{t}, u_\varrho)d\varrho$$
$$+ \int_0^{\mathrm{t}} P_\eta(\mathrm{t})Bv(\varrho)d\varrho + \sum_{0<\mathrm{t}_i<\mathrm{t}} S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_i)I_i(u(\mathrm{t}_i^-)),$$

where $\mathrm{t} \in \mathfrak{Y}$,

$$Q_\eta(\mathrm{t}) = \int_0^\infty \eta\psi M(\psi)S(\mathrm{t}^\eta\psi)d\psi$$

and

$$P_\eta(\mathrm{t}) = \mathrm{t}^{\eta-1}Q_\eta(\mathrm{t}); S_{\zeta,\eta}(\mathrm{t}) = I_{0^+}^{\zeta(1-\eta)}(\mathrm{t})\mathrm{t}^{\eta-1}Q_\eta(\mathrm{t}).$$

Definition 2.10. *A continuous function* $u : (-\infty, p] \to U$ *is defined as a mild solution of H-FIDEs* (1.1)–(1.3) *if* $u_0 = \phi(0) \in \mathcal{P}_{\mathrm{g}}$ *on* $(-\infty, 0]$ *that satisfies*

$$u(\mathrm{t}) = S_{\zeta,\eta}(\mathrm{t})[\phi(0) + \mathrm{g}(y_{\mathrm{t}_1}, y_{\mathrm{t}_2}, y_{\mathrm{t}_3}, \cdots, y_{\mathrm{t}_m})] + \int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)\mathfrak{F}(\varrho, u_\varrho)d\varrho$$
$$+ \int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)Bv(\varrho)d(\varrho)$$
$$+ \sum_{0<\mathrm{t}_i<\mathrm{t}} S_{\zeta,\eta}((\mathrm{t}-\mathrm{t}_i))I_i(u(\mathrm{t}_i^-)), \tag{2.1}$$

where $\mathrm{t} \in \mathfrak{Y}$,

$$S_{\zeta,\eta}(\mathrm{t}) = \int_0^\infty \chi_\eta(\psi)M(\mathrm{t}^\eta\psi)d\psi,$$
$$Q_\eta = \eta\int_0^\infty \psi\chi_\eta(\psi)M(\mathrm{t}^\eta\psi)d\psi$$

are the characteristic solution operators and for $\psi \in (0, \infty)$*,*

$$\chi_\eta(\psi) = \frac{1}{\eta}\psi^{-1-\frac{1}{\eta}}w_\eta(\psi^{-\frac{1}{\eta}}) \geq 0,$$
$$\overline{w}_\eta(\psi) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{n-1}\psi^{-n\eta-1}\frac{\Gamma(n\eta+1)}{n}\sin(n\pi\eta).$$

Here, χ_η *is a probalility denisty function(pdf) defined on* $(0, \infty)$*, that is,* $\chi_\eta(\psi) \geq 0, \psi \in (0, \infty)$ *and* $\int_0^\infty \chi_\eta(\psi)d\psi = 1$.

Remark 2.11. *For $v \in [0,1]$, we have*

$$\int_0^\infty \psi^v \chi_\eta(\psi) d\psi = \int_0^\infty \psi^{-\eta v} \overline{\psi}_\eta(\psi) d\psi = \frac{\Gamma(1+v)}{\Gamma(1+\eta v)}.$$

Lemma 2.12. *The functions $S_{\zeta,\eta}$ and Q_η satisfy the following:*

(i) Any fixed $t \geq 0, S_{\zeta,\eta}$ and Q_η are linear and bounded that is, for any $u \in U$,

$$\|S_{\zeta,\eta}(\mathrm{t})u\| \leq \frac{M\mathrm{t}^{\gamma-1}}{\Gamma(\zeta(1-\eta)+\eta)}\|u\| \quad \textit{and} \quad \|Q_{\eta(\mathrm{t})}u\| \leq \frac{M}{\Gamma(\eta)}\|u\|,$$

where $S_{\zeta,\eta}(\mathrm{t}) = I_{0^+}^{\zeta(1-\eta)} P_\eta(\mathrm{t})$ and $P_\eta(\mathrm{t}) = \mathrm{t}^{\eta-1} Q_\eta(\mathrm{t})$;

(ii) $\{S_{\zeta,\eta}(\mathrm{t})\}_{\mathrm{t}\geq 0}$ and $\{Q_\eta(\mathrm{t})\}_{\mathrm{t}\geq 0}$ are strongly continuous functions.

Lemma 2.13. *The H-FIDEs* (1.1)–(1.3) *are said to be controllable on $\mathfrak{Y}$ for every $\phi \in \mathcal{P}_{\mathfrak{g}}, u_1 \in U$; there exists $v \in L^2(\mathfrak{Y}, V)$ such that the mild solution $u(\mathrm{t})$ of (1.1)–(1.3) satisfies $u(p) = u_1$.*

Lemma 2.14. *$\{Q_\eta(\mathrm{t})\}_{\mathrm{t}\geq 0}$ and $\{S_{\zeta,\eta}(\mathrm{t})\}_{\mathrm{t}\geq 0}$ are strongly continuous, that is, for any $u \in U$, $0 < \mathrm{t}' < \mathrm{t}'' \leq p$,*

$$\|(\mathrm{t}')^{\eta-1} Q_\eta(\mathrm{t}')u - (\mathrm{t}'')^{\eta-1} Q_\eta(\mathrm{t}''u\| \to 0$$

and $\|S_{\zeta,\eta}(\mathrm{t}')u - S_{\zeta,\eta}(\mathrm{t}'')u\| \to 0$ as $\mathrm{t}'' \to \mathrm{t}'$.

We now present the basic result on measures of non-compactness (MNCs).

