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Research article

Application of fixed point results in the setting of $\mathcal{F}$-contraction and simulation function in the setting of bipolar metric space

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Abstract: In the present work, simulation function is applied to establish fixed point results of $\mathcal{F}$-contraction in the setting of Bi-polar metric space. Our results are extensions or generalizations of results proved in the literature. The derived results are substantiated with suitable examples and an application to find the solution to the integral equation.

Keywords: simulation function; fixed point; BPMS
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1. Introduction

The famous Banach Contraction Principle [1] of 1922 laid the foundation for the Metric Fixed Point theory. Over the years, plenty of generalizations have been made by various researchers to the Banach Contraction Principle using different types of contractive conditions in various topological spaces which are both Hausdorff and non-Hausdorff in nature and presented analytical applications of the derived results. In the sequel, in 2012, [2] introduced $\mathcal{F}$-contraction and established fixed point results in the setting of complete metric spaces. In 2004, P. Dhivya et al. [3] analyzed the coupled fixed point and best proximity points involving simulation functions. Later, in 2015, Khojasteh, Shukla and Radenović [5], introduced the concept of simulation function and proved fixed point theorems using these functions. Subsequently, various fixed point results have been proved using various contractive conditions and simulation functions by numerous researchers in metric and metric-like spaces [4–14].

As the distance function for a pair of points is always nonnegative real, metric fixed point theory has varied applications. The generalization of metric and metric-like spaces and the study of their properties have always been a matter of interest to researchers. As a result, Gahler [15] adopted metrics that are non-negative reals (i.e., $[0, +\infty)$) and presented the idea of 2-metric spaces. In metric spaces, different types of distance functions are considered. Still, we can see the distance arising between the elements of two different sets where in distance between the same type of points is either unknown or undefined due to the non-availability of information. The distance between points and lines of Euclidean space, and the distance between sets and points of a metric space are these types of distances to name a few.

Formalizing these types of distances, in the year 2016, Mutlu et al. [16] introduced the concept of Bi-polar metric space and established fixed point theorems in these spaces but without analyzing the topological structure in detail. In the recent past, many researchers have established various fixed point results using various types of contractions in the setting of Bi-polar metric spaces. One can refer to [17–26] and references there on for better understanding. The existence of fixed points of contraction mappings in Bi-polar metric spaces is currently an important topic in fixed point theory, which can be considered as a generalization of the Banach contraction principle.

Moreover, varied applications of metric fixed point theory have been reported in different areas such as variational inequalities, differential, and integral equations, fractal calculus and dynamical systems and space science, etc.

Inspired by the scope and its varied applications, the study is performed to examine the following:

- To analyze the existence of unique fixed point in the setting of Bi-polar metric spaces using $\mathcal{F}$-contraction.

- To analyze the existence of unique fixed point in the setting of Bi-polar metric spaces using Simulation Functions.

- To apply the derived results to find solutions to integral equations.

Accordingly, the rest of the paper is organized as follows. In Section 2, we review some definitions and concepts present in literature and some monograph. In Section 3, we establish fixed point results using $\mathcal{F}$-contraction and simulation functions in the setting of Bi-polar metric space and supplement the derived results with examples. We have also applied the derived results to find the analytical solution of integral equations.

Before proceeding further, we present a notation table listing the symbols and their meanings that are frequently used in this manuscript (see Table 1):

Table 1. List of symbols used in this article.

Symbols	Description
Υ, Ω	Sets
$\mathfrak{d}$	Metric distance function
$\mathcal{M}$	Mapping
$\mathfrak{F}$	Set of all functions $\mathcal{F} : (0, +\infty) \to \mathbb{R}$
$\mathbb{R}$	Real Number
β	Element of Ω
$\aleph$	Element of Υ
σ	Element of $\mathbb{N}$
ϱ	Bi-polar distance function
ξ	Simulation Function
Θ	Set of all simulation functions $\xi : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$

2. Preliminaries

The following are required in the sequel. Let us begin with the concept of a $\mathcal{F}$-contraction introduced by Wardowski [2].

Definition 2.1. [2] Let $(\Upsilon, \mathfrak{d})$ be a metric space. A mapping $\mathcal{M} : \Upsilon \to \Upsilon$ is called an $\mathcal{F}$-contraction if there exist $\tau > 0$ and $\mathcal{F} \in \mathfrak{F}$ such that

$$\tau + \mathcal{F}(\mathfrak{d}(\mathcal{M}\aleph, \mathcal{M}\aleph_1)) \leq \mathcal{F}(\mathfrak{d}(\aleph, \aleph_1))$$

holds for any $\aleph, \aleph_1 \in \Upsilon$ with $\mathfrak{d}(\mathcal{M}\aleph, \mathcal{M}\aleph_1) > 0$, where $\mathfrak{F}$ is the set of all functions $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

(A1) $\mathcal{F}$ is strictly increasing;

(A2) For each sequence $\{\aleph_{\mathfrak{n}}\}$ of positive numbers, we have

$$\lim_{\mathfrak{n}\to+\infty} \aleph_{\mathfrak{n}} = 0 \text{ iff } \lim_{\mathfrak{n}\to+\infty} \mathcal{F}(\aleph_{\mathfrak{n}}) = -\infty;$$

(A3) There exists $\mathfrak{k} \in (0, 1)$ such that $\lim_{\aleph\to 0^+} \aleph^{\mathfrak{k}}\mathcal{F}(\aleph) = 0$.

Definition 2.2. [16] Let Υ and Ω be non-void sets and $\varrho : \Upsilon \times \Omega \to [0, +\infty)$ be a function, such that

(a) $\varrho(\aleph, \beta) = 0$ if and only if $\aleph = \beta$, for all $(\aleph, \beta) \in \Upsilon \times \Omega$;

(b) $\varrho(\aleph, \beta) = \varrho(\beta, \aleph)$, for all $(\aleph, \beta) \in \Upsilon \cap \Omega$;

(c) $\varrho(\aleph, \beta) \leq \varrho(\aleph, \gamma) + \varrho(\aleph_1, \gamma) + \varrho(\aleph_1, \beta)$, for all $\aleph, \aleph_1 \in \Upsilon$ and $\gamma, \beta \in \Omega$.

The pair $(\Upsilon, \Omega, \varrho)$ is called a Bi-polar metric space.

Example 2.3. Let $\Phi = [0, 1]$, $\Psi = [-1, 1]$ and $\varphi : \Phi \times \Psi \to [0, +\infty)$ be defined by

$$\varphi(\eta, \sigma) = |\eta - \sigma|$$

for all $\eta \in \Phi$ and $\sigma \in \Psi$. Then $(\Upsilon, \Omega, \varrho)$ is a Bi-polar metric space.

Definition 2.4. [16] Let $\mathcal{M} : \Upsilon_1 \cup \Omega_1 \to \Upsilon_2 \cup \Omega_2$ be a mapping, where (Υ_1, Ω_1) and (Υ_2, Ω_2) pairs of sets.

(H1) If $\mathcal{M}(\Upsilon_1) \subseteq \Upsilon_2$ and $\mathcal{M}(\Omega_1) \subseteq \Omega_2$, then $\mathcal{M}$ is called a covariant map, or a map from $(\Upsilon_1, \Omega_1, \varrho_1)$ to $(\Upsilon_2, \Omega_2, \varrho_2)$ and this is written as $\mathcal{M} : (\Upsilon_1, \Omega_1, \varrho_1) \rightrightarrows (\Upsilon_2, \Omega_2, \varrho_2)$.