Definition 2.15. *([26]). Assume F^+ is the positive cone of ordered Banach space $(F, \leq)$. The value of F^+ is said to be an MNC on U of D defined on the set of all bounded subsets of U iff $D(\overline{co}\alpha) = D(\alpha)$ for all bounded subsets $\alpha \in U$, where $\overline{co}\phi$ is a closed convex hull of α. The measure of non-compactness ϕ is said to be the following:*

(i) Monotone iff, for all bounded subsets α,α_1, α_2 of U we have $(\alpha_1 \subseteq \phi_2) \Rightarrow (D(\alpha_1) \leq D(\alpha_2))$;
(ii) Non-singular iff $D(\{c\} \cup \alpha) = D(\alpha)$ for every $c \in U, \alpha \subset U$;
(iii) Regular iff $D(\alpha) = 0$ iff α is relatively compact in U.
The MNC of the Hausdorff $\mathbb{R}$ is defined on each bounded subset α of U by

$$\mathbb{R}(\alpha) = \inf\{D > 0 : \alpha \textit{ can be covered by a finite number of balls of radii smaller than } D\}$$

for all bounded subsets $\alpha, \alpha_1, \alpha_2$ of U;
(iv) $\mathbb{R}(\alpha_1 + \alpha_2) \leq \mathbb{R}(\alpha_1) + \mathbb{R}(\alpha_2)$, where $\alpha_1 + \alpha_2 = \{x_1 + x_2 : x_1 \in \alpha_1, x_2 \in \alpha_2\}$;
(v) $\mathbb{R}(\alpha_1 + \alpha_2) \leq \max\{\mathbb{R}(\alpha_1), \mathbb{R}(\alpha_2\}$;
(vi) $\mathbb{R}(\rho\alpha) \leq |\rho|\mathbb{R}(\alpha)$ for any $\rho \in \mathbb{R}$;
(vii) Let Z be a Banach space. If Q is Lipschitz continuous with the mapping $Q : E(Q) \subseteq U \to Z$ with $i > 0$, then $\mathbb{R}_Z(Q\alpha) \leq i\mathbb{R}(\alpha)$ for any bounded subset $\alpha \subseteq E(Q)$.

Lemma 2.16. *If $\mathbb{H} \subset C(\mathfrak{Y}, U)$ is bounded and eqicontinuous, then $\mathrm{t} \to \mathbb{R}(\mathbb{H}(\mathrm{t}))$ is continuous for any $\mathrm{t} \in \mathfrak{Y}$,*

$$\mathbb{R}(\mathbb{H}) = \sup_{\mathrm{t}\in\mathfrak{Y}}\{\mathbb{R}(\mathbb{H}(\mathrm{t})), \mathrm{t} \in \mathfrak{Y}\},$$

where $\mathbb{H}(\mathrm{t}) = \{u(\mathrm{t}) : u \in \mathbb{H}\} \subseteq U$.

Theorem 2.17. $\{v_m\}_{m=1}^{\infty}$ *is a sequence of Bochner integrable functions from* $\mathfrak{Y} \rightarrow U$ *with the estimation* $\|v_m(\mathrm{t})\| \leq \epsilon(\mathrm{t})$ *for almost all* $\mathrm{t} \in \mathfrak{Y}$ *and every* $m \geq 1$*, where* $\epsilon \in L'(\mathfrak{R}, R)$*; then,* $\alpha(\mathrm{t}) = \mathbb{R}(\{v_m(\mathrm{t}) : m \geq 1\}) \in L^1(\mathfrak{R}, R)$ *and satisfies* $\mathbb{R}\left(\{\int_0^{\mathrm{t}} v_m(\varrho)d\varrho : m \geq 1\}\right) \leq 2\int_0^{\mathrm{t}} \phi(\varrho)d\varrho$.

Lemma 2.18. *Suppose* F *is a closed convex subset of* U *and* $\mathrm{t} \in F$*,* $X : E \rightarrow \mathfrak{Y}$ *is continuous which fulfills Monch's cndition, i.e.,* $P \subseteq F$ *is countable,* $P \subseteq \overline{co}(0) \cup G(P)) \Rightarrow \overline{P}$ *is compact. Then,* X *has a fixed point in* F.

3. Controllability results

This section is mainly focusing on the mild solutions of H-FIDEs (1.1)–(1.3). Consider the following assumptions for the discussion of H-FIDEs (1.1)–(1.3):

(H_0) For all bounded subsets $F \subset U$ and $u \in F$,

$$\|T(\mathrm{t}_2{}^{\eta}\varrho)u - T(\mathrm{t}_1{}^{\eta}\varrho)u\| \rightarrow 0, as \mathrm{t}_2 \rightarrow \mathrm{t}_1$$

for each fixed $\varrho \in (0, \infty)$.

(H_1) $\mathfrak{F} : [0, p] \times \mathcal{P}_h \rightarrow U$ fulfiles the following:

(i) Let $\mathfrak{F}(\cdot, \phi)$ be a measurable function $\forall \phi \in \mathcal{P}_{\mathrm{g}}$ and $\mathfrak{F}(\mathrm{t}, \cdot)$ be continuous for $\mathrm{t} \in \mathfrak{Y}$ and for $u \in \mathcal{P}_{\mathrm{g}}, G(\cdot, \cdot) : [0, T] \rightarrow U$ is strongly measurable.

(ii) $\exists\, q_1 \in (0, \eta), \eta \in (0, 1)$ and $l_1 \in L^{\frac{1}{q_1}}(U, \mathbb{R}^+)$ and $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \ni \|G(\mathrm{t}, \phi)\| \leq l_1(\mathrm{t})\psi(\mathrm{t}^{1-\gamma}\|\phi\|\mathcal{P}_{\mathrm{g}}$, for all $(\mathrm{t}, \phi) \in \mathfrak{Y} \times \mathcal{P}_{\mathrm{g}}$ where Φ satisfies $\liminf_{m\rightarrow\infty} \frac{\Phi(m)}{m} = 0$.

(iii) $\exists\, q_2 \in (0, \eta)$ and $l_2 \in L^{\frac{1}{q_1}}(\mathfrak{Y}, \mathbb{R}^+)$ such that for any bounded subset $G_1 \subset \mathcal{P}_{\mathrm{g}}$,

$$\mathbb{R}(\mathfrak{F}(\mathrm{t}, G_1)) \leq l_2(\mathrm{t})\left[\sup_{-\infty<\alpha\leq 0} \mathbb{R}(G_1(\rho))\right]$$

for a.e. $\mathrm{t} \in \mathfrak{Y}$, where $G_1(\rho) = \{D(\rho) \in E_1\}$ and $\mathbb{R}$ is the Hausdroff measure of non-compactness.

(iv) Let $I_i : F \mapsto F$ denote continuous functions and there exists a constant $N > 0$ such that, for all $\mathrm{t} \in \mathfrak{Y}$, we have $\|I_i(u_1) - I_i(u_2)\| \leq N\|u_1 - u_2\|$.

(H_2) The operator $\mathcal{W} : L^2(\mathfrak{Y}, V) \rightarrow U$ which is bounded and defined by

$$\mathcal{W}v = \int_0^p (p - \varrho)^{\eta-1} Q_\eta(\mathrm{t} - \varrho)Bv(\varrho)d\varrho,$$

satisfies the following:

(i) The bounded linear operator $\mathcal{W}$ having an inverse$\mathcal{W}^{-1}$ takes value in $L^2(\mathfrak{Y}, V)/Ker\mathcal{W}$; there exist $N_b > 0$and $N_w > 0$, such that $\|B\| \leq N_b$ and $\|\mathcal{W}^{-1}\| \leq N_w$.

(ii) For $q_3 \in (0, \eta)$ and for every bounded subset $F \in U, \exists\, l_2 \in L^{\frac{1}{q_3}}(J, \mathbb{R}^+)$ such that $\mathbb{R}((\mathcal{W}^{-1})(\mathrm{t})) \leq l_3(\mathrm{t})\mathbb{R}(F)$. Here, $l_i \in L^{\frac{1}{q_3}}(J, \mathbb{R}^+) and\ q_i \in (0, \eta),\ i = 1, 2, 3$.

(H_3) The function $\mathrm{g} : \mathcal{P}^n \rightarrow \mathcal{P}$ is continuous; there exists $L_i(\mathrm{g}) > 0$ such that

$$\|\mathrm{g}(v_1, v_2, \cdots, v_n) - \mathrm{g}(w_1, w_2, \cdots, w_n)\| \leq \sum_{i=1}^{m} L_i(\mathrm{g})\|v_i - w_i\|_{\mathcal{P}_{\mathrm{g}}},$$

for all $v_i, w_i \in \mathcal{P}_{\mathrm{g}}$ and consider $N_{\mathrm{g}} = \sup\{\|\mathrm{g}(v_1, v_2, \cdots, v_m)\| : v_i \in \mathcal{P}_{\mathrm{g}}\}$.

Let us introduce

$$N_1 = k_1\|l_1\|_{L^{\frac{1}{q_1}}(\mathfrak{Y},\mathbb{R}^+)}, N_2 = k_2\|l_2\|_{L^{\frac{1}{q_2}}(\mathfrak{Y},\mathbb{R}^+)},$$
$$N_3 = k_3\|l_1\|_{L^{\frac{1}{q_3}}(\mathfrak{Y},\mathbb{R}^+)},$$
$$k_i = \left[\left(\tfrac{1-q_i}{\eta-q_i}\right)p^{\frac{\eta-q_i}{1-q_i}}\right]^{1-q_i}, i = 1,2,3, K = \tfrac{\eta-1}{1-q},$$
$$N^* = \frac{p^{(1+K)(1-q)}}{(1+K)^{(1-q)}}, q, q_i \in (0,\eta).$$

Theorem 3.1. *Suppose* (H_0)–(H_2) *are satisfied;then, the H-FIDEs* (1.1)–(1.3) *are controllable on* $[0,p]$ *if*

$$C^*\frac{2NN_2p^{1-\gamma}}{\Gamma(\eta)}\left[1+\frac{2NN_bN_3}{\Gamma(\eta)}\right] < 1 for some \frac{1}{2} < \eta < 1. \tag{3.1}$$

Proof. By using (H_2), we define the control $v_u(\mathrm{t})$ by

$$v_u(\mathrm{t}) = \mathcal{W}^{-1}$$
$$\times\left[u_1 - S_{\zeta,\eta}[\phi(0)+\mathrm{g}(y_{\mathrm{t}_1},y_{\mathrm{t}_2},y_{\mathrm{t}_3},\cdots,y_{\mathrm{t}_m}) - \int_0^p(p-\varrho)^{\eta-1}Q_\eta(p-\varrho)\mathfrak{F}(\varrho,u_\varrho)d\varrho + \sum_{0<\mathrm{t}_i<\mathrm{t}}S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_i)I_i(u(\mathrm{t}_i^-))\right](\mathrm{t}).$$

Let $\alpha : \mathcal{P}_{\mathrm{g}}{}' \to \mathcal{P}_{\mathrm{g}}{}'$ be defined by

$$\alpha u(\mathrm{t}) = \begin{cases}\phi(\mathrm{t}), \quad \mathrm{t}\in(-\infty,0]\\ S_{\zeta,\eta}(\mathrm{t})[\phi(0)+\mathrm{g}(y_{\mathrm{t}_1},y_{\mathrm{t}_2},y_{\mathrm{t}_3},\cdots,y_{\mathrm{t}_m})] + \int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)\mathfrak{F}(\varrho,u_\varrho)d\varrho\\ +\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)Bv_u(\varrho)d\varrho\\ +\sum_{0<\mathrm{t}_i<\mathrm{t}}S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_i)I_i(u(\mathrm{t}_i^-)), \quad \mathrm{t}\in\mathfrak{Y}.\end{cases} \tag{3.2}$$

For $\phi \in \mathcal{P}_{\mathrm{g}}$, we define ϕ by

$$\hat{\phi}(\mathrm{t}) = \begin{cases}\phi(\mathrm{t}), \quad \mathrm{t}\in(-\infty,0]\\ S_{\zeta,\eta}(\mathrm{t})\phi(0), \quad \mathrm{t}\in\mathfrak{Y};\end{cases}$$

then $\phi \in \mathcal{P}_{\mathrm{g}}{}'$. Let $u(\mathrm{t}) = \mathrm{t}^{1-\gamma}[w(\mathrm{t})+\widehat{\phi}(\mathrm{t})]$, $-\infty < \mathrm{t} \le p$. It can be easily shown that u from (2.1) iff w satisfied $w_0 = 0$ and

$$\begin{aligned}\mathcal{W}(\mathrm{t}) = &\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)\mathfrak{F}(\varrho,\varrho^{1-\gamma}[w_\varrho+\widehat{\phi_\varrho}])d\varrho\\ &+\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)Bv_w(\varrho)d\varrho\\ &+\sum_{0<\mathrm{t}_i<\mathrm{t}}S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_i)I_i(u(\mathrm{t}_i^-)),\end{aligned}$$

where

$$v_w(\mathrm{t}) = \mathcal{W}^{-1}[u_1 - S_{\zeta,\eta}(p)[\phi(0) + \mathrm{g}(y_{\mathrm{t}_1}, y_{\mathrm{t}_2}, y_{\mathrm{t}_3}, \cdots, y_{\mathrm{t}_m})] - \int_0^p (p-\varrho)^{\eta-1} Q_\eta(p-\varrho)$$
$$\times \mathfrak{F}(\varrho, \varrho^{1-\gamma}[\mathcal{W}_\varrho + \widehat{\phi_\varrho}])d\varrho + \sum_{0<\mathrm{t}_i<\mathrm{t}} S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_i) I_i(u(\mathrm{t}_i^-))](\mathrm{t}).$$