(H2) If $\mathcal{M}(\Upsilon_1) \subseteq \Omega_2$ and $\mathcal{M}(\Omega_1) \subseteq \Upsilon_2$, then $\mathcal{M}$ is called a contravariant map from $(\Upsilon_1, \Omega_1, \varrho_1)$ *to* $(\Upsilon_2, \Omega_2, \varrho_2)$ and this is denoted as $\mathcal{M} : (\Upsilon_1, \Omega_1, \varrho_1) \leftrightarrows (\Upsilon_2, \Omega_2, \varrho_2)$.

Definition 2.5. [16] Let $(\Upsilon, \Omega, \varrho)$ be a Bi-polar metric space.

(B1) A point $\aleph \in \Upsilon \cup \Omega$ is said to be a left point if $\aleph \in \Upsilon$, a right point if $\aleph \in \Omega$ and a central point if both hold.

(B2) A sequence $\{\aleph_\sigma\} \subset \Upsilon$ is called a left sequence and a sequence $\{\beta_\sigma\} \subset \Omega$ is called a right sequence.

(B3) A sequence $\{\eta_\sigma\} \subset \Upsilon \cup \Omega$ is said to converge to a point η if and only if $\{\eta_\sigma\}$ is a left sequence, η is a right point and $\lim_{\sigma \to +\infty} \varrho(\eta_\sigma, \eta) = 0$ or $\{\eta_\sigma\}$ is a right sequence, η is a left point and $\lim_{\sigma \to +\infty} \varrho(\eta, \eta_\sigma) = 0$.

(B4) A sequence $\{(\aleph_\sigma, \beta_\sigma)\} \subset \Upsilon \times \Omega$ is called a bisequence. If the sequences $\{\aleph_\sigma\}$ and $\{\beta_\sigma\}$ both converge then the bisequence $\{(\aleph_\sigma, \beta_\sigma)\}$ is called convergent in $\Upsilon \times \Omega$.

(B5) If $\{\aleph_\sigma\}$ and $\{\beta_\sigma\}$ both converge to a point $\beta \in \Upsilon \cap \Omega$ then the bisequence $\{(\aleph_\sigma, \beta_\sigma)\}$ is called biconvergent. A sequence $\{(\aleph_\sigma, \beta_\sigma)\}$ is a Cauchy bisequence if $\lim_{\sigma, \zeta \to +\infty} \varrho(\aleph_\sigma, \beta_\zeta) = 0$.

(B6) A Bi-polar metric space is said to be complete if every Cauchy bisequence is convergent.

Definition 2.6. [5] Let $\xi : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ be a mapping, then ξ is called a simulation function if

(ξ1) $\xi(0, 0) = 0$;

(ξ2) $\xi(\delta, \eta) < \eta - \delta$ for all $\delta, \eta > 0$;

(ξ3) if $\{\delta_\sigma\}, \{\eta_\sigma\}$ are sequences in $(0, +\infty)$ such that $\lim_{\sigma \to +\infty} \delta_\sigma = \lim_{\sigma \to +\infty} \eta_\sigma > 0$, then

$$\lim_{\sigma \to +\infty} \sup \xi(\delta_\sigma, \eta_\sigma) < 0.$$

We denote the set of all simulation functions by Θ.

Example 2.7. [5] Let $\xi_\mathfrak{i} : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$, $\mathfrak{i} = 1, 2, 3$ be defined by

(G1) $\xi_1(\delta, \eta) = \psi(\eta) - \phi(\delta)$ for all $\delta, \eta \in [0, +\infty)$, where $\phi, \psi : [0, +\infty) \to [0, +\infty)$ are two continuous functions such that $\psi(\delta) = \phi(\delta) = 0$ if and only if $\delta = 0$ and $\psi(\delta) < \delta \leq \phi(\delta)$ for all $\delta > 0$.

(G2) $\xi_2(\delta,\eta) = \eta - \frac{\mathfrak{f}(\delta,\eta)}{\mathfrak{g}(\delta,\eta)}\delta$ for all $\delta,\eta \in [0,+\infty)$ where $\mathfrak{f},\mathfrak{g} : [0,+\infty) \to [0,+\infty)$ are two continuous functions with respect to each variable such that $\mathfrak{f}(\delta,\eta) > \mathfrak{g}(\delta,\eta)$ for all $\delta,\eta > 0$.

(G3) $\xi_3(\delta,\eta) = \eta - \phi(\eta) - \delta$ for all $\delta,\eta \in [0,+\infty)$, where $\phi : [0,+\infty) \to [0,+\infty)$ is a continuous function such that $\phi(\delta) = 0$ if and only if $\delta = 0$.

Then $\xi_{\mathfrak{i}}$ for $\mathfrak{i} = 1,2,3$ are simulation functions.

Definition 2.8. Let $(\varUpsilon,\varOmega,\varrho)$ be a Bi-polar metric space, $\mathcal{M} : (\varUpsilon,\varOmega,\varrho) \rightrightarrows (\varUpsilon,\varOmega,\varrho)$ a mapping and $\xi \in \Theta$. Then $\mathcal{M}$ is called a Θ-contraction with respect to ξ if

$$\xi(\varrho(\mathcal{M}\aleph, \mathcal{M}\beta), \varrho(\aleph,\beta)) \geq 0 \ \text{ for all } \aleph \in \varUpsilon, \beta \in \varOmega.$$

Now we present our main results.

3. Main results

In 2016, Mutlu et al. [16], introduced the cocept of a contraction and proved the following theorem.

Definition 3.1. Let $(\varUpsilon_1,\varOmega_1,\varrho_1)$ *to* $(\varUpsilon_2,\varOmega_2,\varrho_2)$ be bipolar metric spaces and $\lambda > 0$. A covariant map $\mathcal{M}$: $(\varUpsilon_1,\varOmega_1,\varrho_1) \rightrightarrows (\varUpsilon_2,\varOmega_2,\varrho_2)$ such that $\varrho_2(\mathcal{M}(\aleph),\mathcal{M}(\beta)) \leq \lambda\varrho_1(\aleph,\beta)$ for all $\aleph \in \varUpsilon_1, \beta \in \varOmega_1$, or a contravariant map $\mathcal{M} : (\varUpsilon_1,\varOmega_1,\varrho_1) \leftrightarrows (\varUpsilon_2,\varOmega_2,\varrho_2)$ such that $\varrho_2(\mathcal{M}(\aleph),\mathcal{M}(\beta)) \leq \lambda\varrho_1(\aleph,\beta)$ for all $\aleph \in \varUpsilon_1, \beta \in \varOmega_1$ is called Lipschitz continuous. If $\lambda = 1$, then this covariant or contravariant map is said to be non-expansive, and if $\lambda \in (0,1)$, it is called a contraction.

Theorem 3.2. *Let $(\varUpsilon,\varOmega,\varrho)$ be a complete Bi-polar metric space and given a contraction $\mathcal{M}$: $(\varUpsilon,\varOmega,\varrho) \rightrightarrows (\varUpsilon,\varOmega,\varrho)$. Then the function $\mathcal{M}$: $\varUpsilon \cup \varOmega \to \varUpsilon \cup \varOmega$ has a unique fixed point.*

Motivated by the above theorem, we prove fixed point theorems on Bi-polar metric space using $\mathcal{F}$-contraction and simulation functions.

Now we present our first fixed point theorem on Bi-polar metric space using $\mathcal{F}$-contraction function.