Let $\mathcal{P}_{\mathrm{g}}'' = \{w \in \mathcal{P}_{\mathrm{g}}' : w_0 = 0 \in \mathcal{P}_{\mathrm{g}}\}$. For any $w \in \mathcal{P}_{\mathrm{g}}''$,

$$\begin{aligned}\|w\|_p &= \|w_0\|_{\mathcal{P}_{\mathrm{g}}} + \sup\{\|w(\varrho)\|\} : 0 \le \varrho \le p\} \\ &= \sup\{\|w(\varrho)\| : 0 \le \varrho \le p\}.\end{aligned}$$

Hence, $(\mathcal{P}_{\mathrm{g}}'', \|.\|_p)$ is a Banach space. Now, $q > 0$; choose $G_q = \{w \in \mathcal{P}_{\mathrm{g}}'' : \|w\|_p \le q\}$; then, $G_q \subseteq \mathcal{P}_{\mathrm{g}}''$ is uniformly bounded, and for $w \in G_q$, in view of Lemma 2.6,

$$\|w_{\mathrm{t}} + \|\widehat{\phi_{\mathrm{t}}}\|_{\mathcal{P}_{\mathrm{g}}} \le \|w_{\mathrm{t}}\|_{\mathcal{P}_{\mathrm{g}}} + \|\widehat{\phi_{\mathrm{t}}}\|_{\mathcal{P}_{\mathrm{g}}} \le j\left(q + \frac{M|\phi|}{\Gamma(\zeta(1-\eta)+\eta)}\right) + \|\phi\|_{\mathcal{P}_{\mathrm{g}}} = q'. \tag{3.3}$$

Let us introduce an operator $\tilde{\Phi} : \mathcal{P}_{\mathrm{g}}'' \to \mathcal{P}_{\mathrm{g}}''$, defined by

$$\tilde{\Phi}w(\mathrm{t}) = \begin{cases} 0, \mathrm{t} \in (-\infty, 0], \\ \int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{\eta-1} Q_\eta(\mathrm{t}-\varrho)\mathfrak{F}(\varrho, \varrho^{1-\gamma}[w_\varrho + \widehat{\phi_\varrho}])d\varrho \\ + \int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{\varrho-1} Q_\eta(\mathrm{t}-\varrho) B v_w(\varrho) d\varrho + \\ \sum_{0<\mathrm{t}_i<\mathrm{t}} S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_i) I_i(u(\mathrm{t}_i^-)), \mathrm{t} \in \mathfrak{Y}. \end{cases} \tag{3.4}$$

Next, to prove that $\tilde{\Phi}$ has a fixed point, our proof contains the subsequent four steps.

Step 1. Let us prove that there exists a $q > 0$ such that $\tilde{\Phi}(G_q) \subseteq G_q$. If not, then $\exists w^q \in G_q$. But $\tilde{\Phi}(w^q) \notin G_q$ that is $\|(\tilde{\Phi}w^q)(\mathrm{t})\| > q$ for all $\mathrm{t} \in \mathfrak{Y}$.

Choose $q > 0$, and let $\{G_q = u \in C : \|u\|_\gamma \le q\}$. Obviously, G_q is a closed, bounded and convex set of C. Therefore,

$$\|\tilde{\Phi}(u^q)\|_\gamma \equiv \sup\{\mathrm{t}^{1-\gamma}\|\tilde{\Phi}(u^q)(\mathrm{t})\|, \mathrm{t} \in \mathfrak{Y} : \|\tilde{\Phi}(u^q)(\mathrm{t})\| > q\}.$$

By using Hölder's-inequality, Lemma 2.12, (H_1) and (H_2), we get

$$\begin{aligned} q &< \sup_{\mathrm{t}\in\mathfrak{F}} \mathrm{t}^{1-\gamma}\|\tilde{\Phi}(\omega^q)(\mathrm{t})\| \\ &\le p^{1-\gamma}\left\|\int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{\eta-1} Q_\eta(\mathrm{t}-\varrho)\mathfrak{F}(\varrho, \varrho^{1-\gamma}[w_\varrho{}^q + \widehat{\phi_\varrho}])d\varrho\right\| \\ &+ p^{1-\gamma}\left\|\int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{\eta-1} Q_\eta(\mathrm{t}-\varrho) B v_{\omega^q}(\varrho) d\varrho\right\| \\ &+ p^{1-\gamma} \sum_{0<\mathrm{t}_k<\mathrm{t}} \|S_{\zeta,\eta}(\mathrm{t}) I_i(u(\mathrm{t}_i^-))\| \\ &\le \frac{Np^{1-\gamma}}{\Gamma(\eta)}\left\|\int_0^{\mathrm{t}} (\mathrm{t}-\varrho)^{\eta-1}\mathfrak{F}(\varrho, \varrho^{1-\gamma}[\omega_\varrho{}^q + \tilde{\phi}_\varrho])d\varrho\right\| \end{aligned}$$

$$
\begin{aligned}
&+\frac{Np^{1-\gamma}}{\Gamma(\eta)}\left\|\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}Bv_{\omega^q}(\varrho)d\varrho\right\| \\
&+\frac{Nt^{\beta-1}p^{1-\gamma}}{\Gamma(\zeta(1-\eta)+\eta)}\sum_{0<\mathrm{t}_i<\mathrm{t}}\|I_i(u(\mathrm{t}_i^-))\| \\
&\leq \frac{Np^{1-\gamma}}{\Gamma(\eta)}\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}l_1\Phi(q')d\varrho \\
&+\frac{Np^{1-\gamma}}{\Gamma(\eta)}\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}\|B\mathcal{W}^{-1}(\times)[u(p)-S_{\zeta,\eta}(\mathrm{t})[\phi(0)+\mathrm{g}(y_{\mathrm{t}_1},y_{\mathrm{t}_2},y_{\mathrm{t}_3},\cdots,y_{\mathrm{t}_m})] \\
&-\int_0^p(p-\varrho)^{\eta-1}Q_\eta(p-\varrho)\mathfrak{F}(\varrho,\varrho^{1-\gamma}[\mathcal{W}_\varrho^q+\hat{\phi}_\varrho])d\varrho\|](\varrho)d\varrho+\frac{NN^{'}t^{\beta-1}p^{1-\gamma}}{\Gamma(\zeta(1-\eta)+\eta)}\|u\| \\
&\leq \frac{Np^{1-\gamma}}{\Gamma(\eta)}\|\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}l_1\psi(q')d\varrho+\frac{NN_bN_\omega[p^{1-\gamma}}{\Gamma(\eta)}\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}\|u_1\| \\
&+\frac{Np^{\gamma-1}}{\Gamma(\zeta(1-\eta)+\eta)}\|\phi(0)\|+\frac{N}{\Gamma(\eta)}\int_0^d(d-\varrho)^{\eta-1}l_1\Phi(q')d\varrho]d\varrho+\frac{NN^{'}t^{\beta-1}p^{1-\gamma}}{\Gamma(\zeta(1-\eta)+\eta)}\|u\| \\
&\leq \frac{NN_1p^{1-\gamma}}{\Gamma(\eta)}\Phi(q')+\frac{NN_bM_\omega}{\Gamma(\eta)}N^*\Big[p^{1-\eta}\|u^1\| \\
&+\frac{N}{\Gamma(\zeta(1-\eta)+\eta)}\|\phi)(0)\|+\frac{NN_1p^{1-\gamma}}{\Gamma(\eta)}\Phi(q')\Big]+\frac{NN^{'}t^{\beta-1}p^{1-\gamma}}{\Gamma(\zeta(1-\eta)+\eta)}\|u\|\mathrm{t}\in\mathfrak{Y}. \qquad (3.5)
\end{aligned}
$$

Divide (3.5) by q, and letting $q \to \infty$, we have

$$1 \leq \frac{NN_1p^{1-\gamma}}{\Gamma(\eta)}\Phi(q')\Big(1+\frac{NN_bN_\omega}{\Gamma(\eta)}N^*\Big), \mathrm{t}\in\mathfrak{Y}. \qquad (3.6)$$

and then by (H_1)(ii), (3.6) is a contradiction. Hence, $\tilde{\Phi}(G_q) \subseteq G_q$

Step 2. $\tilde{\Phi}$ is continuous on G_q. For any $\omega^m, \omega \in G_q(\mathfrak{Y}),\ m = 0, 1, 2, \cdots$ with $\lim_{m\to\infty}\omega^m = \omega$, then we have $\lim_{m\to\infty}\omega^m = \omega(\mathrm{t})$ and

$$\lim_{m\to\infty}\mathrm{t}^{1-\gamma}\omega^m(\mathrm{t}) = \mathrm{t}^{1-\gamma}\omega(\mathrm{t}).$$

Let $u(\mathrm{t}) = \mathrm{t}^{1-\gamma}[\omega(\mathrm{t})+\hat{\phi}(\mathrm{t})]$; then, $\{\omega^m+\hat{\phi}\} \subset G_q$ with $\omega^m+\hat{\phi} \to \omega+\hat{\phi}$ in G_q as $m \to \infty$. Then, we have

$$
\begin{aligned}
&\mathfrak{F}(\mathrm{t},u_m(\mathrm{t})) = \mathfrak{F}(\mathrm{t},\mathrm{t}^{1-\gamma}[w^m(\mathrm{t})+\hat{\phi}(\mathrm{t})]) \to \\
&\qquad \mathfrak{F}(\mathrm{t},\mathrm{t}^{1-\gamma}[w(\mathrm{t})+\hat{\phi}(\mathrm{t})]) = \mathfrak{F}(\mathrm{t},u(\mathrm{t})), \text{as}, m\to\infty,
\end{aligned}
$$

where $\mathfrak{F}(\mathrm{t},\mathrm{t}^{1-\gamma}[\omega^m(\mathrm{t})+\hat{\phi}(\mathrm{t})]) = G_m(\varrho)$ and $\mathfrak{F}(\mathrm{t},\mathrm{t}^{1-\gamma}[\omega(\mathrm{t})+\hat{\phi}(\mathrm{t})]) = G(\varrho)$. Then, by using the hypotheses (H_1) and Lebesgue's dominated convergence theorem, we have

$$\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}\|G_m(\varrho)-G(\varrho)\|d\varrho \to 0 \ as\ m\to\infty, \mathrm{t}\in\mathfrak{Y}. \qquad (3.7)$$

Now, by (H_1),

$$
\begin{aligned}
\|\tilde{\Phi}\omega^m - \tilde{\Phi}w\|_C &\leq p^{1-\gamma}\|\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)[\mathfrak{F}(\varrho,\varrho^{1-\gamma}[\omega_\varrho^m+\hat{\phi}_\varrho])-\mathfrak{F}(\varrho,\varrho^{1-\gamma}[\omega_\varrho+\hat{\phi}_\varrho])]d\varrho\| \\
&+p^{1-\gamma}\|B\|\|\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}Q_\eta(\mathrm{t}-\varrho)B[v_{\omega^m}(\varrho)-v_\omega(\varrho)]d\varrho\| \\
&+p^{1-\gamma}\|S_{\zeta,\eta}(\mathrm{t})\|\sum_{0<\mathrm{t}_i<\mathrm{t}}\|I_i(u_m(\mathrm{t}_i^-))-I_i(u(\mathrm{t}_i^-))\| \\
&\leq \frac{Np^{1-\gamma}}{\Gamma(\eta)}\Big\|\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}[\mathfrak{F}(\varrho,\varrho^{1-\gamma}[\omega_\varrho^m+\tilde{\phi}_\varrho])]-\mathfrak{F}(\varrho,\varrho^{1-\gamma}[w_\varrho+\tilde{\phi})]d\varrho\Big\| \\
&+\frac{NN_bp^{1-\gamma}}{\Gamma\eta}\Big\|\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}[v_{\omega^m}(\varrho)-v_\omega(\varrho)]d\varrho\| \\
&+\frac{NN't^{\beta-1}p^{1-\gamma}}{\Gamma(\zeta(1-\eta)+\eta)}\|u_m(\mathrm{t}_i^-)-u(\mathrm{t}_i^-)\| \\
&\leq \frac{Np^{1-\gamma}}{\Gamma(\eta)}\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1}[G_m(\varrho)-G(\varrho)]d\varrho \\
&+\frac{N^2N_bN_\omega p^{1-\eta}}{\Gamma(\eta)^2}\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\eta-1} \\
&(\times)(\int_0^p(p-\varrho)^{\eta-1})|G_m(\varrho)-G(\varrho)]d\varrho)d\varrho+\frac{NN't^{\beta-1}p^{1-\gamma}}{\Gamma(\zeta(1-\eta)+\eta)}\|u_m(\varrho)-u(\varrho)\|. \quad (3.8)
\end{aligned}
$$