Theorem 3.3. *Let $(\varUpsilon,\varOmega,\varrho)$ be a complete Bi-polar metric space. Suppose $\mathcal{M}$: $(\varUpsilon,\varOmega,\varrho) \rightrightarrows (\varUpsilon,\varOmega,\varrho)$ is a covariant mapping and there exists $\tau > 0$ such that*

$$\tau + \mathcal{F}(\varrho(\mathcal{M}(\aleph),\mathcal{M}(\beta))) \leq \mathcal{F}(\varrho(\aleph,\beta)) \textit{ for all } \aleph \in \varUpsilon, \beta \in \varOmega,$$

holds for any $\aleph \in \varUpsilon$, $\beta \in \Omega$ with $\varrho(\mathcal{M}(\aleph),\mathcal{M}(\beta)) > 0$. Then the function $\mathcal{M}$: $\varUpsilon \cup \varOmega \to \varUpsilon \cup \varOmega$ has a unique fixed point.

Proof. Let $\aleph_0 \in \varUpsilon$ and $\beta_0 \in \varOmega$. For each $\sigma \in \mathbb{N}$, define $\mathcal{M}(\aleph_\sigma) = \aleph_{\sigma+1}$ and $\mathcal{M}(\beta_\sigma) = \beta_{\sigma+1}$. Then $(\{\aleph_\sigma\}, \{\beta_\sigma\})$ is a bisequence on $(\varUpsilon,\varOmega,\varrho)$ and $\aleph_\sigma \neq \beta_\sigma$. By hypothesis of the theorem, we have

$$\begin{aligned}
\mathcal{F}(\varrho(\aleph_\sigma,\beta_\sigma)) &= \mathcal{F}(\varrho(\mathcal{M}(\aleph_{\sigma-1}),\mathcal{M}(\beta_{\sigma-1}))) \\
&\leq \mathcal{F}(\varrho(\aleph_{\sigma-1},\beta_{\sigma-1})) - \tau \\
&\vdots \\
&\leq \mathcal{F}(\varrho(\aleph_0,\beta_0)) - \sigma\tau.
\end{aligned} \tag{3.1}$$

As $\sigma \to +\infty$, we have

$$\lim_{\sigma\to+\infty} \mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) = -\infty.$$

Using (A2), we derive that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_\sigma, \beta_\sigma) = 0. \tag{3.2}$$

Using (A3), there exists $\mathfrak{k} \in (0, 1)$ such that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) = 0. \tag{3.3}$$

Using (3.1), for all $\sigma \in \mathbb{N}$

$$\begin{aligned}
\varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) - \varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_0, \beta_0))
&\leq \varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} (\mathcal{F}(\varrho(\aleph_0, \beta_0)) - \sigma\tau) - \varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_0, \beta_0)) \\
&= -\varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} \sigma\tau \\
&\leq 0.
\end{aligned} \tag{3.4}$$

As $\sigma \to +\infty$ in (3.4), and using (3.2) and (3.3), we derive that

$$\lim_{\sigma\to+\infty} \sigma\varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} = 0. \tag{3.5}$$

Now, let us observe that from (3.5) there exists σ_1 such that $\sigma\varrho(\aleph_\sigma, \beta_\sigma)^{\mathfrak{k}} \leq 1$ for all $\sigma \geq \sigma_1$. Consequently we have

$$\varrho(\aleph_\sigma, \beta_\sigma) \leq \frac{1}{\sigma^{\frac{1}{\mathfrak{k}}}}, \quad \text{for all } \sigma \geq \sigma_1.$$

Also,

$$\begin{aligned}
\mathcal{F}(\varrho(\aleph_\sigma, \beta_{\sigma+1})) &= \mathcal{F}(\varrho(\mathcal{M}(\aleph_{\sigma-1}), \mathcal{M}(\beta_\sigma))) \\
&\leq \mathcal{F}(\varrho(\aleph_{\sigma-1}, \beta_\sigma)) - \tau \\
&\vdots \\
&\leq \mathcal{F}(\varrho(\aleph_0, \beta_1)) - \sigma\tau.
\end{aligned} \tag{3.6}$$

As $\sigma \to +\infty$, we have

$$\lim_{\sigma\to+\infty} \mathcal{F}(\varrho(\aleph_\sigma, \beta_{\sigma+1})) = -\infty.$$

Using (A2), we derive that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_\sigma, \beta_{\sigma+1}) = 0. \tag{3.7}$$

Using (A3), there exists $\mathfrak{k} \in (0, 1)$ such that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_\sigma, \beta_{\sigma+1})^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_\sigma, \beta_{\sigma+1})) = 0. \tag{3.8}$$

Using (3.6), for all $\sigma \in \mathbb{N}$

$$\begin{aligned}
\varrho(\aleph_\sigma,\beta_{\sigma+1})^{\mathfrak{k}}\mathcal{F}(\varrho(\aleph_\sigma,\beta_{\sigma+1})) - \varrho(\aleph_\sigma,\beta_\sigma)^{\mathfrak{k}}\mathcal{F}(\varrho(\aleph_0,\beta_1))&\\
&\leq \varrho(\aleph_\sigma,\beta_{\sigma+1})^{\mathfrak{k}}(\mathcal{F}(\varrho(\aleph_0,\beta_1)) - \sigma\tau) - \varrho(\aleph_\sigma,\beta_{\sigma+1})^{\mathfrak{k}}\mathcal{F}(\varrho(\aleph_0,\beta_1))\\
&= -\varrho(\aleph_\sigma,\beta_{\sigma+1})^{\mathfrak{k}}\sigma\tau\\
&\leq 0. \qquad (3.9)
\end{aligned}$$

As $\sigma \to +\infty$ in (3.9), and using (3.7) and (3.8), we derive that

$$\lim_{\sigma\to+\infty} \sigma\varrho(\aleph_\sigma,\beta_{\sigma+1})^{\mathfrak{k}} = 0. \qquad (3.10)$$

Now, let us observe that from (3.10) there exists σ_2 such that $\sigma\varrho(\aleph_\sigma,\beta_{\sigma+1})^{\mathfrak{k}} \leq 1$ for all $\sigma \geq \sigma_2$. Consequently we have

$$\varrho(\aleph_\sigma,\beta_{\sigma+1}) \leq \frac{1}{\sigma^{\frac{1}{\mathfrak{k}}}}, \text{ for all } \sigma \geq \sigma_2.$$

Let $\sigma = \max\{\sigma_1,\sigma_2\}$.

$$\begin{aligned}
\varrho(\aleph_{\sigma+\mathfrak{p}},\beta_\sigma) &\leq \varrho(\aleph_{\sigma+\mathfrak{p}},\beta_{\sigma+1}) + \varrho(\aleph_\sigma,\beta_{\sigma+1}) + \varrho(\aleph_\sigma,\beta_\sigma)\\
&\leq \varrho(\aleph_{\sigma+\mathfrak{p}},\beta_{\sigma+2}) + \varrho(\aleph_{\sigma+1},\beta_{\sigma+2}) + \varrho(\aleph_{\sigma+1},\beta_{\sigma+1}) + \varrho(\aleph_\sigma,\beta_{\sigma+1}) + \varrho(\aleph_\sigma,\beta_\sigma)\\
&\vdots\\
&\leq \varrho(\aleph_{\sigma+\mathfrak{p}},\beta_{\sigma+\mathfrak{p}}) + \cdots + \varrho(\aleph_\sigma,\beta_{\sigma+1}) + \varrho(\aleph_\sigma,\beta_\sigma)\\
&\leq \sum_{\mathfrak{i}=\sigma}^{+\infty} \frac{1}{\mathfrak{i}^{\frac{1}{\mathfrak{k}}}}.
\end{aligned}$$

Since $\mathfrak{k} \in (0,1)$, the series $\sum_{\mathfrak{i}=\sigma}^{+\infty} \frac{1}{\mathfrak{i}^{\frac{1}{\mathfrak{k}}}}$ is convergent. Therefore, $(\{\aleph_\sigma\},\{\beta_\sigma\}))$ is a Cauchy bisequence. Since $(\Upsilon,\Omega,\varrho)$ is complete, then $\{\aleph_\sigma\} \to \mathfrak{u}$ and $\{\beta_\sigma\} \to \mathfrak{u}$, where $\mathfrak{u} \in \Upsilon \cap \Omega$ and

$$\{\mathcal{M}(\beta_\sigma)\} = \{\beta_{\sigma+1}\} \to \mathfrak{u} \in \Upsilon \cap \Omega.$$

Since $\mathcal{M}$ is continuous $\mathcal{M}(\beta_\sigma) \to \mathcal{M}(\mathfrak{u})$, so $\mathcal{M}(\mathfrak{u}) = \mathfrak{u}$. Hence $\mathfrak{u}$ is a fixed point of $\mathcal{M}$. If $\mathfrak{v}$ is any fixed point of $\mathcal{M}$, then $\mathcal{M}(\mathfrak{v}) = \mathfrak{v}$ implies that $\mathfrak{v} \in \Upsilon \cap \Omega$ and

$$\tau \leq \mathcal{F}(\varrho(\mathcal{M}(\mathfrak{u}),\mathcal{M}(\mathfrak{v})) - \mathcal{F}(\varrho(\mathfrak{u},\mathfrak{v})) = 0,$$

which is absurd. Hence $\mathfrak{u} = \mathfrak{v}$. □

Remark 3.4. If we take $\Upsilon = \Omega$, then our result is reduced to Theorem 2.1 in [2].

Theorem 3.5. *Let $(\Upsilon,\Omega,\varrho)$ be a complete Bi-polar metric space. Suppose $\mathcal{M}$: $(\Upsilon,\Omega,\varrho) \leftrightarrows (\Upsilon,\Omega,\varrho)$ is a contravariant mapping and there exists $\tau > 0$ such that*

$$\tau + \mathcal{F}(\varrho(\mathcal{M}(\varpi),\mathcal{M}(\aleph))) \leq \mathcal{F}(\varrho(\aleph,\varpi)) \text{ for all } \aleph \in \Upsilon, \varpi \in \Omega,$$

holds for any $\aleph \in \Upsilon$, $\beta \in \Omega$ with $\varrho(\mathcal{M}(\aleph),\mathcal{M}(\beta)) > 0$. Then the function $\mathcal{M}$: $\Upsilon \cup \Omega \to \Upsilon \cup \Omega$ has a unique fixed point.

Proof. Let $\aleph_0 \in \Upsilon$. For each $\sigma \in \mathbb{N}$, define $\mathcal{M}(\aleph_\sigma) = \beta_\sigma$ and $\mathcal{M}(\beta_\sigma) = \aleph_{\sigma+1}$. Then $(\{\aleph_\sigma\}, \{\beta_\sigma\})$ is a bisequence on $(\Upsilon, \Omega, \varrho)$ and $\aleph_\sigma \neq \beta_\sigma$. Then

$$\begin{aligned}\mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) &= \mathcal{F}(\varrho(\mathcal{M}(\beta_{\sigma-1}), \mathcal{M}(\aleph_\sigma))) \\ &\leq \mathcal{F}(\varrho(\aleph_\sigma, \beta_{\sigma-1})) - \tau \\ &= \mathcal{F}(\varrho(\mathcal{M}(\beta_{\sigma-1}), \mathcal{M}(\aleph_{\sigma-1}))) - \tau \\ &\leq \mathcal{F}(\varrho(\aleph_{\sigma-1}, \beta_{\sigma-1}) - 2\tau \\ &\vdots \\ &\leq \mathcal{F}(\varrho(\aleph_0, \beta_0)) - 2\sigma\tau. \end{aligned} \tag{3.11}$$

As $\sigma \to +\infty$, we have

$$\lim_{\sigma\to+\infty} \mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) = -\infty.$$

Using (A2), we derive that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_\sigma, \beta_\sigma) = 0. \tag{3.12}$$

Using (A3), there exists $\text{ŧ} \in (0, 1)$ such that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} \mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) = 0. \tag{3.13}$$

Using (3.11), for all $\sigma \in \mathbb{N}$

$$\begin{aligned}&\varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} \mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) - \varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} \mathcal{F}(\varrho(\aleph_0, \beta_0)) \\ &\quad\leq \varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} (\mathcal{F}(\varrho(\aleph_0, \beta_0)) - 2\sigma\tau) - \varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} \mathcal{F}(\varrho(\aleph_0, \beta_0)) \\ &\quad= -\varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} 2\sigma\tau \\ &\quad\leq 0. \end{aligned} \tag{3.14}$$

As $\sigma \to +\infty$ in (3.14), and using (3.12) and (3.13), we derive that

$$\lim_{\sigma\to+\infty} 2\sigma\varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} = 0. \tag{3.15}$$

Now, let us observe that from (3.15) there exists σ_1 such that $2\sigma\varrho(\aleph_\sigma, \beta_\sigma)^{\text{ŧ}} \leq 1$ for all $\sigma \geq \sigma_1$. Consequently we have

$$\varrho(\aleph_\sigma, \beta_\sigma) \leq \frac{1}{(2\sigma)^{\frac{1}{\text{ŧ}}}}, \text{ for all } \sigma \geq \sigma_1.$$

Also,

$$\begin{aligned}\mathcal{F}(\varrho(\aleph_{\sigma+1}, \beta_\sigma)) &= \mathcal{F}(\varrho(\mathcal{M}(\beta_\sigma), \mathcal{M}(\aleph_\sigma))) \\ &\leq \mathcal{F}(\varrho(\aleph_\sigma, \beta_\sigma)) - \tau \\ &\leq \mathcal{F}(\varrho(\aleph_0, \beta_0)) - (2\sigma + 1)\tau. \end{aligned} \tag{3.16}$$

As $\sigma \to +\infty$, we have

$$\lim_{\sigma\to+\infty} \mathcal{F}(\varrho(\aleph_{\sigma+1},\beta_\sigma)) = -\infty.$$

Using (A2), we derive that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_{\sigma+1},\beta_\sigma) = 0. \tag{3.17}$$

Using (A3), there exists $\mathfrak{k} \in (0,1)$ such that

$$\lim_{\sigma\to+\infty} \varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_{\sigma+1},\beta_\sigma)) = 0. \tag{3.18}$$

Using (3.16), for all $\sigma \in \mathbb{N}$

$$\begin{aligned}
\varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_{\sigma+1},\beta_\sigma)) - \varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_0,\beta_0)) &\\
\leq \varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} (\mathcal{F}(\varrho(\aleph_0,\beta_0) - (2\sigma+1)\tau) &\\
- \varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} \mathcal{F}(\varrho(\aleph_0,\beta_0)) &\\
= -\varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} (2\sigma+1)\tau &\\
\leq 0. &
\end{aligned} \tag{3.19}$$