Observing (3.7) and (3.8), we have $\|\tilde{\Phi}\omega^n-\tilde{\Phi}\omega\|_C \to 0, m \to \infty$, Therefore, $\tilde{\Phi} \in \Phi(G_q)$ is continuous on G_q.

Step 3. $\tilde{\Phi}(G_q)$ is equi-continuous on $\mathfrak{Y}$. for all $\alpha \in \tilde{\Phi}(G_q)$ such that $\|\alpha(\mathrm{t}_2)-\alpha(\mathrm{t}_1)\| \to 0$ as $\mathrm{t}_2 \to \mathrm{t}_1$.

$$
\begin{aligned}
\alpha(\mathrm{t}) &= S_{\zeta,\eta}(\mathrm{t})[\phi_0+\mathrm{g}(y_{\mathrm{t}_1},y_{\mathrm{t}_2},y_{\mathrm{t}_3},\cdots,y_{\mathrm{t}_m})]+\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\mu-1}Q_\eta(\mathrm{t}-\varrho)G(\varrho)d\varrho \\
&+\int_0^{\mathrm{t}}(\mathrm{t}-\varrho)^{\mu-1}Q_\eta(\mathrm{t}-\varrho)Bv_\omega(\varrho)d\varrho \\
&+\sum_{0<\mathrm{t}_k<\mathrm{t}}S_{\zeta,\eta}(\mathrm{t}-\mathrm{t}_k)I_i(u(\mathrm{t}_i^-)).
\end{aligned}
$$

Let $0<\epsilon<\mathrm{t}$ and $0<\mathrm{t}_1<\mathrm{t}_2<p$. Then, $\tilde{\Phi}(G_q)$ is equicontinuous on $\mathfrak{Y}$.

$$
\begin{aligned}
\|\alpha(\mathrm{t}_2)-\alpha(\mathrm{t}_1)\| &= \|\mathrm{t}_2^{1-\gamma}\int_0^{\mathrm{t}_2}(\mathrm{t}_2-\varrho)^{\eta-1}Q_\eta(\mathrm{t}_2-\varrho)[G(\varrho)+Bv_\omega(\varrho)]d\varrho \\
&-\mathrm{t}_1^{1-\gamma}\int_0^{\mathrm{t}_1}(\mathrm{t}_1-\varrho)^{\eta-1}Q_\eta(\mathrm{t}_1-\varrho)[G(\varrho)+Bv_\omega(\varrho)]d\varrho\| \\
&+\sum_{0<\mathrm{t}_i<\mathrm{t}_2-\mathrm{t}_1}\|S_{\zeta,\eta}(\mathrm{t}_y)-S_{\zeta,\eta}(\mathrm{t}_g)\|\|I_iu(\mathrm{t}_i^-)\| \\
&\leq \mathrm{t}_2^{1-\gamma}\|\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-\varrho)^{\eta-1}Q_\eta(\mathrm{t}_2-\varrho)[G(\varrho)+Bv_\omega(\varrho)]d\varrho\|
\end{aligned}
$$

$$+\|\int_{t_1-\epsilon}^{t_1} t_2^{1-\gamma}(t_2-\varrho)^{\eta-1}[Q_\eta$$
$$\times(t_2-\varrho)-Q_\eta(t_1-\varrho)][G(\varrho)+Bv_\omega(\varrho)]d\varrho\|$$
$$+\|\int_{t_1-\epsilon}^{t_1}[t_2^{1-\gamma}(t_2-\varrho)^{\eta-1}-t_1^{1-\gamma}(t_1-\varrho)^{\eta-1}]$$
$$\times Q_\eta(t_1-\varrho)[G(\varrho)+Bv_\omega(\varrho)]d\varrho\|$$
$$+\|\int_0^{t_1-\epsilon} t_2^{1-\gamma}(t_2-\varrho)^{\eta-1}[Q_\eta(t_2-\varrho)-Q_\eta(t_1-\varrho)]$$
$$\times[G(\varrho)+Bv_\omega(\varrho)]d\varrho\|+\|\int_0^{t_1-\epsilon}[t_2^{1-\gamma}(t_2-\varrho)^{\eta-1}-t_1^{1-\gamma}$$
$$\times(t_1-\varrho)^{\eta-1})[Q_\eta(t_1-\varrho)[G(\varrho)+Bv_\omega(\varrho)]d\varrho\|++\frac{NN'}{\Gamma(\zeta(1-\eta)+\eta)}(t_2^{\gamma-1}-t_1^{\gamma-1})\|u\|.$$

$\|\alpha(t_2)-\alpha(t_1)\|$ becomes zero as $t_2-t_1\to 0$ by using absolute continuity of the Lebesgue dominance theorem. Hence, $\tilde{\Phi}(G_q)$ is equicontinuous on $\mathfrak{Y}$.

Step 4. Let us verify Mönch's condition.

Let $\omega^0(t)+\hat{\phi}(t)=t^{1-\gamma}S_{\zeta,\eta}(t)\hat{\phi}_0$ for all $t\in\mathfrak{Y}$ and $w^{n+1}+\hat{\phi}(t)=\tilde{\Phi}[w^n+\hat{\phi(t)}], n=0,1,2,3,\cdots$ and $\tilde{\Phi}$ be relatively compact.

Assume $\mathbb{H}\subset\mathcal{P}_q$ is countable and $\mathbb{H}\subseteq conv\{0\}\cup\tilde{\psi}(\mathbb{H})$. Our aim here is to show that $\mathbb{R}(\mathbb{H})=0$, where $\mathbb{R}$ is the Hausdroff measure of non compactness. Suppose $\mathbb{H}=\{\omega^n+\{\phi\}_{n=1}^\infty\}$. Now we have to show that $\tilde{\Phi}(\mathbb{H})(t)$ is relatively compact in $\mathfrak{Y}$, for all $t\in\mathfrak{Y}$. From Theorem 2.17

$$\begin{aligned}\mathbb{R}(\mathbb{H}(t)) &= \mathbb{R}(\{(w^n+\phi)(t)\}_{n=0}^\infty)\\ &= \mathbb{R}(\{(w^0+\phi)(t)\}\cup\{(w^n+\phi)(t)\}_{n=1}^\infty)\\ &= \mathbb{R}(\{w^n(t)+\phi(t)\}_{n=1}^\infty),\end{aligned}$$

and

$$\begin{aligned}\mathbb{R}(\{\tilde{\psi}w_n(t)\}_{n=1}^\infty) &= \mathbb{R}(\{t^{1-\gamma}\int_0^t(t-\varrho)^{\eta-1}\\ &\quad\times Q_\eta(t-\varrho)[G_n(\varrho)+Bv_{u^n}(\varrho)]d\varrho\}_{n=1}^\infty)\\ &= I_1+I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \frac{2Nd^{1-\gamma}}{\Gamma(\eta)}\int_0^t(t-\varrho)^{\eta-1}\mathbb{R}(\{G_n(\varrho)\}_{n=1}^\infty)d\varrho\\ &\le \frac{2Np^{1-\gamma}}{\Gamma(\eta)}\int_0^t(t-\varrho)^{\eta-1}\mathbb{R}(\{\mathfrak{F}(\varrho,\varrho^{1-\gamma}[w_\varrho^n+\hat{\phi}_\varrho])\}_{n=1}^\infty d\varrho\\ &\le \frac{2Np^{1-\gamma}}{\Gamma(\eta)}\int_0^t(t-\varrho)^{\eta-1}l_2(\varrho)\sup_{-\infty<\theta\le 0}\mathbb{R}(\{\mathfrak{F}(\varrho^{1-\gamma}[\omega^n(\varrho+\varphi)+\widehat{\phi}(\varrho+\varphi)])\}_n^\infty))d\varrho\\ &\le \frac{2Np^{1-\gamma}}{\Gamma(\eta)}\int_0^t(t-\varrho)^{\eta-1}l_2(\varrho)\sup_{0\le\psi\le\varrho}\mathbb{R}(\mathbb{H}(\psi))d\varrho,\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{2NN_b p^{1-\gamma}}{\Gamma(\eta)} \int_0^{t} (t-\varrho)^{\eta-1} \mathbb{R}(\{v_{u_n}(\varrho)\}_{n=1}^{\infty}) d\varrho \\
&\leq \frac{2NN_b p^{1-\gamma}}{\Gamma(\eta)} \int_0^{t} (t-\varrho)^{\eta-1} [\frac{2N}{\Gamma(\eta)} \\
&\times \int_0^{p} (p-\varrho)^{\eta-1} \mathbb{R}(\{\mathfrak{F}(\varrho, \varrho^{1-\gamma}))[\omega_{\varrho}^{n} + \{(\hat{\phi})\}_{n=1}^{\infty}\} d\varrho] d\varrho \\
&\leq \frac{4N^2 N_b p^{1-\gamma}}{\Gamma(\eta)} \int_0^{t} (t-\varrho)^{\eta-1} l_3(\varrho) \\
&\times \int_0^{t} (t-\varrho)^{\eta-1} l_2(\varrho) \sup_{0<\psi\leq\varrho} \mathbb{R}(\mathbb{H}(\psi)) d\varrho] d\varrho,
\end{aligned}$$

$$\begin{aligned}
I_1 + I_2 &= [\frac{2NN_2 p^{1-\gamma}}{\Gamma(\eta)} + \frac{4N^2 N_b N_2 N_3 p^{1-\gamma}}{\Gamma(\eta)^2}] \sup_{0<\theta\leq\varrho} \mathbb{R}(\mathbb{H}(\psi)) \\
&\leq \frac{2NN_2 p^{1-\gamma}}{\Gamma(\eta)} \Big[1 + \frac{2NN_3 N_b p^{1-\gamma}}{\Gamma(\eta)}\Big] \\
&\times \sup_{0<\psi\leq\varrho} \mathbb{R}(\mathbb{H}(\psi)).
\end{aligned}$$

From Lemma 2.16, $\mathbb{R}(\tilde{\Phi}(\mathbb{H})) \leq C^* \mathbb{R}(\mathbb{H})$, where C^* is defined in 3.1. Then, from Mönch's condition,

$$\begin{aligned}
\mathbb{R}(\mathbb{H}) &\leq (conv\{0\} \bigcup (\widetilde{\Phi}))) \\
&= \mathbb{R}(\widetilde{\Phi}) \\
&\leq C^* \mathbb{R}(\mathbb{H}),
\end{aligned}$$

$\mathbb{R}(\mathbb{H}) = 0$ and then $\mathbb{H}$ is relatively compact. From Lemma 2.18, $\tilde{\Phi}$ has a fixed point ω in G_q. Therefore, $u = \omega + \widehat{\phi}$ is a mild solution of the H-FIDEs (1.1–1.3) satisfying $u(p) = u_1$. Hence, the systems (1.1–1.3) is controllable on $\mathfrak{Y}$, and the proof is completed. □