As $\sigma \to +\infty$ in (3.19), and using (3.17) and (3.18), we derive that

$$\lim_{\sigma\to+\infty} (2\sigma+1)\varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} = 0. \tag{3.20}$$

Now, let us observe that from (3.20) there exists σ_1 such that $(2\sigma+1)\varrho(\aleph_{\sigma+1},\beta_\sigma)^{\mathfrak{k}} \leq 1$ for all $\sigma \geq \sigma_1$. Consequently we have

$$\varrho(\aleph_{\sigma+1},\beta_\sigma) \leq \frac{1}{(2\sigma+1)^{\frac{1}{\mathfrak{k}}}}, \text{ for all } \sigma \geq \sigma_1.$$

Now,

$$\begin{aligned}
\varrho(\aleph_{\sigma+\mathfrak{p}},\beta_\sigma) &\leq \varrho(\aleph_{\sigma+\mathfrak{p}},\beta_{\sigma+1}) + \varrho(\aleph_{\sigma+1},\beta_{\sigma+1}) + \varrho(\aleph_{\sigma+1},\beta_\sigma)\\
&\leq \varrho(\aleph_{\sigma+\mathfrak{p}},\beta_{\sigma+1}) + \frac{1}{(2(\sigma+1))^{\frac{1}{\mathfrak{k}}}} + \frac{1}{(2\sigma+1)^{\frac{1}{\mathfrak{k}}}}\\
&\leq \varrho(\aleph_{\sigma+\mathfrak{p}},\beta_{\sigma+2}) + \varrho(\aleph_{\sigma+2},\beta_{\sigma+2}) + \varrho(\aleph_{\sigma+2},\beta_{\sigma+1})\\
&\quad + \frac{1}{(2(\sigma+1))^{\frac{1}{\mathfrak{k}}}} + \frac{1}{(2\sigma+1)^{\frac{1}{\mathfrak{k}}}}\\
&\leq \varrho(\aleph_{\sigma+\mathfrak{p}},\beta_{\sigma+2}) + \frac{1}{(2(\sigma+2))^{\frac{1}{\mathfrak{k}}}} + \frac{1}{(2\sigma+3)^{\frac{1}{\mathfrak{k}}}} + \frac{1}{(2(\sigma+1))^{\frac{1}{\mathfrak{k}}}} + \frac{1}{(2\sigma+1)^{\frac{1}{\mathfrak{k}}}}\\
&\vdots\\
&\leq \sum_{\mathfrak{i}=2\sigma+1}^{+\infty} \frac{1}{\mathfrak{i}^{\frac{1}{\mathfrak{k}}}}.
\end{aligned}$$

Since $\mathfrak{k} \in (0,1)$, the series $\sum_{\mathfrak{i}=\sigma}^{+\infty} \frac{1}{\mathfrak{i}^{\frac{1}{\mathfrak{k}}}}$ is convergent. Therefore, $(\{\aleph_\sigma\}, \{\beta_\sigma\}))$ is a Cauchy bisequence. Since $(\Upsilon, \Omega, \varrho)$ is complete, then $\{\aleph_\sigma\} \to \mathfrak{u}$ and $\{\beta_\sigma\} \to \mathfrak{u}$ where $\mathfrak{u} \in \Upsilon \cap \Omega$ and

$$\{\mathcal{M}(\beta_\sigma)\} = \{\beta_{\sigma+1}\} \to \mathfrak{u} \in \Upsilon \cap \Omega.$$

Since $\mathcal{M}$ is continuous $\mathcal{M}(\beta_\sigma) \to \mathcal{M}(\mathfrak{u})$, so $\mathcal{M}(\mathfrak{u}) = \mathfrak{u}$. Hence $\mathfrak{u}$ is a fixed point of $\mathcal{M}$. If $\mathfrak{v}$ is any fixed point of $\mathcal{M}$, then $\mathcal{M}(\mathfrak{v}) = \mathfrak{v}$ implies that $\mathfrak{v} \in \Upsilon \cap \Omega$ and

$$\tau \leq \mathcal{F}(\varrho(\mathcal{M}(\mathfrak{u}), \mathcal{M}(\mathfrak{v})) - \mathcal{F}(\varrho(\mathfrak{u}, \mathfrak{v})) = 0,$$

which is absurd. Hence $\mathfrak{u} = \mathfrak{v}$. □

Example 3.6. Let $\Upsilon = [0,1]$ and $\Omega = [1,2]$ be equipped with $\varrho(\aleph, \beta) = |\aleph - \beta|$ for all $\aleph \in \Upsilon$ and $\beta \in \Omega$. Then, $(\Upsilon, \Omega, \varrho)$ is a complete Bi-polar metric space. Define $\mathcal{M} : \Upsilon \cup \Omega \rightrightarrows \Upsilon \cup \Omega$ by

$$\mathcal{M}(\aleph) = \frac{\aleph + 4}{5},$$

for all $\aleph \in \Upsilon \cup \Omega$. Let $\aleph \in \Upsilon$ and $\beta \in \Omega$. Now, let us consider the mapping $\mathcal{F}$ defined by $\mathcal{F}(\delta) = \ln \delta$. Let $\tau > 0$. Note that if $\varrho(\mathcal{M}\aleph, \mathcal{M}\beta) > 0$ implies

$$\tau + \mathcal{F}(\varrho(\mathcal{M}\aleph, \mathcal{M}\beta)) \leq \mathcal{F}(\varrho(\aleph, \beta)), \ \forall \aleph \in \Upsilon \textit{ and } \beta \in \Omega,$$

is equivalent to

$$\varrho(\mathcal{M}\aleph, \mathcal{M}\beta) \leq e^{-\tau}(\varrho(\aleph, \beta)), \ \forall \aleph \in \Upsilon \textit{ and } \beta \in \Omega.$$

Then

$$\varrho(\mathcal{M}\aleph, \mathcal{M}\beta) = \varrho\Big(\frac{\aleph + 4}{5}, \frac{\beta + 4}{5}\Big) = \Big|\frac{\aleph}{5} - \frac{\beta}{5}\Big| \leq \frac{1}{3}\varrho(\aleph, \beta),$$

which implies that

$$\varrho(\mathcal{M}\aleph, \mathcal{M}\beta) \leq e^{-\tau}(\varrho(\aleph, \beta)), \ \forall \aleph \in \Upsilon \textit{ and } \beta \in \Omega. \tag{3.21}$$

Therefore, all the conditions of Theorem 3.8 are satisfied. Hence we can conclude that $\mathcal{M}$ has a unique fixed point, which is $\aleph = 1$.

Now we examine the existence and unique solution to an integral equation as an application of Theorem 3.3.

Theorem 3.7. *Let us consider the integral equation*

$$\aleph(\delta) = \mathfrak{b}(\delta) + \int_{\mathcal{E}_1 \mathcal{E}_2} \mathcal{G}(\delta, \eta, \aleph(\eta)) d\eta, \ \delta \in \mathcal{E}_1 \cup \mathcal{E}_2,$$

where $\mathcal{E}_1 \cup \mathcal{E}_2$ is a Lebesgue measurable set. Suppose

(T1) $\mathcal{M}: (\mathcal{E}_1^2 \cup \mathcal{E}_2^2) \times [0, +\infty) \to [0, +\infty)$ *and* $b \in L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2)$*;*

(T2) There is a continuous function $\theta : \mathcal{E}_1^2 \cup \mathcal{E}_2^2 \to [0, +\infty)$ *and* $\tau > 0$ *such that*

$$|\mathcal{G}(\delta, \eta, \aleph(\eta)) - \mathcal{G}(\delta, \eta, \beta(\eta)| \leq e^{-\tau}|\theta(\delta, \eta)|(|\aleph(\eta) - \beta(\eta)|,$$

for $\delta, \eta \in \mathcal{E}_1^2 \cup \mathcal{E}_2^2$*;*

(T3) $\| \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \theta(\delta, \eta) d\eta \|_\infty \leq 1$ *i.e* $\sup_{\delta \in \mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\theta(\delta, \eta)| d\eta \leq 1$.

Then the integral equation has a unique solution in $L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2)$*.*

Proof. Let $\Upsilon = L^\infty(\mathcal{E}_1)$ and $\Omega = L^\infty(\mathcal{E}_2)$ be two normed linear spaces, where $\mathcal{E}_1, \mathcal{E}_2$ are Lebesgue measurable sets and $m(\mathcal{E}_1 \cup \mathcal{E}_2) < \infty$.

Consider ϱ: $\Upsilon \times \Omega \to [0, +\infty)$ to be defined by $\varrho(\aleph, \beta) = \|\aleph - \beta\|_\infty$ for all $(\aleph, \beta) \in \Upsilon \times \Omega$. Then $(\Upsilon, \Omega, \varrho)$ is a complete Bi-polar metric space.

Define the covariant mapping $\mathcal{M} : L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2) \to L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2)$ by

$$\mathcal{M}(\aleph(\delta)) = \mathfrak{b}(\delta) + \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\delta, \eta, \aleph(\eta)) d\eta, \ \ \delta \in \mathcal{E}_1 \cup \mathcal{E}_2.$$

Now, we have

$$\begin{aligned}
\varrho(\mathcal{M}\aleph(\delta), \mathcal{M}\beta(\delta)) &= \|\mathcal{M}\aleph(\delta) - \mathcal{M}\beta(\delta)\| \\
&= \left| \mathfrak{b}(\delta) + \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\delta, \eta, \aleph(\eta)) d\eta - \left(\mathfrak{b}(\delta) + \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\delta, \eta, \aleph(\eta)) d\eta \right) \right| \\
&\leq \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\mathcal{G}(\delta, \eta, \aleph(\eta)) - \mathcal{G}(\delta, \eta, \beta(\eta))| d\eta \\
&\leq \int_{\mathcal{E}_1 \cup \mathcal{E}_2} e^{-\tau} |\theta(\delta, \eta)| (|\aleph(\eta) - \beta(\eta)|) d\eta \\
&\leq e^{-\tau} (\|\aleph(\eta) - \beta(\eta)\|) \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\theta(\delta, \eta)| d\eta \\
&\leq e^{-\tau} (\|\aleph(\eta) - \beta(\eta)\|) \sup_{\delta \in \mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\theta(\delta, \eta)| d\eta \\
&\leq e^{-\tau} (\|\aleph(\eta) - \beta(\eta)\|) \\
&= e^{-\tau} \varrho(\aleph, \beta).
\end{aligned}$$

Hence, all the hypothesis of a Theorem 3.3 are satisfied with $\mathcal{F}(\delta) = \ln \delta$ and consequently, the integral equation has a unique solution. □

Here we present a fixed point theorem on Bi-polar metric space using simulation function.

Theorem 3.8. *Let* $(\Upsilon, \Omega, \varrho)$ *be a complete Bi-polar metric space and given a* Θ*-contraction* $\mathcal{M}$*:* $(\Upsilon, \Omega, \varrho) \rightrightarrows (\Upsilon, \Omega, \varrho)$*. Then the function* $\mathcal{M}$*:* $\Upsilon \cup \Omega \to \Upsilon \cup \Omega$ *has a unique fixed point.*

Proof. Let $\aleph_0 \in \Upsilon$ and $\beta_0 \in \Omega$. For each $\sigma \in \mathbb{N}$, define $\mathcal{M}(\aleph_\sigma) = \aleph_{\sigma+1}$ and $\mathcal{M}(\beta_\sigma) = \beta_{\sigma+1}$. Then $(\{\aleph_\sigma\}, \{\beta_\sigma\})$ is a bisequence on $(\Upsilon, \Omega, \varrho)$. Since $\mathcal{M}$ is a Θ-contraction, we have

$$\begin{aligned}0 &\leq \xi(\varrho(\mathcal{M}\aleph_{\sigma-1}, \mathcal{M}\beta_{\sigma-1}), \varrho(\aleph_{\sigma-1}, \beta_{\sigma-1}))\\ &= \xi(\varrho(\aleph_{\sigma}, \beta_{\sigma}), \varrho(\aleph_{\sigma-1}, \beta_{\sigma-1}))\\ &< \varrho(\aleph_{\sigma-1}, \beta_{\sigma-1}) - \varrho(\aleph_{\sigma}, \beta_{\sigma}),\end{aligned}$$

which implies that

$$\varrho(\aleph_{\sigma}, \beta_{\sigma}) < \varrho(\aleph_{\sigma-1}, \beta_{\sigma-1}), \text{ for all } \sigma \in \mathbb{N}.$$

Therefore, the sequence $\{\varrho(\aleph_{\sigma-1}, \beta_{\sigma-1})\}$ is nonincreasing bisequence and so we can find $\mathfrak{r} \geq 0$ satisfying $\lim_{\sigma\to+\infty} \varrho(\aleph_{\sigma-1}, \beta_{\sigma-1}) = \mathfrak{r}$. Assume that $\mathfrak{r} \neq 0$. Let $\delta_{\sigma} = \varrho(\aleph_{\sigma}, \beta_{\sigma})$ and $\eta_{\sigma} = \varrho(\aleph_{\sigma-1}, \beta_{\sigma-1})$, then $\lim_{\sigma\to+\infty} \delta_{\sigma} = \lim_{\sigma\to+\infty} \eta_{\sigma} = \mathfrak{r} > 0$ and $\delta_{\sigma} < \eta_{\sigma}$, for all $\sigma \in \mathbb{N}$. Therefore,

$$0 \leq \lim_{\sigma\to+\infty} \sup \xi(\delta_{\sigma}, \eta_{\sigma}) < 0,$$

which is a contradiction. Thus,

$$\lim_{\sigma\to+\infty} \varrho(\aleph_{\sigma-1}, \beta_{\sigma-1}) = 0.$$

Since $\mathcal{M}$ is a Θ-contraction, we have

$$\begin{aligned}0 &\leq \xi(\varrho(\mathcal{M}\aleph_{\zeta-1}, \mathcal{M}\beta_{\sigma-1}), \varrho(\aleph_{\zeta-1}, \beta_{\sigma-1}))\\ &= \xi(\varrho(\aleph_{\zeta}, \beta_{\sigma}), \varrho(\aleph_{\zeta-1}, \beta_{\sigma-1}))\\ &< \varrho(\aleph_{\zeta-1}, \beta_{\sigma-1}) - \varrho(\aleph_{\zeta}, \beta_{\sigma}),\end{aligned}$$

which implies that

$$\varrho(\aleph_{\zeta}, \beta_{\sigma}) < \varrho(\aleph_{\zeta-1}, \beta_{\sigma-1}), \text{ for all } \sigma \in \mathbb{N}.$$