4. Example

Now, analyze the following problem:

$$D_{0^+}^{\zeta, \frac{2}{3}} u(t, \mu) = \frac{\partial^2}{\partial \mu^2} u(t, \mu) + \mathbb{W}_{\vartheta}(t, \mu) + \vartheta(t, \int_{-\infty}^{t} \vartheta_1(\sigma - t) u(\sigma, \mu) d\sigma), \tag{4.1}$$

$$\Delta u|_{t=t_i} = I_i(u(t_i^-)), i = 1, 2, \cdots n, \tag{4.2}$$

$$I^{(1-\zeta)\frac{1}{3}}[u(t, \mu)]|_{\mu=0} = u_0(\mu), \mu \in [0, \pi], \tag{4.3}$$

$$u(t, 0) = u(t, \pi) = 0, t \geq 0, \tag{4.4}$$

$$u(0, \mu) = \phi(t, \mu), 0 \leq \mu \leq \pi. \tag{4.5}$$

From previous equations, $D_{0^+}^{\zeta, \frac{2}{3}}$ denotes the Hilfer FD of order $\eta = \frac{2}{3}$, and type $\zeta, I^{(1-\zeta)\frac{1}{3}}$ is the (R-L) integral of order $(1-\zeta)\frac{1}{3}$, $\phi \in \mathcal{P}_h$ and $\vartheta : J \times [0, 1]$ is continuous. To change this frame-work into the abstract structure (1.1) and (1.2), let $U = L^2[0,\pi]$ be endowed with the norm $\|\cdot\|_{L^2}$ and $A : D(A) \subset U \to U$ be given by $A\mathbb{E} = \mathbb{E}''$ along with

$$D(A) = \{\mathbb{E}, \mathbb{E}'' \in \mathfrak{Y} : \mathbb{E}, \mathbb{E}'' \text{are absolutely continous}, \mathbb{E}'' \in \mathfrak{Y}, +\mathbb{E}(0) = \mathbb{E}(\pi) = 0\}. \tag{4.6}$$

Here, A is an infinitesimal generator of a semigroup $\{T(\mathrm{t}), \mathrm{t} \geq 0\}$ in where $\mathfrak{Y}$ and it is given by $T(\mathrm{t})\omega(\sigma) = w(\mathrm{t}+\sigma)$; for $\omega \in U$, $T(\mathrm{t})$ is not compact on U and $\mathbb{R}(T(\mathrm{t})H) \leq \mathbb{R}(H)$, where $\mathbb{R}$ is the Hausdorff MNC, and there exists $N \geq 1$ such that $\sup_{\mathrm{t}\in\mathfrak{Y}} \|T(\mathrm{t})\| \leq N$. Furthermore, $\mathrm{t} \to \omega(\mathrm{t}^{\frac{2}{3}} + \sigma)u$ is equicontinuous for $\mathrm{t} \geq 0$ and $\mu \in (0,\infty)$. Let $\mathfrak{F} : [0,\pi] \times U \to U$ by

$$\mathfrak{F}(\mathrm{t},\pi)(\mu) = \vartheta(\mathrm{t}, \int_{-\infty}^{\mathrm{t}} \vartheta_1(\sigma - \mathrm{t})u(\sigma,\mu)d\sigma),$$

and

$$D_{0^+}^{\zeta,\frac{2}{3}} u(\mathrm{t})(\mu) = \frac{\partial^{\frac{2}{3}}}{\partial\mu^{\frac{2}{3}}} u(\mathrm{t},\mu), u(\mathrm{t})(\mu) = u(\mathrm{t},\mu).$$

Let $B : V \to V$ be defined by $(Bv)(\mathrm{t})(\mu) = \mathbb{W}_{\vartheta}(\mathrm{t},\mu), 0 < \mu < 1$. By assuming the suitable choices of A,B and $\mathfrak{F}$, the H-FIDEs (4.1)–(4.4) can be rewritten as

$$D_{0^+}^{\zeta,\eta} u(\mathrm{t}) = Au(\mathrm{t}) + \mathfrak{F}(\mathrm{t}, u_{\mathrm{t}}) + Bv(\mathrm{t}), \mathrm{t} \in \mathbb{R} = (0,p], \tag{4.7}$$

$$\Delta u|_{\mathrm{t}=\mathrm{t}_i} = I_i(u(\mathrm{t}_i^-)), i = 1,2,\cdots n, \tag{4.8}$$

$$I_{0^+}^{(1-\zeta)(1-\eta)} u(\mathrm{t})|_{\mathrm{t}=0} = \phi(\mathrm{t}), \mathrm{t} \in (-\infty, 0]. \tag{4.9}$$

For $\mu \in (0,\pi)$, $\mathcal{W}$ is given by

$$\mathcal{W}v(\mu) = \int_0^1 (1-\mathrm{t})^{\frac{-1}{3}} Q_\eta(1-\mathrm{t})\mathfrak{F}\vartheta(\mathrm{t},\mu)d\mathrm{t},$$

where

$$Q_{\frac{2}{3}} = \frac{2}{3}\int_0^{-\infty} \mu\chi_{\frac{2}{3}}(\mu)\mathcal{W}(\mathrm{t}^{\frac{2}{3}} + \mu)d\mu,$$

and

$$\chi^{\frac{2}{3}}(\mu) = \frac{3}{2}\mu^{-1-\frac{3}{2}}\bar{w}_{\frac{2}{3}}(\mu^{\frac{-3}{2}}),$$

$$\overline{w}_{\frac{2}{3}}(\mu) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\mathrm{t}^{-\frac{2}{3}n-1}\frac{\Gamma(\frac{2}{3}n+1)}{n!} sin\left(\frac{2n\pi}{3}\right).$$

In the above, $\chi_{\frac{2}{3}}$ is defined on $(0,\infty)$, that is,

$$\chi_{\frac{2}{3}}(\mu) \geq 0, \mu \in (0,\infty) \text{ and } \int_0^\infty \chi_{\frac{2}{3}}(\mu)d\mu = 1.$$

We take $\vartheta(\mathrm{t}, \int_{-\infty}^{\mathrm{t}} \vartheta_1(\sigma-\mathrm{t})u(\sigma,\mu)d\sigma) = C_0 \sin(y(\sigma))$, where C_0 is a constant. Then, $\mathfrak{F}$ and $\mathbb{W}$ satisfy the hypotheses (H_1)–(H_3). This completes the example.

5. Conclusions

In our study, we used non-compactness measures to investigate the controllability of Hilfer fractional impulsive differential systems with infinite delay. We started with the Hilfer fractional impulsive differential systems with controllability and applied Mönch's fixed point theorem for indefinite delay; then, extended our results to the concept of non-local conditions. Finally, an example case was provided to demonstrate the significance of our major findings. In the future, we will use the MNC to investigate the existence and controllability of Sobolov-type Hilfer fractional implusive differential systems with indefinite delay. In addition to this, we can extend our results with integro or implicit terms and we can use integral boundary conditions which has real life applications. Also we can provide some numerical approximations for this considered system.

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Conflict of interest

The authors declare no conflicts of interest.

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