Therefore, the sequence $\{\varrho(\aleph_{\zeta-1}, \beta_{\sigma-1})\}$ is nonincreasing bisequence and so we can find $\mathfrak{r} \geq 0$ satisfying $\lim_{\sigma\to+\infty} \varrho(\aleph_{\zeta-1}, \beta_{\sigma-1}) = \mathfrak{r}$. Assume that $\mathfrak{r} \neq 0$. Let $\delta_{\sigma} = \varrho(\aleph_{\zeta}, \beta_{\sigma})$ and $\eta_{\sigma} = \varrho(\aleph_{\zeta-1}, \beta_{\sigma-1})$, then $\lim_{\sigma\to+\infty} \delta_{\sigma} = \lim_{\sigma\to+\infty} \eta_{\sigma} = \mathfrak{r} > 0$ and $\delta_{\sigma} < \eta_{\sigma}$, for all $\sigma \in \mathbb{N}$. Therefore,

$$0 \leq \lim_{\sigma\to+\infty} \sup \xi(\delta_{\sigma}, \eta_{\sigma}) < 0,$$

which is a contradiction. Thus,

$$\lim_{\sigma\to+\infty} \varrho(\aleph_{\zeta-1}, \beta_{\sigma-1}) = 0.$$

Since $\mathcal{M}$ is a Θ-contraction, we have

$$\begin{aligned}0 &\leq \xi(\varrho(\mathcal{M}\aleph_{\zeta-1}, \mathcal{M}\beta_{\sigma}), \varrho(\aleph_{\zeta-1}, \beta_{\sigma}))\\ &= \xi(\varrho(\aleph_{\zeta}, \beta_{\sigma+1}), \varrho(\aleph_{\zeta-1}, \beta_{\sigma}))\\ &< \varrho(\aleph_{\zeta-1}, \beta_{\sigma}) - \varrho(\aleph_{\zeta}, \beta_{\sigma+1}),\end{aligned}$$

which implies that

$$\varrho(\aleph_{\zeta},\beta_{\sigma+1}) < \varrho(\aleph_{\zeta-1},\beta_{\sigma}), \text{ for all } \sigma \in \mathbb{N}.$$

Therefore, the sequence $\{\varrho(\aleph_{\zeta-1},\beta_{\sigma})\}$ is nonincreasing bisequence and so we can find $\mathfrak{r} \geq 0$ satisfying $\lim\limits_{\sigma\to+\infty} \varrho(\aleph_{\zeta-1},\beta_{\sigma}) = \mathfrak{r}$. Assume that $\mathfrak{r} \neq 0$. Let $\delta_{\sigma} = \varrho(\aleph_{\zeta},\beta_{\sigma+1})$ and $\eta_{\sigma} = \varrho(\aleph_{\zeta-1},\beta_{\sigma})$, then $\lim\limits_{\sigma\to+\infty} \delta_{\sigma} = \lim\limits_{\sigma\to+\infty} \eta_{\sigma} = \mathfrak{r} > 0$ and $\delta_{\sigma} < \eta_{\sigma}$, for all $\sigma \in \mathbb{N}$. Therefore,

$$0 \leq \lim_{\sigma\to+\infty} \sup \xi(\delta_{\sigma}, \eta_{\sigma}) < 0,$$

which is a contradiction. Thus,

$$\lim_{\sigma\to+\infty} \varrho(\aleph_{\zeta-1},\beta_{\sigma}) = 0.$$

Now, we show that $(\{\aleph_{\sigma}\}, \{\beta_{\sigma}\})$ is a Cauchy bisequence. On the contrary, assume that $(\{\aleph_{\sigma}\}, \{\beta_{\sigma}\})$ is not a Cauchy bisequence. Then, there exists an $\epsilon > 0$ for which we can find two subsequences $\{\aleph_{\sigma_{\mathfrak{k}}}\}$ of $\{\aleph_{\sigma}\}$ and $\{\beta_{\zeta_{\mathfrak{k}}}\}$ of $\{\beta_{\zeta}\}$ such that $\sigma_{\mathfrak{k}} > \zeta_{\mathfrak{k}} > \mathfrak{k}$, for all $\mathfrak{k} \in \mathbb{N}$ and

$$\varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}}}) \geq \epsilon. \tag{3.22}$$

Suppose that $\sigma_{\mathfrak{k}}$ is the least integer exceeding $\zeta_{\mathfrak{k}}$ satisfying inequality (3.22). Then,

$$\varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}-1}}) < \epsilon. \tag{3.23}$$

Using (3.22),(3.23) and (c), we obtain

$$\begin{aligned}\epsilon &\leq \varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}}}) \\ &\leq \varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}}-1}) + \varrho(\aleph_{\zeta_{\mathfrak{k}}-1},\beta_{\sigma_{\mathfrak{k}}-1}) + \varrho(\aleph_{\zeta_{\mathfrak{k}}-1},\beta_{\sigma_{\mathfrak{k}}}).\end{aligned}$$

As $\mathfrak{k} \to +\infty$, we obtain

$$\lim_{\mathfrak{k}\to+\infty} \varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}}}) = \epsilon.$$

Since $\mathcal{M}$ is a Θ-contraction, we have

$$\begin{aligned}0 &\leq \xi(\varrho(\mathcal{M}\aleph_{\zeta_{\mathfrak{k}}-1}, \mathcal{M}\beta_{\sigma_{\mathfrak{k}}-1}), \varrho(\aleph_{\zeta_{\mathfrak{k}}-1},\beta_{\sigma_{\mathfrak{k}}-1})) \\ &= \xi(\varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}}}), \varrho(\aleph_{\zeta_{\mathfrak{k}}-1},\beta_{\sigma_{\mathfrak{k}}-1})) \\ &< \varrho(\aleph_{\zeta_{\mathfrak{k}}-1},\beta_{\sigma_{\mathfrak{k}}-1}) - \varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}}}),\end{aligned}$$

which implies that

$$\varrho(\aleph_{\zeta_{\mathfrak{k}}},\beta_{\sigma_{\mathfrak{k}}}) < \varrho(\aleph_{\zeta_{\mathfrak{k}}-1},\beta_{\sigma_{\mathfrak{k}}-1}), \text{ for all } \sigma_{\mathfrak{k}} \in \mathbb{N}.$$

As $\sigma \to +\infty$, we obtain

$$\epsilon < 0.$$

Therefore $(\{\aleph_\sigma\}, \{\beta_\sigma\})$ is a Cauchy bisequence. Since $(\Upsilon, \Omega, \varrho)$ is complete, then $\{\aleph_\sigma\} \to \rho$ and $\{\beta_\sigma\} \to \rho$ where $\rho \in \Upsilon \cap \Omega$. Since $\mathcal{M}$ is continuous, $\mathcal{M}(\beta_\sigma) \to \mathcal{M}(\rho)$, so $\mathcal{M}(\rho) = \rho$. Hence ρ is a fixed point of $\mathcal{M}$. Next, we prove that $\mathcal{M}$ has a unique fixed point. Suppose not, assume that there are two fixed points such that $\varrho(\rho, \upsilon) = \varrho(\mathcal{M}(\rho), \mathcal{M}(\upsilon)) > 0$. Since $\mathcal{M}$ is a Θ-contraction, we have

$$\begin{aligned}0 &\leq \xi(\varrho(\mathcal{M}(\rho), \mathcal{M}(\upsilon)), \varrho(\rho, \upsilon)) \\ &= \xi(\varrho(\rho, \upsilon), \varrho(\rho, \upsilon)) \\ &< \varrho(\rho, \upsilon) - \varrho(\rho, \upsilon).\end{aligned}$$

Therefore, $\mathcal{M}$ has a unique fixed point. □

Example 3.9. Let $\Upsilon = [0, 1]$ and $\Omega = [1, 2]$ be equipped with $\varrho(\aleph, \beta) = |\aleph - \beta|$ for all $\aleph \in \Upsilon$ and $\beta \in \Omega$. Then, $(\Upsilon, \Omega, \varrho)$ is a complete Bi-polar metric space. Define $\mathcal{M} : \Upsilon \cup \Omega \rightrightarrows \Upsilon \cup \Omega$ by

$$\mathcal{M}(\aleph) = \frac{\aleph + 4}{5},$$

for all $\aleph \in \Upsilon \cup \Omega$. We now show that Θ-contraction with respect to $\xi \in \Theta$, where

$$\xi(\delta, \eta) = \frac{\eta}{\eta + 1} - \delta,$$

for all $\delta, \eta \in [0, +\infty)$. Then

$$\begin{aligned}\xi(\varrho(\mathcal{M}\aleph, \mathcal{M}\beta), \varrho(\aleph, \beta)) &= \frac{\varrho(\aleph, \beta)}{1 + \varrho(\aleph, \beta)} - \varrho(\mathcal{M}\aleph, \mathcal{M}\beta) \\ &= \frac{|\aleph - \beta|}{5 + |\aleph - \beta|} - \left|\frac{\aleph}{5} - \frac{\beta}{5}\right| \geq 0.\end{aligned}$$

Therefore, conditions of Theorem 3.8 are fulfilled and $\mathcal{M}$ has a unique fixed point $\aleph = 1$.

Now we examine the application of the derived result in Theorem 3.8.

Theorem 3.10. *Let*

$$\aleph(\delta) = \mathrm{b}(\delta) + \int_{\mathcal{A}_1 \cup \mathcal{A}_2} \mathcal{H}(\delta, \eta, \aleph(\eta)) d\eta, \ \ \delta \in \mathcal{A}_1 \cup \mathcal{A}_2,$$

where $\mathcal{A}_1 \cup \mathcal{A}_2$ is a Lebesgue measurable set. Suppose

(T1) $\mathcal{H} : (\mathcal{A}_1^2 \cup \mathcal{A}_2^2) \times [0, +\infty) \to [0, +\infty)$ and $b \in L^\infty(\mathcal{A}_1) \cup L^\infty(\mathcal{A}_2)$;

(T2) There is a continuous function $\theta : \mathcal{A}_1^2 \cup \mathcal{A}_2^2 \to [0, +\infty)$ and $\lambda \in (0, 1)$ such that

$$|\mathcal{H}(\delta, \eta, \aleph(\eta)) - \mathcal{H}(\delta, \eta, \beta(\eta)| \leq \lambda |\theta(\delta, \eta)| (|\aleph(\eta) - \beta(\eta)|,$$

for $\delta, \eta \in \mathcal{A}_1^2 \cup \mathcal{A}_2^2$;

(T3) $\| \int_{\mathcal{A}_1 \cup \mathcal{A}_2} \theta(\delta, \eta) d\eta \|_\infty \leq 1$ i.e $\sup_{\delta \in \mathcal{A}_1 \cup \mathcal{A}_2} \int_{\mathcal{A}_1 \cup \mathcal{A}_2} |\theta(\delta, \eta)| d\eta \leq 1$.

Then the integral equation has a unique solution in $L^{\infty}(\mathcal{A}_1) \cup L^{\infty}(\mathcal{A}_2)$.

Proof. Let $\Upsilon = L^{\infty}(\mathcal{A}_1)$ and $\Omega = L^{\infty}(\mathcal{A}_2)$ be two normed linear spaces, where $\mathcal{A}_1, \mathcal{A}_2$ are Lebesgue measurable sets and $m(\mathcal{A}_1 \cup \mathcal{A}_2) < \infty$.

Consider $\varrho : \Upsilon \times \Omega \to [0, +\infty)$ defined by $\varrho(\aleph, \beta) = \|\aleph - \beta\|_{\infty} = \sup_{\delta \in \mathcal{A}_1 \cup \mathcal{A}_2} |\aleph(\delta) - \beta(\delta)|$ for all $(\aleph, \beta) \in \Upsilon \times \Omega$. Then $(\Upsilon, \Omega, \varrho)$ is a complete Bi-polar metric space.

Define $\mathcal{M} : L^{\infty}(\mathcal{A}_1) \cup L^{\infty}(\mathcal{A}_2) \rightrightarrows L^{\infty}(\mathcal{A}_1) \cup L^{\infty}(\mathcal{A}_2)$ by

$$\mathcal{M}(\aleph(\delta)) = \mathfrak{b}(\delta) + \int_{\mathcal{A}_1 \cup \mathcal{A}_2} \mathcal{H}(\delta, \eta, \aleph(\eta)) d\eta, \;\; \delta \in \mathcal{A}_1 \cup \mathcal{A}_2.$$

Now,

$$\begin{aligned}
\varrho(\mathcal{M}\aleph(\delta), \mathcal{M}\beta(\delta)) &= \|\mathcal{M}\aleph(\delta) - \mathcal{M}\beta(\delta)\| \\
&= \left| \mathfrak{b}(\delta) + \int_{\mathcal{A}_1 \cup \mathcal{A}_2} \mathcal{H}(\delta, \eta, \aleph(\eta)) d\eta - \left(\mathfrak{b}(\delta) + \int_{\mathcal{A}_1 \cup \mathcal{A}_2} \mathcal{H}(\delta, \eta, \aleph(\eta)) d\eta \right) \right| \\
&\leq \int_{\mathcal{A}_1 \cup \mathcal{A}_2} |\mathcal{H}(\delta, \eta, \aleph(\eta)) - \mathcal{H}(\delta, \eta, \beta(\eta))| d\eta \\
&\leq \int_{\mathcal{A}_1 \cup \mathcal{A}_2} \lambda |\theta(\delta, \eta)| (|\aleph(\eta) - \beta(\eta)|) d\eta \\
&\leq \lambda (\|\aleph(\eta) - \beta(\eta)\|) \int_{\mathcal{A}_1 \cup \mathcal{A}_2} |\theta(\delta, \eta)| d\eta \\
&\leq \lambda (\|\aleph(\eta) - \beta(\eta)\|) \sup_{\delta \in \mathcal{A}_1 \cup \mathcal{A}_2} \int_{\mathcal{A}_1 \cup \mathcal{A}_2} |\theta(\delta, \eta)| d\eta \\
&\leq \lambda \|\aleph(\eta) - \beta(\eta)\| \\
&= \lambda \varrho(\aleph, \beta),
\end{aligned}$$

which implies that

$$\lambda \varrho(\aleph, \beta) - \varrho(\mathcal{M}\aleph(\delta), \mathcal{M}\beta(\delta)) \geq 0.$$

We consider the simulation function as $\xi(\delta, \eta) = \lambda \eta - \delta$. Then

$$\xi(\varrho(\mathcal{M}\aleph(\delta), \mathcal{M}\beta(\delta)), \varrho(\aleph, \beta)) \geq 0.$$

Therefore, all the hypothesis of a Theorem 3.8 are fulfilled. Hence, the integral equation has a unique solution. □

4. Conclusions

In the present work, we established fixed point results using $\mathcal{F}$-contraction and simulation functions. The derived results have been supported with suitable example and application to find analytical solution of integral equation. Our results are extensions/generalisation of some proven results in the past. Readers can explore extending the results in the setting of Bi-polar p-metric space, complex metric space etc.

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Conflict of interests

The authors declare no conflicts of interest.

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